On an inequality of G. H. Hardy for convex function with fractional integrals and fractional derivatives

Sajid Iqbal[1], Kristina Krulić Himmelreich[2], Josip Pečarić[1,3]

[1] Abdus Salam School of Mathematical Sciences, GC University Lahore, Pakistan

[2] Faculty of Textile Technology, University of Zagreb, Prilaz baruna Filipovića 28a, 10000 Zagreb, Croatia

[3] Faculty of Textile Technology, University of Zagreb, Prilaz baruna Filipovića 28a, 10000 Zagreb, Croatia

E-mail: sajid_uos2000@yahoo.com, kkrulic@ttf.hr, pecaric@element.hr

Abstract

The main goal of this paper is to give applications of Hardy-type inequalities. We construct new inequalities of G. H. Hardy for convex function using different types of fractional integrals and fractional derivatives.

2000 Mathematics Subject Classification. **26D15**. 26D10.
Keywords. Convex function, kernel, fractional derivative, fractional integrals..

1 Introduction

In recent years applications of fractional calculus in mathematical inequalities have great importance. Many authors use fractional integrals and fractional derivatives to construct new integral inequalities. The Hardy inequality has fundamental importance in the mathematical analysis and lot of rich literature and information concerning Hardy-type inequalities and related results for Riemann-Liouville operators can be found in [4], [6], [9], [12], [15], [16], [17], [18], [20] including the references cited therein. Many mathematicians gave generalizations and improvements of Hardy's inequalities. In this paper, we establish some more general inequalities of G. H. Hardy for different kinds of fractional integrals and fractional derivatives like Riemann-Liouville fractional integrals, Caputo fractional derivative, fractional integral of a function with respect to an increasing function, Hadamard-type fractional integrals and Erdélyi-Kober fractional integrals. We will use different weights in this construction to obtain new inequalities of G. H. Hardy for convex functions. Such type of results are discussed in [9](see also [6]). Our particular interest is to give inequalities of G. H. Hardy and discover results which involve fractional integrals and fractional derivatives.

Let us recall some facts about fractional derivatives needed in the sequel, for more details see e.g. [1], [7].

Let $0 < a < b \leq \infty$. By $C^m([a,b])$ we denote the space of all functions on $[a,b]$ which have continuous derivatives up to order m, and $AC([a,b])$ is the space of all absolutely continuous functions on $[a,b]$. By $AC^m([a,b])$ we denote the space of all functions $g \in C^{m-1}([a,b])$ with $g^{(m-1)} \in AC([a,b])$. For any $\alpha \in \mathbb{R}$ we denote by $[\alpha]$ the integral part of α (the integer k satisfying $k \leq \alpha < k+1$) and $\lceil \alpha \rceil$ is the ceiling of α ($\min\{n \in \mathbb{N}, n \geq \alpha\}$). By $L_1(a,b)$ we denote the space of all functions integrable on the interval (a,b), and by $L_\infty(a,b)$ the set of all functions measurable and essentially bounded on (a,b). Clearly, $L_\infty(a,b) \subset L_1(a,b)$.

Let us recall the well known definitions of Riemann-Liouville fractional integrals, see [13] and [5].

Tbilisi Mathematical Journal 6 (2013), pp. 1–12.
Tbilisi Centre for Mathematical Sciences & College Publications.

Received by the editors: 25 April 2012.
Accepted for publication: 08 February 2013.

Let $[a,b], (-\infty < a < b < \infty)$ be a finite interval on real axis $\mathbb{R}$. The Riemann-Liouville fractional integrals $I^{\alpha}_{a^+}f$ and $I^{\alpha}_{b^-}f$ of order $\alpha > 0$ are defined by

$$I^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(y)(x-y)^{\alpha-1}dy, \quad (x > a),$$

and

$$I^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b f(y)(y-x)^{\alpha-1}dy, \quad (x < b).$$

Here $\Gamma(\alpha)$ is the Gamma function. These integrals are called the left-sided and right-sided fractional integrals respectively. It is known that the fractional integral operators are bounded in $L_p(a,b)$, $-\infty < a < b < \infty$, $1 \leq p \leq \infty$, that is

$$\|I^{\alpha}_{a^+}f\|_p \leq K\|f\|_p, \qquad \|I^{\alpha}_{b^-}f\|_p \leq K\|f\|_p \tag{1.1}$$

where

$$K = \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}.$$

G. H. Hardy proved the inequality (1.1) involving left-sided fractional integral in one of his initial paper, see [8]. The calculation for the constant K is hidden inside the proof. The inequality (1.1) has been investigated in many ways for convex function as well as superquadratic functions (see [9], [10] and [11]). Inequality (1.1) refers to as inequality of G. H. Hardy.

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with σ-finite measures and A_k be an integral operator defined by

$$A_k f(x) := \frac{1}{K(x)} \int_{\Omega_2} k(x,y)f(y)d\mu_2(y), \tag{1.2}$$

where $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is measurable and non-negative kernel, f is measurable function on Ω_2, and

$$K(x) := \int_{\Omega_2} k(x,y)d\mu_2(y), \qquad x \in \Omega_1. \tag{1.3}$$

Throughout the paper, we consider that $K(x) > 0$ a.e. on Ω_1.

The following Theorem is given in [14].

Theorem 1.1. Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with σ-finite measures, u be a weight function on Ω_1, k be a non-negative measurable function on $\Omega_1 \times \Omega_2$ and K be defined on Ω_1 by (1.3). Let $0 < p \leq q < \infty$ and that the function $x \mapsto u(x)\left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}}$ is integrable on Ω_1 for each fixed $y \in \Omega_2$, and that v is defined on Ω_2 by

$$v(y) := \left(\int_{\Omega_1} u(x) \left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}} d\mu_1(x) \right)^{\frac{p}{q}} < \infty. \tag{1.4}$$

If Φ is a non-negative convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\left(\int_{\Omega_1} u(x) [\Phi(A_k f(x))]^{\frac{q}{p}} d\mu_1(x) \right)^{\frac{1}{q}} \leq \left(\int_{\Omega_2} v(y) \Phi(f(y)) \, d\mu_2(y) \right)^{\frac{1}{p}}, \tag{1.5}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$, such that $Imf \subseteq I$, where A_k is defined by (1.2).

Throughout this paper, all measures are assumed to be positive, all functions are assumed to be positive and measurable and expressions of the form $0 \cdot \infty$, $\frac{\infty}{\infty}$ and $\frac{0}{0}$ are taken to be equal to zero. Moreover, by a weight $u = u(x)$ we mean a nonnegative measurable function on the actual interval or more general set.

The paper is organized as follows: After introduction, in Section 2, we prove some new inequalities of G. H. Hardy using different kind of fractional derivatives and fractional integrals.

2 The Main Results

Using Theorem 1.1, we will give some special cases for different fractional integrals and fractional derivatives to establish new inequalities of G. H. Hardy.

We continue with definitions and some properties of the *fractional integrals of a function f with respect to given function g*. For details see e.g. [13, p. 99]:

Let (a, b), $-\infty \leq a < b \leq \infty$ be a finite or infinite interval of the real line $\mathbb{R}$ and $\alpha > 0$. Also let g be an increasing function on (a, b) and g' be a continuous function on (a, b). The left- and right-sided fractional integrals of a function f with respect to another function g in $[a, b]$ are given by

$$(I^{\alpha}_{a+;g} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t) f(t) dt}{[g(x) - g(t)]^{1-\alpha}}, \quad x > a$$

and

$$(I^{\alpha}_{b-;g} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t) f(t) dt}{[g(t) - g(x)]^{1-\alpha}}, \quad x < b,$$

respectively.

Our first result involving fractional integral of f with respect to another increasing function g is given in the following theorem and from this we recover the case of Riemann–Liouville fractional integrals and Hadamard fractional integrals.

Theorem 2.1. Let $0 < p \leq q < \infty$, $f \geq 0$, u be a weight function on (a, b), g be increasing function on (a, b) such that g' be continuous on (a, b), $I^{\alpha}_{a+;g} f$ denotes the left sided fractional integral of f

with respect to another increasing function g. Let v be defined on (a,b) by

$$v(y) := \alpha g'(y) \left(\int_y^b u(x) \left(\frac{(g(x)-g(y))^{\alpha-1}}{(g(x)-g(a))^{\alpha}} \right)^{\frac{q}{p}} dx \right)^{\frac{p}{q}} < \infty. \tag{2.1}$$

If Φ is a non-negative convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\left(\int_a^b u(x) \left[\Phi\left(\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x) \right) \right]^{\frac{q}{p}} dx \right)^{\frac{1}{q}} \leq \left(\int_a^b v(y) \Phi\left(f(y)\right) dy \right)^{\frac{1}{p}} \tag{2.2}$$

holds for all measurable functions $f : (a,b) \to \mathbb{R}$, such that $Imf \subseteq I$.

Proof. Applying Theorem 1.1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \frac{g'(y)}{\Gamma(\alpha)(g(x)-g(y))^{1-\alpha}}, & a \leq y \leq x\,; \\ 0, & x < y \leq b, \end{cases}$$

we get that $K(x) = \frac{1}{\Gamma(\alpha+1)}(g(x)-g(a))^{\alpha}$, $A_k f(x) = \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^{\alpha}} I^{\alpha}_{a+;g} f(x)$ and the inequality in (1.5) reduces to (2.2) with v defined by (2.1). Q.E.D.

Corollary 2.2. Let $0 < p \leq q < \infty$, $s \geq 1$, $\alpha > 1 - \frac{p}{q}$, $f \geq 0$, g be increasing function on (a,b) such that g' be continuous on (a,b), $I^{\alpha}_{a+;g} f$ denotes the left sided fractional integral of f with respect to another increasing function g. Then the inequality

$$\left(\int_a^b g'(x) (I^{\alpha}_{a+;g} f(x))^{\frac{sq}{p}} dx \right)^{\frac{1}{q}} \leq \frac{\alpha^{\frac{1}{p}} (g(b)-g(a))^{\frac{q(\alpha s-1)+p}{pq}}}{((\alpha-1)\frac{q}{p}+1)^{\frac{1}{q}} (\Gamma(\alpha+1))^{\frac{s}{p}}} \left(\int_a^b g'(y) f^s(y) dy \right)^{\frac{1}{p}} \tag{2.3}$$

holds.

Proof. For particular convex function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$, $\Phi(x) = x^s, s \geq 1$ and weight function $u(x) = g'(x)(g(x)-g(a))^{\frac{\alpha q}{p}}$, $x \in (a,b)$ in (2.2), we get $v(y) = (\alpha g'(y)(g(b)-g(y))^{\alpha-1+\frac{p}{q}})/(((\alpha-1)\frac{q}{p}+1)^{\frac{p}{q}})$ and (2.2) becomes

$$\left(\int_a^b g'(x)(g(x)-g(a))^{\frac{\alpha q}{p}(1-s)} (I^{\alpha}_{a+;g} f(x))^{\frac{sq}{p}} dx \right)^{\frac{1}{q}}$$

$$\leq \frac{\alpha^{\frac{1}{p}}}{((\alpha-1)\frac{q}{p}+1)^{\frac{1}{q}} (\Gamma(\alpha+1))^{\frac{s}{p}}} \left(\int_a^b g'(y)(g(b)-g(y))^{\alpha-1+\frac{p}{q}} f^s(y) dy \right)^{\frac{1}{p}}.$$

Since $(g(x)-g(a))^{\frac{\alpha q}{p}(1-s)} \geq (g(b)-g(a))^{\frac{\alpha q}{p}(1-s)}$ and $(g(b)-g(y))^{\alpha-1+\frac{p}{q}} \leq (g(b)-g(a))^{\alpha-1+\frac{p}{q}}$, $\alpha > 1 - \frac{p}{q}$ we obtain (2.6). Q.E.D.

Remark 2.3. Similar result can be obtained for the right sided fractional integral of f with respect to another increasing function g, but here we omit the details.

Here, we give a first special case for the Riemman-Liouville fractional integral.
If $g(x) = x$, then $I^{\alpha}_{a+;x} f(x)$ reduces to $I^{\alpha}_{a+} f(x)$ left-sided Riemann-Liouville fractional integral, so the following result follows.

Corollary 2.4. Let $0 < p \le q < \infty$, $f \ge 0$, u be a weight function on (a, b), $I^{\alpha}_{a+} f$ denotes the left-sided Riemann-Liouville fractional integral of f. Let v be defined on (a, b) by

$$v(y) := \alpha \left(\int_y^b u(x) \left(\frac{(x-y)^{\alpha-1}}{(x-a)^{\alpha}} \right)^{\frac{q}{p}} dx \right)^{\frac{p}{q}} < \infty. \tag{2.4}$$

If Φ is a non-negative convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\left(\int_a^b u(x) \left[\Phi \left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}} I^{\alpha}_{a+} f(x) \right) \right]^{\frac{q}{p}} dx \right)^{\frac{1}{q}} \le \left(\int_a^b v(y) \Phi \left(f(y) \right) dy \right)^{\frac{1}{p}} \tag{2.5}$$

holds for all measurable functions $f : (a, b) \to \mathbb{R}$, such that $Imf \subseteq I$.

Corollary 2.5. Let $0 < p \le q < \infty$, $s \ge 1$, $\alpha > 1 - \frac{p}{q}$, $f \ge 0$, $I^{\alpha}_{a+} f$ denotes the left-sided Riemann-Liouville fractional integral of f. Then the inequality

$$\left(\int_a^b \left(I^{\alpha}_{a+} f(x) \right)^{\frac{sq}{p}} dx \right)^{\frac{1}{q}} \le \frac{\alpha^{\frac{1}{p}} (b-a)^{\frac{q(\alpha s-1)+p}{pq}}}{\left((\alpha-1)\frac{q}{p} + 1 \right)^{\frac{1}{q}} \left(\Gamma(\alpha+1) \right)^{\frac{s}{p}}} \left(\int_a^b f^s(y) dy \right)^{\frac{1}{p}} \tag{2.6}$$

holds.

Now we continue with the definition of Hadamard-type fractional integrals.
Let (a, b) be finite or infinite interval of $\mathbb{R}^+$ and $\alpha > 0$. The left and right-sided Hadamard-type fractional integrals of order $\alpha > 0$ is given by

$$(J^{\alpha}_{a+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{y} \right)^{\alpha-1} \frac{f(y)dy}{y}, \quad x > a$$

and

$$(J^{\alpha}_{b-} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left(\log \frac{y}{x} \right)^{\alpha-1} \frac{f(y)dy}{y}, \quad x < b$$

respectively.

Notice that Hadamard fractional integrals of order α are special case of the left- and right-sided fractional integrals of a function f with respect to another function $g(x) = \log(x)$ in $[a, b]$ where $0 \le a < b \le \infty$, the following result follows.

Corollary 2.6. Let $0 < p \le q < \infty,\ s \ge 1,\ \alpha > 1 - \frac{p}{q},\ f \ge 0,\ J^{\alpha}_{a+} f$ denotes the Hadamard-type fractional integrals of f. Then the following inequality holds:

$$\left(\int_a^b (J^{\alpha}_{a+;g} f(x))^{\frac{sq}{p}} \frac{dx}{x} \right)^{\frac{1}{q}} \le \frac{\alpha^{\frac{1}{p}} (\log b - \log a)^{\frac{q(\alpha s-1)+p}{pq}}}{((\alpha-1)\frac{q}{p}+1)^{\frac{1}{q}} (\Gamma(\alpha+1))^{\frac{s}{p}}} \left(\int_a^b f^s(y) \frac{dy}{y} \right)^{\frac{1}{p}}. \tag{2.7}$$

Next we give result with respect to the generalized Riemann–Liouville fractional derivative. Let us recall the definition, for details see [2].

Let $\alpha > 0$ and $n = [\alpha] + 1$ where $[\cdot]$ is the integral part and we define the *generalized Riemann-Liouville fractional derivative of* f of order α by

$$(D^{\alpha}_a f)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx} \right)^n \int_a^x (x-y)^{n-\alpha-1} f(y)\, dy\,.$$

In addition, we stipulate

$$D^0_a f := f =: I^0_a f, \quad I^{-\alpha}_a f := D^{\alpha}_a f \text{ if } \alpha > 0.$$

If $\alpha \in \mathbb{N}$ then $D^{\alpha}_a f = \frac{d^{\alpha} f}{dx^{\alpha}}$, the ordinary α-order derivative.

The space $I^{\alpha}_a(L(a,b))$ is defined as the set of all functions f on $[a,b]$ of the form $f = I^{\alpha}_a \varphi$ for some $\varphi \in L(a,b)$, [19, Chapter 1, Definition 2.3]. According to Theorem 2.3 in [19, p. 43], the latter characterization is equivalent to the condition

$$I^{n-\alpha}_a f \in AC^n[a,b]\,, \tag{2.8}$$

$$\frac{d^j}{dx^j} I^{n-\alpha}_a f(a) = 0\,, \quad j = 0, 1, \ldots, n-1.$$

A function $f \in L(a,b)$ satisfying (2.8) is said to have an *integrable fractional derivative* $D^{\alpha}_a f$, [19, Chapter 1, Definition 2.4].

The following lemma summarizes conditions in identity for generalized Riemann-Liouville fractional derivative. For details see [2].

Lemma 2.7. Let $\beta > \alpha \ge 0$, $n = [\beta] + 1$, $m = [\alpha] + 1$. Identity

$$D^{\alpha}_a f(x) = \frac{1}{\Gamma(\beta-\alpha)} \int_a^x (x-y)^{\beta-\alpha-1} D^{\beta}_a f(y)\, dy\,, \quad x \in [a,b], \tag{2.9}$$

is valid if one of the following conditions holds:

(*i*) $f \in I^{\beta}_a(L(a,b))$.

(*ii*) $I^{n-\beta}_a f \in AC^n[a,b]$ and $D^{\beta-k}_a f(a) = 0$ for $k = 1, \ldots n$.

(*iii*) $D^{\beta-k}_a f \in C[a,b]$ for $k = 1, \ldots, n$, $D^{\beta-1}_a f \in AC[a,b]$ and $D^{\beta-k}_a f(a) = 0$ for $k = 1, \ldots n$.

(iv) $f \in AC^n[a,b]$, $D_a^\beta f \in L(a,b)$, $D_a^\alpha f \in L(a,b)$, $\beta-\alpha \notin \mathbb{N}$, $D_a^{\beta-k}f(a)=0$ for $k=1,\ldots,n$ and $D_a^{\alpha-k}f(a)=0$ for $k=1,\ldots,m$.

(v) $f \in AC^n[a,b]$, $D_a^\beta f \in L(a,b)$, $D_a^\alpha f \in L(a,b)$, $\beta-\alpha=l \in \mathbb{N}$, $D_a^{\beta-k}f(a)=0$ for $k=1,\ldots,l$.

(vi) $f \in AC^n[a,b]$, $D_a^\beta f \in L(a,b)$, $D_a^\alpha f \in L(a,b)$ and $f(a)=f'(a)=\cdots=f^{(n-2)}(a)=0$.

(vii) $f \in AC^n[a,b]$, $D_a^\beta f \in L(a,b)$, $D_a^\alpha f \in L(a,b)$, $\beta \notin \mathbb{N}$ and $D_a^{\beta-1}f$ is bounded in a neighborhood of $t=a$.

Theorem 2.8. Let $0<p\le q<\infty$, u be a weight function on (a,b), $\beta>\alpha\ge 0$, $D_a^\alpha f\ge 0$, $D_a^\beta f\ge 0$ and let assumptions of Lemma 2.7 be satisfied. Let v be defined on (a,b) by

$$v(y):=(\beta-\alpha)\left(\int_y^b u(x)\left(\frac{(x-y)^{\beta-\alpha-1}}{(x-a)^{\beta-\alpha}}\right)^{\frac{q}{p}}dx\right)^{\frac{p}{q}}<\infty. \tag{2.10}$$

If Φ is a non-negative convex function on the interval $I\subseteq\mathbb{R}$, then the inequality

$$\left(\int_a^b u(x)\left[\Phi\left(\frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}}D_a^\alpha f(x)\right)\right]^{\frac{q}{p}}dx\right)^{\frac{1}{q}}\le\left(\int_a^b v(y)\Phi\left(D_a^\beta f(y)\right)dy\right)^{\frac{1}{p}} \tag{2.11}$$

holds for all measurable functions $f:(a,b)\to\mathbb{R}$, such that $Imf\subseteq I$.

Proof. Applying Theorem 1.1 with $\Omega_1=\Omega_2=(a,b)$, $d\mu_1(x)=dx, d\mu_2(y)=dy$,

$$k(x,y)=\begin{cases}\frac{(x-y)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}, & a\le y\le x\,;\\ 0, & x<y\le b,\end{cases}$$

we get that $K(x)=\frac{(x-a)^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}$. Replace f by $D_a^\beta f$. Then $A_k f(x)=\frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}}D_a^\alpha f(x)$ and the inequality given in (1.5) reduces to (2.11). Q.E.D.

If we take $\Phi(x)=x^s, s\ge 1$ and $u(x)=(x-a)^{\frac{(\beta-\alpha)q}{p}}$, $x\in(a,b)$, similar to the proof of Corollary 2.2 we obtain the following result.

Corollary 2.9. Let $0<p\le q<\infty,\ s\ge 1,\ \beta-\alpha>1-\frac{p}{q}$ and let assumption of Lemma 2.7 be satisfied. Then for non-negative functions $D_a^\alpha f$ and $D_a^\beta f$ the following inequality holds:

$$\left(\int_a^b (D_a^\alpha f(x))^{\frac{sq}{p}}dx\right)^{\frac{1}{q}}\le\frac{(\beta-\alpha)^{\frac{1}{p}}(b-a)^{\frac{q((\beta-\alpha)s-1)+p}{pq}}}{((\beta-\alpha-1)\frac{q}{p}+1)^{\frac{1}{q}}(\Gamma(\beta-\alpha+1))^{\frac{s}{p}}}\left(\int_a^b (D_a^\beta f(y))^s dy\right)^{\frac{1}{p}}.$$

Q.E.D.

Now we define Canavati-type fractional derivative of f over $[a,b]$ ($\nu-$fractional derivative of f), for details see [3]. We consider

$$C_a^\nu([a,b])=\{f\in C^n([a,b]):I^{1-\bar{\nu}}f^{(n)}\in C^1([a,b])\},$$

$\nu > 0$, $n = [\nu], [.]$ is the integral part, and $\bar{\nu} = \nu - n, 0 \leq \bar{\nu} < 1$.
For $f \in C_a^\nu([a,b])$, the Canavati-ν fractional derivative of f is defined by

$$D_a^\nu f = D I_a^{1-\bar{\nu}} f^{(n)},$$

where $D = d/dx$.

The following lemma gives conditions in composition rule for Canavati fractional derivative.

Lemma 2.10. Let $\nu > \gamma > 0$, $n = [\nu]$, $m = [\gamma]$. Let $f \in C_a^\nu([a,b])$, be such that $f^{(i)}(a) = 0$, $i = m, m+1, ..., n-1$. Then
(i) $\quad f \in C_a^\gamma([a,b])$
(ii) $\quad (D_a^\gamma f)(x) = \frac{1}{\Gamma(\nu-\gamma)} \int\limits_a^x (x-t)^{\nu-\gamma-1} (D_a^\nu f)(t) dt,$
for every $x \in [a,b]$.

In the following Theorem, we will construct new inequality for the Canavati-type fractional derivative.

Theorem 2.11. Let $0 < p \leq q < \infty$, $\nu > \gamma > 0$, $D_a^\gamma f \geq 0$, $D_a^\nu f \geq 0$, u be a weight function on (a,b) and assumptions in Lemma 2.10 be satisfied, $D_a^\gamma f$ denotes the Canavati-type fractional derivative of f. Let v be defined on (a,b) by

$$v(y) := (\nu - \gamma) \left(\int\limits_y^b u(x) \left(\frac{(x-y)^{\nu-\gamma-1}}{(x-a)^{\nu-\gamma}} \right)^{\frac{q}{p}} dx \right)^{\frac{p}{q}} < \infty. \tag{2.12}$$

If Φ is a non-negative convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\left(\int\limits_a^b u(x) \left[\Phi \left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_a^\gamma f(x) \right) \right]^{\frac{q}{p}} dx \right)^{\frac{1}{q}} \leq \left(\int\limits_a^b v(y) \Phi \left(D_a^\nu f(y) \right) dy \right)^{\frac{1}{p}} \tag{2.13}$$

holds for all measurable functions $f : (a,b) \to \mathbb{R}$, such that $Imf \subseteq I$.

Proof. Applying Theorem 1.1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \frac{(x-y)^{\nu-\gamma-1}}{\Gamma(\nu-\gamma)}, & a \leq y \leq x\,; \\ 0, & x < y \leq b \end{cases}$$

we get that $K(x) = \frac{(x-a)^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}$. Replace f by $D_a^\nu f$. Then the inequality given in (1.5) reduces to (2.13). Q.E.D.

Example 2.12. If we take $\Phi(x) = x^s, s \geq 1$, $\nu - \gamma > 1 - \frac{p}{q}$, $D_a^\gamma f \geq 0$, $D_a^\nu f \geq 0$ and weight function $u(x) = (x-a)^{\frac{(\nu-\gamma)q}{p}}$, $x \in (a,b)$ in (2.13), after some calculations we obtain

$$\left(\int\limits_a^b (D_a^\gamma f(x))^{\frac{sq}{p}} dx \right)^{\frac{1}{q}} \leq \frac{(\nu-\gamma)^{\frac{1}{p}} (b-a)^{\frac{q((\nu-\gamma)s-1)+p}{pq}}}{((\nu-\gamma-1)\frac{q}{p}+1)^{\frac{1}{q}} (\Gamma(\nu-\gamma+1))^{\frac{s}{p}}} \left(\int\limits_a^b (D_a^\nu f(y))^s dy \right)^{\frac{1}{p}}.$$

Next, we give the result for Caputo fractional derivative, for details see [1, p. 449]. The Caputo fractional derivative is defined as:
Let $\alpha \geq 0$, $n = [\alpha] + 1$, $g \in AC^n([a,b])$. The Caputo fractional derivative is given by

$$D^{\alpha}_{*a} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{g^{(n)}(y)}{(x-y)^{\alpha-n+1}} dy,$$

for all $x \in [a,b]$. The above function exists almost everywhere for $x \in [a,b]$.

Using the above definition, we will prove the following result as a special case of Theorem 1.1.

Theorem 2.13. Let $0 < p \leq q < \infty$, $f^{(n)} \geq 0$, u be a weight function on (a,b) and $D^{\alpha}_{*a} f$ denotes the Caputo fractional derivative of f. Let v be defined on (a,b) by

$$v(y) := (n-\alpha) \left(\int_y^b u(x) \left(\frac{(x-y)^{n-\alpha-1}}{(x-a)^{n-\alpha}} \right)^{\frac{q}{p}} dx \right)^{\frac{p}{q}} < \infty. \tag{2.14}$$

If Φ is a non-negative convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\left(\int_a^b u(x) \left[\Phi \left(\frac{\Gamma(n-\alpha+1)}{(x-a)^{n-\alpha}} D^{\alpha}_{*a} f(x) \right) \right]^{\frac{q}{p}} dx \right)^{\frac{1}{q}} \leq \left(\int_a^b v(y) \Phi \left(f^{(n)}(y) \right) dy \right)^{\frac{1}{p}} \tag{2.15}$$

holds for all measurable functions $f : (a,b) \to \mathbb{R}$, such that $Imf \subseteq I$.

Proof. Applying Theorem 1.1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \frac{(x-y)^{n-\alpha-1}}{\Gamma(n-\alpha)}, & a \leq y \leq x\,; \\ 0, & x < y \leq b \end{cases}$$

we get that $K(x) = \frac{(x-a)^{n-\alpha}}{\Gamma(n-\alpha+1)}$. Replace f by $f^{(n)}$. Then the inequality given in (1.5) reduces to (2.15). Q.E.D.

Example 2.14. If we take $\Phi(x) = x^s, s \geq 1$, $n - \alpha > 1 - \frac{p}{q}$, $f^{(n)} \geq 0$ and weight function $u(x) = (x-a)^{\frac{(n-\alpha)q}{p}}$, $x \in (a,b)$, in (2.15), after some calculations we obtain

$$\left(\int_a^b (D^{\alpha}_{*a} f(x))^{\frac{sq}{p}} dx \right)^{\frac{1}{q}} \leq \frac{(n-\alpha)^{\frac{1}{p}} (b-a)^{\frac{q((n-\alpha)s-1)+p}{pq}}}{((n-\alpha-1)\frac{q}{p}+1)^{\frac{1}{q}} (\Gamma(n-\alpha+1))^{\frac{s}{p}}} \left(\int_a^b (f^{(n)}(y))^s dy \right)^{\frac{1}{p}}.$$

Now we present definitions and some properties of the *Erdélyi-Kober type fractional integrals.* Some of these definitions and results were presented in Samko et al. in [19].

Let (a,b) , $(0 \leq a < b \leq \infty)$ be a finite or infinite interval of the half-axis $\mathbb{R}^+$. Also let $\alpha > 0, \sigma > 0$, and $\eta \in \mathbb{R}$. We consider the left- and right-sided integrals of order $\alpha \in \mathbb{R}$ defined by

$$(I^{\alpha}_{a_+;\sigma;\eta} f)(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^x \frac{t^{\sigma\eta+\sigma-1} f(t)dt}{(x^\sigma - t^\sigma)^{1-\alpha}}, \quad x > a, \tag{2.16}$$

and

$$(I^{\alpha}_{b_-;\sigma;\eta} f)(x) = \frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)} \int_x^b \frac{t^{\sigma(1-\eta-\alpha)-1} f(t)dt}{(t^\sigma - x^\sigma)^{1-\alpha}}, \qquad x < b, \tag{2.17}$$

respectively. Integrals (2.16) and (2.17) are called the Erdélyi–Kober type fractional integrals.

Now, we give the following result.

Theorem 2.15. Let $0 < p \leq q < \infty$, $f \geq 0$, u be a weight function on (a,b), $I^{\alpha}_{a_+;\sigma;\eta} f$ denotes the Erdélyi–Kober type fractional integrals of f, ${}_2F_1(a,b;c;z)$ denotes the hypergeometric function. Let v be defined on (a,b) by

$$v(y) := \alpha \left(\int_y^b u(x) \left(\frac{\sigma x^{-\sigma\eta} y^{\sigma\eta+\sigma-1}}{(x^\sigma - y^\sigma)^{1-\alpha} (x^\sigma - a^\sigma)^\alpha} \right)^{\frac{q}{p}} dx \right)^{\frac{p}{q}} < \infty. \tag{2.18}$$

If Φ is a non-negative convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\left(\int_a^b u(x) \left[\Phi \left(\frac{\Gamma(\alpha+1)}{\left(1 - \left(\frac{a}{x}\right)^\sigma\right)^\alpha {}_2F_1(x)} I^{\alpha}_{a_+;\sigma;\eta} f(x) \right) \right]^{\frac{q}{p}} dx \right)^{\frac{1}{q}} \leq \left(\int_a^b v(y) \Phi\left(f(y)\right) dy \right)^{\frac{1}{p}} \tag{2.19}$$

holds for all measurable functions $f : (a,b) \to \mathbb{R}$, such that $Imf \subseteq I$.

Proof. Applying Theorem 1.1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \frac{1}{\Gamma(\alpha)} \frac{\sigma x^{-\sigma(\alpha+\eta)}}{(x^\sigma - y^\sigma)^{1-\alpha}} y^{\sigma\eta+\sigma-1}, & a \leq y \leq x\,; \\ 0, & x < y \leq b, \end{cases}$$

we get that $K(x) = \frac{1}{\Gamma(\alpha+1)} \left(1 - \left(\frac{a}{x}\right)^\sigma\right)^\alpha {}_2F_1(-\eta, \alpha; \alpha+1; 1 - \left(\frac{a}{x}\right)^\sigma)$. Then the inequality (1.5) becomes (2.19). Q.E.D.

Example 2.16. If we take $\Phi(x) = x^s, s \geq 1$, $f \geq 0$, and weight function $u(x) = x^{\sigma-1}\left((x^\sigma - a^\sigma)^\alpha {}_2F_1(x)\right)^{\frac{q}{p}}$, $x \in (a,b)$ in (2.19), after some calculations we obtain

$$\left(\int_a^b ({}_2F_1(x))^{\frac{q}{p}(1-s)} \left(I^{\alpha}_{a_+;\sigma;\eta} f(x) \right)^{\frac{sq}{p}} dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b ({}_2F_1(y)) f^s(y) dy \right)^{\frac{1}{p}} \tag{2.20}$$

where

$$C = \frac{\alpha^{\frac{1}{p}}\sigma^{\frac{q-p}{pq}}b^{\frac{\sigma-1}{p}}(b^\sigma - a^\sigma)^{\frac{q(\alpha s-1)+p}{pq}}}{a^{\frac{p\sigma-p+qs\sigma\alpha}{pq}}((\alpha-1)\frac{q}{p}+1)^{\frac{1}{q}}(\Gamma(\alpha+1))^{\frac{s}{p}}},$$

$${}_2F_1(x) = {}_2F_1\left(-\eta, \alpha; \alpha+1; 1-\left(\frac{a}{x}\right)^\sigma\right) \text{ and } {}_2F_1(y) = {}_2F_1\left(\eta, \alpha; \alpha+1; 1-\left(\frac{b}{y}\right)^\sigma\right).$$

Remark 2.17. Similar result can be obtained for the right sided Erdélyi-Kober type fractional integrals, but here we omit the details.

Acknowledgements. This research is partially funded by Higher Education Commission (HEC) of Pakistan. The research of the second and third author was supported by the Croatian Ministry of Science, Education and Sports, under the Research Grant 117-1170889-0888.

References

[1] G. A. Anastassiou, *Fractional Differentiation Inequalities*, Springer Science-Businness Media, LLC, Dordrecht, the Netherlands, (2009).

[2] M. Andrić, J. Pečarić, I. Perić, *A multiple Opial type inequality for the Riemann-Liouville fractional derivatives*, J. Math. Inequal., **7**(1) (2013), 139-150.

[3] M. Andrić, J. Pečarić, I. Perić, *Improvements of composition rule for Canavati fractional derivative and applications to Opial-type inequalities,* Dynam. Systems Appl., **20** (2011), 383–394.

[4] A. Čižmešija, K. Krulić, J. Pečarić, *Some new refined Hardy-type inequalities with kernels*, J. Math. Inequal., **4**(4) (2010), 481-503.

[5] D. E. Edmunds, V. Kokilashvili, A. Meskhi, *Bounded and Compact Integral Operators*, Kluwer Academic Publishers, 101, Philip Drive, Norwell, MA02061, USA.

[6] N. Elezović, K. Krulić, J. Pečarić, *Bounds for Hardy type differences*, Acta Math. Sinica, (Engl. Ser.), **27**(4), 671-684, (2011).

[7] G. D. Handley, J. J. Koliha, J. Pečarić, *Hilbert-Pachpatte type integral inequalities for fractional derivatives*, Fract. Calc. Appl. Anal., **4** (2001), 37-46.

[8] G. H. Hardy, *Notes on some points in the integrral calculus*, Messenger. Math, **47**(10), 1918, 145-150.

[9] S. Iqbal, K. Krulić, J. Pečarić, *On an inequality of H. G. Hardy*, J. Inequal. Appl., vol. 2010. Artical ID 264347, 2010.

[10] S. Iqbal, K. Krulić, J. Pečarić, *Improvement of an inequality of G. H. Hardy*, Tamkang J. Math., **43**(3) (2012), 399-416.

[11] S. Iqbal, K. Krulić, J. Pečarić, *Improvement of an Inequality of G. H. Hardy via Superquadratic functions*, Panamer. Math. J., **22**(2), (2012), 77–97.

[12] S. Kaijser, L. Nikolova, L-E. Persson, A. Wedestig, *Hardy type inequalities via convexity,* Math. Inequal. Appl. **8(3)** (2005), 403–417.

[13] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Application of Fractinal Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier, New York-London, (2006).

[14] K. Krulić, J. Pečarić, L. E. Persson, *Some new Hardy type inequalities with general kernels,* Math. Inequal. Appl., **12** (2009) 473-485.

[15] A. Kufner, L. Maligranda, L.-E. Persson, *The Hardy Inequality. About its History and Some Related Results*, Vydavatelsky Servis Publishing House, Pilsen, (2007).

[16] A. Kufner, L.-E. Persson, *Weighted Inequalities of Hardy Type*, World Scientic Publishing Co, Singapore/ New Jersey/ London/Hong Kong, (2003).

[17] B. Opic, A. Kufner, *Hardy-type inequalities.*,Pitman Research Notes in Mathematics Series, 219. Longman Scientific and Technical, Harlow, xii+333 pp. ISBN: 0-582-05198-3, (1990).

[18] J. Pečarić, F. Proschan, Y. L. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Inc. (1992).

[19] S. G. Samko, A. A. Kilbas, O. J. Marichev, *Fractional Integral and Derivatives : Theory and Applications*, Gordon and Breach Science Publishers, Switzerland, (1993).

[20] V. D. Stepanov, *Weighted inequalities of Hardy type for Riemann-Liouville fractional integrals*, (Russian) Sibirsk. Mat. Zh. **31**(3) (1990), 186–197, 219; translation in Siberian Math. J. **31**(3) (1990), 513522 (1991).