Some results on p-calculus

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Abstract

Our aim is to present some new properties of functions in p-calculus. The effects of a convex or monotone function on the p-derivative and vice versa and also the behavior of p-derivative in a neighborhood of a local extreme point are expressed. Moreover, mean value theorems for p-derivatives and p-integrals are proved.

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1 Introduction and preliminaries

Quantum calculus is usually known as "calculus without limit". There are several types of quantum calculus such as h-calculus, q-calculus and Hahn calculus. The following three expressions,

$$\begin{aligned} D_q f(x) &= \frac{f(qx)-f(x)}{(q-1)x}, \\ D_h f(x) &= \frac{f(x+h)-f(x)}{h}, \\ D_{q,h} f(x) &= \frac{f(qx+h)-f(x)}{(q-1)x+h}, \end{aligned}$$

are called the q-derivative, the h-derivative and q,h-derivative, respectively, of the function $f(x)$, where q is a fixed number different from 1, and h a fixed number different from 0. The h-derivative of the function $f(x)$ is also known as finite difference operator. Taylor's "Methods Incrementorum" is considered the first reference of the h-calculus or the calculus of finite differences [14], but it is Jacob Stirling who is considered the founder of the h-calculus [13]. In 1750 Euler proved the pentagonal number theorem which was the first example of a q-series and, in some sense, he introduced the q-calculus. The q-derivative was (re)introduced by Jackson in the early twentieth century [7]. Another type of quantum calculus is the Hahn quantum calculus which can be seen as a generalization of both q-calculus and h-calculus. Although Hahn defined this operator in 1949, only in 2009 Aldwoah constructed its inverse operator [1, 2]. For more details about quantum calculus, we refer the readers to [3, 4, 6, 8, 12]. Applications of q-calculus to problems in physics and chemical physics abound [6, 9, 10]. Also, it has developed into an interdisciplinary subject and has a lot of applications in different mathematical areas such as orthogonal polynomials, analytic number theory, basic hyper-geometric functions, combinatorics, etc. A history of the q-calculus was given by T.Ernst [5].

Throughout this paper, we assume that p is a fixed number different from 1 and domain of function $f(x)$ is $[0, +\infty)$. In this section, we recall some definitions and fundamental results on p-calculus that is needed to prove our results (see [11]).

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Definition 1.1. Consider an arbitrary function f(x). Its p-derivative is defined as

$$D_p f(x) = \frac{f(x^p) - f(x)}{x^p - x}, \quad \text{if} \;\; x \neq 0, 1,$$

and

$$D_p f(0) = \lim_{x \to 0^+} D_p f(x), \quad D_p f(1) = \lim_{x \to 1} D_p f(x).$$

Corollary 1.2. If $f(x)$ is differentiable, then $\lim\limits_{p \to 1} D_p f(x) = f'(x)$, and also if $f'(x)$ exists in a neighborhood of $x = 0$, $x = 1$ and is continuous at $x = 0$ and $x = 1$, then we have

$$D_p f(0) = f'_+(0), \quad D_p f(1) = f'(1).$$

Definition 1.3. The p-derivative of higher order of function $f(x)$ is defined by

$$(D_p^0 f)(x) = f(x), \quad (D_p^n f)(x) = D_p(D_p^{n-1} f)(x), n \in N.$$

Notice that the p-derivative is a linear operator, i.e., for any constants a and b, and arbitrary functions $f(x)$ and $g(x)$, we have

$$D_p(af(x) + bg(x)) = aD_p f(x) + bD_p g(x).$$

Also, the p-derivative of the product and the quotient of $f(x)$ and $g(x)$ are computed as follows.

$$\begin{aligned} D_p(f(x)g(x)) &= \frac{f(x^p)g(x^p) - f(x)g(x)}{x^p - x} \\ &= \frac{f(x^p)g(x^p) - f(x)g(x^p) + f(x)g(x^p) - f(x)g(x)}{x^p - x} \\ &= \frac{(f(x^p) - f(x))g(x^p) + f(x)(g(x^p) - g(x))}{x^p - x}, \end{aligned}$$

thus

$$D_p(f(x)g(x)) = g(x^p)D_p f(x) + f(x)D_p g(x). \tag{1.1}$$

Similarly, we can interchange f and g, and obtain

$$D_p(f(x)g(x)) = g(x)D_p f(x) + f(x^p)D_p g(x), \tag{1.2}$$

by changing $f(x)$ to $\dfrac{f(x)}{g(x)}$ in (1.1), we have

$$D_p f(x) = D_p(\frac{f(x)}{g(x)} g(x)) = g(x^p) D_p(\frac{f(x)}{g(x)}) + \frac{f(x)}{g(x)} D_p g(x),$$

then,

$$D_p(\frac{f(x)}{g(x)}) = \frac{g(x)D_p f(x) - f(x)D_p g(x)}{g(x)g(x^p)},$$

using (1.2) with functions $\frac{f(x)}{g(x)}$ and $g(x)$, we obtain

$$D_p(\frac{f(x)}{g(x)}) = \frac{g(x^p)D_pf(x) - f(x^p)D_pg(x)}{g(x)g(x^p)}.$$

Now let us define the definite p-integral. We consider the following three cases. Then, the definite p-integral related to each case is given.

Case 1. Let $1 < a < b$ and $p \in (0,1)$. Notice that for any $j \in \{0,1,2,3,...\}$, we have $b^{p^j} \in (1,b]$. We now define the definite p-integral of $f(x)$ on interval $(1,b]$.

Definition 1.4. The definite p-integral of $f(x)$ on the interval $(1,b]$ is defined as

$$\int_1^b f(x)d_px = \lim_{N\to\infty}\sum_{j=0}^{N}(b^{p^j} - b^{p^{j+1}})f(b^{p^j}) = \sum_{j=0}^{\infty}(b^{p^j} - b^{p^{j+1}})f(b^{p^j}), \tag{1.3}$$

and

$$\int_a^b f(x)d_px = \int_1^b f(x)d_px - \int_1^a f(x)d_px.$$

Note 1.5. Geometrically, the integral in (1.3) corresponds to the area of the union of an infinite number of rectangles. On $[1+\varepsilon, b]$, where ε is a small positive number, the sum consists of finitely many terms, and is a Riemann sum. Therefore, as $p \to 1$, the norm of partition approaches zero, and the sum tends to the Riemann integral on $[1+\varepsilon, b]$. Since ε is arbitrary, provided that $f(x)$ is continuous in the interval $[1,b]$, thus we have

$$\lim_{p\to 1}\int_1^b f(x)d_px = \int_1^b f(x)dx.$$

Case 2. Let $0 < b < 1$ and $p \in (0,1)$. It should be noted that for any $j \in \{0,1,2,3,...\}$, we have $b^{p^j} \in [b,1)$ and $b^{p^j} < b^{p^{j+1}}$. We will define the definite p-integral of $f(x)$ on interval $[b,1)$ as follows.

Definition 1.6. The definite p-integral of $f(x)$ on the interval $[b,1)$ is defined as

$$\int_b^1 f(x)d_px = \lim_{N\to\infty}\sum_{j=0}^{N}(b^{p^{j+1}} - b^{p^j})f(b^{p^j}) = \sum_{j=0}^{\infty}(b^{p^{j+1}} - b^{p^j})f(b^{p^j}).$$

Note 1.7. The above two definite p-integrals are also denoted by

$$\int_1^b f(x)d_px = I_{p^+}f(b), \quad \int_b^1 f(x)d_px = I_{p^-}f(b).$$

Case 3. Let $0 < a < b < 1$ and $p \in (0,1)$. For any $j \in \{0,1,2,3,...\}$, we have $b^{p^{-j}} \in (0,b]$ and $b^{p^{-j-1}} < b^{p^{-j}}$. We will define the definite p-integral of $f(x)$ on interval $(0,b]$ as follows.

Definition 1.8. The definite p-integral of $f(x)$ on the interval $(0, b]$ $(b < 1)$ is defined as

$$I_p f(b) = \int_0^b f(x) d_p x = \lim_{N\to\infty} \sum_{j=0}^{N} (b^{p^{-j}} - b^{p^{-j-1}}) f(b^{p^{-j-1}}) = \sum_{j=0}^{\infty} (b^{p^{-j}} - b^{p^{-j-1}}) f(b^{p^{-j-1}}),$$

and

$$\int_a^b f(x) d_p x = \int_0^b f(x) d_p x - \int_0^a f(x) d_p x.$$

Note 1.9. We can also apply Note 1.5 for the p-integrals defined in the cases 2 and 3 on the intervals $[b, 1-\varepsilon]$ and $[\varepsilon, b]$ respectively, and by it define the Riemann integral.

Remark 1.10. If $p \in (0,1)$, then for any $j \in \{0, \pm 1, \pm 2, ...\}$, we have $p^{p^j} \in (0,1)$, $p^{p^j} < p^{p^{j+1}}$ and

$$\int_0^1 f(x) d_p x = \sum_{j=-\infty}^{\infty} \int_{p^{p^j}}^{p^{p^{j+1}}} f(x) d_p x = \sum_{j=-\infty}^{\infty} (p^{p^{j+1}} - p^{p^j}) f(p^{p^j}).$$

Property 1.11. Suppose $0 \le a < 1 < b$. Then by Note 1.5 and Note 1.9, we have

$$\int_a^b f(x) d_p x = \int_a^1 f(x) d_p x + \int_1^b f(x) d_p x.$$

Corollary 1.12. Suppose $0 \le a < 1 < b$ and function $f(x)$ is continuous on $[a, b]$. Then by Note 1.5 and Note 1.9 and also property 1.11 ,we have

$$\lim_{p\to 1} \int_a^b f(x) d_p x = \lim_{p\to 1} \left(\int_a^1 f(x) d_p x + \int_1^b f(x) d_p x \right) = \int_a^1 f(x) dx + \int_1^b f(x) dx = \int_a^b f(x) dx.$$

Definition 1.13. The p-integral of higher order of function $f(x)$ is given by

$$(I_p^0 f)(x) = f(x), \quad (I_p^n f)(x) = I_p(I_p^{n-1} f)(x), \ \ n \in N.$$

2 Mean value theorems for p-derivatives

In this section we establish and prove some p-mean value theorems. Before we get p-mean value theorems, we describe the behavior of p-derivative in a neighborhood of a local extreme point.

Theorem 2.1. Let $0 < a < b$ and $f(x)$ be a continuous function on $[a, b]$. If f assumes a local maximum at $c \in (a, b)$ with $c \neq 1$, then there exists $p' \in (0, 1)$, such that for every $p \in (p', 1) \cup (1, \frac{1}{p'})$ there exists $\delta \in (a, b)$ such that $(D_p f)(\delta) = 0$.

Proof. We consider the following three cases.
Case 1. Suppose $1 < a < b$. Since c is a point of local maximum of the function $f(x)$, there exists $\varepsilon > 0$, such that $f(c) \ge f(x)$, for all $x \in (c - \varepsilon, c + \varepsilon)$. Let $p_0 \in (0, 1)$ such that $c^{p_0} \in (c - \varepsilon, c)$. Thus, for all $p \in (p_0, 1)$, we have $c^p < c$ and $f(c^p) \le f(c)$. Therefore, $(D_p f)(c) \ge 0$. Similarly, there exists $p_1 \in (0, 1)$, such that for all $p \in (1, \frac{1}{p_1})$, we have $c^p \in (c, c + \varepsilon)$ and $f(c) \ge f(c^p)$ and thus, $(D_p f)(c) \le 0$. Now let us choose $p' = \max\{p_0, p_1\}$. Suppose $p \in (p', 1)$. If $\eta = c^{p^{-1}}$, then

$f(c) \geq f(\eta)$ and $(D_pf)(\eta) \leq 0$. On the other hand, in this case we have $(D_pf)(c) \geq 0$ and by the continuity $(D_pf)(x)$ on (a,b), it implies that there exists $\delta \in (c,\eta) \subset (a,b)$, such that $(D_pf)(\delta) = 0$. Now suppose $p \in (1, \frac{1}{p'})$. If $\eta = c^{p^{-1}}$, then $\eta \in (c-\varepsilon, c)$ and thus $(D_pf)(\eta) \geq 0$. On the other hand, in this case we have $(D_pf)(c) \leq 0$ and by the continuity $(D_pf)(x)$ on (a,b), it implies that there exists $\delta \in (c,\eta) \subset (a,b)$, such that $(D_pf)(\delta) = 0$.

Case 2. Suppose $0 < a < b < 1$. Let $p_0 \in (0,1)$ such that $c^{p_0} \in (c, c+\varepsilon)$. Thus, for all $p \in (p_0, 1)$, we have $c^p \in (c, c+\varepsilon)$ and $f(c) \geq f(c^p)$. Therefore, $(D_pf)(c) \leq 0$. Similarly, there exists $p_1 \in (0,1)$, such that for all $p \in (1, \frac{1}{p_1})$, we have $c^p \in (c-\varepsilon, c)$ and thus $(D_pf)(c) \geq 0$. Let $p' = \max\{p_0, p_1\}$. Suppose $p \in (p', 1)$. If $\eta = c^{p^{-1}}$, then $(D_pf)(\eta) \geq 0$. On the other hand, in this case we have $(D_pf)(c) \leq 0$ and by the continuity $(D_pf)(x)$ on (a,b), it implies that there exists $\delta \in (c,\eta) \subset (a,b)$, such that $(D_pf)(\delta) = 0$. If $p \in (1, \frac{1}{p'})$, then the proof is similar to the above process.

Case 3. Suppose $0 < a < 1 < b$. If $a < 1 < c < b$, then the proof is similar to the proof of case 1 and if $a < c < 1 < b$, then the proof is similar to the proof of case 2.

Note 2.2. If in Theorem 2.1, $f'(x)$ exists in a neighborhood of $x = 1$ and is continuous at $x = 1$ and also if $c = 1$ is a point of local maximum of the function $f(x)$ on (a,b), then for every $p \in (0,1)$, we have $(D_pf)(1) = f'(1) = 0$.

Note 2.3. Theorem 2.1 is also true if c is a point of local minimum of the function $f(x)$.

Remark 2.4. Suppose $0 < a < b$ and $f(x)$ is differentiable on (a,b). If c is a point of local extreme of $f(x)$, then by Corollary 1.2, we have $\lim\limits_{p\to 1} D_pf(c) = f'(c) = 0$.

Example 2.5. Consider $f(x) = -x^2 + 5x - 4$. Its maximum is at $c = 2.5$ and

$$(D_pf)(x) = \frac{f(x^p) - f(x)}{x^p - x} = \frac{-x^{2p} + 5x^p + x^2 - 5x}{x^p - x}.$$

If $\varepsilon = 0.5$ and $p_0 = \frac{1}{1.2}$, then $c^p \in (c-\varepsilon, c)$ for all $p \in (p_0, 1)$ and also, if $p_1 = \frac{1}{1.1}$, then $c^p \in (c, c+\varepsilon)$ for all $p \in (1, \frac{1}{p_1})$. Let $p' = \max\{p_0, p_1\} = \frac{1}{1.1}$. For $p = \frac{1}{1.01}$, we have $(D_{\frac{1}{1.01}}f)(c) = (D_{\frac{1}{1.01}}f)(2.5) = \frac{4}{3} > 0$. If $\eta = c^{p^{-1}} = 2.52$, then we have $(D_{\frac{1}{1.01}}f)(\eta) = -0.043 < 0$. Therefore, there exists $\delta \in (2.5, 2.52)$, such that $(D_pf)(\delta) = 0$.

We now are in position to state and prove some p-mean value theorems.

Theorem 2.6. Let $0 < a < b$ and $f(x)$ be a continuous function on $[a,b]$ satisfying $f(a) = f(b)$ and also $f'(x)$ exists in a neighborhood of $x = 1$ and be continuous at $x = 1$. Then there exists $p' \in (0,1)$, such that for every $p \in (p', 1) \cup (1, \frac{1}{p'})$, there exists $\delta \in (a,b)$ such that $(D_pf)(\delta) = 0$.

Proof. If $f = const$, then the result is obvious. If f is not a constant function on $[a,b]$, then it attains its extreme value in some point in (a,b). If $c = 1$ is a point of local extreme of $f(x)$, then by Note 2.2, the result holds, and if the point of local extreme of $f(x)$ is different from 1, then by Theorem 2.1 and Note 2.3, the statement follows.

Theorem 2.7. Let $0 < a < b$ and $f(x)$ be a continuous function on $[a,b]$ and also $f'(x)$ exists in a neighborhood of $x = 1$ and be continuous at $x = 1$. Then there exists $p' \in (0,1)$, such that for every $p \in (p', 1) \cup (1, \frac{1}{p'})$, there exists $\delta \in (a,b)$ such that $f(b) - f(a) = (D_pf)(\delta)(b-a)$.

Proof. Let $g(x)$ be a function defined on $[a,b]$ by $g(x) = f(x) - x\frac{f(b)-f(a)}{b-a}$. Clearly, $g(x)$ is a continuous function on $[a,b]$ with $g(a) = g(b)$ and also $g^{'}(x)$ exists in a neighborhood of $x = 1$ and is continuous at $x = 1$. Hence, by Theorem 2.6, the statement follows.

3 Monotone or convex function and its p-derivative

In this section we study relations between monotone or convex function and p-derivatives.

Definition 3.1. A function $f(x)$ is called increasing on an interval I if $f(b) \geq f(a)$ for all $b > a$, whenever $a, b \in I$. Also, a function $f(x)$ is called decreasing on an interval I if $f(b) \leq f(a)$ for all $b > a$ with $a, b \in I$.

Theorem 3.2. Let $p \in R^+ - \{1\}$ and $f(x)$ be a function defined on $I = (0, +\infty)$. Then,

(i) If $f(x)$ is an increasing function on I, then $(D_p f)(x) \geq 0$, for all $x \in I - \{1\}$.

(ii) If $f(x)$ is a decreasing function on I, then $(D_p f)(x) \leq 0$, for all $x \in I - \{1\}$.

Proof. Since the proofs of (i) and (ii) are very similar, we will expose only the first one. Since $f(x)$ is increasing function on I, hence for every $x \in I - \{1\}$, if $x^p < x$, then $f(x^p) \leq f(x)$ and thus, $(D_p f)(x) \geq 0$, and if $x^p > x$, then $f(x^p) \geq f(x)$ and we conclude $(D_p f)(x) \geq 0$.

Note 3.3. We can generalize the results of Theorem 3.2 to interval $I = (0, +\infty)$ if $f^{'}(x)$ exists in a neighborhood of $x = 1$ and is continuous at $x = 1$, because in this case if $f(x)$ is an increasing function, then $(D_p f)(1) = f^{'}(1) \geq 0$, and if $f(x)$ is a decreasing function, then $(D_p f)(1) = f^{'}(1) \leq 0$.

Theorem 3.4. Let $0 < a < b$ and $p \in R^+ - \{1\}$ and $f(x)$ be a continuous function on $[a,b]$ such that for every $p \in R^+ - \{1\}$, we have $(D_p f)(x) \geq 0$ on (a,b). Then $f(x)$ is an increasing function on (a,b).

Proof. Suppose $a < x_1 < x_2 < b$. By Theorem 2.7, there exists $p \in R^+ - \{1\}$ and $\delta \in (x_1, x_2)$ such that $f(x_2) - f(x_1) = (D_p f)(\delta)(x_2 - x_1)$. Since the right side of equality is nonnegative, we have $f(x_2) \geq f(x_1)$. Hence, the proof is complete.

Theorem 3.5. Let $0 < a < b$ and $p \in R^+ - \{1\}$ and $f(x)$ be a continuous function on $[a,b]$ such that for every $p \in R^+ - \{1\}$, we have $(D_p f)(x) \leq 0$ on (a,b). Then $f(x)$ is a decreasing function on (a,b).

Proof. The proof is similar to the proof of Theorem 3.4.

Definition 3.6. Let $f(x)$ be a real value function defined on (a,b) where $-\infty \leq a < b \leq \infty$. Then, $f(x)$ is called convex if for any two point x and y in (a,b) and any λ where $0 \leq \lambda \leq 1$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$$

Lemma 3.7. If $f(x)$ is a convex function on (a,b) and $a < s < t < u < b$, then we have

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

Theorem 3.8. Let $p \in R^+ - \{1\}$ and $f(x)$ be a convex function on $I = (0, +\infty)$. Then $(D_p f)(x)$ is increasing on $I - \{1\}$.

Proof. We prove the result only for the case $p \in (0, 1)$. The proof for the case $p > 1$ is similar. We consider the following three cases.
Case 1. Let $0 < x < y < 1$. Thus, we have $0 < x < x^p < y^p < 1$ and $0 < x < y < y^p < 1$. By Lemma 3.7, we have

$$\frac{f(x^p) - f(x)}{x^p - x} \leq \frac{f(y^p) - f(x)}{y^p - x} \leq \frac{f(y^p) - f(y)}{y^p - y}.$$

Therefore, $(D_p f)(x) \leq (D_p f)(y)$.
Case 2. Let $1 < x < y < \infty$. Thus, we have $1 < x^p < x < y < \infty$ and $1 < x^p < y^p < y < \infty$. By Lemma 3.7, we have

$$\frac{f(x) - f(x^p)}{x - x^p} \leq \frac{f(y) - f(x^p)}{y - x^p} \leq \frac{f(y) - f(y^p)}{y - y^p}.$$

Therefore, $(D_p f)(x) \leq (D_p f)(y)$.
Case 3. Let $0 < x < 1 < y < \infty$. Thus, we have $0 < x < x^p < y^p < y < \infty$ and

$$\frac{f(x^p) - f(x)}{x^p - x} \leq \frac{f(y^p) - f(x^p)}{y^p - x^p} \leq \frac{f(y) - f(y^p)}{y - y^p}.$$

Therefore, $(D_p f)(x) \leq (D_p f)(y)$. This complete the proof.

Corollary 3.9. Let $p \in R^+ - \{1\}$ and $f(x)$ be a convex function on $I = (0, +\infty)$. Then $D_p^2 f(x) \geq 0$ for all $x \in I - \{1\}$.

Proof. (The first way). By Theorem 3.8 and also Theorem 3.2 the statement follows.
(The second way). We prove the result only the case $p \in (0, 1)$. The case when $p > 1$ can be proved in a similar way. By the definition of p-derivative, we have

$$D_p^2 f(x) = D_p(D_p f)(x) = \frac{D_p f(x^p) - D_p f(x)}{x^p - x} = \frac{1}{x^p - x}\left(\frac{f(x^{p^2}) - f(x^p)}{x^{p^2} - x^p} - \frac{f(x^p) - f(x)}{x^p - x}\right).$$

If $0 < x < 1$, then $0 < x < x^p < x^{p^2} < 1$ and by Lemma 3.7, we have

$$\frac{f(x^p) - f(x)}{x^p - x} \leq \frac{f(x^{p^2}) - f(x^p)}{x^{p^2} - x^p},$$

and it implies, $D_p^2 f(x) \geq 0$. If $x > 1$, then $1 < x^{p^2} < x^p < x$ and

$$\frac{f(x^p) - f(x^{p^2})}{x^p - f(x^{p^2})} \leq \frac{f(x) - f(x^p)}{x - x^p},$$

and therefore, $D_p^2 f(x) \geq 0$.

Definition 3.10. Let $0 < a < b$ and $p \in R^+ - \{1\}$ and $f(x)$ be a real value function defined on (a, b). Then,

(i) Operator $D_p f$ is increasing respect to p on (a,b) if $D_{p_1} f(x) \leq D_{p_2} f(x)$ for all $p_1 < p_2$.

(ii) Operator $D_p f$ is decreasing respect to p on (a,b) if $D_{p_1} f(x) \geq D_{p_2} f(x)$ for all $p_1 < p_2$.

Theorem 3.11. Let $p \in R^+ - \{1\}$ and $f(x)$ be a convex function on $I = (1, +\infty)$. Then $D_p f$ is increasing respect to p on $(1, +\infty)$.

Proof. We prove the result only the case $0 < p_1 < p_2 < 1$. Cases when $1 < p_1 < p_2$ or $p_1 < 1 < p_2$, can be proved in a similar way. For every $x > 1$, we have $1 < x^{p_1} < x^{p_2} < x$, and by Lemma 3.7, we have $\frac{f(x) - f(x^{p_1})}{x - x^{p_1}} \leq \frac{f(x) - f(x^{p_2})}{x - x^{p_2}}$, and it implies $D_{p_1} f(x) \leq D_{p_2} f(x)$.

Theorem 3.12. Let $p \in R^+ - \{1\}$ and $f(x)$ be a convex function on $I = (0,1)$. Then $D_p f$ is decreasing respect to p on $(0,1)$.

Proof. The proof is similar to the proof of Theorem 3.11.

4 Mean value theorems for p-integrals

In this section we present mean value theorems for p-integrals.

Theorem 4.1. Let $f(x)$ be a continuous function on $[0,b]$ $(b > 0)$. Then for every $p \in (0,1)$, there exists $\delta \in [0,b]$ such that $\frac{1}{b} \int_0^b f(x) d_p x = f(\delta)$.

Proof. It is sufficient to prove the result for the case $b > 1$. Since $f(x)$ is a continuous function $[0,b]$, there exist m and M such that for each $x \in [0,b]$, $m \leq f(x) \leq M$. Let $p \in (0,1)$. Then, for any $j \in \{0, \pm 1, \pm 2, \pm 3, ...\}$, we have $p^{p^j} \in (0,1)$ and $m \leq f(p^{p^j}) \leq M$, and also for any $j \in \{0, 1, 2, 3, ...\}$, we have $b^{p^j} \in (1, b]$ and $m \leq f(b^{p^j}) \leq M$. Hence,

$$\begin{aligned}\sum_{j=-\infty}^{\infty} (p^{p^{j+1}} - p^{p^j}) f(p^{p^j}) + \sum_{j=0}^{\infty} (b^{p^j} - b^{p^{j+1}}) f(b^{p^j}) &\leq \sum_{j=-\infty}^{\infty} (p^{p^{j+1}} - p^{p^j}) M + \sum_{j=0}^{\infty} (b^{p^j} - b^{p^{j+1}}) M \\ &= M + M(b-1) = Mb,\end{aligned}$$

also,

$$\begin{aligned}\sum_{j=-\infty}^{\infty} (p^{p^{j+1}} - p^{p^j}) f(p^{p^j}) + \sum_{j=0}^{\infty} (b^{p^j} - b^{p^{j+1}}) f(b^{p^j}) &\geq \sum_{j=-\infty}^{\infty} (p^{p^{j+1}} - p^{p^j}) m + \sum_{j=0}^{\infty} (b^{p^j} - b^{p^{j+1}}) m \\ &= m + m(b-1) = mb.\end{aligned}$$

Now by Remark 1.10 and Property 1.11 , we have $m \leq \frac{1}{b} \int_0^b f(x) d_p x \leq M$. By the intermediate value theorem, there exists $\delta \in [0,b]$ such that $\frac{1}{b} \int_0^b f(x) d_p x = f(\delta)$.

Theorem 4.2. Let $f(x)$ be a continuous function on $[a,b]$ $(a > 0)$. Then, there exists $p' \in (0,1)$ such that for every $p \in (p', 1)$ there exists $\delta \in (a,b)$ such that $\frac{1}{b-a} \int_a^b f(x) d_p x = f(\delta)$.

Proof. By Note 1.5 and Note 1.9 and also Corollary 1.12, we have $\lim_{p\to 1}\int_a^b f(x)d_px = \int_a^b f(x)dx$. Thus, for every $\varepsilon > 0$, there exists $p_0 \in (0,1)$ such that for all $p \in (p_0, 1)$, we have

$$\int_a^b f(x)dx - \varepsilon < \int_a^b f(x)d_px < \int_a^b f(x)dx + \varepsilon.$$

By the mean value theorem for integrals, there exists $c \in (a,b)$ such that $\dfrac{1}{b-a}\int_a^b f(x)dx = f(c)$. Let $\varepsilon \leq (b-a)\min\{M - f(c), f(c) - m\}$, where m and M are the minimum and maximum of $f(x)$ on $[a,b]$, respectively. Hence, there exists $p^{'} \in (0,1)$ such that for all $p \in (p^{'}, 1)$,

$$f(c) - \frac{\varepsilon}{b-a} < \frac{1}{b-a}\int_a^b f(x)d_px < f(c) + \frac{\varepsilon}{b-a}.$$

It implies, $m < \dfrac{1}{b-a}\int_a^b f(x)d_px < M$ and therefore, there exists $\delta \in (a,b)$ such that

$$\frac{1}{b-a}\int_a^b f(x)d_px = f(\delta).$$

proving the intended result.

Theorem 4.3. Let $f(x)$ and $g(x)$ be some continuous functions on $[a,b]$ $(a \geq 0)$. Then, there exists $p^{'} \in (0,1)$ such that for all $p \in (p^{'}, 1)$ there exists $\delta \in (a,b)$ such that

$$\int_a^b f(x)g(x)d_px = g(\delta)\int_a^b f(x)d_px.$$

Proof. Suppose f is not constant function zero. By the second mean value theorem for integrals, there exists $c \in (a,b)$ such that $\int_a^b f(x)g(x)dx = g(c)\int_a^b f(x)dx$. Hence, we have

$$\lim_{p\to 1}\int_a^b f(x)g(x)d_px = g(c)\lim_{p\to 1}\int_a^b f(x)d_px,$$

or

$$\lim_{p\to 1}\frac{\int_a^b f(x)g(x)d_px}{\int_a^b f(x)d_px} = g(c).$$

Thus, for every $\varepsilon > 0$, there exists $p_0 \in (0,1)$ such that for $p \in (p_0, 1)$, we have

$$g(c) - \varepsilon < \frac{\int_a^b f(x)g(x)d_px}{\int_a^b f(x)d_px} < g(c) + \varepsilon.$$

Since $g(x)$ is a continuous function on $[a,b]$, there exist m_g and M_g such that for each $x \in [a,b]$, $m_g \leq g(x) \leq M_g$. Let $\varepsilon \leq \min\{M_g - g(c), g(c) - m_g\}$. Hence, there exists $p^{'} \in (0,1)$ such that for all $p \in (p^{'}, 1)$, $m_g < \dfrac{\int_a^b f(x)g(x)d_px}{\int_a^b f(x)d_px} < M_g$. Therefore, there exists $\delta \in (a,b)$ such that

$$\int_a^b f(x)g(x)d_px = g(\delta)\int_a^b f(x)d_px.$$

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