PF-rings of skew generalized power series

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Abstract

Let R be a ring which is S-compatible and (S,ω)-Armendariz. In this paper, we investigate that the skew generalized power series ring $R[[S,\omega]]$ is a PF-ring if and only if for any two S-indexed subsets P and Q of R such that $Q \subseteq ann_R(P)$ and there exists $a \in ann_R(P)$ such that $qa = q$ for all $q \in Q$. Further, we prove that if R be a Noetherian ring then $R[[S,\omega]]$ is a PP-ring if and only if R is a PP-ring.

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1 Introduction

In 1974, Armendariz [4] proved that if the product of two polynomials over a reduced ring (that is, a ring without nonzero nilpotent elements) is zero, then the products of their coefficients are all zero, that is, for the polynomial ring $R[x]$ the following holds:

$$\text{if } f(x)g(x) = 0 \text{ for any } f(x), g(x) \in R[x]$$
$$\text{then } a_ib_j = 0 \text{ for all } i, j, \text{ where } a_i, b_j \in R. \tag{τ}$$

This property (τ) is known as the Armendariz property. In 1997, Rage and Chhawchharia [18] used this property and coined the definition of Armendariz ring. According to the definition, a ring R satisfying the property(τ) is called Armendariz ring. Some basic properties, examples and various generalization of Armendariz rings were studied by several authors in [4, 6, 7, 8, 18]. Recently, Marks et al. [15, 16] introduced the concept of Armendariz property for skew generalized power series ring $R[[S,\omega]]$, which is known as (S,ω)-Armendariz property; and also studied the properties of skew generalized power series ring $R[[S,\omega]]$ related to semicommutative and abelian rings using (S,ω)-Armendariz property. (S,ω)-Armendariz property can also be used to study other properties of skew generalized power series ring $R[[S,\omega]]$ related to ring structures such as PF-ring and PP-ring. A ring R is called a PF-ring (resp. PP-ring) if every principal ideal is flat (resp. projective); further, if R is a Noetherian ring then these two notations are equal [21, Corollary 4.3].

Al-Ezeh [1] proved that a ring R is a PF-ring if and only if $ann_R(m)$ for each $m \in R$ is a pure ideal (that is, for all $b \in ann_R(r)$ there exists $c \in ann_R(r)$ such that $bc = b$). In [2], Al-Ezeh also proved that the power series ring $R[[X]]$ is a PF-ring if and only if for any two countable subsets $P = \{p_0, p_1, p_2, \dots\}$ and $Q = \{q_1, q_2, \dots\}$ of R such that $Q \subseteq ann_R(P)$, there exists $r \in ann_R(P)$ such that $qr = q$ for all $q \in Q$. Further, J.H. Kim [9] showed that for a Noetherian ring R, $R[[X]]$ is a PF (resp. PP)-ring if and only if R is a PF (resp. PP)-ring. Later, Liu and Ahsan [13] investigated that the ring $[[R^{S,\leq}]]$ of generalized power series is a PP-ring if and only if R is a PP-ring and every S-indexed subset C of $B(R)$ (set of all idempotents of R) has a least upper bound in $B(R)$.

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Afterwards, Kim and Kwon [10] studied that if R is a commutative ring with identity and $(S,\leq)$ is a strictly totally ordered monoid, then the ring $[[R^{S,\leq}]]$ is a PF-ring if and only if for any two S-indexed subset P and Q of R such that $P \subseteq ann_R(Q)$, there exists $c \in ann_R(Q)$ such that $ac = a$ for any $a \in P$, and that for a Noetherian ring R, $[[R^{S,\leq}]]$ is a PP-ring if and only if R is a PP-ring.

In this paper, R and S denote an associative ring with identity and monoid, respectively. We here extend the above results to skew generalized power series ring $R[[S,\omega]]$ using (S,ω)-Armendariz property.

2 Preliminaries and Definitions

This section deals with the fundamentals of the skew generalized power series rings, (S,ω)-Armendariz rings and S-compatible rings.

Firstly, in order to define skew generalized power series ring, we need to give the following definitions. Let $(S,\leq)$ be a partial ordered set, $(S,\leq)$ is called artinian if every strictly decreasing sequence of elements of S is finite, and $(S,\leq)$ is called narrow if every subset of pairwise order-incomparable elements of S is finite. Thus, $(S,\leq)$ is artinian and narrow if and only if every nonempty subset of S has at least one but only a finite number of minimal elements.

An ordered monoid is a pair $(S,\leq)$ consisting of a monoid S and an order $\leq$ on S such that for all $a,b,c \in S$, $a \leq b$ implies $ca \leq cb$ and $ac \leq bc$. An ordered monoid $(S,\leq)$ is said to be strictly ordered if for all $a,b,c \in S$, $a < b$ implies $ca < cb$ and $ac < bc$ (for more details see [19, 20]).

Let R be a ring, $(S,\leq)$ a strictly ordered monoid and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. For $s \in S$, let ω_s denote the image s under ω, that is, $\omega_s = \omega(s)$. Let A be a set of functions $\alpha : S \to R$ such that the $\mathrm{supp}(\alpha) = \{s \in S : \alpha(s) \neq 0\}$ is an artinian and narrow. Then for any $s \in S$ and $\alpha,\beta \in A$ the set $X_s(\alpha,\beta) = \{(x,y) \in \mathrm{supp}(\alpha) \times \mathrm{supp}(\beta) : s = xy\}$ is finite. Thus, one can define the product of $\alpha\beta : S \to R$ of $\alpha,\beta \in A$ as follows:

$$(\alpha\beta)(s) = \sum_{(x,y)\in X_s(\alpha,\beta)} \alpha(x)\omega_x(\beta(y));$$

with point-wise addition and multiplication as defined above, A becomes a ring, called the ring of skew generalized power series with coefficients in R and exponents in S, denoted by $R[[S,\omega,\leq]]$ (or by $R[[S,\omega]]$). The skew generalized power series ring is common generalization of (skew) polynomial rings, (skew) power series rings, (skew) Laurent polynomial rings, (skew) Laurent power series rings, (skew) group rings, (skew) monoid rings, Mal'cev-Neumann Laurent series rings, and generalized power series rings. We will use the symbol 1 to denote the identity elements of S, the ring R, and the ring $R[[S,\omega]]$ as well as the trivial monoid homomorphism $1 : S \to End(R)$ that send the every element of S to identity endomorphism.

Let $r \in R$ and $s \in S$, and define a mappings $c_r, e_s \in R[[S,\omega]]$ as follows:

$$c_r(x) = \begin{cases} r & \text{if } x = 1 \\ 0 & \text{if } x \in S \setminus \{1\}, \end{cases} \quad \text{and} \quad e_s(x) = \begin{cases} 1 & \text{if } x = s \\ 0 & \text{if } x \in S \setminus \{s\}, \end{cases}$$

then it is clear that the ring R is canonically embedded as a subring of $R[[S,\omega]]$, and S is canonically embedded as a submonoid of $(R[[S,\omega]] - \{1\})$, and $e_s c_r = c_{\omega_s(r)} e_s$.

Moreover, for each nonempty subset X of R we write $X[[S,\omega]] = \{\alpha \in R[[S,\omega]] : \alpha(x) \in X \cup \{0\}$ for every $s \in S\}$, denotes the subsets of $R[[S,\omega]]$, and for each nonempty subset Y of $R[[S,\omega]]$, $C_Y = \{\beta(t) : \beta \in y \text{ and } t \in S\}$, denotes the subset of R (for more details see [14, 15, 16]).

With the help of the definition of skew generalized power series ring, Marks et al. [15, 16] introduced the definition of (S,ω)-Armendariz ring such as:

Definition 2.1. Let R be a ring, $(S,\leq)$ a strictly ordered monoid, and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. A ring R is (S,ω)-Armendariz if whenever $\alpha\beta = 0$ for $\alpha,\beta \in R[[S,\omega]]$, then $\alpha(s)\omega_s(\beta(t)) = 0$ for all $s,t \in S$. If $S = \{1\}$, then every ring is (S,ω)-Armendariz.

Definition 2.2. An endomorphism σ of a ring R is called compatible if for all $ab \in R$ implies $a\sigma(b) = 0$. Let R be a ring, $(S,\leq)$ a strictly ordered monoid and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. Then R is called S-compatible if ω_s is compatible for all $s \in S$ (for more details [16]).

3 Main results

For the purpose of this section we consider a ring R, $(S,\leq)$ a strictly ordered monoid and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. In this section, we study main results on PF-rings and PP-rings related to the skew generalized power series ring$[[S,\omega]]$. To prove the main results of this section, we need to invoke Lemma 3.1 and Proposition 3.2 due to Marks et al. [16].

Lemma 3.1. Let R be a ring, $(S,\leq)$ a strictly ordered monoid, $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism and $A = R[[S,\omega]]$. The following conditions are equivalent:

(i) R is S-compatible.

(ii) for any $a \in R$ and any nonempty subset $Y \subseteq A$, $a \in ann_R(C_Y) \Leftrightarrow c_a \in ann_A(Y)$.

Proof. See [16, Lemma 3.1]. Q.E.D.

Proposition 3.2. Let R be a ring, $(S,\leq)$ a strictly ordered monoid, $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism and $A = R[[S,\omega]]$. Assume that R is (S,ω)-Armendariz.

(i) if f is an idempotent of A, then $f(1)$ is an idempotent of R and $f = c_{f(1)}$.

(ii) A is abelian.

Proof. See [16, Proposition 4.10]. Q.E.D.

Theorem 3.3. If a ring R is (S,ω)-Armendariz and S-compatible then the skew generalized power series ring $R[[S,\omega]]$ is a PF-ring if and only if for any two S-indexed subsets P and Q of R such that $Q \subseteq ann_R(P)$ and there exists $a \in ann_R(P)$ such that $qa = q$ for all $q \in Q$.

Proof. For convenience of the notation, we write $A = R[[S,\omega]]$. We first established that if R is (S,ω)-Armendariz and S-compatible, and only if part is hold then A is a PF-ring. Let $\beta \in ann_R(\alpha)$, where $\alpha,\beta \in A$. Then $\beta\alpha = 0$. Since R is (S,ω)-Armendariz and S-compatible, $\beta(t)\alpha(s) = 0$ for all $s,t \in S$. Then $\beta(t) \in ann_R(\alpha(s))$. Now, suppose $C_{\{\alpha\}} = P = \{\alpha(s) : s \in \mathrm{supp}(\alpha)\}$ and $C_{\{\beta\}} = Q = \{\beta(t) : t \in \mathrm{supp}(\beta)\}$ are S-indexed subsets of R such that $Q \subseteq ann_R(P)$. There exists $a \in ann_R(P)$ such that $\beta(t)a = \beta(t)$ for all $\beta(t) \in Q$, it follows that $\beta(t)a = \beta(t)$ for all $t \in S$. Thus $\beta c_a = \beta$, and by Lemma 3.1, $c_a \in ann_A(\alpha)$. Therefore for any $\beta \in ann_A(\alpha)$, there exists $c_a \in ann_A(\alpha)$ such that $\beta c_a = \beta$, for all $\beta \in Q$. Hence A is a PF-ring.

Conversely, suppose that A is a PF-ring. Let $P = \{p_s : s \in I\}$ and $Q = \{q_t : t \in J\}$ are two S-indexed subsets of R such that $Q \subseteq ann_R(P)$, where I and J are artinian and narrow subsets of S. Define a mapping $\alpha : R \to S$ and $\beta : R \to S$, respectively, via

$$\alpha(s) = \begin{cases} p_s & \text{if } s \in I \\ 0 & \text{if } s \notin I \end{cases} \quad \text{and} \quad \beta(t) = \begin{cases} q_t & \text{if } t \in J \\ 0 & \text{if } t \notin J\,. \end{cases}$$

Then $\mathrm{supp}(\alpha) = I$ and $\mathrm{supp}(\beta) = J$ are artinian and narrow and so $\alpha, \beta \in A$. Thus $\beta(t) \in Q \subseteq ann_R(P)$, $\beta(t)\alpha(s) = 0$ for all $\alpha(s) \in P$ and $s, t \in S$. Since R is S-compatible, thereby $\beta(t)\omega_t(\alpha) = 0$, then $\beta\alpha = 0$, it implies that $\beta \in ann_A(\alpha)$. Now, by assumption, A is a PF-ring, there exists $\gamma \in ann_A(\alpha)$ such that $\beta\gamma = \beta$. Thus $\gamma\alpha = 0$ and $\beta\gamma = \beta$. Since R is (S, ω)-Armendariz and S-compatible, $\gamma(u)\alpha(s) = 0$ and $\beta(t)(\gamma(1) - c_1(1)) = 0$ for all $u, t, s \in S$. It follows that $\gamma(1) \in ann_R(\alpha(s))$ and $\beta(t)\gamma(1) = \beta(t)c_1(1) = \beta(t)$ for all $\beta(t) \in B$. It proves the Theorem. Q.E.D.

Let $(S, \leq)$ be an ordered monoid. If for any $g_1, g_2, h \in M$, $g_1 < g_2$ implies that $g_1h < g_2h$ and $hg_1 < hg_2$, then $(S, \leq)$ is a called strictly totally ordered monoid.

Corollary 3.4 ([10, Theorem 2.4])**.** Let R be a commutative ring with identity and $(S, \leq)$ a strictly totally ordered monoid. Then $[[R^{S,\leq}]]$ is a PF-ring if and only if for any two S-indexed subsets P and Q such that $Q \subseteq ann_R(P)$, there exists $c \in ann_R(P)$ such that $qc = q$ for all $q \in Q$.

Corollary 3.5 ([10, Corollary 2.5])**.** Let $\mathbb{Q}^+ = \{a \in \mathbb{Q} : a \geq 0\}$ and $\mathbb{R}^+ = \{a \in R : a \geq 0\}$. Then the ring $[[\mathbb{Z}^{\mathbb{N},\leq}]]$, $[[\mathbb{Z}^{\mathbb{R}^+,\leq}]]$, $[[\mathbb{Z}^{\mathbb{Z},\pm}]]$, $[[\mathbb{Z}^{\mathbb{N},\leq}]]$, $[[\mathbb{Z}^{\mathbb{Q},\leq}]]$ and $[[\mathbb{Z}^{\mathbb{R},\leq}]]$ are PF-rings, where $\leq$ is the usual order.

Corollary 3.6 ([10, Corollary 2.7])**.** Let R be a commutative ring. Then $[[R^{N \geq 1,\leq}]]$ is a PF-ring if and only if for any two S-indexed subsets P and Q of R such that $Q \subseteq ann_R(P)$, there exists $c \in ann_R(P)$ such that $qc = q$ for all $q \in Q$.

Corollary 3.7 ([10, Corollary 3.8])**.** Let R be a commutative ring and $(S, \leq)$ a strictly ordered monoid with S being cancellative and torsion-free. If for any two S-indexed subsets P and Q such that $Q \subseteq ann_R(P)$, there exists $c \in ann_R(p)$ such that $qc = q$ for all $q \in Q$ and $(S, \leq)$ is narrow, then $[[R^{S,\leq}]]$ is a PF-ring.

Theorem 3.8. If a ring R is (S, ω)-Armendariz, S-compatible and Noetherian then the skew generalized power series ring $R[[S, \omega]]$ is a PP-ring if and only if R is a PP-ring.

Proof. For convenience of the notation, we write $A = R[[S, \omega]]$. Suppose A is a PP-ring and $a \in R$. Then $c_a \in A$ and $ann_A(c_a) = \varphi A$, where $\varphi^2 = \varphi \in A$. Since R is (S, ω)-Armendariz and S-compatible, by Proposition 3.2, there exists an idempotent $\varphi(1) \in R$ such that $c_{\varphi(1)} = \varphi$. Now we need to show that $ann_R(a) = \varphi(1)R$. Since $ann_R(a) = \varphi A$, $a\varphi = 0$. Therefore $a\varphi(1) = 0$ implies $\varphi(1) \in ann_R(a)$. Thus $\varphi(1)R \subseteq ann_R(a)$. Suppose any $b \in ann_R(a)$, then $ab = 0$, which implies $c_b \in ann_R(c_a)$ since R is S-compatible. Since A is a PP-ring, therefore $c_b = \varphi\alpha$, where $\alpha \in A$. Now,

$$\begin{aligned} b &= c_b(1) \\ &= (\varphi\alpha)(1) \\ &= \varphi(1)\omega_1(\alpha(1)) \\ &= \varphi(1)\alpha(1) \in \varphi(1)R, \end{aligned}$$

therefore $ann_R(a) \subseteq \varphi(1)R$. Hence $ann_R(a) = \varphi(1)R$.

Conversely, suppose that R is a PP-ring. Let $\beta \in ann_A(\alpha)$, where $\alpha \in A$, then $\beta\alpha = 0$. Since R is (S,ω)-Armendariz and S-compatible, $\beta(s)\alpha(t) = 0$ for all $s, t \in S$. By hypothesis R is a Noetherian ring, therefore $C(\alpha)$ is finitely generated, say $C(\alpha) = (\alpha(t_0), \alpha(t_1), \ldots, \alpha(t_n))$. Let $I = ann_R(C(\alpha))$, then

$$\begin{aligned}\beta(s) \in I &= ann_R(C(\alpha)) \\ &= \bigcap_{i=0}^{n} ann_R(\alpha(t_i))\,.\end{aligned}$$

Since R is a PP-ring, $ann_R(\alpha(t_i)) = e_iR$, where $e_i^2 = e_i \in R$ for all $i = 0, 1, \ldots, n$. Thus $\beta(s) \in I = eR$, where $e = e_0e_1 \ldots e_n \in R$. Since R is S-compatible, $\beta \in c_eA$. By hypothesis, $ann_R(C(\alpha)) = eR$, where $C(\alpha)$ is finitely generated. It implies $\beta(s) \in ann_R(C(\alpha))$. Since (S,ω)-Armendariz and S-compatible,so by [16, Theoerm 3.4], $\beta \in ann_A(\alpha)$. Therefore $ann_A(\alpha) = c_eA$. Hence A is a PP-ring. Q.E.D.

Corollary 3.9 ([10, Theorem 2.10])**.** Let R be a Noetherian ring and $(S,\leq)$ a strictly totally ordered monoid. Then $[[R^{S,\leq}]]$ is a PP-ring if and only if R is a PP-ring.

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