Differential and integral equations associated with some hybrid families of Legendre polynomials

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Abstract

The article aims to explore some new classes of differential and integral equations for some hybrid families of Legendre polynomials. Beginning with the recurrence relations and shift operators, the authors derived the differential, integro-differential and partial differential equations for the hybrid Legendre-Appell polynomials. Certain examples are framed for the hybrid Legendre-Bernoulli, Legendre-Euler and Legendre-Genocchi polynomials to show the applications of main results. Further, the homogeneous Volterra integral equations for the hybrid Legendre-Appell and other hybrid families of special polynomials are derived. The inclusion of integral equations is a bonus to this article.

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1 Introduction and preliminaries

It is known that the special polynomials of two variables provided new means of analysis for the solutions of large classes of partial differential equations often encountered in physical problems. The introduction of the 2-variable Legendre polynomials $S_n(x,y)$ [6] is of great interest due to their intrinsic mathematical importance and for their applications in physics.

The 2-variable Legendre polynomials (2VLeP) $S_n(x,y)$ are specified by means of the following generating equation:

$$e^{yt}J_0(2t\sqrt{-x}) = \sum_{n=0}^{\infty} S_n(x,y)\frac{t^n}{n!}. \tag{1.1}$$

where $J_0(xt)$ is the 0^{th} order ordinary Bessel function of first kind [2] defined by

$$J_n(2\sqrt{x}) = \sum_{k=0}^{\infty} \frac{(-1)^k\ (\sqrt{x})^{n+2k}}{k!\ (n+k)!}. \tag{1.2}$$

We also note that

$$\exp(-\alpha D_x^{-1}) = J_0(2\sqrt{\alpha x}), \qquad D_x^{-n}\{1\} := x^n/n! \tag{1.3}$$

is the inverse derivative operator.

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The class of Appell polynomial sequences [3] arise in numerous problems of applied mathematics, theoretical physics, approximation theory and several other mathematical branches. These sequences are defined by the following generating function:

$$R(x,t) := R(t)e^{xt} = \sum_{n=0}^{\infty} R_n(x)\frac{t^n}{n!}, \qquad R_n := R_n(0), \tag{1.4}$$

where $R(t)$ is an analytic function at $t=0$ and is given by

$$R(t) = \sum_{n=0}^{\infty} R_n \frac{t^n}{n!}, \qquad R_0 \neq 0, \qquad R_i \ (i=0,1,2,\cdots) \text{ being real coefficients.} \tag{1.5}$$

The Appell polynomials $R_n(x)$ are defined by the following series expansion:

$$R_n(x) = \sum_{k=0}^{n} \binom{n}{k} R_{n-k}\ x^k, \qquad R_n^{'}(x) = n\ R_{n-1}(x). \tag{1.6}$$

By making appropriate selection of $R(t)$, the members belonging to the Appell polynomials family can be obtained. These are given in Table 1 below.

Table 1. Certain members belonging to the Appell family

S.No.	Name of polynomials	$R(t)$	Generating function	Series definition
I.	Bernoulli polynomials and numbers [8]	$\frac{t}{e^t-1}$	$\left(\frac{t}{e^t-1}\right)e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}$ $\left(\frac{t}{e^t-1}\right) = \sum_{n=0}^{\infty} B_n\frac{t^n}{n!}$ $B_n (:= B_n(0) = B_n(1))$	$B_n(x) = \sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k}$
II.	Euler polynomials and numbers [8]	$\frac{2}{e^t+1}$	$\left(\frac{2}{e^t+1}\right)e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}$ $\frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n\frac{t^n}{n!}$ $E_n := 2^n E_n\left(\frac{1}{2}\right)$	$E_n(x) = \sum_{k=0}^{n}\binom{n}{k}\frac{E_k}{2^k}\left(x-\frac{1}{2}\right)^{n-k}$
III.	Genocchi polynomials and numbers [17]	$\frac{2t}{e^t+1}$	$\left(\frac{2t}{e^t+1}\right)e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}$ $\frac{2t}{e^t+1} = \sum_{n=1}^{\infty} G_n\frac{t^n}{n!}$ $G_n := G_n(0)$	$G_n(x) = \sum_{k=0}^{n}\binom{n}{k}G_k x^{n-k}$

We give first few values of Bernoulli numbers B_n, Euler numbers E_n and Genocchi numbers G_n in Table 2 below, which will be used later.

Table 2. Values of five four B_n, E_n and G_n

n	0	1	2	3	4
B_n	1	$\pm\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$
E_n	1	0	-1	0	5
G_n	0	1	-1	0	1

Note 1. From the above table, we note that the degree of $G_n(x)$ is $n-1$, however the degree of all other Appell polynomials is n. Therefore, $G_n(x)$ is considered in the class of polynomial sequences

which are not Appell in the strong sense, see for details [1].

In 2012, Khan and Raza [11] constructed and studied a hybrid class of the Legendre-Sheffer polynomials ${}_Ss_n(x,y)$, which are defined by the generating function of the form:

$$R(t)\ e^{yH(t)}\ J_0(2H(t)\sqrt{-x}) = \sum_{n=0}^{\infty} {}_Ss_n(x,y)\frac{t^n}{n!}. \tag{1.7}$$

As, for $H(t) = t$ the Sheffer polynomials $s_n(x)$ [18] reduce to the Appell polynomials $R_n(x)$. Therefore, by taking $H(t) = t$ in equation (1.7), we obtain the hybrid Legendre-Appell polynomials (LeAP), which are defined by

$$R(t)\ e^{yt}\ J_0(2t\sqrt{-x}) = \sum_{n=0}^{\infty} {}_SR_n(x,y)\frac{t^n}{n!}, \tag{1.8}$$

or, equivalently

$$R(t)\ e^{yt}\ e^{D_x^{-1}t^2} = \sum_{n=0}^{\infty} {}_SR_n(x,y)\frac{t^n}{n!}. \tag{1.9}$$

The hybrid LeAP ${}_SR_n(x,y)$ are defined by the following series expansion:

$${}_SR_n(x,y) = n!\sum_{k=0}^{[n/2]} \frac{R_{n-2k}(y)x^k}{(n-2k)!(k!)^2}. \tag{1.10}$$

Based on appropriate selection for $R(t)$, different members belonging to the family of hybrid LeAP can be obtained. These members are given in Table 3 below.

Table 3. Certain members belonging to the HLeAP family

S. No.	Name of hybrid polynomials	$R(t)$	Generating function	Series definition
I.	Hybrid Legendre-Bernoulli polynomials	$\frac{t}{e^t-1}$	$\left(\frac{t}{e^t-1}\right) e^{yt}\ J_0(2t\sqrt{-x}) = \sum_{n=0}^{\infty} {}_SB_n(x,y)\frac{t^n}{n!}$	${}_SB_n(x,y) = n!\sum_{k=0}^{[n/2]} \frac{B_{n-2k}(y)x^k}{(n-2k)!(k!)^2}$
II.	Hybrid Legendre-Euler polynomials	$\frac{2}{e^t+1}$	$\left(\frac{2}{e^t+1}\right) e^{yt}\ J_0(2t\sqrt{-x}) = \sum_{n=0}^{\infty} {}_SE_n(x,y)\frac{t^n}{n!}$	${}_SE_n(x,y) = n!\sum_{k=0}^{[n/2]} \frac{E_{n-2k}(y)x^k}{(n-2k)!(k!)^2}$
III.	Hybrid Legendre-Genocchi polynomials	$\frac{2t}{e^t+1}$	$\left(\frac{2t}{e^t+1}\right) e^{yt}\ J_0(2t\sqrt{-x}) = \sum_{n=0}^{\infty} {}_SG_n(x,y)\frac{t^n}{n!}$	${}_SG_n(x,y) = n!\sum_{k=0}^{[n/2]} \frac{G_{n-2k}(y)x^k}{(n-2k)!(k!)^2}$

Note 2. In view of the fact given in Note 1, we can say that the hybrid LeGP ${}_SG_n(x,y)$ do not belong to the class of hybrid LeAP ${}_SR_n(x,y)$ in a strong sense.

The study of differential equations is a wide field in pure and applied mathematics, physics and engineering. Differential equations play an important role in modelling virtually every physical, technical, or biological process, from celestial motion to bridge design, to interactions between neurons. The differential equations and other characterizations of Appell and hybrid Appell polynomials are considered in [5, 7, 9, 10, 12, 13, 15, 16, 19–22].

Let $\{p_n(x)\}_{n=0}^{\infty}$ be a sequence of polynomials such that $\deg(p_n(x)) = n,\ (n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\})$. The differential operators Θ_n^- and Θ_n^+ satisfying the properties

$$\Theta_n^-\{p_n(x)\} = p_{n-1}(x) \qquad \text{and} \qquad \Theta_n^+\{p_n(x)\} = p_{n+1}(x), \tag{1.11}$$

are called derivative and multiplicative operators, respectively. The polynomial sequence $\{p_n(x)\}_{n=0}^{\infty}$ satisfying equation (1.11) is then called quasi-monomial. Obtaining the derivative and multiplicative operators of a given family of polynomials give rise to some useful properties such as

$$(\Theta_{n+1}^-\Theta_n^+)\{p_n(x)\} = p_n(x) \qquad \text{and} \qquad (\Theta_{n-1}^+\Theta_{n-2}^+\ldots\Theta_2^+\Theta_1^+\Theta_0^+)\{p_0(x)\} = p_n(x). \tag{1.12}$$

If Θ_n^- and Θ_n^+ are given by differential realizations, then the above equations give rise to the differential equation satisfied by $p_n(x)$. The technique used in obtaining differential equations via (1.12) is known as the factorization method [10].

The article is organized as follow. In Section 2, the recurrence relations and shift operators for the hybrid Legendre-Appell polynomials are established followed by differential, integro-differential and partial differential equations via factorization method. In Section 3, certain applications are framed to give the results for the hybrid Legendre-Bernoulli, Legendre-Euler and Legendre-Genocchi polynomials. In section 4, the integral equations associated with hybrid Legendre-Appell and other hybrid special polynomials are derived.

2 Recurrence relations and differential equations

This section is followed by deriving recurrence relation, shift operators and differential equations for the hybrid Legendre-Appell polynomials. To derive the recurrence relations for the hybrid LeAP ${}_SR_n(x, y)$, we prove the following result:

Theorem 2.1. The hybrid Legendre-Appell polynomials satisfy the following recurrence relation:

$${}_SR_{n+1}(x, y) = (y + \alpha_0)\,{}_SR_n(x, y) + \sum_{k=1}^{n} \binom{n}{k} \alpha_k\ {}_SR_{n-k}(x, y) + 2nD_x^{-1}\,{}_SR_{n-1}(x, y), \tag{2.1}$$

where the coefficients $\{\alpha_k\}_{k\in\mathbb{N}_0}$ are given by following expansion:

$$\frac{R'(t)}{R(t)} = \sum_{k=0}^{\infty} \alpha_k \frac{t^k}{k!}. \tag{2.2}$$

Proof. Differentiation of both sides of generating relation (1.9) with respect to t and then rearranging the terms, it follows that

$$\left(y + 2D_x^{-1}t + \frac{R'(t)}{R(t)}\right) R(t)e^{yt}e^{D_x^{-1}t^2} = \sum_{n=0}^{\infty} {}_RS_{n+1}(x, y)\frac{t^n}{n!}, \tag{2.3}$$

which on using of equations (1.9) and (2.2) and then applying the Cauchy product rule in the l.h.s. of the resultant equation, gives

$$\sum_{n=0}^{\infty}\left(y\ {}_SR_n(x,y)+\sum_{k=0}^{n}\binom{n}{k}\alpha_k\ {}_SR_{n-k}(x,y)+2nD_x^{-1}{}_SR_{n-1}(x,y)\right)=\sum_{n=0}^{\infty}{}_SR_{n+1}(x,y)\frac{t^n}{n!}. \quad (2.4)$$

Equating the coefficients of like powers of t on both sides of the above equation and then interchanging the sides of resultant equation yields assertion (2.1). Q.E.D.

Theorem 2.2. The shift operators for the hybrid Legendre-Appell polynomials are given by

$${}_y\pounds_n^- := \frac{1}{n}D_y, \quad (2.5)$$

$${}_x\pounds_n^- := \frac{1}{n}D_y^{-1}D_x, \quad (2.6)$$

$${}_y\pounds_n^+ := y+\alpha_0+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_y^k+2D_x^{-1}D_y, \quad (2.7)$$

$${}_x\pounds_n^+ := y+\alpha_0+\sum_{k=1}^{n}\frac{\alpha_k}{k!}D_y^{-k}D_x^k+2D_y^{-1}, \quad (2.8)$$

where

$$D_x := \frac{\partial}{\partial x},\ D_y := \frac{\partial}{\partial y} \text{ and } D_x^{-1} := \int_0^x f(\xi)d\xi.$$

Proof. Differentiating both sides of generating relation (1.9) with respect to y and then equating the coefficients of like powers of t on both sides of the resultant equation, it follows that

$$\frac{\partial}{\partial y}\{{}_SR_n(x,y)\} = n\ {}_SR_{n-1}(x,y). \quad (2.9)$$

Consequently, we have

$${}_y\pounds_n^-\{{}_SR_n(x,y)\} = \frac{1}{n}D_y = {}_SR_{n-1}(x,y), \quad (2.10)$$

which proves assertion (2.5).

Again, differentiating both sides of generating relation (1.9) with respect to x and then equating the coefficients of like powers of t on both sides of the resultant equation, it follows that

$$\frac{\partial}{\partial x}\{{}_SR_n(x,y)\} = n(n-1){}_SR_{n-2}(x,y).$$

The above equation can also be written as

$$\frac{\partial}{\partial x}\{{}_SR_n(x,y)\} = n\frac{\partial}{\partial y}\{{}_SR_{n-1}(x,y)\}, \quad (2.11)$$

which finally gives

$$ {}_x\pounds_n^-\{{}_SR_n(x,y)\} = \frac{1}{n}D_y^{-1}D_x = {}_SR_{n-1}(x,y). \tag{2.12} $$

Thus assertion (2.6) is proved.

In order to derive the expression for raising operator (2.7), the following relation is used:

$$ {}_SR_{n-k}(x,y) = ({}_y\pounds_{n-k+1}^-\ {}_y\pounds_{n-k+2}^- \cdots {}_y\pounds_{n-1}^-\ {}_y\pounds_n^-)\{{}_SR_n(x,y)\}, \tag{2.13} $$

which in view of equation (2.5) can be simplified as:

$$ {}_SR_{n-k}(x,y) = \frac{(n-k)!}{n!}D_y^k\{{}_SR_n(x,y)\}. \tag{2.14} $$

Making use of equation (2.14) in recurrence relation (2.1) and using ${}_y\pounds_n^+\{{}_SR_n(x,y)\} = {}_SR_{n+1}(x,y)$, we find

$$ {}_y\pounds_n^+ := \left(y + \alpha_0 + \sum_{k=1}^{n}\frac{\alpha_k}{k!}D_y^k + 2D_x^{-1}D_y\right)\{{}_SR_n(x,y)\} = {}_SR_{n+1}(x,y), \tag{2.15} $$

which proves assertion (2.7).

Next, to find the raising operator ${}_x\pounds_n^+$, the following relation is used:

$$ {}_SR_{n-k}(x,y) = ({}_x\pounds_{n-k+1}^-\ {}_x\pounds_{n-k+2}^- \cdots {}_x\pounds_{n-1}^-\ {}_x\pounds_n^-)\{{}_SR_n(x,y)\}, \tag{2.16} $$

which in view of equation (2.6) can be written as:

$$ {}_SR_{n-k}(x,y) = \frac{(n-k)!}{n!}D_y^{-k}D_x^k\{{}_SR_n(x,y)\}, \tag{2.17} $$

Making use of equation (2.17) in recurrence relation (2.1) and using ${}_x\pounds_n^+\{{}_SR_n(x,y)\} = {}_SR_{n+1}(x,y)$, we find

$$ {}_x\pounds_n^+ := \left(y + \alpha_0 + \sum_{k=1}^{n}\frac{\alpha_k}{k!}D_y^{-k}D_x^k + 2D_y^{-1}\right)\{{}_SR_n(x,y)\} = {}_SR_{n+1}(x,y), \tag{2.18} $$

which proves assertion (2.8). Q.E.D.

Theorem 2.3. The hybrid Legendre-Appell polynomials satisfy the following differential equation:

$$ \left((y+\alpha_0)D_y + \sum_{k=1}^{n}\frac{\alpha_k}{k!}D_y^{k+1} + 2xD_x - n\right){}_SR_n(x,y) = 0. \tag{2.19} $$

Proof. Using expressions (2.5) and (2.7) of the shift operators in the following factorization relation:

$$ {}_y\mathcal{L}_{n+1}^-\ {}_y\mathcal{L}_n^+\{{}_SR_n(x,y)\} = {}_SR_n(x,y), \tag{2.20} $$

we find

$$ \left(\left(y+\alpha_0\right)D_y + \sum_{k=1}^{n}\tfrac{\alpha_k}{k!}D_y^{k+1} + 2D_x^{-1}D_y^2 - n\right){}_SR_n(x,y) = 0, $$

which on using relation $\frac{\partial^2}{\partial y^2} = \frac{\partial}{\partial x}x\frac{\partial}{\partial x}$, yields assertion (2.19). Q.E.D.

Theorem 2.4. The hybrid Legendre-Appell polynomials satisfy the following integro-differential equation:

$$\left(\left(y+\alpha_0\right)D_x+\sum_{k=1}^{n}\tfrac{\alpha_k}{k!}D_y^{-k}D_x^{k+1}+2D_xD_y^{-1}-(n+1)D_y\right){}_SR_n(x,y)=0. \tag{2.21}$$

Proof. Use of expressions (2.6) and (2.8) of the shift operators in the following factorization relation:

$${}_x\mathcal{L}_{n+1}^{-}\ {}_x\mathcal{L}_{n}^{+}\{{}_SR_n(x,y)\}={}_SR_n(x,y), \tag{2.22}$$

yields assertion (2.21). Q.E.D.

Theorem 2.5. The hybrid Legendre-Appell polynomials satisfy the following partial differential equation:

$$\left(\left(y+\alpha_0\right)D_y^n\ D_x+nD_y^{n-1}D_x+\sum_{k=1}^{n}\tfrac{\alpha_k}{k!}D_y^{n-k}D_x^{k+1}+2D_xD_y^{n-1}-(n+1)D_y^{n+1}\right){}_SR_n(x,y)=0. \tag{2.23}$$

Proof. Differentiation of integro-differential equation (2.21) n-times with respect to y yields assertion (2.23). Q.E.D.

In the next section, certain examples are considered as applications of the results derived above.

3 Applications

In this section, the recurrence relation, shift operators, differential, integro-differential and partial differential equations for some members (given in Table 2) belonging to the hybrid LeAP are derived by considering the following examples:

Example 3.1. Taking $R(t)=\left(\frac{t}{e^t-1}\right)$ (that is when the hybrid LeAP ${}_SR_n(x,y)$ reduce to the hybrid LeBP ${}_SB_n(x,y)$) and in view of equation (2.2) and equations (Table 1(I)), we have

$$\alpha_n=-\frac{B_{n+1}(1)}{n+1};\ \alpha_0=-\frac{1}{2}. \tag{3.1}$$

Finally, on substituting the values from equation (3.1) in equations (2.1), (2.5)-(2.8), (2.19), (2.21) and (2.23), we find the corresponding results for the hybrid LeBP ${}_SB_n(x,y)$. These results are given in Table 4 below.

Table 4. Results for the hybrid LeBP $_SB_n(x,y)$

S.No.	Results	Expressions
I.	Recurrence relation	${}_SB_{n+1}(x,y)=\left(y-\frac{1}{2}\right){}_SB_n(x,y)-\sum\limits_{k=1}^{n}\binom{n}{k}\frac{B_{k+1}(1)}{k+1}\,{}_SB_{n-k}(x,y)+2nD_x^{-1}\,{}_SB_{n-1}(x,y)$
II.	Shift operators	${}_y\pounds_n^-:=\frac{1}{n}D_y,$ ${}_x\pounds_n^-:=\frac{1}{n}D_y^{-1}D_x$ ${}_y\pounds_n^+:=\left(y-\frac{1}{2}\right)-\sum\limits_{k=1}^{n}\frac{B_{k+1}(1)}{(k+1)!}D_y^k+2D_x^{-1}D_y$ ${}_x\pounds_n^+:=\left(y-\frac{1}{2}\right)-\sum\limits_{k=1}^{n}(-1)^k\frac{B_{k+1}(1)}{(k+1)!}D_y^{-k}D_x^k+2D_y^{-1}$
III.	Differential equation	$\left((y-\frac{1}{2})D_y+\sum\limits_{k=1}^{n}\frac{B_{k+1}}{(k+1)!}D_y^{k+1}+2xD_x-n\right){}_SB_n(x,y)=0$
IV.	Integro-differential equation	$\left(\left(y-\frac{1}{2}\right)D_x-\sum\limits_{k=1}^{n}\frac{B_{k+1}}{(k+1)!}D_y^{-k}D_x^{k+1}+2D_xD_y^{-1}-(n+1)D_y\right){}_SB_n(x,y)=0$
V.	Partial differential equation	$\left(\left(y-\frac{1}{2}\right)D_y^nD_x+n\,D_y^{n-1}\,D_x-\sum\limits_{k=1}^{n}\frac{B_{k+1}}{(k+1)!}D_y^{n-k}\,D_x^{k+1}+2D_xD_y^{n-1}-(n+1)D_y^{n+1}\right){}_SB_n(x,y)=0$

Example 3.2. Taking $R(t)=\left(\frac{2}{e^t+1}\right)$ (that is when the hybrid LeAP $_SR_n(x,y)$ reduce to the hybrid LeEP $_SE_n(x,y)$) and in view of equation (2.2) and equations (Table 1(II)), we have

$$\alpha_n=\frac{\mathcal{E}_n}{2};\ \alpha_0=-\frac{1}{2}. \tag{3.2}$$

where the numerical coefficients $\mathcal{E}_k$ $(k=1,2,\ldots,n-2,n-1)$ are linked to the Euler numbers E_k by $\mathcal{E}_n=\frac{-1}{2^n}\sum\limits_{k=0}^{n}\binom{n}{k}E_{n-k}$.

Finally, on substituting the values from equation (3.2) in equations (2.1), (2.5)-(2.8), (2.19), (2.21) and (2.23), we find the corresponding results for the hybrid LeEP $_SE_n(x,y)$. These results are given in Table 5 below.

Table 5. Results for the hybrid LeEP $_SE_n(x,y)$

S.No.	Results	Expressions
I.	Recurrence relation	${}_SE_{n+1}(x,y)=\left(y-\frac{1}{2}\right){}_SE_n(x,y)+\frac{1}{2}\sum\limits_{k=1}^{n}\binom{n}{k}\mathcal{E}_k\,{}_SE_{n-k}(x,y)+2nD_x^{-1}\,{}_SE_{n-1}(x,y)$
II.	Shift operators	${}_y\pounds_n^-:=\frac{1}{n}D_y$ ${}_x\pounds_n^-:=\frac{1}{n}D_y^{-1}D_x$ ${}_y\pounds_n^+:=\left(y-\frac{1}{2}\right)+\frac{1}{2}\sum\limits_{k=1}^{n}\frac{\mathcal{E}_k}{k!}D_y^k+2D_x^{-1}D_y$ ${}_x\pounds_n^+:=\left(y-\frac{1}{2}\right)+\frac{1}{2}\sum\limits_{k=1}^{n}\frac{\mathcal{E}_k}{k!}D_y^{-k}D_x^k+2D_y^{-1}$
III.	Differential equation	$\left((y-\frac{1}{2})D_y+\frac{1}{2}\sum\limits_{k=1}^{n}\frac{\mathcal{E}_k}{k!}D_y^{k+1}+2xD_x-n\right){}_SE_n(x,y)=0$
IV.	Integro-differential equation	$\left(\left(y-\frac{1}{2}\right)D_x+\frac{1}{2}\sum\limits_{k=1}^{n}\frac{\mathcal{E}_k}{k!}D_y^{-k}D_x^{k+1}+2D_x-D_y^{-1}-(n+1)D_y\right){}_SE_n(x,y)=0$
V.	Partial differential equation	$\left(\left(y-\frac{1}{2}\right)D_y^n\,D_x+nD_y^{n-1}\,D_x+\frac{1}{2}\sum\limits_{k=1}^{n}\frac{\mathcal{E}_k}{k!}D_x^{k+1}D_y^{n-k}+2D_xD_y^{n-1}-(n+1)D_y^{n+1}\right){}_SE_n(x,y)=0$

Example 3.3. Taking $R(t) = \left(\frac{2t}{e^t+1}\right)$ (that is when the hybrid LeAP ${}_SR_n(x,y)$ reduce to the hybrid LeGP ${}_SG_n(x,y)$) and in view of equation (2.2) and equations (Table 1(III)), we have

$$\alpha_n = \frac{G_n}{2};\ \alpha_1 = -1;\ \alpha_0 = 1. \tag{3.3}$$

Finally, on substituting the values from equation (3.3) in equations (2.1), (2.5)-(2.8), (2.19), (2.21) and (2.23), we find the corresponding results for the hybrid LeGP ${}_SG_n(x,y)$. These results are given in Table 6 below.

Table 6. Results for the hybrid LeGP ${}_SG_n(x,y)$

S.No.	Results	Expressions
I.	Recurrence relation	${}_SG_{n+1}(x,y) = (y+1){}_SG_n(x,y) - n{}_SG_{n-1}(x,y) + \frac{1}{2}\sum\limits_{k=2}^{n}\binom{n}{k}G_k\ {}_SG_{n-k}(x,y) + 2nD_x^{-1}{}_SG_{n-1}(x,y)$
II.	Shift operators	${}_y\pounds_n^- := \frac{1}{n}D_y$
		${}_x\pounds_n^- := \frac{1}{n}D_y^{-1}D_x$
		${}_y\pounds_n^+ := y+1-D_y+\frac{1}{2}\sum\limits_{k=2}^{n}\frac{G_k}{k!}D_y^k + 2D_x^{-1}D_y$
		${}_x\pounds_n^+ := y+1-D_y^{-1}D_x+\frac{1}{2}\sum\limits_{k=2}^{n}\frac{G_k}{k!}D_y^{-k}D_x^k + 2D_y^{-1}$
III.	Differential equation	$\left((y+1)D_y - D_y^2 + \frac{1}{2}\sum\limits_{k=2}^{n}\frac{G_k}{k!}D_y^{k+1} + 2xD_x - n\right){}_SG_n(x,y) = 0$
IV.	Integro-differential equation	$\left((y+1)D_x - D_y^{-1}\ D_x^2 + \frac{1}{2}\sum\limits_{k=2}^{n}\frac{G_k}{k!}D_y^{-k}\ D_x^{k+1} + 2D_x\ D_y^{-1} - (n+1)D_y\right){}_SG_n(x,y) = 0$
V.	Partial differential equation	$\Big((y+1)D_y^n\ D_x + n\ D_y^{n-1}\ D_x - D_y^{n-1}\ D_x^2 + \frac{1}{2}\sum\limits_{k=2}^{n}\frac{G_k}{k!}D_y^{n-k}\ D_x^{k+1} + 2D_x\ D_y^{n-1}$ $-(n+1)D_y^{n+1}\Big){}_SG_n(x,y) = 0$

In the next section, the homogeneous volterra integral equations for the Hybrid LeAP ${}_SR_n(x,y)$ and for the members belonging to this family are explored.

4 Volterra integral equations

First, we derive the integral equation for the hybrid LeAP ${}_SR_n(x,y)$ by proving the following result:

Theorem 4.1. For the hybrid Legendre-Appell polynomials, the following homogeneous Volterra integral equation holds true:

$$\varphi(y) = -\frac{1}{\alpha_1}(2xD_x - n)\left(n\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{R_{n-1-k}}{k+1}\ y^{k+1} + \sum_{k=0}^{n}\binom{n}{k}R_{n-k}\ y^k\right) - \frac{y+\alpha_0}{y+\alpha_1}\ n\sum_{k=0}^{n-1}\binom{n-1}{k}$$
$$R_{n-1-k}y^k + \int_0^y\left(-\frac{1}{\alpha_1}\Big(2xD_x - n\Big)(y-\xi) + \frac{y+\alpha_0}{\alpha_1}\right)\varphi(\xi)d\xi. \tag{4.1}$$

Proof. Consider the second order differential equation for the hybrid LeAP ${}_SR_n(x,y)$ in the following form:

$$\left(D_y^2 + \frac{1}{\alpha_0}(y+\alpha_0)D_y + \frac{1}{\alpha_1}\Big(2xD_x - n\Big)\right){}_SR_n(x,y) = 0. \tag{4.2}$$

The initial conditions are obtained as follows:

$$ {}_SR_n(0,y) = R_n(y) = \sum_{k=0}^{n} \binom{n}{k} R_{n-k}\; y^k, \tag{4.3} $$

$$ \frac{d}{dy}{}_SR_n(0,y) = n\; R_{n-1}(y) = n\sum_{k=0}^{n-1} \binom{n-1}{k} R_{n-1-k}\; y^k. \tag{4.4} $$

Next, consider

$$ D_y^2{}_SR_n(x,y) = \varphi(y). \tag{4.5} $$

Integrating the above equation and using initial conditions (4.3) and (4.4), we have

$$ D_y{}_SR_n(x,y) = \int_0^y \varphi(\xi)d\xi + n\sum_{k=0}^{n-1} \binom{n-1}{k} R_{n-1-k}\; y^k, \tag{4.6} $$

$$ {}_SR_n(x,y) = \int_0^y \varphi(\xi)d\xi^2 + n\sum_{k=0}^{n-1} \binom{n-1}{k} \frac{R_{n-1-k}}{k+1}\; y^{k+1} + \sum_{k=0}^{n} \binom{n}{k} R_{n-k}\; y^k. \tag{4.7} $$

Q.E.D.

Use of expressions (4.6) and (4.7) in equation (4.2) yields assertion (4.1).

Further, we find the volterra integral equations for the hybrid LeBP, hybrid LeEP and hybrid LeGP. For this, we consider the following remarks:

Remark 4.1 Substituting the values of coefficients $\alpha_0 = -\frac{1}{2}$, $\alpha_1 = -\frac{B_2(1)}{2} = -\frac{1}{12}$; $\alpha_n = -\frac{B_{n+1}(1)}{n+1}$ in integral equation (4.1) of the hybrid LeAP, we deduce the following consequence of Theorem 4.1:

Corollary 4.2. For the hybrid Legendre-Bernoulli polynomials, the following homogeneous Volterra integral equation holds true:

$$ \varphi(y) = 12\,(2xD_x - n)\left(n\sum_{k=0}^{n-1} \binom{n-1}{k} \frac{B_{n-1-k}}{k+1}\; y^{k+1} + \sum_{k=0}^{n} \binom{n}{k} B_{n-k}\; y^k\right) + 12(y-\frac{1}{2})\; n\sum_{k=0}^{n-1} \binom{n-1}{k} $$
$$ \times\; B_{n-1-k}y^k + \int_0^y \left(12\Big(2xD_x - n\Big)(y-\xi) - 12(y-\frac{1}{2})\right)\varphi(\xi)d\xi. \tag{4.8} $$

Remark 4.2 Substituting the values of coefficients $\alpha_0 = -\frac{1}{2}$, $\alpha_1 = \frac{\mathcal{E}_1}{2} = -\frac{1}{4}$; $\alpha_n = \frac{\mathcal{E}_n}{2}$ in integral equation (4.1) of the hybrid LeAP, we deduce the following consequence of Theorem 4.1:

Corollary 4.3. For the hybrid Legendre-Euler polynomials, the following homogeneous Volterra integral equation holds true:

$$ \varphi(y) = 4\,(2xD_x - n)\left(n\sum_{k=0}^{n-1} \binom{n-1}{k} \frac{E_{n-1-k}}{k+1}\; y^{k+1} + \sum_{k=0}^{n} \binom{n}{k} E_{n-k}\; y^k\right) + 4(y-\frac{1}{2})\; n\sum_{k=0}^{n-1} \binom{n-1}{k} $$
$$ \times\; E_{n-1-k}\; y^k + \int_0^y \left(4\Big(2xD_x - n\Big)(y-\xi) + 4(y-\frac{1}{2})\right)\varphi(\xi)d\xi. \tag{4.9} $$

Remark 4.3 Substituting the values of coefficients $\alpha_0 = 1$, $\alpha_1 = -1$; $\alpha_n = \frac{G_n}{2}$ in integral equation (4.1) of the hybrid LeAP, we deduce the following consequence of Theorem 4.1:

Corollary 4.4. For the hybrid Legendre-Genocchi polynomials, the following homogeneous Volterra integral equation holds true:

$$\varphi(y) = (2xD_x - n)\left(n\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{G_{n-1-k}}{k+1}\ y^{k+1} + \sum_{k=0}^{n}\binom{n}{k}G_{n-k}\ y^k\right)(y-1)\ n\sum_{k=0}^{n-1}\binom{n-1}{k} \times\ G_{n-1-k}\ y^k + \int_0^y \left(\left(2xD_x - n\right)(y-\xi) + (y-1)\right)\varphi(\xi)d\xi. \tag{4.10}$$

In view of relation $S_n(x,y) = H_n(y, D_x^{-1})$, the Legendre polynomials $S_n(x,y)$ reduce to the 2-variable Hermite Kampé de Fériet polynomials $H_n(y, D_x^{-1})$ [4], which are defined by

$$e^{yt+D_x^{-1}t^2} = \sum_{n=0}^{\infty} H_n(y, D_x^{-1})\frac{t^n}{n!}. \tag{4.11}$$

From the above fact, we find that the results derived in this paper for the hybrid LeAP ${}_SR_n(x,y)$ reduce to the results for the hybrid Hermite-Appell polynomials ${}_HR_n(y, D_x^{-1})$ [14, 21].

The second form of the hybrid Legendre-Appell polynomials $\frac{{}_RR_n(x,y)}{n!}$ is defined by

$$R(t)J_0(2\sqrt{xt})\ J_0(2\sqrt{-yt}) = \sum_{n=0}^{\infty}\frac{{}_RR_n(x,y)}{n!}\frac{t^n}{n!}. \tag{4.12}$$

By using the similar lines of proof, we can obtain the recurrence relation, shift operators and differential equations for the second form of the hybrid Legendre-Appell polynomials $\frac{{}_RR_n(x,y)}{n!}$.

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