On algebraic K-functors of crossed group rings and its applications

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Abstract

Let $R[\pi,\sigma,\rho]$ be a crossed group ring. An induction theorem is proved for the functor $G_0^R(R[\pi,\sigma,\rho])$ and the Swan-Gersten higher algebraic K-functors $K_i(R[\pi,\sigma,\rho])$. Using this result, a theorem on reduction is proved for the discrete normalization ring R with the field of quotients K: If P and Q are finitely generated $R[\pi,\sigma,\rho]$-projective modules and $K\bigotimes_R P \simeq K\bigotimes_R Q$ as $K[\pi,\sigma,\rho]$-modules, then $P \simeq Q$. Under some restrictions on $n=(\pi:1)$ it is shown that finitely generated $R[\pi,\sigma,\rho]$-projective modules are decomposed into the direct sum of left ideals of the ring $R[\pi,\sigma,\rho]$. More stronger results are proved when $\sigma = id$.

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1 Introduction

In 1960, R. G. Swan proved [1] that for a Dedekind domain R of characteristic 0 and a finite group π any finitely generated projective $R[\pi]$-module is the direct sum of left ideals of $R[\pi]$ if no prime divider of $(\pi:1)$ is invertible in R. In [1] it was also proved that this direct sum may be replaced by the direct sum of a free $R[\pi]$ module and an ideal of $R[\pi]$, which generalizes the properties of projective modules over Dedekind domains. Swan's results were based on two theorems, each having an independent value: on the induction theorem for the functors $G_0^R(R\pi)$ and $K_0(R\pi)$, and on the "reduction" theorem.

In 1968, T.Y. Lam [2] proved an induction theorem for $K_1(R\pi)$ and in 1973 A.I. Nemytov [3] proved that $K_m(R\pi)$, $m\geq 2$, functors are Frobenius modules on $G_0^R(R[\pi])$ and that the induction theorem is valid for Swan-Gersten algebraic K-functors ([4], [5]) $K_m(R\pi)$, $m \geq 2$. Induction theorems for some kinds of algebraic K-functors of group rings were obtained in 1986 by K. Kawakubo [6] and in 2005 by A. Bartels and W. Luck [7].

In the first section of this paper the induction theorem is generalized for Swan-Gersten algebraic K-functors $K_m(R[\pi,\sigma,\rho])$ (Theorem 2.4) for a crossed group ring $R[\pi,\sigma,\rho]$. In the second section, using the induction theorem for $K_0(R[\pi,\sigma,\rho])$ the "reduction" theorem is proved for finitely generated projective $R[\pi,\sigma,\rho]$-modules if R is a discrete valuation ring (Theorem 2.1). In Section 3 we prove the theorems on the structure of finitely generated projective $R[\pi,\sigma,\rho]$- and $R[\pi,\rho]$-modules which generalize Swan's theorem.

Let R be a commutative ring with identity, π a group, $\sigma:\pi\to AutR$ a group morphism, $U(R)$ a set of invertible elements of R and $\rho:\pi\times\pi\to U(R)$ be such a mapping, that

$$\rho(x,y)\rho(xy,z)=\rho(y,z)^x\rho(x,yz).$$

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Then a crossed group ring $R[\pi,\sigma,\rho]$ (see [8], [9]) is a free R-module with the set of free generators π and with multiplication

$$r_1\overline{x_1}r_2\overline{x_2} = r_1 r_2^{x_1}\rho(x_1,x_2)\overline{x_1x_2},$$

where $\overline{x}$ is the image of $x\in\pi$ via a mapping $\pi\to R[\pi,\sigma,\rho]$ and $r_1,r_2\in R$. If $\sigma(\pi)=id$ and $\rho\sim 1$ (i.e. $\rho(x,y)=\alpha(x)\alpha(y)\alpha(xy)^{-1}$ for some $\alpha:\pi\to U(R)$), then $R[\pi,\sigma,\rho]\simeq R[\pi]$.

In this paper all modules are left modules, $\underline{M}(A)$ and $\underline{P}(A)$ denote respectively the categories of finitely generated A-modules and finitely generated projective A-modules (A is a ring); $\underline{M}^R(R[\pi,\sigma,\rho])$ is the category of finitely generated R-projective $R[\pi,\sigma,\rho]$-modules; $G_0^R(R[\pi,\sigma,\rho])$ is a Grothendieck group of the category $\underline{M}^R R[\pi,\sigma,\rho]$.

Further, π will always be the finite group.

The main results of the paper are Theorems 3.1 and 3.2. These theorems were proved by author in the particular case when $\rho\sim 1$ in [10], [11] and [12]; a general case for any ρ was announced in [12] and its proof was the subject of the authors doctoral thesis in 1981. These theorems are similar to the results of Kawakubo [6] which were obtained later in 1986 for some kinds of algebraic K-functors of group rings and particular cases of crossed group rings.

2 Inductive theorems

Let $\underline{G}$ be a category, $\underline{Rings}$ a category of rings and $G:\underline{G}\to\underline{Rings}$ a contravariant functor. Suppose to each morphism $i:\pi^{'}\to\pi$ in $\underline{G}$ there corresponds a morphism $i_*:G(\pi^{'})\to G(\pi)$ in $\underline{Rings}$ such that $Id_*=Id$ and $(ij)_*=i_*j_*$ whenever ij makes sense in $\underline{G}$. Let us denote $i^*=G(i):G(\pi)\to G(\pi^{'})$. The functor G is called a **Frobenius functor** [2] if it satisfies the Frobenius reciprocity formula

$$i_*(i^*a\cdot b)=a\cdot i_*b.$$

Let $\underline{Ab}$ be a category of commutative groups. A contravariant functor $K:\underline{G}\to\underline{Ab}$ is called a **Frobenius module** [2] on the Frobenius functor G if it satisfies the following conditions:

(i) $K(\pi)$ is a module over $G(\pi)$.

(ii) For each morphism of groups $i:\pi^{'}\to\pi$ there exists a morphism $i_\#:K(\pi^{'})\to K(\pi)$ (whenever ij makes sense) such that

$$(ij)_\# = i_\# j_\#. \tag{2.1}$$

(iii) i_*, i^*, $i_\#$ and $i^\#$ are related to each other by the relations

$$\begin{array}{rcl} i_\#(y\cdot i^\#(a) & = & i_*(y)\cdot a,\\ i_\#(i^*(x)\cdot b) & = & x\cdot i_\#(b), \end{array} \tag{2.2}$$

where $i^\#=K(i)$, $x\in G(\pi)$, $y\in G(\pi^{'})$, $a\in K(\pi)$, $b\in G(\pi^{'})$.

Let $\underline{G}(\pi)$ denote a category whose objects are all subgroups $\pi^{'}\subseteq\pi$ and morphisms are monomorphisms $i:\pi^{'}\to\pi^{''}$. Then the functors $G_0^S(S[-])$ and $K_m(R[-,\sigma,\rho])$ (σ and ρ are defined respectively on π and $\pi\times\pi$ and are fixed for the category $\underline{G}(\pi)$) are contavariant functors from the category $\underline{G}(\pi)$ to the categories $\underline{Rings}$ and $\underline{Ab}$ respectively.

It is known [1] that $G_0^S(S[\pi])$ is a Frobenius functor.

Let us denote $R^\pi=\{r\in R|(\forall x\in\pi)r^x=r\}$.

Theorem 2.1. Let R^π be an algebra over the commutative ring S with identity. Then the functors $G_0^R(R[-,\sigma,\rho])$ and $K_m(R[-,\sigma,\rho])$, $m = 0, 1, ...,$ are Frobenius modules on the Frobenius functor $G_0^S(S[\pi])$.

Let us remark that in [10] instead of the functor $G_0^R(R[-,\sigma,\rho])$ it is considered a functor $G_0^{R^\pi}(R[-,\sigma,\rho])$ - the Grothendieck group of R^π-finitely generated and R^π-projective $R[\pi,\sigma,\rho]$-modules.

To prove Theorem 2.1 we need some propositions.

If R^π is an algebra over S, then R is an S-algebra by the action $sr = (s \cdot 1)r$, $1 \in R$. Let us construct the morphisms of rings

$$\alpha_1 : R[\pi,\sigma,\rho] \to S[\pi] \otimes_S R[\pi,\sigma,\rho],$$

$$\alpha_2 : R[\pi,\sigma,\rho] \to R[\pi,\sigma,\rho] \otimes_S S[\pi]$$

in this way: $\alpha_1(r\bar{x}) = \bar{x} \otimes r\bar{x}$, $\alpha_2(r\bar{x}) = r\bar{x} \otimes \bar{x}$. Then for any $S[\pi]$-module M and $R[\pi,\sigma,\rho]$-module P the modules $M \otimes_S P$ and $P \otimes_S M$ become $R[\pi,\sigma,\rho]$-modules via the action

$$r\bar{x}(m \otimes p) = \alpha_1(r\bar{x})(m \otimes p) = \bar{x}m \otimes r\bar{x}p,$$

$$r\bar{x}(p \otimes m) = \alpha_2(r\bar{x})(m \otimes p) = r\bar{x}p \otimes \bar{x}m.$$

It is clear that the $R[\pi,\sigma,\rho]$-modules $M \otimes_S P$ and $P \otimes_S M$ are isomorphic.

Proposition 2.2. If a $S[\pi]$-module M is S-projective and a $R[\pi,\sigma,\rho]$-module P is $R[\pi,\sigma,\rho]$-projective, then $M \otimes_S P$ is $R[\pi,\sigma,\rho]$-projective.

Proof. If M is a free S-module, then $M \otimes_S R[\pi,\sigma,\rho] \simeq \oplus_{x\in\pi} M \otimes R\bar{x}$ as R-modules. But if $\{e_\alpha\}$ is a S-basis of M, then $\{\bar{x}e_\alpha\}$ is also a free S-basis because x induces an automorphism on M. Therefore $\{e_\alpha \otimes 1\}$ is a free $R[\pi,\sigma,\rho]$-basis of $M \otimes_S R[\pi,\sigma,\rho]$.

Suppose $P^{'}$ is a $R[\pi,\sigma,\rho]$-module,such that $P \oplus P^{'}$ is a free $R[\pi,\sigma,\rho]$ -module, and M is such a S-module that $M \oplus M^{'}$ is a free S-module. If we define the action of $S\pi$ on M as $s\bar{x}(m) = sm$, then $(M \oplus M^{'}) \otimes_S (P \oplus P^{'})$ will be a free $R[\pi,\sigma,\rho]$-module and since

$$(M \oplus M^{'}) \otimes_S (P \oplus P^{'}) \simeq (M \otimes_S P) \oplus (M^{'} \otimes_S P) \oplus (M \otimes_S P^{'}) \oplus (M^{'} \otimes_S P^{'}),$$

$(M \otimes_S P)$ will be $R[\pi,\sigma,\rho]$-projective. Q.E.D.

Proposition 2.3. Let R^π be an algebra over S, $\pi^{'} \subseteq \pi$ a subgroup, $M \in S\pi - \underline{Mod}$, $M^{'} \in S\pi^{'} - \underline{Mod}$, $P \in R[\pi,\sigma,\rho] - \underline{Mod}$ and $P^{'} \in R[\pi^{'},\sigma,\rho] - \underline{Mod}$. Then there exist isomorphisms of $R[\pi,\sigma,\rho]$-modules

$$b) R[\pi,\sigma,\rho] \otimes_{R[\pi^{'},\sigma,\rho]} (M^{'} \otimes_S P) \simeq (R[\pi,\sigma,\rho] \otimes_{R[\pi^{'},\sigma,\rho]} M^{'}) \otimes_S P. \quad (2.3)$$

$$a) R[\pi,\sigma,\rho] \otimes_{R[\pi^{'},\sigma,\rho]} (P^{'} \otimes_S M) \simeq (R[\pi,\sigma,\rho] \otimes_{R[\pi^{'},\sigma,\rho]} P^{'}) \otimes_S M, \quad (2.4)$$

The modules on the left side in the brackets and on the right sides are endowed with the structure of $R[\pi,\sigma,\rho]$-modules in the case a) by α_1 and in the case b) by α_2. The left sides are endowed with the structure of $R[\pi,\sigma,\rho]$-modules by multiplication by $R[\pi,\sigma,\rho]$.

Proof. In the case a) the isomorphism is constructed by the inverse mappings

$$r\bar{x}\otimes(p^{'}\otimes m)\to(r\bar{x}\otimes p^{'})\otimes\bar{x}m,$$

$$(r\bar{x}\otimes p^{'})\otimes m\to r\bar{x}\otimes(p^{'}\otimes\bar{x}^{-1}m).$$

In the case b) the isomorphism is constructed by the inverse mappings

$$r\bar{x}\otimes(m^{'}\otimes p)\to(r\bar{x}\otimes m^{'})\otimes r\bar{x}p,$$

$$(s\bar{x}\otimes m^{'})\otimes p\to r\bar{x}\otimes(m^{'}\otimes s\bar{x}^{-1}p).$$

Q.E.D.

Proof of Theorem 2.1. Let $\pi^{'}\subseteq\pi$ be a subgroup and let $i:\pi^{'}\to\pi$ be an imbedding. Let us consider the additive functors

$$I^{\#}:\underline{P}(R[\pi,\sigma,\rho])\to\underline{P}(R[\pi^{'},\sigma,\rho]),\;\; I^{\#}(P)=Res_{R[\pi,\sigma,\rho]}P;$$

$$I_{\#}:\underline{P}(R[\pi^{'},\sigma,\rho])\to\underline{P}(R[\pi,\sigma,\rho]),\;\; I_{\#}(P)=Ind(P^{'})=R[\pi,\sigma,\rho]\otimes_{R[\pi^{'},\sigma,\rho]}P^{'}.$$

For any module $M\in\underline{M}^{S}(S[\pi]$ assume

$$J_M(P)=M\otimes_S P,\;\; P\in\underline{P}(R[\pi,\sigma,\rho]),$$

$$J_M^{'}(P)=M\otimes_S N,\;\; N\in\underline{M}^{R}(R[\pi,\sigma,\rho]).$$

From Proposition 2.2 it follows that the functors $J_M(-)$ and $J_M^{'}(-)$ take the values in the categories $\underline{P}(R[\pi,\sigma,\rho])$ and $\underline{M}^{R}(R[\pi,\sigma,\rho])$, respectively. It is known that Swan-Gersten's K-functors $K_m(R[\pi,\sigma,\rho])$ and Quillen's K-functors are isomorphic. Therefore from [13] it follows that the functors $I^{\#}$, $I_{\#}$, J_M and $J_M^{'}$ define the morphisms of abelian groups

$$\begin{aligned} i_m^{\#}&:K_m(R[\pi,\sigma,\rho])\to K_m(R[\pi^{'},\sigma,\rho]),m\geq 0;\\ i_{\#}^{m}&:K_m(R[\pi^{'},\sigma,\rho])\to K_m(R[\pi,\sigma,\rho]),m\geq 0;\\ j_m&:G_0^S(S\pi)\otimes K_m(R[\pi^{'},\sigma,\rho])\to K_m(R[\pi^{'},\sigma,\rho]),m\geq 0;\\ j_0^{'}&:G_0^S(S\pi)\otimes G_0^R(R[\pi^{'},\sigma,\rho])\to G_0^R(R[\pi^{'},\sigma,\rho]),m\geq 0.\end{aligned}\tag{2.5}$$

Let us recall that the existence of morphisms i_m and $i_{\#}^{m}$ for $m\geq 2$ follows also from [2]. Using the of results from [13] and [3] it is easy to show that conditions (2.1) and (2.2) for morphisms (2.5) are consequences of isomorphisms (2.3) and (2.4).

Suppose M is some family of objects from $\underline{G}$. Let us denote for $\pi\in\underline{G}$

$$K(\pi)_M=\sum_{\pi^{'},i}\{Im(i_{\#}:K(\pi^{'})\to K(\pi))|i:\pi^{'}\to\pi,\pi^{'}\subseteq M\}.$$

Let $A\subseteq B$ be abelian groups. A natural number n is called an index of A in B if $nB\subseteq A$. Q.E.D.

Theorem 2.4. Let $c(\pi)$ be a set of all cyclic subgroups of the group π. Then $K_m(R[\pi,\sigma,\rho])_{c(\pi)}$ and $G_0^R(R[\pi,\sigma,\rho])_{c(\pi)}$ have the index n^2 in $K_m(R[\pi,\sigma,\rho])$ and $G_0^R(R[\pi,\sigma,\rho])$, respectively for all $m \geq 0$. If R^π is an algebra over the field, then n^2 may be replaced by n.

Proof. It is known that an index of $K(\pi)_M$ in $K(\pi)$ is equal to an index of $G(\pi)_M$ in $G(\pi)$ if K is a Frobenius module over a Frobenius functor G. Therefore by Theorem 1.1 it suffices to prove our statement for the functor $G_0^S(S\pi)$. Suppose in Theorem 1.1 we have $S = Z$. Then the first part of our statement follows from the fact that an index of $G_0^Z(Z\pi)_{c(\pi)}$ in $G_0^Z(Z\pi)$ is n^2 [1]. If $S = k$ is a field, in [1] it was proved that an index of $G_0(k\pi)_{c(\pi)}$ in $G_0(k\pi)$ is n. Q.E.D.

3 Reduction theorem

Let R be an integral domain with quotient field K and $R[\pi,\sigma,\rho]$ be the crossed group ring. It is clear that we may construct the crossed group ring $K[\pi,\sigma,\rho]$ where $\sigma : \pi \to Aut(K)$ is induced from $\sigma : \pi \to Aut(R)$ and $\rho : \pi \times \pi \to U(R) \subseteq K$.

Theorem 3.1. Let R be a discrete valued ring with quotient field K and $P, Q \in \underline{P}(R[\pi,\sigma,\rho])$. Suppose $K \otimes_R P \simeq K \otimes_R Q$ as $K[\pi,\sigma,\rho]$-modules. Then $P \simeq Q$ as $R[\pi,\sigma,\rho]$-modules.

Remark. *$K[\pi,\sigma,\rho]$ acts on $K \otimes_R P$ as $\bar{x}(\alpha \otimes p) = \alpha^x \otimes xp$.*

This theorem was proved by Swan [1] in the case $\sigma = id, \rho = id$, i.e. for group rings.

Let us first prove several necessary assertions.

Let us remark that if $\mathbf{m}$ is a maximal ideal in R, then it is possible to construct in a natural way the ring $R/\mathbf{m}[\pi,\sigma,\rho]$ from the ring $R[\pi,\sigma,\rho]$ because from the uniqueness of the maximal ideal it follows that $\sigma(\mathbf{m}) \subseteq \mathbf{m}$ for any $\sigma \in Aut(R)$.

Proposition 3.2. Let R be a discrete valued ring with a field of quotients K and $M_1, M_2 \in \underline{M}(R[\pi,\sigma,\rho])$. Suppose $K \otimes_R M_1 \simeq K \otimes_R M_2$ as $K[\pi,\sigma,\rho]$-modules. Then $[M_1/\mathbf{m}M_1] = [M_2/\mathbf{m}M_2]$ in $G_0^R(R/\mathbf{m}[\pi,\sigma,\rho])$.

Proof. Let t be a generator of the ideal $\mathbf{m}$. Then for any $x \in \pi$ there is $t^x = tu$ for some invertible $u \in R$. Therefore if $M \in \underline{M}(R[\pi,\sigma,\rho])$, then $tM \in \underline{M}(R[\pi,\sigma,\rho])$ and $K \otimes_R M \simeq K \otimes_R tM$ as $K[\pi,\sigma,\rho]$-modules. Indeed, if $m = tm' \in tM$, then $\bar{x}m = \bar{x}tm' = t^x\bar{x}m' = t(u\bar{x}m') = tm'' \in tM$. Similarly, if $t^n M_1 \subseteq M_2 \subseteq M_1$, then $M_1' = \{m \in M_1 | t^{n-1}m \in M_2$ is again a finitely generated $R[\pi,\sigma,\rho]$-module and $K \otimes_R M_1 \simeq K \otimes_R M_1'$ as $K[\pi,\sigma,\rho]$-modules.

Let $\mathbf{m}M_1 \subseteq M_2 \subseteq M_1$ (note that $\mathbf{m}M_1 = tRM_1 = tM_1$). Denote $T = M_2/M_1, \bar{M}_i = M_i/\mathbf{m}M_i$. It is clear that T is also the $R/\mathbf{m}[\pi,\sigma,\rho]$-module. Let us construct a sequence

$$0 \to T \xrightarrow{\psi} \overline{M}_2 \to T \xrightarrow{\alpha} \overline{M}_1 \to T \xrightarrow{\varphi} T \to 0,$$

where $\psi(m_1 + M_2) = tm_1 + tM_2$, $\alpha(e_2 + tM_2) = e_2 + tM_1$, $\varphi(e_1 + tM_1) = e_1 + M_2$. This sequence is exact, therefore $[M_1] = [M_2]$ in $G_0(R/\mathbf{m}[\pi,\sigma,\rho])$. In particular $[\overline{M}] = [\overline{tM}]$ in $G_0(R/\mathbf{m}[\pi,\sigma,\rho])$, because $tM \subseteq tM \subseteq M$. Taking this into account, because all modules are finitely generated, we may conclude that $M \subseteq M_1$ and there exists an integer $n > 0$, such that $t^n M_1 \subseteq M_2$. Indeed, let us identify M_1 and M_2 and the corresponding $R[\pi,\sigma,\rho]$-modules in $K \otimes_R M_1 \simeq K \otimes_R M_2$. Then there exists $k > 0$ such that $M_2 \subseteq t^k M_1$. There is: $[t^k\overline{M}_1] = [t^{k-1}\overline{M}_1] = ... = [t\overline{M}_1] = \overline{M}_1$. Therefore, without loss of generality, we may assume that $M_2 \subseteq M_1$. Analogously we proceed with respect to the second assumption. Hence $t^nM_1 \subseteq M_2 \subseteq M_1$. Let us denote $M_1^{'} = \{m \in M_1 | t^{n-1}m \in M_2\}$. There is $tM_1 \subseteq M_1^{'} \subseteq M_1$, $t^{n-1}M_1^{'} \subseteq M_2 \subseteq M_1^{'}$. The induction on n proves our statement because we have already proved that $[\overline{M_1^{'}}] = [\overline{M_1}]$. Q.E.D.

Corollary 3.3. There exists a homomorphism

$$G_0(K[\pi,\sigma,\rho]) \to G_0(R/\mathbf{m}[\pi,\sigma,\rho]).$$

Suppose $E \in \underline{M}(K[\pi,\sigma,\rho])$ and $E \simeq K \otimes_R M$, where $M \in \underline{M}(R[\pi,\sigma,\rho])$. Then from Proposition 3.2 follows that the mapping $[E] \to [M/\mathbf{m}M]$ is well defined.

Proposition 3.4. Suppose that the conditions of Theorem 3.1 are satisfied and the Cartan mapping

$$\chi : K_0(R/\mathbf{m}[\pi,\sigma,\rho]) \to G_0(R/\mathbf{m}[\pi,\sigma,\rho]),$$

which is induced by the embedding $\underline{P}(R/\mathbf{m}[\pi,\sigma,\rho]) \to \underline{M}(R/\mathbf{m}[\pi,\sigma,\rho])$ is a monomorphism. Then the conclusion of Theorem 3.1 is true.

Proof. Let us consider the $R/\mathbf{m}[\pi,\sigma,\rho]$ as $R[\pi,\sigma,\rho]$-module by the epimorphism $R[\pi,\sigma,\rho] \xrightarrow{\varphi} R/\mathbf{m}[\pi,\sigma,\rho]$. Since $P/\mathbf{m}P \simeq R/\mathbf{m}[\pi,\sigma,\rho] \otimes_{R[\pi,\sigma,\rho]} P$ and P is projective over $R[\pi,\sigma,\rho]$, we have that $P/\mathbf{m}P$ is projective over $R/\mathbf{m}[\pi,\sigma,\rho]$. Similarly, $Q/\mathbf{m}Q \in \underline{P}(R/\mathbf{m}[\pi,\sigma,\rho])$. Cosequently,

$$[P/\mathbf{m}P], [Q/\mathbf{m}Q] \in K_0(R/\mathbf{m}[\pi,\sigma,\rho]).$$

Proposition 2.2 implies $[P/\mathbf{m}P] = [Q/\mathbf{m}Q])$ in $G_0(R/\mathbf{m}[\pi,\sigma,\rho])$. This means that $\chi([\overline{P}]) = \chi([\overline{Q}])$. The mapping χ is monomorphic, and therefore $[\overline{P}] = [\overline{Q}]$ in $K_0(R/\mathbf{m}[\pi,\sigma,\rho])$. Therefore $\overline{P} \oplus F \simeq \overline{Q} \oplus F$ for some free finitely generated $R/\mathbf{m}[\pi,\sigma,\rho]$-module F. $R/\mathbf{m}$ is a field, therefore $R/\mathbf{m}[\pi,\sigma,\rho]$ is an Artinian ring and the Krull-Schmidt theorem holds for it and consequently $\overline{P} \simeq \overline{Q}$ as $R/\mathbf{m}[\pi,\sigma,\rho]$-modules. Let $f^{'} : \overline{P} \to \overline{Q}$ be any $R/\mathbf{m}[\pi,\sigma,\rho]$-isomorphism of $R/\mathbf{m}[\pi,\sigma,\rho]$-modules. We may consider $f^{'}$ as an isomorphism of $R[\pi,\sigma,\rho]$-modules by the epimorphism φ. Consider the diagram in $\underline{M}(R[\pi,\sigma,\rho])$

$$\begin{array}{ccccc} P & \to & P/\mathbf{m}P & \to & 0 \\ \downarrow f & & \downarrow f^{'} & & \\ Q & \to & Q/\mathbf{m}Q & \to & 0. \end{array}$$

Since P is $R[\pi,\sigma,\rho]$-projective and is mapped on $Q/\mathbf{m}Q$, there exists a $R[\pi,\sigma,\rho]$-morphism f such that diagramm (6) is commutative. Then we have $f(P) + \mathbf{m}Q = Q$. But $\mathbf{m} = rad(R)$ and by the lemma of Nakayama $f(P) = Q$, i.e. f is an epimorphism. Since Q is projective and f is epimorphic, therefore $P \simeq Q \oplus Kerf$. Hence $Kerf = Q^{'}$ is projective and finitely generated. From (3) it follows that $Q^{'}/\mathbf{m}Q^{'} \subseteq Kerf^{'}$. From $Kerf^{'} = 0$ it follows that $Q^{'}/\mathbf{m}Q^{'} = 0$ and again from the lemma of Nakayama it follows that $Q^{'} = 0$, i.e. f is an isomorphism. The theorem is proved. Q.E.D.

Since $R/\mathbf{m}$ is the field, by Proposition 3.4, to prove Theorem 3.1 it suffices to prove

Theorem 3.5. Let k be the field. Then the Cartan homomorphism

$$\chi : K_0(k[\pi,\sigma,\rho]) \to G_0(k[\pi,\sigma,\rho]).$$

is injective.

Proof. Since by Theorem 1.1 $K_0(k[\pi,\sigma,\rho])$ and $G_0([\pi,\sigma,\rho])$ are Frobenius modules over the Frobenius functor $G_0(k^\pi[\pi])$, the $Ker\chi$ functor will also be a Frobenius module over $G_0(k^\pi[\pi])$. Therefore an index of $(Ker\chi)_{c(\pi)}$ in $Ker\chi$ is equal to an index of $G_0(k^\pi[\pi])_{c(\pi)}$ in $G_0(k^\pi[\pi])$, namely it is $n = (\pi : 1)$. This means that $nKer\chi \subseteq (Ker\chi)_{c(\pi)}$. The ring $k[\pi,\sigma,\rho]$ is Artinian and therefore $K_0(k[\pi,\sigma,\rho])$ and its subgroup $Ker\chi$ are finitely generated free commutative groups. If we proved that χ is monomorphic for cyclic groups, then we would have that $(Ker\chi)_{c(\pi)} = 0$ and $nKer\chi = 0$. From the freeness of the group $Ker\chi$ it would follow that $Ker\chi = 0$. But if π is cyclic with a generator a, then $k[\pi,\sigma,\rho] \simeq k[x,\sigma]/(x^n - \alpha)$, there $k[x,\sigma]$ is a ring of skew polynomials of x and σ is the automorphism $\sigma(a) \in Aut(k)$, $n = (\pi : 1)$ and $\alpha \in k^\pi$. The ring $k[x,\sigma]$ is a principal (noncommutative) ideal domain, σ has a finite index and any ideal in $k[x,\sigma]$ is bounded [8]. Therefore from the next theorem it follows that χ is monomorphic for a cyclic group π Q.E.D.

Theorem 3.6. Let A be a (noncommutative) principal ideal domain, in which each ideal is bounded. If $I \subseteq A$ is a two sided ideal, $K_0(A/I)$ and $G_0(A/I)$ are Grothendieck groups of the categories $\underline{P}(A/I)$ and $\underline{M}(A/I)$ respectively, then the Cartan homomorphism

$$\chi : K_0(A/I) \to G_0(A/I)$$

is injective.

Proof. So A is a noncommutative integral domain in which any right and left ideal is principal. We say that two elements a_1 and a_2 are similar if $A/a_1 \simeq A/a_2$ as A-modules. An ideal is bounded if it contains a nonzero two sided ideal, and such a maximal ideal is called a boundary of A.

We recall that since A/I is Artinian, $G_0(A/I)$ is well defined.

We will carry out the proof in several steps.

Step 1. $I = Aa^*$ splits into the product of coprime maximal two sided ideals (the asterisk over the letter indicates that we deal with the generator of the ideal):

$$I = Aa^* = (Ap_1^*)^{e_1}(Ap_2^*)^{e_1}...(Ap_r^*)^{e_r}.$$

Step 2. It is clear that

$$K_0(A/I) \simeq \oplus_{i=1}^r K_0(A/(Ap_i^*)^{e_i}),$$

$$G_0(A/I) \simeq \oplus_{i=1}^r G_0(A/(Ap_i^*)^{e_i})$$

and these isomorphisms and χ commute with each other. Therefore it suffices to prove that χ is monomorphic if $I = A/(Ap*)^e$.

Step 3. $J = Ap^*/(Ap^*)^e$ is a radical of the ring $\overline{A} = A/(Ap^*)^e$ and $\overline{A}/J \simeq A/Ap^*$.

Step 4. Since the radical of the ring $\overline{A} = A/(Ap^*)^e$ acts trivially on simple modules, simple $\overline{A} = A/(Ap^*)^e$-modules will be simple as modules over $\overline{A} = A/(Ap^*)$. But because $\overline{A} = A/(Ap^*)$ is a simple ring, simple modules are direct summands of the ring $\overline{A} = A/(Ap^*)$.

Step 5. Using Zorn's lemma it is easy to prove that Ap^* is contained in some maximal ideal Ap. If $Ap \supseteq Aq^* \supseteq Ap^*$, then $Ap^* = Aq^*$ since Ap^* is maximal, i.e. Ap is the boundary of the ideal Ap. Since Ap is the maximal left ideal, A/Ap is a simple A-module. From Theorem 3.20, [8] it follows that A/Ap^* splits as an A-module into the direct sum of simple A-modules, which are isomorphic to A/Ap:

$$A/AP^* \simeq \oplus_i Aq_i/Ap^*, \quad Aq_i/Ap^* \simeq A/Ap.$$

Therefore the direct summand Aq_i/Ap^* is indecomposable as an A-module and, consequently, as an $A/(AP^*)^e$-module. Since A/AP^* is a simple ring, it has a single simple module, i.e. all Aq_i/Ap^* are isomorphic as A/AP^*-modules. Let q be one of q_i. We may conclude that $Aq_i/(Ap^*)^e$ has the single simple module Aq/Ap^* which is A-isomorphic to A/Ap.

Step 6. Let us find all indecomposable projective $Aq_i/(Ap^*)^e$-modules. Since $Aq_i/(Ap^*)^e$ is Artinian, such a modules are exhausted by the direct summands of $Aq_i/(Ap^*)^e$. Further, as follows from the proof of Theorem 3.21, [8], $Aq_i/(Ap^*)^e$ splits as a A-module into the direct sum of indecomposable A-modules, which is isomorphic to the A-module $A/p_1p_2...p_e$:

$$Aq_i/(Ap^*)^e \overset{\varphi}{\simeq} \oplus Ar_i/(Ap^*)^e, \;\; Ar_i/(Ap^*)^e \simeq A/p_1p_2...p_e.$$

Therefore if r is one of r_i then $A/(Ap^*)^e$ has a single indecomposable projective module $Ar/(Ap^*)^e$, which is isomorphic to $A/p_1p_2...p_e$.

Step 7. From steps 5 and 6 it follows that $G_0(A/(Ap^*)^e)$ and $K_0(A/(Ap^*)^e)$ are free commutative groups with one generator; the generator for $G_0(A/(Ap^*)^e)$ is $[Aq/Ap^*]$, and the generator for $K_0(A/(Ap^*)^e)$ is $[Ar/Ap^*]$. It is clear that $(Ap^*)^e = A(p^*)^e$. Since $A(p^*)^e \subseteq Ar$, we have $(p^*)^e = r^{'}r$. Therefore, as A-modules

$$Ar/Ap^* = Ar/Ar^{'}r \simeq A/Ar^{'}.$$

So $A/Ar^{'} \simeq A/Ap_1p_2...p_e$, therefore $r^{'} = p_1^{'}p_2^{'}...p_1^{'}$, $p_i^{'} \sim p$.

For $Ar/(Ap^*)^e$ there exists a composition row of $A/(Ap^*)^e$-modules

$$Ar/(Ap^*)^e = Ar/p_1^{'}p_2^{'}...p_e^{'}r \supseteq Ap_e^{'}r/p_1^{'}p_2^{'}...p_e^{'}r \supseteq ... \supseteq Ap_2^{'}p_e^{'}r/p_1^{'}p_2^{'}...p_e^{'}r \supseteq 0$$

whose factors are A-isomorphic to an A-module A/Ap. It is clear that all these factors are $A/(Ap^*)^e$-isomorphic to the simple module Aq/Ap^*. Therefore

$$\chi([Ar/(Ap^*)^e]) = [Ar/(Ap^*)^e] = e[Aq/Ap^*]$$

and χ is monomorphic. Q.E.D.

4 Projective modules

Let ω be a subgroup of π which contains just an elements of π which acts trivially on R, i.e. $\omega = Ker(\sigma : \pi \to Aut(R))$. If $\sigma(\pi) = id$, then we denote $R[\pi, \sigma, \rho] := R[\pi, \rho]$.

Theorem 4.1. Let R be a Dedekind domain of characteristic zero. Suppose that no one divider of $n = (\pi : 1)$ is invertible in R, and $\sigma : \pi \to Aut(R)$ is a morphism such that (i) *R is projective over R^π;*

(ii) *if $\boldsymbol{p} \in spec(R)$, $\boldsymbol{p}|(n)$, then $\sigma(\pi)(\boldsymbol{p}) \subseteq \boldsymbol{p}$;*

(iii) *if p is a prime divider of (n), $p \in \boldsymbol{p} \in spec(R)$ and π_p is a Sylov p-subgroup of π, then π_p acts trivially on $R/\boldsymbol{p}$;*

(iv) $\rho(\pi \times \pi) \subseteq R^\pi$. Then any finitely generated projective $R[\pi, \sigma, \rho]$-module splits into the direct sum of left ideals of the ring $R[\pi, \sigma, \rho]$.

For the particular case $\sigma(\pi) = id$, we may prove a stronger result.

Theorem 4.2. Let R be the Dedekind domain of characteristic zero. Suppose that no one divider of $n = (\pi : 1)$ is invertible in R. Then any finitely generated projective $R[\pi, \rho]$-module is the direct sum of the free $R[\pi, \rho]$-module and left ideal $I \subseteq R[\pi, \rho]$. For any nonzero ideal $\mathbf{j} \subseteq R$ we may choose an ideal I in such a way that I and $\mathbf{j}$ would be coprime ideals.

Let us denote $R \cap I = (I : R[\pi, \rho]) = \{r \in R | rR[\pi, \rho] \subseteq I\}$.

First, we must prove some useful propositions.

Let us denote $(\omega : 1) = h$. It is clear that $n = hm$ and $\sigma(x)^m = id$ for any $x \in \pi$.

Lemma 4.3. Suppose k is a field, $char(k) = p$, π is a cyclic group, $(p, h) = 1$, and $\rho(\omega \times \omega) \subseteq k^\pi$. Then any simple $k[\pi, \sigma, \rho]$-module splits as a $k^\pi[\omega, \rho]$ module into the direct sum isomorphic simple $k^\pi[\omega, \rho]$ modules: $M = N \oplus N \oplus N \oplus ... \oplus N$. The relation $M \to N$ induces a bijection between the isomorphism classes of simple $k[\pi, \sigma, \rho]$ and simple $k^\pi[\omega, \rho]$ modules.

Proof. It is clear that $k[\pi, \sigma, \rho] \simeq k[x, \sigma]/(x^{mh} - \alpha)$, where $m = (\pi : \omega)$, $\alpha \in k^\pi \backslash 0$. It is known that two-side ideals of $k[x, \sigma]$ are generated by elements of the form $x^t \varphi(x^m \gamma)$, where $\varphi(x) \in k^\pi(x)$, $\gamma \in k$. Since $\alpha \neq 0$, two-sided maximal ideals which divide the two-sided ideal $k[x, \sigma]/(x^{mh} - \alpha)$ must have the form $\varphi(x^m) k[x, \sigma]$, where $\varphi(x) \in k^\pi[x]$ and $\varphi(x)$ is indecomposable in $k^\pi[x]$. I.e.

$$(x^{mh} - \alpha) = (\varphi_1(x^m)^{i_1} ... (\varphi_r(x^m)^{i_r}),$$

where if $i \neq j$, then $\varphi_i(x^m) \not\simeq \varphi_j(x^m)$. Since $(p, h) = 1$, the rings $k[\pi, \sigma, \rho]$ and $k^\pi[\omega, \rho]$ are semisimple [8]. Thus $i_1 = ... = i_r = 1$. Therefore

$$k[x, \sigma]/(x^{mh} - \alpha) \simeq k[x, \sigma]/(\varphi_1(x^m)) \oplus ... \oplus k[x, \sigma]/(\varphi_r(x^m)).$$

Since $k[x, \sigma]/(\varphi_i(x^m))$ is a simple ring, it has a single simple module M_i and $M_i \not\simeq M_j$ for different i and j. On the other hand,

$$k^\pi[\omega, \rho] \simeq k^\pi[x, \sigma]/(x^{mh} - \alpha) \simeq k^\pi[x, \sigma]/(\varphi_1(x)) \oplus ... \oplus k^\pi[x, \sigma]/(\varphi_r(x)).$$

The fields $N_i = k^\pi[x]/(\varphi(x))$ are simple $k^\pi[\omega, \rho]$-modules. From the embedding

$$k^\pi[x]/(\varphi(x)) \to k[x, \sigma]/(\varphi_i(x^m), \quad [f(x)] \to [f(x^m)]$$

it follows that $k[x, \sigma]/(\varphi(x^m)$ is a free $k^\pi[x]/(\varphi_i(x))$-module with a basis $[\alpha_j x^k]$, $j = 1, 2, ..., m$, $k = 0, 1, ..., m - 1$, where $\alpha_1, ..., \alpha_m$ is a k^π-basis of the field k. Therefore $k[x, \sigma]/(\varphi_i(x^m))$ splits as a $k^\pi[\omega, \rho]$-module into the direct sum of $k^\pi[\omega, \rho]$-modules which are isomorphic to the $k^\pi[\omega, \rho]$-module N_i. Since M_i is the direct summand of $k[x, \sigma]/(\varphi_i(x^m))$, by the Krull-Schmidt theorem M_i splits too as a $k^\pi[\omega, \rho]$-module into the direct sum of $k^\pi[\omega, \rho]$-modules which are isomorphic to a simple $k^\pi[\omega, \rho]$-module N_i. The correspondence $M_i \to N_i$ proves the lemma. Q.E.D.

Proposition 4.4. Let R be an integral domain with quotient field K, such that $char(k) = p$, $(p, h) = 1$, $\rho(\omega \times \omega) \subseteq U(R^\pi)$, R is projective and finitely generated over R^π and the following condition holds: (*) *For any cyclic subgroup $\omega_0 \subseteq \omega$ and any $Q_1, Q_2 \in \underline{P}(R^\pi[\omega, \rho])$ from $rk_{K^\pi} Q_1 = rk_{K^\pi} Q_2$ it follows that $K^\pi \otimes_{R^\pi} Q_1 \simeq K^\pi \otimes_{R^\pi} Q_2$ as $K^\pi[\omega_0, \rho]$-modules.* Then for any $P_1, P_2 \in \underline{P}(R[\pi, \sigma, \rho])$ from $rk_K(P_1) = rk_K(P_2)$ it follows that $K \otimes_R P_1 \simeq K \otimes_R P_2$ as $K[\pi, \sigma, \rho]$-modules.

Proof. $R^\pi \subseteq R$ is an integral extension of rings and $K \simeq K^\pi \otimes_{R^\pi} R$. Therefore $K \otimes_R P \simeq K^\pi \otimes_{R^\pi} R \otimes_R P \simeq K^\pi \otimes_{R^\pi} P$. Consequently, $rk_K(P_1) = rk_K(P_2) \simeq rk_{K^\pi}(P_1) = rk_{K^\pi}(P_2)$. Let $\pi_0 \subseteq \pi$ be a cyclic subgroup. Let us denote $\omega_0 = Ker(\sigma : \pi_0 \to Aut(R))$. Since $R[\omega_0, \rho] \simeq R^\pi[\omega_0, \rho] \otimes_{R^\pi} R$ as $R[\omega_0, \rho]$ modules, $R[\omega_0, \rho]$ is projective as a $R^\pi[\omega_0, \rho]$-module. Since $R[\pi, \sigma, \rho]$ is free over $R[\omega_0, \rho]$, $R[\pi, \sigma, \rho]$ is also projective over $R[\omega_0, \rho]$. Therefore $P_1, P_2 \in \underline{P}(R[\pi, \sigma, \rho])$, we have $P_1, P_2 \in \underline{P}(R^\pi[\omega_0, \rho])$. By the condition $rk_K(P_1) = rk_K(P_2)$. As we have already noted $rk_{K^\pi}(P_1) = rk_{K^\pi}(P_2)$. Then by the condition (*) we have $K^\pi \otimes_{R^\pi} P_1 \simeq K^\pi \otimes_{R^\pi} P_2$ as $K^\pi[\omega_0, \rho]$-modules or, what is the same, $K \otimes_R P_1 \simeq K \otimes_R P_2$ as $K^\pi[\omega_0, \rho]$)-modules. If we suppose in Lemma 4.3 that $\pi = \pi_0$, $\omega = \omega_0$, it follows that $K \otimes_R P_1$ and $K \otimes_R P_2$ contain N_i as a direct summand the same number of times (recall that $K[\pi, \sigma, \rho]$, $K^\pi[\omega, \rho]$, $K[\pi_0, \sigma, \rho]$ and $K^\pi[\omega_0, \rho]$ are semisimole rings). By Lemma 4.3 N_i is contained as a direct summand only in M_i, and M_i does not contain other summands. Therefore $K \otimes_R P_1$ and $K \otimes_R P_2$ must contain M_i as a direct summand the same number of times. Therefore $K \otimes_R P_1 \simeq K \otimes_R P_2$ as $K[\pi_0, \sigma, \rho]$-modules. Suppose χ_i is a character of $K[\pi, \sigma, \rho]$-modules $K \otimes_R P_i$, $i = 1, 2$, $a = \sum_{x \in \pi} \alpha_x \bar{x}$ and $\pi_x \subseteq \pi$ is the cyclic subgroup generated by x. Then $K \otimes_R P_1 \simeq K \otimes_R P_2$ as $K[\pi_x, \sigma, \rho]$)-modules and therefore $\chi_1(\alpha_x \bar{x}) = \chi_2(\alpha_x \bar{x})$. Hence $\chi_1(a) = \chi_2(a)$. From the equality of characters it follows that $K \otimes_R P_1 \simeq K \otimes_R P_2$ as $K[\pi, \sigma, \rho]$-modules. Q.E.D.

Lemma 4.5. Under the conditions of Theorem 3.1 the rank $rk_K(P)$ is divided into n.

Proof. Let $n = \prod p^{\mu_p}$. Then $(\pi_p : 1) = p^{\mu_p}$. p is not convertible in R, therefore there exists $\mathbf{p} \in spec(R)$ such that $p \in \mathbf{p}$. The group π_p acts trivially on R_p. Therefore $R/\mathbf{p}[\pi_p, \bar{\sigma}, \bar{\rho}] = R/\mathbf{p}[\pi_p, \bar{\rho}]$. $R/\mathbf{p}$ is a field of characteristic $p \geq 0$ and π_p is the finite p-group, thus $R/\mathbf{p}[\pi_p, \bar{\rho}]$ is a local ring [14]. Consequently, the module $P/\mathbf{p}P$ is not only projective, but also free over $R/\mathbf{p}[\pi_p, \bar{\rho}]$. Therefore $p_p^\mu | (P/\mathbf{p}P : R/\mathbf{p})$. Since R is the Dedekind domain, we have $p_p^\mu | rk_K(P)$. Since that is true for all $p|n$, we have $n | rk_K(P)$. Q.E.D.

Theorem 4.6. Under the conditions of Theorem 3.1 the module $K \otimes_R P$ is free over $K[\pi, \sigma, \rho]$.

Proof. Let us first prove the theorem for the cyclic group π. More precisely, we must prove that if π is a finite cyclic group, $\sigma(\pi) = id$, R is the Dedekind domain of characteristic 0, prime dividers of n are not invertible in R and $P \in \underline{P}(R[\pi, \rho])$, then the module $K \otimes_R P$ is a free $R[\pi, \rho]$-module.

Step 1. Let M be any simple $R[\pi, \rho]$-module. From $char(K) = 0$ it follows that $K[\pi, \rho]$ is a semisimple K-algebra. Suppose that $K \otimes_R P$ contains M n-times as a direct summand. Then $Hom_{K[\pi,\rho]}(M, K \otimes_R P)$ is isomorphic to the direct sum of r summands, which are isomorphic to $Hom_K(M, K)$. The consideration of bases and comparison of dimensions show that the mapping $\varphi(f \otimes v)(m) := f(m)(v)$, where $v \in K \otimes_R P$, is an isomorphism of K-modules

$$\varphi : M^* \otimes_K (K \otimes_R P) \simeq Hom_K(M, K \otimes_R P).$$

Step 2. It is clear that $M^* \otimes_K (K \otimes_R P)$ is a $K[\pi]$-module if we suppose $x(f \otimes v) = f\bar{x}^{-1} \otimes \bar{x}v$, where $x \in K[\pi]$ and $\bar{x}, \bar{x}^{-1} \in K[\pi, \rho]$. Similarly, $Hom_K(M, K \otimes_R P)$ is a left $K[\pi]$-module via the action $(xf)(m) = \bar{x}f(\bar{x}^{-1}m)$ and

$$Hom_K(M, K \otimes_R P)^\pi := \{f \in Hom_K(M, K \otimes_R P) | (\forall x \in \pi) xf = x\} =$$

$$= Hom_{K[\pi,\rho]}(M, K \otimes_R P).$$

Let us prove that φ from Step 1 is an isomorphism of left $K[\pi]$-modules:

$$\varphi(x(f\otimes v))(m)=\varphi(f\bar{x}^{-1}\otimes\bar{x}v)(m)=f(\bar{x}^{-1}m)\bar{x}v=$$

$$=\bar{x}f(\bar{x}^{-1}m)v=[x\varphi(f\otimes v)](m).$$

Therefore

$$Hom_{K[\pi,\rho]}(M,K\otimes_R P)\simeq[M^*\otimes_K(K\otimes_R P)]^\pi.$$

Step 3. In our conditions there exists a finitely generated $R[\pi,\rho]$-module Q such that Q is projective, i.e. it is torsion-free over R and $M\simeq K\otimes_R Q$. Indeed, suppose $0\neq m\in M$. Let $Q=R[\pi,\rho]m\subseteq M$; it is clear that $M\simeq K\otimes_R Q$ because M is a semisimple $R[\pi,\rho]$-module. If $q\in Q$ and $rq=0$, $r\in R$, then $r^{-1}(rq)=q$, i.e. Q is torsion-free over R. Since R is a Noetherian ring and Q is a finitely generated R-module, then $M^*\simeq(K\otimes_R Q)^*\simeq K\otimes_R Q^*$, $Q^*=Hom_R(Q,R)$. Therefore by Lemma 8.2 [1], from step 2 it follows that

$$Hom_{R[\pi,\rho]}(M,K\otimes_R P)\simeq(K\otimes_R Q^*\otimes_R P)^\pi\simeq K\otimes_R(Q^*\otimes_R P)^\pi,$$

where $R[\pi]$ acts on $Q^*\otimes_R P$ as $x(f\otimes p)=f\bar{x}^{-1}\otimes\bar{x}p$.

Step 4. $Q^*\otimes_R P$ is a $R[\pi]$-projective module. Then by Lemma 8.3, from [1] we have

$$rk(Q^*\otimes_R P)^\pi=\frac{1}{n}rk(Q^*\otimes_R P)=\frac{1}{n}rk(Q)rk(P),$$

where $rk(M)=dim_K(K\otimes_R M)$, $M\in R-\underline{Mod}$. Consequently,

$$\frac{1}{n}rk(Q)rk(P)=dim_K(Hom_{K[\pi,\rho]}(M,K\otimes_R P)),$$

i.e. r depends only on $rk(P)$. From Lemma 3.5 it follows that there exists a free $R[\pi,\rho]$-module F such that $rk(F)=rk(P)$. Therefore M is contained in $K\otimes_R F$ and $K\otimes_R P$ the same number of times, and, consequently, $K\otimes_R P\simeq K\otimes_R F$, i.e. $K\otimes_R P$ is a free $K[\pi,\rho]$-module.

Let us prove the theorem in the general case. It is well known that R^π is a Dedekind domain and R is finitely generated over R^π. Thus P is projective and finitely generated over $R^\pi[\omega,\rho]$. As we have already proved $K\otimes_R P$ is a free $R^\pi[\omega_0,\rho]$-module for all cyclic subgroups $\omega_0\subseteq\omega$. By Proposition 3.4 $K\otimes_R P$ is uniquely determined as a $K[\pi,\sigma,\rho]$-module by $rk(P)$. By Lemma 3.5 there exists a free $R[\pi,\sigma,\rho]$-module F such that $rk(P)=rk(F)$, therefore $K\otimes_R P\simeq K\otimes_R F$, i.e. $K\otimes_R P$ is a free $K[\pi,\sigma,\rho]$-module. Q.E.D.

Theorem 4.7. Let R be a Dedekind domain, $\mathbf{m}\subseteq R$ be a nonzero ideal, $\mathbf{m}=\prod_i\mathbf{p}_i^{\nu_i}$, $\mathbf{p}_i\in spec(R)$ and $\pi(\mathbf{p}_i)\subseteq\mathbf{p}_i$ for all i. If $P\in\underline{P}(R[\pi,\sigma,\rho])$ and $K\otimes_R P$ is a free $K[\pi,\sigma,\rho]$-module, then P contains a free module F such that ω and F are coprime ideals.

Proof. If $\omega=R$ then we may suppose that F is equal to any free submodule of P. Let us now suppose that $\omega\neq R$. Let first $\omega=\mathbf{p}\in spec(R)$. By the condition $K\otimes_R P\simeq K\otimes_{R_\mathbf{p}}R_\mathbf{p}\otimes_R P\simeq K\otimes_{R_\mathbf{p}}P_\mathbf{p}$ is free over $K[\pi,\sigma,\rho]$, i.e. $K\otimes_{R_\mathbf{p}}P_\mathbf{p}\simeq K\otimes_{R_\mathbf{p}}F_0$ for some free $R_\mathbf{p}[\pi,\sigma,\rho]$-module F_0. $R_\mathbf{p}$ is a discrete valued ring. Consequently, by Theorem 3.1 $P_\mathbf{p}$ is a free $R_\mathbf{p}[\pi,\sigma]$-module. Since $R_\mathbf{p}/\mathbf{p}\simeq R/\mathbf{p}$,

$$P_\mathbf{p}/(\mathbf{p}P_\mathbf{p})\simeq R_\mathbf{p}/\mathbf{p}\otimes_{R_\mathbf{p}}/P_\mathbf{p}\simeq R_\mathbf{p}/\mathbf{p}\otimes_{R_\mathbf{p}}R_\mathbf{p}\otimes_R P\simeq R_\mathbf{p}\otimes_R P\simeq P/(\mathbf{p}P).$$

Therefore P/($\mathbf{p}$P) is a free $(R/\mathbf{p})[\pi, \bar{\sigma}, \bar{\rho}]$-module.

Let us consider the general case. As we already have proved $P/(\mathbf{p}_i P)$ are free $(R/\mathbf{p}_i)[\pi, \bar{\sigma}, \bar{\rho}]$-modules for all i. Let $\bar{a}_1^{(i)}, ..., \bar{a}_k^{(i)}$ be a free basis of $P/(\mathbf{p}_i P)$. Since $rk_K(P) = (P/(\mathbf{p}_i P) : R/\mathbf{p}_i)$, $k_1 = k_2 = ... = k$. By the Chinese remainder theorem there exist elements $r_i \in R$ such that $r_i \equiv \delta_{ij}(mod \mathbf{p}_j)$. Let $a_s^{(i)}$ be a coimages of the elements $\bar{a}_s^{(i)}$ with the respect to a morphism $P \to P/(\mathbf{p}_i P)$. Let us denote $a_s = \sum_i \alpha_i a_s^{(i)}$. Then for any i, the images of elements in $P/(\mathbf{p}_i P)$ coincide with the basis $\bar{a}_1^{(i)}, ..., \bar{a}_k^{(i)}$.

Let F be a $R[\pi, \sigma, \rho]$-submodule of P generated by elements $a_1, ..., a_k$. Let us prove that F is a free $R[\pi, \sigma, \rho]$-module with a basis $a_1, ..., a_k$. Otherwise in F there would exist a nontrivial relation between the elements $a_1, ..., a_k$ and we would have in $K \otimes_R F$ that $rk_K F < nk$, $n = (\pi : 1)$. On the other hand, $(F/(\mathbf{p}_i F) : R/\mathbf{p}_i) = (P/(\mathbf{p}_i P) : R/\mathbf{p}_i) = nk$ because $F/(\mathbf{p}F) \to P/(\mathbf{p}P)$ is surjective. But $rk(F) = (F/(\mathbf{p}_i F) : R/\mathbf{p}_i)$, a contradiction. Thus F is a free module.

Since $(F/(\mathbf{p}_i F) \simeq (P/(\mathbf{p}_i P)$ we have $(F : P) + \mathbf{p}_i = R$. R is the Dedekind domain; consequently, $(F : P) + \omega = R$. Q.E.D.

Corollary 4.8. Under the conditions of Theorem 4.7 the module $P/(\omega P)$ is free over $R/\omega[\pi, \bar{\sigma}, \bar{\rho}]$.

Proof. Indeed, because $(F : P) + \omega = R$, $F/(\omega F) \to P/(\omega P)$ is an isomorphism. Q.E.D.

Proposition 4.9. Under the conditions of Theorem 3.1 there exists an embedding of the module P in the free $R[\pi, \sigma, \rho]$-module F such that $(P : F) + (n) = R$, $(P : F)_{R^\pi} + nR^\pi = R^\pi$.

Proof. Let us suppose in Corollary 3.8 that $\omega = (n) = nR$. In the proof of Corollary 3.8 it was shown that $F/(nRF) \simeq P/(nRP)$. But $F/(nRF) = F/(nF) = F/(nR^\pi F)$ and, similarly, for P. Therefore $F/(nR^\pi F) \simeq P/(nR^\pi P)$ and $(F : P)_{R^\pi} + nR^\pi = R^\pi$ because R^π is a Dedekind domain. Therefore there exists $a \in nR^\pi$, $b \in (F : P)_{R^\pi}$ such that $a + b = 1$. $nR^\pi \neq R^\pi$ and therefore $b \neq 0$. From $b \in R^\pi$ it follows that b is contained in the center of the ring $R[\pi, \sigma, \rho]$. Consequently, bP is a $R[\pi, \sigma, \rho]$-module and as P is R-torsion-free (finitely generated modules are torsion-free over Dedekind domains), $P \simeq bP$ as $R[\pi, \sigma, \rho]$-modules. $bP \subseteq F$ since $b \in (F : P)_{R^\pi}$. It is clear that $b \in (F : P)$. Let us identify P and bP. Then F will be the desired free module and $bP \subseteq F$ will be the desired embedding for which $(P : F)_{R^\pi} + nR^\pi = R^\pi$ and a fortiori $(P : F) + (n) = R$. Q.E.D.

Proposition 4.10. Let $M \in \underline{M}(R[\pi, \sigma, \rho])$. If $ann_R M + (n) = R$, $n = (\pi : 1)$, $ann_R M = \{r \in R | rM = 0\}$, then M is $R[\pi, \sigma, \rho]$-projective.

Proof. By condition there exist $a, b \in R$ such that $an + b = 1$ and $bM = 0$. Let us define $(n) : M \to M$, $(n) : m \to nm$. Let $nm = 0$. Then $anm + bm = 0 = m$, i.e. this morphism is injective. If $m \in M$, then by the equality $m = anm + bm = anm$ we have $am \xrightarrow{(n)} m$, i.e. (n) is surjective. Therefore (n) is an isomorphism. On the other hand, $R[\pi, \sigma, \rho] \supseteq R$ is a free Frobenius extension with dual bases $\{\bar{x}\}_{x \in \pi}$ and $\{\bar{x}^{-1}\}_{x \in \pi}$. As we have already proved $\sum_{x \in \pi} \bar{x}\bar{x}^{-1} = n$ is an isomorphism and therefore from the properties of Frobenius extensions it follows that M is $(R[\pi, \sigma, \rho], R)$-projective. Q.E.D.

Proposition 4.11. Let $I \subseteq R[\pi, \sigma, \rho]$ be an ideal such that $(I : R[\pi, \sigma, \rho]) + (n) = R$ and let R be a Dedekind domain. Then I is a $R[\pi, \sigma, \rho]$-projective module.

Proof. Let us consider a $R[\pi,\sigma,\rho]$-module $M = R[\pi,\sigma,\rho]/I$. Since R is a Dedekind domain, $dimpr_R(M) \leq 1$. Since $(I : R[\pi,\sigma,\rho]) = ann_R(M)$, $ann_R(M) + (n) = R$ and by Proposition 3.10 the module M is $R([\pi,\sigma,\rho],R)$-projective. Since $R[\pi,\sigma,\rho]$ is free over R, $dimpr_{R[\pi,\sigma,\rho]}(M) \leq 1$. But then there exists an exact sequence

$$0 \to I \to R[\pi,\sigma,\rho] \to M \to 0,$$

from which implies that I is $R[\pi,\sigma,\rho]$-projective. Q.E.D.

Proof of Theorem 4.1. Let F be the free module from Proposition 3.9 with $R[\pi,\sigma,\rho]$-basis $a_1, a_2, ..., a_k$. Let us consider the morphisms of $R[\pi,\sigma,\rho]$-modules

$$\varphi_1 : F \to R[\pi,\sigma,\rho], \ \sum_i \mu_i a_i \to \mu_1.$$

An image $\varphi_1(P) = I_1$ of this morphism is an ideal in $R[\pi,\sigma,\rho]$. Since $rF \subseteq P \Rightarrow rR[\pi,\sigma,\rho] \subseteq I_1$, $(P : F) \subseteq (I_1 : R[\pi,\sigma,\rho])$. Therefore from $(P : F)+(n) = R$ it follows that $(I_1 : R[\pi,\sigma,\rho])+(n) = R$. Then by Proposition 3.11 the ideal I_1 is $R[\pi,\sigma,\rho]$-projective. $\varphi : P \to I_1$ is surjective, therefore $P \simeq P^{'} \oplus I_1$. Now the theorem is easy to prove by mathematical induction with respect to $rk_K(P)$. Q.E.D.

Example for Theorem 4.1. Let $d \neq 0$ be a natural number which does not contain a square of a prime number as a multiplier and such that $d \equiv 2 \vee 3 (mod 4)$. Then the ring of integers for the field $Q(\sqrt{d})$ will be $Z([\sqrt{d}])$. Let us suppose that a natural number $n > 0$ satisfies the following condition: If $p \neq 2$ is a prime number and $p|n$, then $(\frac{D}{p}) = 0 \vee -1$ where $D = 4d$ is a discriminant of the field and $(\frac{D}{p})$ is a quadratic residue symbol. If $(\pi : 1) = n$, then any crossed group ring $Z[\sqrt{d}][\pi,\sigma,\rho]$ satisfies the conditions of Theorem 3.1 for any σ and ρ.

Indeed if $2|n$, then $2 = \mathbf{p}^2$ for some $\mathbf{p} \in spec(Z[\sqrt{d}])$, [15]. If $p \neq 2$, $p|n$, then from $(\frac{D}{p}) = 0 \vee -1$ it follows that either $(p) = \mathbf{p}^2$ for some $\mathbf{p} \in spec(Z[\sqrt{d}])$ or (p) is prime in $Z[\sqrt{d}]$, [15]. It is clear that in all these cases the group $Aut(Z[\sqrt{d}])$, $\sigma(\sqrt{d}) = -\sqrt{d}$ satisfies the condition (ii) of Theorem 3.1 and a fortiori this is true for the group π. Further, $(2) = \mathbf{p}^2 \Rightarrow Z[\sqrt{d}]/\mathbf{p} \simeq F_2$, but F_2 has only one, identity authomorphism and condition (iii) is satisfied too. Q.E.D.

Proposition 4.12. Let $\omega \subseteq R$ be a nonzero ideal. Then under the conditions of Theorem 3.2 there exists an embedding of the module P in the free $R[\pi,\rho]$-module F such that $(P : F) + \omega = R$.

Proof. Proved similarly to Proposition 4.9. Q.E.D.

Proposition 4.13. Under the conditions of Theorem 3.2 any module P is isomorphic to the direct sum $\sum I_i$ of ideals of $R[\pi,\rho]$; in addition the ideals I_i can be chosen in such a way that for all i

$$(I_i : R[\pi,\rho]) + \omega = R.$$

Proof. By condition, $(\pi : 1) = n \neq 0$ in R. Let us choose in Proposition 3.12 a free $R[\pi,\rho]$-module with the basis $a_1, a_2, ..., a_k$ in such a way that $n\omega + (P : F) = R$. Let us consider the morphism of $R[\pi,\rho]$-modules $\varphi_1 : F \to R[\pi,\rho]$, $\sum_i \mu_i a_i \to \mu_1$. The image $\varphi_1(P) = I_1$ is the left ideal in $R[\pi,\rho]$. Since $rF \subseteq P \Rightarrow rR[\pi,\rho] \subseteq I_1$, $(P : F) \subseteq (I_1 : R[\pi,\rho])$. From $n\omega + (P : F) = R$ it follows that $(I_1 : R[\pi,\rho]) + (P : F) = R$ and thus the ideals (n) and ω are coprime with the respect to $(I_1 : R[\pi,\rho])$. Then from Proposition 4.11 it follows that I_1 is $R[\pi,\rho]$-projective. Since $\varphi_1 : P \to I_1$ is an epimorphism, $P = P' + I_1$ and the proposition is easy to prove by mathematical induction with the respect to $rk_K(P)$. Q.E.D.

Remark 4.14. We may suppose that $K \otimes_R I_i \simeq K[\pi,\rho]$ for all i. Indeed, let $\omega \subseteq R$ be an improper ideal. Then $\omega(R[\pi,\rho]I_i) = R[\pi,\rho]I_i$ and by Lemma 7.4, [1] there exists $a \in \omega$ such that $(1-a)R[\pi,\rho]/I_i = 0$. Since ωR is an improper ideal, $1 - a \neq 0$ and thus $K \otimes_R I \simeq K \otimes_R R[\pi,\rho] \simeq K[\pi,\rho]$.

Proof of Theorem 4.2. By Proposition 4.13 and Remark 4.14 it is suffices to prove the following: let $I_1, I_2 \subseteq R[\pi,\rho]$ be a projective ideals such that $(I_1 : R[\pi,\rho])$ and $(I_2 : R[\pi,\rho])$ are coprime with the respect to ω and $K \otimes_R I_1 \simeq K \otimes_R I_2 \simeq K[\pi,\rho]$; then $I_1 \oplus I_2 \simeq R[\pi,\rho] \oplus I$, where $I \subseteq R[\pi,\rho]$ is a left ideal and $(I : R[\pi,\rho]) + \omega = R$.

Let $\omega_1 = (I_1 : R[\pi,\rho])$. From Proposition 4.13 it follows that there exists $I_2' \subseteq R[\pi,\rho]$ such that $I_2 \simeq I_2'$ and $(I_2' : R[\pi,\rho]) + \omega\omega_1 = R$. Let us replace I_2 by I_2'. Therefore we may assume that there exist $b_1 \in (I_1 : R[\pi,\rho])$ and $b_2 \in (I_2 : R[\pi,\rho])$ such that $b_1 + b_2 = 1$.

Let F be the free $R[\pi,\rho]$-module with two free generators e_1, e_2 and $V = I_1e_1 + I_2e_2 \subseteq F$. Then $A \simeq I_1 + I_2$ and $(V : F) + \omega = R$. It is clear that the elements $e_1' = e_1b_1 + e_2b_2$ and $e_2' = e_1 - e_2$ are also free generators of F, because $e_1 = e_1' + b_2e_2'$, $e_2 = e_1' - b_2e_2'$. But $e_1' \in V$ because $b_1 \in I_1$, $b_2 \in I_2$. Consequently, $V = R[\pi,\rho]e_1' + Ie_2'$ where $I = \{a \in R[\pi,\rho] | re_2' \in V\}$. It is also clear that $(I : R[\pi,\rho]) + \omega = R$ because $(I : R[\pi,\rho]) = (V : F)$. Q.E.D.

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