Two step Newton and Steffensen type methods for solving non-linear equations

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Abstract

Families of Newton type methods for solving non-linear equations are obtained. These families consist of second and third order methods. Concrete two step methods are presented which are obtained as a unification of some existing methods in the literature and the standard secant method. It is shown that by this unification, the order of convergence of the methods increases. All the methods presented in this paper are well supported by examples.

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1 Introduction

Non-linear equations arise in almost all areas of sciences, in particular, in physical and mathematical sciences. Practically, it is rarely possible to solve a non-linear equation analytically. So iterative methods are generally employed in such situations. The most common among such methods for solving a non-linear equation $f(x) = 0$ are the Newton method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

and the secant method

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} f(x_n).$$

Over the years, tremendous variants of these methods have appeared showing one or the other advantages over these methods in some sense. In [6], Sharma studied the following variant of the Newton method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) - pf(x_n)}, \tag{1.1}$$

where p is a real number. For $p = 0$, this method coincides with the Newton method. It was shown in [6] that method (1.1) is of order 2 for general p and if $p = \frac{f''(x_n)}{2f'(x_n)}$ then it is of order 3. In fact, the corresponding method is the well known Halley method.

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In the present paper, we begin our study by reinvestigating method (1.1). We produce a family of second order methods of which (1.1) is a member. We go on further with our procedure and produce another family around (1.1) consisting of third order methods. The motivation of the construction of such families comes from the work of Gander [2] and Jain [3].

Next, in order to increase the order of Newton method, Kasturiarachi [4] used iterations, alternatively, from Newton method and from secant method. The resulting method was proved to be of order 3. Also, in [3], Jain mixed iterations from the well known Steffensen method with the secant method and proved his method to be of third order as compared to the quadratic convergent Steffensen method. We shall give the same treatment to method (1.1) by mixing its iterations with the secant method and show that the corresponding method is of order 3 (see Theorem 3).

Also, in this paper, we discuss a Steffensen type method which is obtained by replacing $f'(x)$ in (1.1) by the expression $\frac{f(x+f(x))-f(x)}{f(x)}$ so that the method becomes derivative free. It is proved that the method is still of second order (see Theorem 4) and if its iterations are mixed with secant method, it becomes of order 3 (See Theorem 5). All the methods presented here are supported by examples.

2 A family of Newton type methods

We use the idea of Gander [2] and Jain [3] to prove the following

Proposition 2.1. Let the functions f and G have sufficient number of continuous derivatives in a neighbourhood of α which is a simple zero of f. Consider the function

$$F(x) = x - \frac{f(x)}{f'(x) - pf(x)} G(x),$$

p being a real number. Then the iteration formula

$$x_{n+1} = F(x_n), \tag{2.1}$$

is of second order if and only if $G(\alpha) = 1$.

Proof. Let us write

$$F(x) = x - A(x)G(x),$$

where we have denoted

$$A(x) = \frac{f(x)}{f'(x) - pf(x)}. \tag{2.2}$$

Clearly, $A(\alpha) = 0$ and also it can be easily checked that $A'(\alpha) = 1$. Since

$$F'(x) = 1 - A(x)G'(x) - A'(x)G(x)$$

we find that

$$F'(\alpha) = 1 - A(\alpha)G'(\alpha) - A'(\alpha)G(\alpha) = 1 - G(\alpha). \tag{2.3}$$

It is known, see e.g., [5], that a numerical method

$$x_{n+1} = J(x_n)$$

is of second order if and only if $J'(\beta) = 0$, where β is a simple zero of the function J. Consequently, in our case, the method (2.1) is of second order if and only if $F'(\alpha) = 0$ which in view of (2.3) gives that

$$G(\alpha) = 1$$

and we are done. Q.E.D.

Remark 2.2. In order to make use of Proposition 1, the zero α of f needs to be known in advance which is not generally the case in practice. To handle this situation, we argue as follows. Note that $A(\alpha) = 0$. Thus if a function G_1 is chosen such that

$$G(x) = G_1(A(x)),$$

then $G(\alpha) = G_1(0)$. Consequently, the condition $G(\alpha) = 1$ can be replaced by $G_1(0) = 1$.

In view of Remark 1, Proposition 1 leads to the following theorem.

Theorem 2.3. Let f be a differentiable function and G_1 be any function with $G_1(0) = 1$. Then the method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) - pf(x_n)} G_1(A(x_n))$$

is of second order, where the function A is as defined by (2.2).

Remark 2.4. The method described in Theorem 1 represents a family of second order methods. If G_1 is chosen to be the constant function $G_1(x) = 1$, then this method is the one considered by Sharma [6] which for $p = 0$ becomes the standard Newton method. Note that there could be infinitely many possibilities for choosing G_1, e.g., the polynomial function

$$G_1(x) = 1 + a_0 x + a_1 x^2 + ... + a_n x^n$$

works for any choice $n \geq 0$ and $a_0, a_1, ..., a_n$.

Next, we generate a family of third order methods. We prove the following.

Proposition 2.5. Let f, F, G, α be as in Proposition 1. Then the iteration formula (2.1) is of third order if and only if

$$G(\alpha) = 1 \quad \text{and} \quad G'(\alpha) = \frac{1}{2}\left(\frac{f''(\alpha) - 2pf'(\alpha)}{f'(\alpha)}\right).$$

Proof. As in the proof of Proposition 1, we write

$$F(x) = x - A(x)G(x),$$

where A is as given in (2.2). Recall that

$$A(\alpha) = 0 \quad \text{and} \quad A'(\alpha) = 1$$

so that

$$F'(\alpha) = 1 - G(\alpha). \tag{2.4}$$

Also, it can be calculated that

$$A''(\alpha) = \frac{-f''(\alpha) + 2pf'(\alpha)}{f'(\alpha)}$$

which along with the above calculations gives that

$$F''(\alpha) = -2G'(\alpha) + \frac{f''(\alpha) - 2pf'(\alpha)}{f'(\alpha)}. \tag{2.5}$$

Now, it is known, see e.g., [5], that a numerical method

$$x_{n+1} = J(x_n)$$

is of third order if and only if

$$J'(\beta) = 0 \quad \text{and} \quad J''(\beta) = 0,$$

where β is a simple zero of the function J. In our case, in view of (2.4) and (2.5), the method (2.1) is of third order if and only if

$$G(\alpha) = 1 \quad \text{and} \quad G'(\alpha) = \frac{1}{2}\left(\frac{f''(\alpha) - 2pf'(\alpha)}{f'(\alpha)}\right).$$

This proves the assertion. Q.E.D.

Remark 2.6. As in Remark 1, Proposition 2 cannot be used unless the zero of f is known. Therefore some more work is to be done before this proposition could be used effectively. In this regard, we prove the following.

Theorem 2.7. Let f be a function which is differentiable at least twice and H be a function satisfying $H(0) = 1$ and $H'(0) = \frac{1}{2}$. Then the iteration method

$$x_{n+1} = F(x_n),$$

where

$$F(x) = x - \frac{f(x)}{f'(x) - pf(x)} H(B(x)),$$

with

$$B(x) = \frac{f(x)(f''(x) - 2pf'(x))}{f'(x)} \tag{2.6}$$

is of third order.

Proof. Clearly $B(\alpha) = 0$. Also, it can be calculated that

$$B'(\alpha) = \frac{(f''(\alpha) - 2pf'(\alpha))}{f'(\alpha)}.$$

Using this setting, the conditions $G(\alpha) = 1$ and $G'(\alpha) = \frac{1}{2}\left(\frac{f''(\alpha)-2pf'(\alpha)}{f'(\alpha)}\right)$ in Proposition 2 can be replaced respectively by $H(0) = 1$ and $H'(0) = \frac{1}{2}$. The assertion thus follows. Q.E.D.

3 A method of order 3

Recall that in [6], the following method was studied:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) - pf(x_n)}, \tag{3.1}$$

which is of order 2. We shall prove that if we use iterates alternatively from the method (3.1) and the standard secant method, then the resulting method will be of order 3. Thus we propose the following method:

$$x_{n+1} = x_n - \frac{\overline{x}_n - x_n}{f(\overline{x}_n) - f(x_n)} f(x_n), \tag{3.2}$$

where

$$\overline{x}_n = x_n - \frac{f(x_n)}{f'(x_n) - pf(x_n)}. \tag{3.3}$$

Such techniques have been used by Kasturiarachi [4] and Jain [3]. Kasturiarachi mixed the standard Newton method with the secant method and proved that his method is of order 3, i.e., one order higher than that of Newton method. Similarly, Jain mixed the standard Steffensen method with the secant method and proveed his method to be of order 3, again, one order higher than that of the Steffensen method. We prove the following:

Theorem 3.1. Let the function f have sufficient number of continuous derivatives in a neighbourhood of α which is a simple zero of f, i.e., $f'(\alpha) \neq 0$. Then the method (3.2) is of order 3 whenever $\frac{pf(x_n)}{f'(x_n)} < 1$, x_n being the iterates of zeros of f.

Proof. The expression (3.3) can be written as

$$\begin{aligned}
\overline{x}_n &= x_n - \frac{f(x_n)}{f'(x_n)} \frac{1}{\left[1 - p\frac{f(x_n)}{f'(x_n)}\right]} \\
&= x_n - \frac{f(x_n)}{f'(x_n)} \left[1 - p\frac{f(x_n)}{f'(x_n)}\right]^{-1} \\
&= x_n - \frac{f(x_n)}{f'(x_n)} \left[1 + p\frac{f(x_n)}{f'(x_n)} + \left(p\frac{f(x_n)}{f'(x_n)}\right)^2 + \ldots\right] \\
&= x_n - \frac{f(x_n)}{f'(x_n)} - p\left(\frac{f(x_n)}{f'(x_n)}\right)^2 - p^2\left(\frac{f(x_n)}{f'(x_n)}\right)^3 - \ldots \qquad (3.4)
\end{aligned}$$

Now, let e_n be the error in the term x_n, i.e., let $x_n = \alpha + e_n$. Then it is easy to see using Taylor's expansion that

$$\frac{f(x_n)}{f'(x_n)} = \frac{f(\alpha + e_n)}{f'(\alpha + e_n)} = e_n - Ae_n^2 + o(e_n^3), \qquad (3.5)$$

where $A = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}$. Thus, if $\overline{e}_n$ is the error in the term $\overline{x}_n$, i.e., if $\overline{x}_n = \alpha + \overline{e}_n$, then by using (3.5) in (3.4), the error in (3.3) can be calculated as

$$\begin{aligned}
\overline{e}_n &= e_n - \left[e_n - Ae_n^2 + o(e_n^3)\right] - p\left[e_n - Ae_n^2 + o(e_n^3)\right]^2 + o(e_n^3) \\
&= (A - p)e_n^2 + o(e_n^3). \qquad (3.6)
\end{aligned}$$

Next, it is easy to see that

$$f(\overline{x}_n) - f(x_n) = (\overline{e}_n - e_n)f'(\alpha)\left[1 + (\overline{e}_n + e_n)A + o(e_n^2)\right]$$

so that the error equation for (3.2) can be written as

$$\begin{aligned}
e_{n+1} &= \overline{e}_n - \frac{(\overline{e}_n - e_n)\left[e_n f'(\alpha) + \frac{e_n^2}{2}f''(\alpha) + o(e_n^3)\right]}{(\overline{e}_n - e_n)f'(\alpha)\left[1 + (\overline{e}_n + e_n)A + o(e_n^2)\right]} \\
&= \overline{e}_n - \frac{\left[e_n + Ae_n^2 + o(e_n^3)\right]}{1 + (\overline{e}_n + e_n)A + o(e_n^2)} \\
&= \left[e_n + Ae_n^2 + o(e_n^3)\right]\left[1 + (\overline{e}_n + e_n)A + o(e_n^2)\right]^{-1} \\
&= A\overline{e}_n e_n + e_n^2(\overline{e}_n + e_n)A^2 + o(e_n^3). \qquad (3.7)
\end{aligned}$$

Now, using the value of $\overline{e}_n$ from (3.6) in the last equation, we get

$$e_{n+1} \approx o(e_n^3). \qquad (3.8)$$

Consequently, the method (3.2) is of order 3. Q.E.D.

Remark 3.2. When $p = 0$, method (3.2) coincides with the method of Kasturiarachi [4]. Therefore our method (3.2) unifies both the method of Kasturiarchi as well as of Sharma [6].

Remark 3.3. Although, theoretically method (3.2) works very well but because of the machine limitations, there could be situations when in the denominator of (3.3), we have subtraction of two almost equal floating point numbers. This observation was also made by Sharma [6] for his method. To avoid such situations, we choose $p > 0$ or $p < 0$ accordingly as $f(x)f'(x) \leq 0$ or $f(x)f'(x) \geq 0$ respectively.

Example 3.4. Consider the non-linear equation

$$\cos x - xe^x = 0.$$

This equation has a simple root in the interval $(0, 1)$. Table 1 gives a comparison of a root of the above equation as obtained by Newton method, the method (3.1) given by Sharma [6] and our method (3.2) for $p = 1, -1, 0.5$.

Table 1.

Method	n	x_n
Newton	1	1
	2	0.6530794035261767
	3	0.5313433676065809
	4	0.5179099131356748
	5	0.5177573831648338
	6	0.5177573636824587
	7	0.5177573636824583
	8	0.5177573636824583
	9	0.5177573636824583
Method (3.1)	1	1
	2	0.5802735237837837
	3	0.5190247092041577
	4	0.5177579050010849
	5	0.5177573636825571
	6	0.5177573636824583
	7	0.5177573636824583
	8	0.5177573636824583
Method (3.2) (with $p = 1$)	1	1
	2	0.501511456095896
	3	0.5177579444077034
	4	0.5177573636824583
Method (3.2) (with $p = -1$)	1	1
	2	0.5859726827620504
	3	0.5181745832943766
	4	0.5177573637940752
	5	0.5177573636824583
Method (3.2) (with $p = 0.5$)	1	1
	2	0.5377490949862871
	3	0.5177595697282191
	4	0.5177573636824583

4 Steffensen type methods of higher order

The standard Steffensen method differ from the Newton method in the sense that the term $f'(x)$ in Newton method is replaced by the expression $\frac{f(x+f(x))-f(x)}{f(x)}$. Consequently the Newton method becomes

$$x_{n+1} = x_n - \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n)}$$

which, in fact, is the Steffensen method. It is already known that this method is of order 2 and, of course, is derivative free.

Let us formulate a Steffensen type method which is obtained from the method (3.1) by replacing the term $f'(x)$ with $\frac{f(x+f(x))-f(x)}{f(x)}$, i.e., the method

$$x_{n+1} = x_n - \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n) - pf(x_n)^2}. \tag{4.1}$$

In the theorem below, we prove that this method is of second order.

Theorem 4.1. Let the function f have sufficient number of continuous derivatives in a neighbourhood of α which is a simple zero of f, i.e., $f'(\alpha) \neq 0$. Then the method (4.1) is of order 2.

Proof. Let e_n be the error in x_n, i.e., $x_n = \alpha + e_n$. The Taylor series expansion around α gives

$$f(\alpha + e_n) = e_n f'(\alpha) + \frac{e_n^2}{2} f''(\alpha) + o(e_n^3).$$

Therefore

$$f(\alpha + e_n)^2 = e_n^2 f'(\alpha)^2 + e_n^3 f'(\alpha) f''(\alpha) + o(e_n^4)$$

and

$$\begin{aligned} f(\alpha + e_n + f(\alpha + e_n)) &= e_n(1 + f'(\alpha)) f'(\alpha) + \frac{e_n^2}{2}\left[f'(\alpha) + (1 + f'(\alpha))^2\right] f''(\alpha) \\ &\quad + o(e_n^3). \end{aligned}$$

Consequently, we get

$$\begin{aligned} & f(\alpha + e_n + f(\alpha + e_n)) - f(\alpha + e_n) - pf(\alpha + e_n)^2 \\ &\quad = e_n(1 + f'(\alpha)) f'(\alpha) + \frac{e_n^2}{2}\left[f'(\alpha) + (1 + f'(\alpha))^2\right] f''(\alpha) - e_n f'(\alpha) - \frac{e_n^2}{2} f''(\alpha) \\ &\qquad - pe_n^2 f'(\alpha)^2 + o(e_n^3) \\ &\quad = e_n f'(\alpha)^2 + \frac{e_n^2}{2} f'(\alpha)^2 \left[f''(\alpha) - 2p + \frac{3f''(\alpha)}{f'(\alpha)}\right] + o(e_n^3) \\ &\quad = e_n f'(\alpha)^2 \left[1 + \frac{e_n}{2}\left(f''(\alpha) - 2p + \frac{3f''(\alpha)}{f'(\alpha)}\right) + o(e_n^2)\right]. \end{aligned}$$

In view of the above equations, the error equation for the method (4.1) becomes

$$\begin{aligned}
e_{n+1} &= e_n - \frac{e_n^2 f'(\alpha)^2 + e_n^3 f'(\alpha) f''(\alpha) + o(e_n^4)}{e_n f'(\alpha)^2 \left[1 + \frac{e_n}{2}\left(f''(\alpha) - 2p + \frac{3f''(\alpha)}{f'(\alpha)}\right) + o(e_n^2)\right]} \\
&= e_n - \left[e_n + e_n^2 \frac{f''(\alpha)}{f'(\alpha)} + o(e_n^3)\right] \\
&\quad \times \left[1 + \frac{e_n}{2}\left(f''(\alpha) - 2p + \frac{3f''(\alpha)}{f'(\alpha)}\right) + o(e_n^2)\right]^{-1} \\
&= e_n - \left[e_n + e_n^2 \frac{f''(\alpha)}{f'(\alpha)} + o(e_n^3)\right] \\
&\quad \times \left[1 - \frac{e_n}{2}\left(f''(\alpha) - 2p + \frac{3f''(\alpha)}{f'(\alpha)}\right) + o(e_n^2)\right] \\
&= e_n - \left[e_n - \frac{e_n^2}{2}\left(f''(\alpha) - 2p + \frac{3f''(\alpha)}{f'(\alpha)}\right) + e_n^2 \frac{f''(\alpha)}{f'(\alpha)} + o(e_n^3)\right] \\
&= e_n^2 \left[\frac{f''(\alpha)}{2f'(\alpha)} + \frac{f''(\alpha)}{2} - p\right] + o(e_n^3). \qquad (4.2)
\end{aligned}$$

This proves the assertion that the method (4.1) is of order 2. Q.E.D.

Remark 4.2. The equation (4.2) suggests that the method (4.1) is at least of order 3, if

$$p = \frac{1}{2} f''(\alpha) \left[1 + \frac{1}{f'(\alpha)}\right].$$

Example 4.3. We demonstrate method (4.1) for the equation

$$\cos x - x e^x = 0.$$

and the corresponding iterations are recorded in Table 2.

Next, it is natural to consider a method similar to (3.2) in which we use iterates alternatively from the secant method and the method (4.1). Precisely, we propose the method

$$x_{n+1} = x_n - \frac{\overline{x}_n - x_n}{f(\overline{x}_n) - f(x_n)} f(x_n), \qquad (4.3)$$

where

$$\overline{x}_n = x_n - \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n) - p f(x_n)^2}. \qquad (4.4)$$

We prove in the following theorem that the method (4.3) is of order 3.

Theorem 4.4. Let the function f have sufficient number of continuous derivatives in a neighbourhood of α which is a simple zero of f, i.e., $f'(\alpha) \neq 0$. Then the method (4.3) is of order 3.

Proof. Let e_n and $\overline{e}_n$ be the errors in x_n and $\overline{x}_n$ respectively, i.e.,

$$x_n = \alpha + e_n \qquad \text{and} \qquad \overline{x}_n = \alpha + \overline{e}_n.$$

Going through the proof of Theorem 4, in view of (4.2), the error equation for (4.4) can be written as

$$\begin{aligned} \overline{e}_n &= e_n^2 \left[\frac{f''(\alpha)}{2f'(\alpha)} + 2f''(\alpha) - p \right] + o(e_n^3) \\ &= e_n^2 \left(A + \frac{f''(\alpha)}{2} - p \right) + o(e_n^3), \end{aligned} \tag{4.5}$$

where $A = \frac{f''(\alpha)}{2f'(\alpha)}$. Also, the error equation of (4.3) in terms of $\overline{e}_n$ is already obtained in Theorem 3 which is given by (3.7). Thus using the value of $\overline{e}_n$ from (4.5) in (3.7), we obtain

$$\begin{aligned} e_{n+1} &= Ae_n \left[e_n^2 \left(A + \frac{f''(\alpha)}{2} - p \right) \right] + Ae_n^2 \left[e_n^2 \left(A + \frac{f''(\alpha)}{2} - p \right) \right] + o(e_n^3) \\ &\approx o(e_n^3). \end{aligned}$$

This shows that the method (4.3) is of third order. Q.E.D.

Example 4.5. We consider the same equation, i.e.,

$$\cos x - xe^x = 0.$$

and carry out method (4.3). The corresponding iterations are recorded in Table 3.

Table 2.

Method	n	x_n
Method (3.1)	1	1
	2	0.5802735237837837
	3	0.5190247092041577
	4	0.5177579050010849
	5	0.5177573636825571
	6	0.5177573636824583
	7	0.5177573636824583
	8	0.5177573636824583
Method (4.1) (with $p = -1$)	1	1
	2	0.3813025230211976
	3	0.5063219067362609
	4	0.5176654097603029
	5	0.5177573576804625
	6	0.5177573636824583
	7	0.5177573636824583
	8	0.5177573636824583
Method (4.1) (with $p = -0.5$)	1	1
	2	0.1041825137254052
	3	0.4052566392923251
	4	0.50455624053796
	5	0.5175500351298677
	6	0.5177573116891044
	7	0.5177573636824551
	8	0.5177573636824583

Table 3.

Method	n	x_n
Method (3.2) (with $p=-1$)	1	1
	2	0.5859726827620504
	3	0.5181745832943766
	4	0.5177573637940752
	5	0.5177573636824583
Method (4.3) (with $p=-1.0$)	1	1
	2	0.4711208386842657
	3	0.517816563944614
	4	0.5177573636823349
	5	0.5177573636824583
Method (4.3) (with $p=-0.5$)	1	1
	2	0.3617555141344706
	3	0.5211770388024592
	4	0.517757323061568
	5	0.5177573636824583

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