Some results associated with distortion bounds and coefficient inequalities for certain new subclasses of analytic functions

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Abstract

The author introduces two new subclasses of functions which are analytic in the open unit disk. He obtains coefficient inequalities for functions belonging to this class. Furthermore, he gives some results associated with distortion bounds.

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1 Introduction

Let A denote the class of functions $f(z)$ normalized in the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $U := \{z : z \in \mathbb{C}, |z| < 1\}$. Further, let S denote the subclass of A consisting of all functions $f(z)$ which are also univalent in U.

We denote by $S^*(\alpha)$ and $K(\alpha)$, the familiar subclasses of A consisting of functions which are, respectively, *starlike of order α in U, convex of order α in U.* Thus, by definition, we have

$$S^*(\alpha) := \left\{ f : f \in A, \Re e\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ 0 \le \alpha < 1, z \in U \right\}, \tag{1.2}$$

$$K(\alpha) := \left\{ f : f \in A, \Re e\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,\ 0 \le \alpha < 1, z \in U \right\}. \tag{1.3}$$

It is easily observed from the definitions (1.2)-(1.3) that

$$f(z) \in K(\alpha) \Leftrightarrow zf'(z) \in S^*(\alpha).$$

(See for details, [2] and [3].)

Silverman [6] gave the following coefficient inequalities for the function classes $S^*(\alpha)$ and $K(\alpha)$.

Theorem 1.1 ([6]) If $f(z) \in A$ satisfies the following coefficient inequality:

$$\sum_{n=2}^{\infty} (n - \alpha)|a_n| \le 1 - \alpha\,(0 \le \alpha < 1), \tag{1.4}$$

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then

$$\left|\frac{zf'(z)}{f(z)}-1\right|<1-\alpha(z\in U,0\leq\alpha<1) \tag{1.5}$$

that is, that $f(z)\in S^*(\alpha)$.

Theorem 1.2 ([6]) If $f(z)\in A$ satisfies the following coefficient inequality:

$$\sum_{n=2}^{\infty}n\,(n-\alpha)\,|a_n|\leq 1-\alpha\ \ (0\leq\alpha<1)\,, \tag{1.6}$$

then

$$\left|\frac{zf''(z)}{f'(z)}\right|<1-\alpha\ (z\in U,0\leq\alpha<1) \tag{1.7}$$

that is, that $f(z)\in K(\alpha)$.

Let $A(n)$ denote the class of functions of the form

$$f(z)=z-\sum_{k=n+1}^{\infty}a_kz^k\quad (a_k\geq 0,\ n\in\mathbb{N}=\{1,2,3,...\}) \tag{1.8}$$

that are analytic in the unit disk U. A function $f(z)\in A(n)$ is said to be in the class $P(n,\lambda,\alpha)$ if it satisfies

$$\Re e\left\{\frac{zf'(z)+\lambda z^2f''(z)}{\lambda zf'(z)+(1-\lambda)\,f(z)}\right\}>\alpha \tag{1.9}$$

for some α $(0\leq\alpha<1)$, λ $(0\leq\lambda\leq 1)$ and for all $z\in U$.

Altıntas [1] gave the following coefficient inequalities for the function class $P(n,\lambda,\alpha)$.

Theorem 1.3 A function $f(z)\in A(n)$ is in the class $P(n,\lambda,\alpha)$ if and only if

$$\sum_{k=n+1}^{\infty}(k-\alpha)\,(\lambda k-\lambda+1)\,a_k\leq 1-\alpha. \tag{1.10}$$

More recently, Owa, Ochiai and Srivastava [4], (see also [5]) considered the subclass $M(\alpha)$ of the class A consisting of functions $f(z)$ such that

$$\left|\frac{f(z)}{zf'(z)}-\frac{1}{2\alpha}\right|<\frac{1}{2\alpha}\quad (z\in U,0<\alpha<1)\,. \tag{1.11}$$

Owa, Ochiai and Srivastava [4] proved the following theorems

Theorem 1.4 Let $0<\alpha<1$. If $f(z)\in A$ satisfies the following coefficient inequality:

$$\sum_{n=2}^{\infty}(n-\alpha)\,|a_n|\leq\frac{1}{2}\,(1-|1-2\alpha|)=\left\{\begin{array}{cc}\alpha; & \left(0<\alpha\leq\frac{1}{2}\right)\\ 1-\alpha; & \left(\frac{1}{2}\leq\alpha<1\right)\end{array}\right.\ , \tag{1.12}$$

then $f(z) \in \mathrm{M}(\alpha)$.

For $f(z)$ belonging to A, Salagean [5] has introduced the following operator called the Salagean operator:

$$
\begin{aligned}
&D^0 f(z) = f(z)\\
&D^1 f(z) = Df(z) = zf^{'}(z) = z + \textstyle\sum_{n=2}^{\infty} na_n z^n\\
&\vdots\\
&D^{\Omega} f(z) = D(D^{\Omega-1} f(z)) = z(z + \textstyle\sum_{n=2}^{\infty} n^{\Omega-1} a_n z^n)^{'} = z + \textstyle\sum_{n=2}^{\infty} n^{\Omega} a_n z^n
\end{aligned}
$$

where $\Omega \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

In this paper, we consider a new subclass $\mathrm{M}(\alpha, \lambda, \Omega)$ of the class A consisting of functions $f(z)$ such that

$$
\left|\frac{(1-\lambda)(D^{\Omega} f(z)) + \lambda(D^{\Omega+1} f(z))}{(1-\lambda) z(D^{\Omega} f(z))' + \lambda z(D^{\Omega+1} f(z))'} - \frac{1+\lambda}{2\alpha}\right| < \frac{1+\lambda}{2\alpha} \tag{1.13}
$$

$z \in U,\ 0 < \alpha < 1,\ 0 \leq \lambda < 1,\ \Omega \in \mathbb{N}_0$.

Now, we shall give a sufficient coefficient inequality for functions to belong to the class $\mathrm{M}(\alpha, \lambda, \Omega)$.

2 The coefficient inequality for the class $M(\alpha, \lambda, \Omega)$

Theorem 2.1 Let $0 < \alpha < 1$ and $0 \leq \lambda < 1$. If $f(z) \in A$ satisfies the following coefficient inequality:

$$
\sum_{n=2}^{\infty} n^{\Omega} (\lambda n + 1 - \lambda) \{|2\alpha - (1+\lambda) n| + (1+\lambda) n\} |a_n| \leq (1+\lambda) - |2\alpha - (1+\lambda)|
$$

$$
= \begin{cases} 2\alpha; & 0 < \alpha \leq \frac{\lambda+1}{2} \\ 2(\lambda + 1 - \alpha); & \frac{\lambda+1}{2} \leq \alpha < 1 + \lambda \end{cases} \tag{2.1}
$$

then $f(z) \in \mathrm{M}(\alpha, \lambda, \Omega)$.

Proof. By virtue of the condition (1.13), we have to show that

$$
\left|\left(\frac{2\alpha}{1+\lambda}\right) \frac{(1-\lambda)(D^{\Omega} f(z)) + \lambda(D^{\Omega+1} f(z))}{(1-\lambda) z(D^{\Omega} f(z))' + \lambda z(D^{\Omega+1} f(z))'} - 1\right| < 1 \tag{2.2}
$$

We first observe that

$$
\left|\left(\frac{2\alpha}{1+\lambda}\right) \frac{(1-\lambda)(D^{\Omega} f(z)) + \lambda(D^{\Omega+1} f(z))}{(1-\lambda) z(D^{\Omega} f(z))' + \lambda z(D^{\Omega+1} f(z))'} - 1\right|
$$

$$
= \left|\frac{2\alpha\left[(1-\lambda)(D^{\Omega} f(z)) + \lambda(D^{\Omega+1} f(z))\right] - (\lambda+1)\left[(1-\lambda) z(D^{\Omega} f(z))^{'} + \lambda z(D^{\Omega+1} f(z))^{'}\right]}{(\lambda+1)\left[(1-\lambda) z(D^{\Omega} f(z))' + \lambda z(D^{\Omega+1} f(z))'\right]}\right|
$$

$$
= \left|\frac{[2\alpha - (1+\lambda)] z + \sum_{n=2}^{\infty} n^{\Omega} [\lambda n + (1-\lambda)] [2\alpha - (1+\lambda) n] a_n z^n}{(1+\lambda) z + \sum_{n=2}^{\infty} n^{\Omega+1} (1+\lambda) [\lambda n + (1-\lambda)] a_n z^n}\right|
$$

$$\leq \frac{|2\alpha-(1+\lambda)|+\sum_{n=2}^{\infty}n^{\Omega}\left[\lambda n+(1-\lambda)\right]|2\alpha-(1+\lambda)n|\,|a_n|\,|z|^{n-1}}{(1+\lambda)-\sum_{n=2}^{\infty}n^{\Omega+1}(1+\lambda)\left[\lambda n+(1-\lambda)\right]|a_n|\,|z|^{n-1}}$$

$$< \frac{|2\alpha-(1+\lambda)|+\sum_{n=2}^{\infty}n^{\Omega}\left[\lambda n+(1-\lambda)\right]|2\alpha-(1+\lambda)n|\,|a_n|}{(1+\lambda)-\sum_{n=2}^{\infty}n^{\Omega+1}(1+\lambda)\left[\lambda n+(1-\lambda)\right]|a_n|}. \tag{2.3}$$

By using the coefficient inequality (2.1), we can write

$$\sum_{n=2}^{\infty}n^{\Omega}(\lambda n+1-\lambda)|2\alpha-(1+\lambda)n|\,|a_n|+\sum_{n=2}^{\infty}n^{\Omega+1}(\lambda n+1-\lambda)(1+\lambda)|a_n|\leq(1+\lambda)-|2\alpha-(1+\lambda)|$$

or

$$\sum_{n=2}^{\infty}n^{\Omega}(\lambda n+1-\lambda)|2\alpha-(1+\lambda)n|\,|a_n|\leq(1+\lambda)-|2\alpha-(1+\lambda)|-\sum_{n=2}^{\infty}n^{\Omega+1}(1+\lambda)(\lambda n+1-\lambda)|a_n|.$$

By using this last inequality in (2.3), we obtain

$$\left|\left(\frac{2\alpha}{1+\lambda}\right)\frac{(1-\lambda)(D^{\Omega}f(z))+\lambda(D^{\Omega+1}f(z))}{(1-\lambda)z(D^{\Omega}f(z))'+\lambda z(D^{\Omega+1}f(z))'}-1\right|$$

$$< \frac{|2\alpha-(1+\lambda)|+(1+\lambda)-|2\alpha-(1+\lambda)|-\sum_{n=2}^{\infty}n^{\Omega+1}(1+\lambda)\{(1-\lambda)+\lambda n\}|a_n|}{(1+\lambda)-\sum_{n=2}^{\infty}n^{\Omega+1}(1+\lambda)\{(1-\lambda)+\lambda n\}|a_n|}=1$$

or

$$\left|\frac{(1-\lambda)(D^{\Omega}f(z))+\lambda(D^{\Omega+1}f(z))}{(1-\lambda)z(D^{\Omega}f(z))'+\lambda z(D^{\Omega+1}f(z))'}-\frac{1+\lambda}{2\alpha}\right|<\frac{1+\lambda}{2\alpha},$$

that is, $f(z)\in \mathrm{M}(\alpha,\lambda,\Omega)$.

Theorem 2.2 If $f(z)\in \mathrm{M}(\alpha,\lambda,\Omega)$, then $Re\left\{\frac{(1-\lambda)z(D^{\Omega}f(z))'+\lambda z(D^{\Omega+1}f(z))'}{(1-\lambda)D^{\Omega}f(z)+\lambda D^{\Omega+1}f(z)}\right\}>\frac{\alpha}{1+\lambda}$.

Proof. Let $\varphi(z)=\frac{zF'(z)}{F(z)}$ and $F(z)=(1-\lambda)D^{\Omega}f(z)+\lambda D^{\Omega+1}f(z)$, $f(z)$ being an element of $\mathrm{M}(\alpha,\lambda,\Omega)$, we write

$$\left|\frac{1}{\varphi(z)}-\frac{1+\lambda}{2\alpha}\right|<\frac{1+\lambda}{2\alpha}.$$

By simple calculation, we can obtain

$$\left|\frac{1}{\varphi(z)}-\frac{1+\lambda}{2\alpha}\right|<\frac{1+\lambda}{2\alpha}\Rightarrow\left|\frac{1}{\varphi(z)}-\frac{1+\lambda}{2\alpha}\right|^2<\left(\frac{1+\lambda}{2\alpha}\right)^2\Rightarrow$$

$$\left|\frac{2\alpha-(1+\lambda)\varphi(z)}{2\alpha\varphi(z)}\right|^2<\left(\frac{1+\lambda}{2\alpha}\right)^2\Rightarrow|2\alpha-(1+\lambda)\varphi(z)|^2<(1+\lambda)^2|\varphi(z)|^2\Rightarrow$$

$$(2\alpha-(1+\lambda)\varphi(z))\left(\overline{2\alpha-(1+\lambda)\varphi(z)}\right)<(1+\lambda)^2\varphi(z)\left(\overline{\varphi(z)}\right)\Rightarrow$$

$$4\alpha^2<2\alpha(1+\lambda)\left\{\varphi(z)+\overline{\varphi(z)}\right\}\Rightarrow$$

$$4\alpha^2 < 2\alpha\,(1+\lambda)\,\{2Re\varphi\,(z)\} \Rightarrow Re\,\{\varphi\,(z)\} > \frac{\alpha}{1+\lambda}.$$

Thus, we obtain $Re\left\{\frac{(1-\lambda)z(D^{\Omega}f(z))'+\lambda z(D^{\Omega+1}f(z))'}{(1-\lambda)D^{\Omega}f(z)+\lambda D^{\Omega+1}f(z)}\right\} > \frac{\alpha}{1+\lambda}$.

Remark 2.3 If $f(z) \in \mathrm{M}\,(\alpha,\lambda,\Omega)$, then $F\,(z) \in S^*\left(\frac{\alpha}{1+\lambda}\right)$.

Remark 2.4 Let denote by $M^*\,(\alpha,\lambda,\Omega)$ the subclass of the class $M\,(\alpha,\lambda,\Omega)$ which satisfies the coefficient inequality (2.1) for some α and which consists of the$f\,(z) \in M\,(\alpha,\lambda,\Omega)$.

3 Some results associated with distortion bounds

In 2006, S.Owa, K.Ochiai and H.M.Srivastava [4] have represented the integro-differential operator for a function $f\,(z) \in A$ which is denoted in the form of $I_k f\,(z)$ and defined as shown below:

$$I_{-1}f\,(z) = f'\,(z)\,, I_0 f\,(z) = f\,(z)$$

and

$$I_k f\,(z) = \int_0^z I_{k-1}f\,(t)\,dt$$

for $k \in \mathbb{N} = \{1,2,3,...\}$.

Using this definition in (1.1), we obtain

$$I_k f(z) = \frac{1}{(k+1)!}z^{k+1} + \sum_{n=2}^{\infty}\frac{n!}{(n+k)!}a_n z^{n+k} \tag{3.1}$$

Now, we should give some results related to the distortion bounds for the functions belonging to the subclass $M^*\,(\alpha,\lambda,\Omega)$ using integro-differential operator.

Theorem 3.1 If $f\,(z) \in M^*\,(\alpha,\lambda,\Omega)$, then we have following inequality:

$$\frac{1}{(k+1)!}\,|z|^{k+1} - \frac{(1+\lambda)-|2\alpha-(1+\lambda)|}{(k+2)!2^{\Omega}\,(1+\lambda)\,\{|\alpha-(1+\lambda)|+(1+\lambda)\}}\,|z|^{k+2} \leq |I_k f\,(z)|$$

$$\leq \frac{1}{(k+1)!}\,|z|^{k+1} + \frac{(1+\lambda)-|2\alpha-(1+\lambda)|}{(k+2)!2^{\Omega}\,(1+\lambda)\,\{|\alpha-(1+\lambda)|+(1+\lambda)\}}\,|z|^{k+2}$$

for $z \in U$, $k \in \mathbb{N}\cup\{-1,0\}$, $0 < \alpha < 1$, $0 \leq \lambda < 1$ and $\Omega \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$.

Proof: If the absolute of both hands of the equality (3.1) is taken and the triangle inequality is applied, we obtain

$$|I_k f(z)| = \left|\frac{1}{(k+1)!}z^{k+1} + \sum_{n=2}^{\infty}\frac{n!}{(n+k)!}a_n z^{n+k}\right| \leq \frac{1}{(k+1)!}\,|z|^{k+1} + |z|^{k+2}\sum_{n=2}^{\infty}\frac{n!}{(n+k)!}\,|a_n|$$

$$< \frac{1}{(k+1)!}\,|z|^{k+1} + |z|^{k+2}\sum_{n=2}^{\infty}\frac{n!}{(n+k)!}\,|a_n|\,. \tag{3.2}$$

Besides, we can write

$$\frac{1}{2!}(k+2)!2^{\Omega}(1+\lambda)\{|2(1+\lambda)-2\alpha|+2(1+\lambda)\}\sum_{n=2}^{\infty}\frac{n!}{(n+k)!}|a_n|$$
$$\leq\frac{1}{2!}(k+2)!2^{\Omega}(1+\lambda)\{|2(1+\lambda)-2\alpha|+2(1+\lambda)\}\frac{2!}{(k+2)!}\sum_{n=2}^{\infty}|a_n|$$
$$\leq\sum_{n=2}^{\infty}n^{\Omega}(\lambda n+(1-\lambda))\{|2\alpha-(1+\lambda)n|+n(1+\lambda)\}|a_n|$$
$$\leq(1+\lambda)-|2\alpha-(1+\lambda)|$$

or

$$\sum_{n=2}^{\infty}\frac{n!}{(n+k)!}|a_n|\leq\frac{(1+\lambda)-|2\alpha-(1+\lambda)|}{(k+2)!2^{\Omega}(1+\lambda)\{|\alpha-(1+\lambda)|+(1+\lambda)\}}.$$

Using this last inequality in (3.2), we obtain

$$|I_k f(z)|\leq\frac{1}{(k+1)!}|z|^{k+1}+\frac{(1+\lambda)-|2\alpha-(1+\lambda)|}{(k+2)!2^{\Omega}(1+\lambda)\{|\alpha-(1+\lambda)|+(1+\lambda)\}}|z|^{k+2}. \qquad (3.3)$$

With similar operations, we can write

$$|I_k f(z)|\geq\frac{1}{(k+1)!}|z|^{k+1}-\frac{(1+\lambda)-|2\alpha-(1+\lambda)|}{(k+2)!2^{\Omega}(1+\lambda)\{|\alpha-(1+\lambda)|+(1+\lambda)\}}|z|^{k+2}. \qquad (3.4)$$

By joining (3.3) and (3.4), we obtain

$$\frac{1}{(k+1)!}|z|^{k+1}-\frac{(1+\lambda)-|2\alpha-(1+\lambda)|}{(k+2)!2^{\Omega}(1+\lambda)\{|\alpha-(1+\lambda)|+(1+\lambda)\}}|z|^{k+2}\leq|I_k f(z)|$$
$$\leq\frac{1}{(k+1)!}|z|^{k+1}+\frac{(1+\lambda)-|2\alpha-(1+\lambda)|}{(k+2)!2^{\Omega}(1+\lambda)\{|\alpha-(1+\lambda)|+(1+\lambda)\}}|z|^{k+2}.$$

Taking $k=-1,0,1$ in the Theorem 3.1, we obtain the following Corollary 3.2.

Corollary 3.2 If$f(z)\in M^{*}(\alpha,\lambda,\Omega)$, then we have the following inequalities:

$$1-\frac{(1+\lambda)-|2\alpha-(1+\lambda)|}{2^{\Omega}(1+\lambda)\{|\alpha-(1+\lambda)|+(1+\lambda)\}}|z|\leq\left|f^{'}(z)\right|\leq1+\frac{(1+\lambda)-|2\alpha-(1+\lambda)|}{2^{\Omega}(1+\lambda)\{|\alpha-(1+\lambda)|+(1+\lambda)\}}|z|$$

for $k=-1$,

$$|z|-\frac{(1+\lambda)-|2\alpha-(1+\lambda)|}{2^{\Omega+1}(1+\lambda)\{|\alpha-(1+\lambda)|+(1+\lambda)\}}|z|^{2}\leq|f(z)|$$
$$\leq|z|+\frac{(1+\lambda)-|2\alpha-(1+\lambda)|}{2^{\Omega+1}(1+\lambda)\{|\alpha-(1+\lambda)|+(1+\lambda)\}}|z|^{2}$$

for $k=0$ and

$$\frac{1}{2}\left|z\right|^{2}-\frac{(1+\lambda)-|2\alpha-(1+\lambda)|}{6.2^{\Omega}\left(1+\lambda\right)\left\{|\alpha-(1+\lambda)|+(1+\lambda)\right\}}\left|z\right|^{3}\leq\left|I_{2}f\left(z\right)\right|$$
$$\leq\frac{1}{2}\left|z\right|^{2}+\frac{(1+\lambda)-|2\alpha-(1+\lambda)|}{6.2^{\Omega}\left(1+\lambda\right)\left\{|\alpha-(1+\lambda)|+(1+\lambda)\right\}}\left|z\right|^{3}$$

for $k=1$.

Putting $\lambda=0$, $\Omega=0$ in Theorem 3.1, we get the Theorem 3.3 given by Owa, Ochiai and Srivastava [4].

Theorem 3.3 If $f(z)\in M^{*}\left(\alpha\right)$, then we have the following inequalities:

$$\frac{1}{(k+1)!}\left|z\right|^{k+1}-\frac{1-|1-2\alpha|}{(k+2)!\left(2-\alpha\right)}\left|z\right|^{k+2}\leq\left|I_{k}f\left(z\right)\right|\leq\frac{1}{(k+1)!}\left|z\right|^{k+1}+\frac{1-|1-2\alpha|}{(k+2)!\left(2-\alpha\right)}\left|z\right|^{k+2}$$

for $z\in U$, $k\in\mathbb{N}\cup\{-1,0\}$.

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