A report on complete mock theta functions of order eight

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Abstract

The aim of the present paper is to establish new representation of complete mock theta functions of order eight, certain relations between complete mock theta functions of order eight and some relations between complete mock theta functions and mock theta functions of order eight

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1 Introduction

The most famous topic in the Ramanujan's Lost Notebook is the mock theta functions that were first introduced by Ramanujan in his last letter to G. H. Hardy. Ramanujan quoted about the mock theta functions of order 3, 5 and 7 in his "Lost Notebook" [19]. G. E. Andrews [3](recently K. Hikami [16, 17]), G. E. Andrews and D. Hickerson [5] and B. Gordon and R. J. McIntosh [14] introduced mock theta functions of order 2, 6 and 8 respectively. Y. S. Choi [10] found mock theta functions of order 10 from the Lost Notebook of Ramanujan. Further exploration on the properties and representation of mock theta functions were carried out by G. N. Watson [22, 23], G. E. Andrews [4, 5], Bruce C. Berndt and Song Heng Chan [7], R. P. Agarwal [1, 2], R. Y. Denis [12], S. N. Singh [12], Pankaj Srivastava [21], S. Ahmad Ali [6], Bhaskar Srivastava [20], S. P. Zwegers [24], Maheshwar Pathak and Pankaj Srivastava [18] and others.

Due to the emerging applications of the mock theta functions in the field of partition theory and quantum invariants theory, people become much more interested to work on mock theta functions. The most important applications of mock theta functions in the field of partition theory come from the study of Dyson's rank [11], Bringmann and Ono [8, 9]. Also, one can easily see the significant connection of mock theta functions with the quantum invariants in the study of K. Hikami [16], while the connection of mock theta functions of order eight with quantum invariants is unavailable in the Hikami [16] study. An interesting report on complete mock theta functions of order 6 was given by Anju Gupta [15]. Later on Bhaskar Srivastava [20] has given some identities for the complete mock theta functions of order 5.

To make the application of mock theta functions of order 8 more convenient, we have established many relations for mock theta functions and partial mock theta functions of order 8 with different order mock theta functions in our article [18]. In this paper, an attempt has been made to develop certain new representations of the complete mock theta functions of order 8, certain relations between complete mock theta functions of order 8 and some relations between complete mock theta functions and mock theta functions of order 8.

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2 Definitions & Notations

We shall use the following q-symbols:

For $|q| < 1$

$$\begin{aligned}
(a;q)_n &= \prod_{s=0}^{n-1}(1-aq^s),\ n \geq 1 \\
(a;q)_{-n} &= \frac{(-q/a)^n}{(q/a;q)_n}\ q^{n(n-1)/2},\ n \geq 1 \\
(a;q)_0 &= 1, \\
(a;q)_\infty &= \prod_{s=0}^{\infty}(1-aq^s), \\
(a_1,a_2,...,a_r;q)_n &= (a_1;q)_n(a_2;q)_n...(a_r;q)_n.
\end{aligned}$$

A generalized basic bilateral hypergeometric series with base q is defined as:

$${}_2\psi_2(a_1,a_2;b_1,b_2;q;z) = {}_2\psi_2\left[\begin{matrix} a_1, & a_2 \\ & & ;q; & z \\ b_1, & b_2 \end{matrix}\right] = \sum_{n=-\infty}^{\infty} \frac{(a_1;q)_n(a_2;q)_n}{(b_1;q)_n(b_2;q)_n}\ z^n,$$

where, $|b_1b_2/a_1a_2| < |z| < 1$, $|q| < 1$.

If $F(q) = \sum_{n=0}^{\infty} f(q,n)$ is a mock theta function, then the corresponding complete or bilateral mock theta function is denoted and defined by the bilateral series,

$$F_b(q) = \sum_{n=-\infty}^{\infty} f(q,n).$$

Definitions and notations of the mock theta functions that shall be used in our analysis are as

Mock theta functions of order 8.

Gordon & McIntosh [14] found eight mock theta functions of order 8 which are given below as:

$$S_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n},$$

$$S_1(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_n}{(-q^2;q^2)_n},$$

$$T_0(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}},$$

$$T_1(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}},$$

$$U_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^4;q^4)_n},$$

$$U_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(-q^2;q^4)_{n+1}},$$

$$V_0(q) = -1 + 2\sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n}$$

$$= -1 + 2\sum_{n=0}^{\infty} \frac{q^{2n^2}(-q^2;q^4)_n}{(q;q^2)_{2n+1}},$$

$$V_1(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{q^{2n^2+2n+1}(-q^4;q^4)_n}{(q;q^2)_{2n+2}}.$$

In our analysis we shall use the notations $S_{0b}, S_{1b}, T_{0b}, T_{1b}, U_{0b}, U_{1b}, V_{0b}, V_{1b}$ for complete mock theta functions of order 8.

To established our main results we have used the following transformations:

(I) Bailey transformation [13, p. 137, (5.20)]:

$${}_2\psi_2\left[\begin{matrix} a_1, & a_2 \\ & & ;q; & z \\ b_1, & b_2 \end{matrix}\right] = \frac{\left(a_1 z, a_2 z, \dfrac{b_1 q}{a_1 a_2 z}, \dfrac{b_2 q}{a_1 a_2 z}; q\right)_\infty}{\left(\dfrac{q}{a_1}, \dfrac{q}{a_2}, b_1, b_2; q\right)_\infty} {}_2\psi_2\left[\begin{matrix} \dfrac{a_1 a_2 z}{b_1}, & \dfrac{a_1 a_2 z}{b_2} \\ & & ;q; & \dfrac{b_1 b_2}{a_1 a_2 z} \\ a_1 z, & a_2 z \end{matrix}\right],$$

where, $|b_1 b_2/a_1 a_2| < |z| < 1$.

And

(II) The particular form of the general transformation of Slater [13, p. 129, (5.4.3)] i.e. for $r = 2$ is given as:

$$\frac{(b_1, b_2, q/a_1, q/a_2, dz, q/dz; q)_\infty}{(c_1, c_2, q/c_1, q/c_2; q)_\infty} {}_2\psi_2\left[\begin{matrix} a_1, & a_2 \\ & & ;q; & z \\ b_1, & b_2 \end{matrix}\right]$$

$$= \frac{q}{c_1} \frac{(c_1/a_1, c_1/a_2, qb_1/c_1, qb_2/c_1, dc_1 z/q, q^2/dc_1 z; q)_\infty}{(c_1, q/c_1, c_1/c_2, qc_2/c_1; q)_\infty}$$

$$\times {}_2\psi_2\left[\begin{matrix} qa_1/c_1, & qa_2/c_1 \\ & & ;q; & z \\ qb_1/c_1, & qb_2/c_1 \end{matrix}\right] + idem(c_1; c_2).$$

where, $d = a_1 a_2/c_1 c_2$ and $|b_1 b_2/a_1 a_2| < |z| < 1$, and the symbol idem $(c_1; c_2)$, after expression, means that the preceding expression is repeated with c_1 and c_2 interchanged.

3 Main Results

3.1 Certain new representation for complete mock theta functions of order eight.

In this section we have provided different representations of complete mock theta functions of order eight with the help of Bailey transformation [13, p. 137, (5.20)] and the general transformation of Slater [13, p. 129, (5.4.3)] respectively.

3.1.1

In order to develop the new representation for complete mock theta functions of order eight in accordance with the Bailey transformation [13, p. 137, (5.20)], we proceed as follows:

Changing the base q to q^2 in (I), we get

$$\text{(III)}\ {}_2\psi_2\left[\begin{matrix} a_1, & a_2 \\ & & ;q^2; \quad z \\ b_1, & b_2 \end{matrix}\right] = \frac{(a_1z, a_2z, b_1q^2/a_1a_2z, b_2q^2/a_1a_2z; q^2)_\infty}{(q^2/a_1, q^2/a_2, b_1, b_2; q^2)_\infty}$$
$$\times\ {}_2\psi_2\left[\begin{matrix} a_1a_2z/b_1, & a_1a_2z/b_2 \\ & & ;q^2; \quad b_1b_2/a_1a_2z \\ a_1z, & a_2z \end{matrix}\right]$$

(a) Taking $a_1 \to \infty, a_2 = -q, b_1 = 0, b_2 = -q^2$ and $z = (-q/a_1)$ in (III), after simplification, we get

$$\sum_{-\infty}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n} = \sum_{-\infty}^{\infty} \frac{q^{n(n+1)}(-1;q^2)_n}{(-q;q^2)_n}$$

hence, we have

$$S_{0b}(q) = \sum_{-\infty}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n} \tag{3.1}$$

$$= \sum_{-\infty}^{\infty} \frac{q^{n(n+1)}(-1;q^2)_n}{(-q;q^2)_n} \tag{3.2}$$

The definition of the complete mock theta function $S_{0b}(q)$ of order eight is given by (3.1) and the new representation of $S_{0b}(q)$ is given by (3.2).

Similarly we have established the definitions and new representations of the other complete mock theta functions of order eight, which are as follows:

(b) Taking $a_1 \to \infty, a_2 = -q, b_1 = 0, b_2 = -q^2$ and $z = (-q^3/a_1)$ in (III), we have

$$S_{1b}(q) = \sum_{-\infty}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_n}{(-q^2;q^2)_n} \tag{3.3}$$

$$= \frac{2}{(1+q)} \sum_{-\infty}^{\infty} \frac{q^{2(n+1)^2}(-q^2;q^2)_n}{(-q^3;q^2)_n} \tag{3.4}$$

(c) Taking $a_1 \to \infty$, $a_2 = -q^2$, $b_1 = 0$, $b_2 = -q^3$ and $z = (-q^4/a_1)$ in (III), we get

$$T_{0b}(q) = \sum_{-\infty}^{\infty} \frac{q^{(n+1)(n+2)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}} \tag{3.5}$$

$$= \frac{1}{2}\sum_{-\infty}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_{n+1}}{(-q^2;q^2)_{n+1}} \tag{3.6}$$

(d) Taking $a_1 \to \infty$, $a_2 = -q^2$, $b_1 = 0$, $b_2 = -q^3$ and $z = (-q^2/a_1)$ in (III), we get

$$T_{1b}(q) = \sum_{-\infty}^{\infty} \frac{q^{n(n+1)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}} \tag{3.7}$$

$$= \frac{1}{2}\sum_{-\infty}^{\infty} \frac{q^{n(n+2)}(-q;q^2)_n}{(-q^2;q^2)_n} \tag{3.8}$$

(e) Taking $a_1 \to \infty$, $a_2 = -q$, $b_1 = iq^2$, $b_2 = -iq^2$ and $z = (-q/a_1)$ in (III), we get

$$U_{0b}(q) = \sum_{-\infty}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(-q^4;q^4)_n} \tag{3.9}$$

$$= \sum_{-\infty}^{\infty} \frac{q^{2n}(-1;q^4)_n}{(-q;q^2)_n} \tag{3.10}$$

(f) Taking $a_1 \to \infty$, $a_2 = -q$, $b_1 = iq^3$, $b_2 = -iq^3$ and $z = (-q^3/a_1)$ in (III), we get

$$U_{1b}(q) = \sum_{-\infty}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(-q^2;q^4)_{n+1}} \tag{3.11}$$

$$= \frac{q}{(1+q)}\sum_{-\infty}^{\infty} \frac{q^{2n}(-q^2;q^4)_n}{(-q^3;q^2)_n} \tag{3.12}$$

(g) Taking $a_1 \to \infty$, $a_2 = -q$, $b_1 = 0$, $b_2 = q$ and $z = (-q/a_1)$ in (III), we get

$$V_{0b}(q) = -1 + 2\sum_{-\infty}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q;q^2)_n} \tag{3.13}$$

$$= -1 + 2\sum_{-\infty}^{\infty} \frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q;q^2)_n} \tag{3.14}$$

(h) Taking $a_1 \to \infty$, $a_2 = -q$, $b_1 = 0$, $b_2 = q^3$ and $z = (-q^3/a_1)$ in (III), we get

$$V_{1b}(q) = \sum_{-\infty}^{\infty} \frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}} \tag{3.15}$$

$$= \sum_{-\infty}^{\infty} \frac{(-1)^n q^{(n+1)^2}(q;q^2)_n}{(-q;q^2)_{n+1}} \tag{3.16}$$

3.1.2

In this section we are using the particular form of the general transformation of Slater [13, p. 129, (5.4.3)] to develop the new representation for complete mock theta functions of order eight. The beauty of this transformation is that c's are absent on the left-hand side so we can choose c's in any appropriate way.

Changing q by q^2 in the (**II**) transformation, we have

$$\begin{aligned}
&\frac{(b_1, b_2, q^2/a_1, q^2/a_2, dz, q^2/dz; q^2)_\infty}{(c_1, c_2, q^2/c_1, q^2/c_2; q^2)_\infty}\, {}_2\psi_2\left[\begin{matrix} a_1, & a_2 \\ & & ; q^2; & z \\ b_1, & b_2 \end{matrix}\right] \\
&= \frac{q^2}{c_1}\frac{(c_1/a_1, c_1/a_2, q^2b_1/c_1, q^2b_2/c_1, dc_1z/q^2, q^4/dc_1z; q^2)_\infty}{(c_1, q^2/c_1, c_1/c_2, q^2c_2/c_1; q^2)_\infty} \\
&\times {}_2\psi_2\left[\begin{matrix} q^2a_1/c_1, & q^2a_2/c_1 \\ & & ; q^2; & z \\ q^2b_1/c_1, & q^2b_2/c_1 \end{matrix}\right] + idem(c_1; c_2). \qquad (3.17)
\end{aligned}$$

(a) Taking $a_1 \to \infty$, $a_2 = -q$, $b_1 = 0$, $b_2 = -q^2$ and $z = (-\frac{q}{a_1})$ in (3.17) and after simplification new representation of $S_{0b}(q)$. is obtained, which is as follows:

$$\begin{aligned}
\frac{(-q^2, -q, q^2/c_1c_2, c_1c_2; q^2)_\infty}{(c_1, c_2, q^2/c_1, q^2/c_2; q^2)_\infty}\, S_{0b}(q) &= \frac{q^2}{c_1}\frac{(-c_1/q, -q^4/c_1, 1/c_2, q^2c_2; q^2)_\infty}{(c_1, q^2/c_1, c_1/c_2, q^2c_2/c_1; q^2)_\infty} \\
&\times \sum_{-\infty}^{\infty} \frac{q^{n(n+2)}(-q^3/c_1; q^2)_n}{c_1^n(-q^4/c_1; q^2)_n} + idem(c_1; c_2). \qquad (3.18)
\end{aligned}$$

Similarly by appropriate selection of parameters, new representations for other complete mock theta functions of order eight have been developed

(b) Taking $a_1 \to \infty$, $a_2 = -q$, $b_1 = 0$, $b_2 = -q^2$ and $z = (-\frac{q^3}{a_1})$ in (3.17), we have new representation of $S_{1b}(q)$.

$$\begin{aligned}
\frac{(-q^2, -q, q^4/c_1c_2, c_1c_2/q^2; q^2)_\infty}{(c_1, c_2, q^2/c_1, q^2/c_2; q^2)_\infty}\, S_{1b}(q) &= \frac{q^2}{c_1}\frac{(-c_1/q, -q^3/c_1, q^2/c_2, c_2; q^2)_\infty}{(c_1, q^2/c_1, c_1/c_2, q^2c_2/c_1; q^2)_\infty} \\
&\times \sum_{-\infty}^{\infty} \frac{q^{n(n+4)}(-q^3/c_1; q^2)_n}{c_1^n(-q^4/c_1; q^2)_n} + idem(c_1; c_2). \qquad (3.19)
\end{aligned}$$

(c) Taking $a_1 \to \infty$, $a_2 = -q^2$, $b_1 = 0$, $b_2 = -q^3$ and $z = (-\frac{q^4}{a_1})$ in (3.17), we have new representation of $T_{0b}(q)$.

$$\frac{(-q^3, -1, q^6/c_1c_2, c_1c_2/q^4; q^2)_\infty}{(c_1, c_2, q^2/c_1, q^2/c_2; q^2)_\infty}\ \frac{(1+q)}{q^2}\ T_{0b}(q) = \frac{q^2}{c_1}\frac{(-c_1/q^2, -q^5/c_1, q^4/c_2, c_2/q^2; q^2)_\infty}{(c_1, q^2/c_1, c_1/c_2, q^2c_2/c_1; q^2)_\infty}$$

$$\times \sum_{-\infty}^{\infty} \frac{q^{n(n+5)}(-q^4/c_1;q^2)_n}{c_1^n(-q^5/c_1;q^2)_n} + idem(c_1;c_2). \tag{3.20}$$

(d) Taking $a_1 \to \infty$, $a_2 = -q^2$, $b_1 = 0$, $b_2 = -q^3$ and $z = (-\frac{q^2}{a_1})$ in (3.17), we have new representation of $T_{1b}(q)$.

$$\frac{(-q^3,-1,q^4/c_1c_2,c_1c_2/q^2;q^2)_\infty}{(c_1,c_2,q^2/c_1,q^2/c_2;q^2)_\infty}\,(1+q)\;T_{1b}(q) = \frac{q^2}{c_1}\frac{(-c_1/q^2,-q^5/c_1,q^2/c_2,c_2;q^2)_\infty}{(c_1,q^2/c_1,c_1/c_2,q^2c_2/c_1;q^2)_\infty}$$
$$\times \sum_{-\infty}^{\infty} \frac{q^{n(n+3)}(-q^4/c_1;q^2)_n}{c_1^n(-q^5/c_1;q^2)_n} + idem(c_1;c_2). \tag{3.21}$$

(e) Taking $a_1 \to \infty$, $a_2 = -q$, $b_1 = iq^2$, $b_2 = -iq^2$ and $z = (-\frac{q}{a_1})$ in (3.17), we have new representation of $U_{0b}(q)$.

$$\frac{(iq^2,-iq^2,-q,q^2/c_1c_2,c_1c_2;q^2)_\infty}{(c_1,c_2,q^2/c_1,q^2/c_2;q^2)_\infty}\,U_{0b}(q) = \frac{q^2}{c_1}\frac{(-c_1/q^2,-q^5/c_1,1/c_2,q^2c_2;q^2)_\infty}{(c_1,q^2/c_1,c_1/c_2,q^2c_2/c_1;q^2)_\infty}$$
$$\times \sum_{-\infty}^{\infty} \frac{q^{n(n+2)}(-q^3/c_1;q^2)_n}{c_1^n(-q^8/c_1^2;q^4)_n} + idem(c_1;c_2). \tag{3.22}$$

(f) Taking $a_1 \to \infty$, $a_2 = -q$, $b_1 = iq^3$, $b_2 = -iq^3$ and $z = (-\frac{q^3}{a_1})$ in (3.17), we have new representation of $U_{1b}(q)$.

$$\frac{(iq^3,-iq^3,-q,q^4/c_1c_2,c_1c_2/q^2;q^2)_\infty}{(c_1,c_2,q^2/c_1,q^2/c_2;q^2)_\infty}\,\frac{(1+q^2)}{q}\;U_{1b}(q) = \frac{q^2}{c_1}\frac{(-c_1/q,iq^5/c_1,-iq^5/c_1,q^2/c_2,c_2;q^2)_\infty}{(c_1,q^2/c_1,c_1/c_2,q^2c_2/c_1;q^2)_\infty}$$
$$\times \sum_{-\infty}^{\infty} \frac{q^{n(n+4)}(-q^3/c_1;q^2)_n}{(-q^{10}/c_1^2;q^4)_n} + idem(c_1;c_2). \tag{3.23}$$

(g) Taking $a_1 \to \infty$, $a_2 = -q$, $b_1 = 0$, $b_2 = q$ and $z = (-\frac{q}{a_1})$ in (3.17), we have new representation of $V_{0b}(q)$.

$$\frac{(q,-q,q^2/c_1c_2,c_1c_2;q^2)_\infty}{(c_1,c_2,q^2/c_1,q^2/c_2;q^2)_\infty}\,\frac{1}{2}\,[V_{0b}(q)+1] = \frac{q^2}{c_1}\frac{(-c_1/q,q^3/c_1,1/c_2,q^2c_2;q^2)_\infty}{(c_1,q^2/c_1,c_1/c_2,q^2c_2/c_1;q^2)_\infty}$$
$$\times \sum_{-\infty}^{\infty} \frac{q^{n(n+2)}(-q^3/c_1;q^2)_n}{c_1^n(q^3/c_1;q^2)_n} + idem(c_1;c_2). \tag{3.24}$$

(h) Taking $a_1 \to \infty$, $a_2 = -q$, $b_1 = 0$, $b_2 = q^3$ and $z = (-\frac{q^3}{a_1})$ in (3.17), we have new representation of $V_{1b}(q)$.

$$\frac{(q^3, -q, q^4/c_1c_2, c_1c_2/q^2; q^2)_\infty}{(c_1, c_2, q^2/c_1, q^2/c_2; q^2)_\infty} \ \frac{(1-q)}{q} \ V_{1b}(q) = \frac{q^2}{c_1} \frac{(-c_1/q, q^5/c_1, q^2/c_2, c_2; q^2)_\infty}{(c_1, q^2/c_1, c_1/c_2, q^2c_2/c_1; q^2)_\infty}$$
$$\times \sum_{-\infty}^{\infty} \frac{q^{n(n+4)}(-q^3/c_1; q^2)_n}{c_1^n (q^5/c_1; q^2)_n} + idem(c_1; c_2). \tag{3.25}$$

4 Certain relations between complete mock theta functions of order eight

In this section relations among the complete mock theta functions of order eight have been developed, which are as follows:

(a)

$$\begin{aligned} S_{0b}(q) &= \sum_{-\infty}^{\infty} \frac{q^{n(n+1)}(-1; q^2)_n}{(-q; q^2)_n} \\ &= \sum_{-\infty}^{\infty} \frac{q^{(n+1)(n+2)}(-1; q^2)_{n+1}}{(-q; q^2)_{n+1}} \\ &= \sum_{-\infty}^{\infty} \frac{q^{(n+1)(n+2)} \ 2 \ (-q^2; q^2)_n}{(-q; q^2)_{n+1}} \\ &= 2\sum_{-\infty}^{\infty} \frac{q^{(n+1)(n+2)}(-q^2; q^2)_n}{(-q; q^2)_{n+1}} \end{aligned}$$

$$S_{0b}(q) = 2 \ T_{0b}(q) \tag{4.1}$$

(b) Putting $c_1 = q^3$ in (3.19), we get

$$\frac{(-q^2, -q, q/c_2, qc_2; q^2)_\infty}{(q^3, c_2, 1/q, q^2/c_2; q^2)_\infty} \ S_{1b}(q) = \frac{1}{q} \frac{(-q^2, -1, q^2/c_2, c_2; q^2)_\infty}{(q^3, 1/q, q^3/c_2, c_2/q; q^2)_\infty} \ S_{0b}(q)$$
$$+ \ \frac{q^2}{c_2} \frac{(-c_2/q, -q^3/c_2, 1/q, q^3; q^2)_\infty}{(c_2, q^2/c_2, c_2/q^3, q^5/c_2; q^2)_\infty} \sum_{-\infty}^{\infty} \frac{q^{n(n+4)}(-q^3/c_2; q^2)_n}{c_2^n(-q^4/c_2; q^2)_n} \tag{4.2}$$

(c) Putting $c_1 = q^3$ in (3.20), we get

$$\frac{(-q^3, -1, q^3/c_2, c_2/q; q^2)_\infty}{(q^3, c_2, 1/q, q^2/c_2; q^2)_\infty} \ \frac{(1+q)}{q^2} \ T_{0b}(q) = \frac{1}{q} \frac{(-q, -q^2, q^4/c_2, c_2/q^2; q^2)_\infty}{(q^3, 1/q, q^3/c_2, c_2/q; q^2)_\infty} \ 2T_{1b}(q)$$
$$+ \ \frac{q^2}{c_2} \frac{(-c_2/q^2, -q^5/c_2, q, q; q^2)_\infty}{(c_2, q^2/c_2, c_2/q^3, q^5/c_1; q^2)_\infty} \sum_{-\infty}^{\infty} \frac{q^{n(n+5)}(-q^4/c_2; q^2)_n}{c_2^n(-q^5/c_2; q^2)_n} \tag{4.3}$$

(d) By putting $c_1 = q^3$ in (3.21) with the expression (3.1), one can obtain a relation between complete mock theta functions $T_{1b}(q)$ and $S_{0b}(q)$ as given below:

$$\frac{(-q^3, -1, q/c_2, qc_2; q^2)_\infty}{(q^3, c_2, 1/q, q^2/c_2; q^2)_\infty} \ (1+q) \ T_{1b}(q) = \frac{1}{q} \frac{(-q, -q^2, q^2/c_2, c_2; q^2)_\infty}{(q^3, 1/q, q^3/c_2, c_2/q; q^2)_\infty} \ S_{0b}(q)$$

$$+\frac{q^2}{c_2}\frac{(-c_2/q^2,-q^5/c_2,1/q,q^3;q^2)_\infty}{(c_2,q^2/c_2,c_2/q^3,q^5/c_2;q^2)_\infty}\sum_{-\infty}^{\infty}\frac{q^{n(n+3)}(-q^4/c_2;q^2)_n}{c_2^n(-q^5/c_2;q^2)_n} \tag{4.4}$$

5 Certain Relations for complete mock theta functions and mock theta functions of order eight

We have

$$\begin{aligned} S_{0b}(q) &= \sum_{n=-\infty}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n} = \sum_{0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n} + \sum_{-1}^{-\infty}\frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n} \\ &= \sum_{0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n} + \sum_{1}^{\infty}\frac{q^{(n-1)^2+3(n-1)+2}(-1;q^2)_n}{(-q;q^2)_n} \\ &= \sum_{0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n} + \frac{2}{(1+q)}\sum_{0}^{\infty}\frac{q^{n^2+3n+2}(-q^2;q^2)_n}{(-q^3;q^2)_n} \\ &= \sum_{0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(-q^2;q^2)_n} + \ 2\sum_{0}^{\infty}\frac{q^{(n+1)(n+2)}(-q^2;q^2)_n}{(-q;q^2)_{n+1}} \end{aligned}$$

In accordance with the definition of mock theta functions of order eight given in the §2, we have

$$S_{0b}(q) = S_0(q) + \ 2\ T_0(q) \tag{5.1}$$

Similarly, we have established other relations for complete mock theta functions and mock theta functions of order eight, which are as follows:

$$S_{1b}(q) = S_1(q) + \ 2\ T_1(q) \tag{5.2}$$

$$T_{0b}(q) = T_0(q) + \frac{1}{2}S_0(q) \tag{5.3}$$

$$T_{1b}(q) = T_1(q) + \frac{1}{2}S_1(q) \tag{5.4}$$

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