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# A random continued fraction in $\mathbb{R}^{d+1}$ with an inverse Gaussian distribution

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A continued fraction in $\mathbb{R}^{d+1}$ is the composition of an infinite number of projectivities of $\mathbb{R}^{d+1}$ which preserve $(0, +\infty) \times \mathbb{R}^d$. We consider a right random walk on the semigroup of such projectivities governed by a special distribution, and we prove that the corresponding random continued fraction has a generalized inverse Gaussian distribution on $\mathbb{R}^{d+1}$. This leads to a characterization of these distributions.

**Keywords:** characterizations; iteration of random functions; random walks on matrices

## 1. Introduction

Let $V$ be a $d$-dimensional Euclidean space, where the scalar product of $\vec{x}$ and $\vec{y}$ is denoted by $\vec{x} \cdot \vec{y}$ and the squared norm of $\vec{x}$ by $\vec{x}^2$. An element $x$ of $\mathbb{R} \times V$ is denoted by $x = (x_0, \vec{x})$, and adopting the notation $x^* = x_0 - \vec{x}^2$, we consider the interior $P$ of a paraboloid of revolution,

$$P = \{x \in \mathbb{R} \times V; x^* > 0\}. \tag{1.1}$$

For $\lambda$ in $\mathbb{R}$, $K_\lambda : \mathbb{R} \to \mathbb{R}$ is the Bessel function defined by (Watson 1966, p. 78)

$$K_\lambda(s) = \int_0^\infty t^{2\lambda-1} \exp\Big(-\frac{s}{2}t^2 - \frac{s}{2}t^{-2}\Big)\mathrm{d}t. \tag{1.2}$$

Let $b$ be in $P$. We define the following two probability distributions on $(0, +\infty) \times V$:

$$\mu_{\lambda,a,b}(\mathrm{d}x) = (2\pi)^{-d/2}\{K_\lambda(\sqrt{ab^*})\}^{-1} a^{-\lambda/2}(b^*)^{\lambda/2} \times x_0^{\lambda-d/2-1} \exp\Big\{-\frac{1}{2x_0}(a + \vec{x}^2) - \frac{b_0 x_0}{2} + \vec{b}\cdot\vec{x}\Big\} 1_{(0,+\infty)}(x_0)\,\mathrm{d}x_0\,\mathrm{d}\vec{x}, \tag{1.3}$$

where $\lambda \in \mathbb{R}$ and $a > 0$; and

$$\gamma_{\lambda,b}(\mathrm{d}y) = (2\pi)^{-d/2}\{\Gamma(\lambda)\}^{-1}(b^*/2)^\lambda y_0^{\lambda-d/2-1} \times \exp\Big\{-\frac{\vec{y}^2}{2y_0} - \frac{b_0 y_0}{2} + \vec{b}\cdot\vec{y}\Big\} 1_{(0,+\infty)}(y_0)\,\mathrm{d}y_0\,\mathrm{d}\vec{y}, \tag{1.4}$$

where $\lambda > 0$.

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1350-4864 

To check that $\mu_{\lambda,a,b}$ is actually a probability distribution, one has to use the formula, for $a$ and $a' > 0$:

$$K_\lambda(\sqrt{aa'}) = \frac{a^{\lambda/2}(a')^{-\lambda/2}}{2}\int_0^\infty x_0^{\lambda-1}\exp\left\{-\frac{1}{2}\left(ax_0+\frac{a'}{x_0}\right)\right\}\mathrm{d}x_0, \tag{1.5}$$

which is easily obtained from (1.2). Integrating (1.3) first with respect to $\vec{x}$, then, using (1.5), with respect to $x_0$, will give the result. Such a computation, *passim*, also yields the marginal distribution of $x_0$ under (1.3), a generalized inverse Gaussian one in $\mathbb{R}$,

$$\{K_\lambda(\sqrt{ab^*})\}^{-1}a^{-\lambda/2}(b^*)^{\lambda/2}x_0^{\lambda-1}\exp\left(-\frac{a}{2x_0}-\frac{b^*}{2}x_0\right)1_{(0,+\infty)}(x_0)\,\mathrm{d}x_0, \tag{1.6}$$

whereas $\vec{x}$ conditioned by $x_0$ is Gaussian distributed with mean $\vec{b}x_0$ and covariance $x_0 I_V$. Similarly, one can verify that (1.4) is a probability distribution, that the marginal distribution of $y_0$ is gamma

$$\{\Gamma(\lambda)\}^{-1}\left(\frac{b^*}{2}\right)^\lambda y_0^{\lambda-1}\exp\left(-\frac{b^*y_0}{2}\right)1_{(0,+\infty)}(y_0)\,\mathrm{d}y_0, \tag{1.7}$$

and that $\vec{y}$ conditioned by $y_0$ is Gaussian distributed.

Because this is well adapted to the context and to the choice of the constants, we shall define in this paper the Laplace transform of any positive measure $\mu$ on $\mathbb{R}\times V$ by

$$L_\mu(\theta) = L_\mu(\theta_0,\vec{\theta}) = \int_{\mathbb{R}\times V}\exp\left(-\frac{\theta_0}{2}x_0+\vec{\theta}\cdot\vec{x}\right)\mu(\mathrm{d}x). \tag{1.8}$$

With this convention, the fact that (1.3) and (1.4) are probability measures gives without further calculation that, for $\theta$ in $P$, one has

$$L_{\mu_{\lambda,a,b}}(\theta) = \{K_\lambda(\sqrt{ab^*})\}^{-1}(b^*)^{\lambda/2}K_\lambda(\sqrt{a(\theta+b)^*})\{(b+\theta)^*\}^{-\lambda/2}, \tag{1.9}$$

$$L_{\gamma_{\lambda,b}}(\theta) = (b^*)^\lambda\{(b+\theta)^*\}^{-\lambda}. \tag{1.10}$$

We are now able to state the two basic observations which motivate this paper. In the following, $*$ indicates convolution in $\mathbb{R}\times V$.

**Proposition 1.1** If $\lambda > 0$, $a > 0$ and $b \in P$, then

$$\gamma_{\lambda,b} * \mu_{-\lambda,a,b} = \mu_{\lambda,a,b}. \tag{1.11}$$

*Proof*
That $K_\lambda(s) = K_{-\lambda}(s)$ is clear from (1.2). Taking the Laplace transforms of both sides of (1.11) and using (1.9) and (1.10) proves (1.11). □

**Proposition 1.2** For $\vec{b}$ and $\vec{c}$ in $V$, define the permutation $h(\cdot;\vec{b},\vec{c})$ of $(0,+\infty)\times V$ by

$$h(x_0,\vec{x};\vec{b},\vec{c}) = \left(\frac{1}{x_0}, -\vec{b}+\frac{\vec{x}+\vec{c}}{x_0}\right). \tag{1.12}$$

Then, for all $(\lambda,b,c)$ in $\mathbb{R}\times P^2$, the image of $\mu_{\lambda,b^*,c}$ by $h(\cdot;\vec{b},\vec{c})$ is $\mu_{-\lambda,c^*,b}$.

*Proof*
One has to use the fact that

$$h^{-1}(\cdot;\vec{b},\vec{c}) = h(\cdot;\vec{c},\vec{b}). \tag{1.13}$$

Thus, if $x' = h(x;\vec{b},\vec{c})$, one gets $x = h(x';\vec{c},\vec{b})$, $\mathrm{d}x = (x'_0)^{-d-2}\,\mathrm{d}x'$, and the image of $\mu_{\lambda,b^*,c}$ by $h(\cdot;\vec{b},\vec{c})$ is easily computed. □

Now fix $\lambda > 0$ and $(b,c)$ in $P^2$; consider a sequence $X_0$, $Y_1$, $Y_2, \ldots$ of independent random variables valued in $(0,+\infty) \times V$ such that, for all $n$ in $\mathbb{N}^* = \{1,2,\ldots\}$,

$$\mathscr{L}(X_0) = \mu_{-\lambda,c^*,b}, \qquad \mathscr{L}(Y_{2n-1}) = \gamma_{\lambda,b}, \qquad \mathscr{L}(Y_{2n}) = \gamma_{\lambda,c}$$

(where $\mathscr{L}(X_0)$ means 'distribution of $X_0$'). Then, from Proposition 1.1, $\mathscr{L}(X_0 + Y_1) = \mu_{\lambda,c^*,b}$.

Let us define $X_1 = h(X_0 + Y_1;\vec{c},\vec{b})$, $X_2 = h(X_1 + Y_2;\vec{b},\vec{c})$, and more generally, for $n$ in $\mathbb{N}^*$

$$X_{2n-1} = h(X_{2n-2} + Y_{2n-1};\vec{c},\vec{b}), \qquad X_{2n} = h(X_{2n-1} + Y_{2n};\vec{b},\vec{c}). \tag{1.14}$$

Then, Propositions 1.1 and 1.2 imply that for all $n$ in $\mathbb{N}^*$

$$\mathscr{L}(X_{2n-1}) = \mu_{-\lambda,b^*,c}, \qquad \mathscr{L}(X_{2n}) = \mu_{-\lambda,c^*,b}. \tag{1.15}$$

Thus, $(X_{2n})_{n=0}^{\infty}$ and $(X_{2n+1})_{n=0}^{\infty}$ are two stationary Markov chains valued in $(0,+\infty) \times V$. Let us now raise the following question: suppose that $X$, $Y_1$ and $Y_2$ are three independent random variables of $(0,+\infty) \times V$ such that $\mathscr{L}(Y_1) = \gamma_{\lambda,b}$, $\mathscr{L}(Y_2) = \gamma_{\lambda,c}$ and such that

$$\mathscr{L}(X) = \mathscr{L}(h(Y_2 + h(Y_1 + X;\vec{c},\vec{b});\vec{b},\vec{c})); \tag{1.16}$$

can we claim that $\mathscr{L}(X) = \mu_{-\lambda,c^*,b}$? One of the results of this paper is to prove that the answer is yes (Corollary 3.2). Of course, if $\vec{b} = \vec{c}$, a similar question arises: if

$$\mathscr{L}(X) = \mathscr{L}(h(Y_1 + X;\vec{b},\vec{b})) \tag{1.17}$$

is it true that $\mathscr{L}(X) = \mu_{-\lambda,b^*,b}$? The answer is again yes. However, the general method of the proof provides a much more interesting result: consider the random projectivities $F_n$ of $\mathbb{R} \times V$ defined for $n$ in $\mathbb{N}^*$ by

$$F_n(x) = h(Y_{2n} + h(Y_{2n-1} + x;\vec{c},\vec{b});\vec{b},\vec{c}). \tag{1.18}$$

Hence

$$X_{2n} = F_n \circ F_{n-1} \circ \cdots \circ F_1(X_0), \tag{1.19}$$

and the Markov chain is the result of the action on $\mathbb{R} \times V$ of the left random walk $(F_n \circ \cdots \circ F_1)_{n \in \mathbb{N}^*}$ on the group of projectivities of $\mathbb{R} \times V$. As usual in these types of question, the delicate point will be to consider the action on $\mathbb{R} \times V$ of the right random walk, and to show that

$$Z_n(x) = F_1 \circ F_2 \circ \cdots \circ F_n(x) \tag{1.19}$$

converges almost surely towards a random variable $Z$ on $(0,+\infty) \times V$ which does not depend on $x$ (note that $Z_n(X_0)$ and $X_{2n}$ are identically distributed, although $(Z(x))_{n=0}^{\infty}$ is not a Markov chain). This convergence of $Z_n(x)$ towards $Z$ will thus imply that $Z$, a kind of continued fraction in $\mathbb{R}^{d+1}$, with random elements following the 'gamma' distribution (1.4), has an 'inverse Gaussian' distribution (1.3). This representation of (1.3) as the law of a 'random continued fraction' is the main result of this paper (Theorem 3.1). Before establishing this (section 3), we give a few references.

## 2. Bibliographical comments

1. For dim $V = 0$, the above problem has been solved by Letac and Seshadri (1983). Probabilistic applications are given in Vallois (1989).
2. The work of Letac and Seshadri (1983) has been generalized by Bernadac (1992). She first replaces the positive line by the cone of positive definite matrices, and considers the random continued fraction

$$X_0 = (Y_1 + (Y_2 + (Y_3 + \dots)^{-1})^{-1})^{-1}. \tag{2.1}$$

She proves its convergence, and is able to compute the law of $X_0$ when the $Y_j$ are Wishart distributed.

Actually, it is easy to consider a generalization of (1.3), (1.4), and Propositions 1.1 and 1.2. One has to replace the gamma distribution and the generalized inverse Gaussian distribution by the Wishart distribution and the generalized inverse Gaussian distribution on the symmetric $d \times d$ real matrices, as defined by Bernadac (1992; 1995). However, the convergence of (2.1) is sufficiently delicate to prevent us from trying to extend Theorem 3.1 below.

A deeper generalization appears in Bernadac (1993; 1995), where the positive definite matrices are replaced by an irreducible symmetric cone, i.e. the set of squares in a simple Euclidean Jordan algebra, where a suitable Wishart distribution can be defined (see Casalis 1990). The proof of the convergence of (2.1), even in a deterministic case, becomes rather difficult and is achieved by beautiful algebraic identities.
3. The distribution (1.4) seems to have been considered for the first time by Casalis (1992; 1993) and is one of the $2d + 6$ simple natural quadratic exponential families in $\mathbb{R}^{d+1}$. For general $\lambda > 0$, the distribution of $\vec{y}$ has no simple expression. However, since

$$K_{1/2}(s) = \sqrt{\frac{\pi}{2s}}\exp(-s), \tag{2.2}$$

as is easily seen from (1.2), the marginal distributions of $\vec{y}$ for $\lambda = (d+1)/2$ and (if $d \geqslant 2$) for $\lambda = (d-1)/2$ are computable and interesting. For $\lambda = (d+1)/2$ it is

$$(2\pi)^{-d/2}\left\{\Gamma\left(\frac{d+1}{2}\right)\right\}^{-1}(b^*)^{(d+1)/2}b_0^{-1/2}\exp\left(-\sqrt{b_0}\|\vec{y}\| + \vec{b}\cdot\vec{y}\right)\mathrm{d}\vec{y}, \tag{2.3}$$

and for $\lambda = (d-1)/2$ it is

$$(2\pi)^{-d/2}\left\{\Gamma\left(\frac{d-1}{2}\right)\right\}^{-1}(b^*)^{(d-1)/2}\|\vec{y}\|^{-1}\exp\left(-\sqrt{b_0}\|\vec{y}\| + \vec{b}\cdot\vec{y}\right)\mathrm{d}\vec{y}. \tag{2.4}$$

If $b$ varies in $P$, (2.3) and (2.4) are general exponential families on $V$ (see Letac 1992), generated by the measures $\mathrm{d}\vec{y}$ and $\mathrm{d}\vec{y}/\|\vec{y}\|$ respectively, and the map $t : V \to \mathbb{R} \times V$, $t(\vec{y}) = (\|\vec{y}\|, \vec{y})$. From (2.3) and (2.4), the Laplace transforms of the images of $\mathrm{d}\vec{y}$ and $\mathrm{d}\vec{y}/\|\vec{y}\|$ can be easily computed; the latter generates the Wishart distribution of the cone of revolution (Letac 1994), an observation made by Barndorff-Nielsen *et al.* (1989, p. 100).

4. Special cases of the distribution (1.3) appear in various places, see in particular Barndorff-Nielsen and Blæsild (1983b). For instance, let $(B_0, \vec{B})$ be a standard Brownian motion in $\mathbb{R} \times V$ (endowed with its canonical Euclidean structure). Denote

$$T = \inf\{t; B_0(t) + t\sqrt{b_0} = \sqrt{a}\}.$$

Then

$$\mathscr{L}(T, \vec{B}(T)) = \mu_{-1/2,a,(b_0,\vec{0})}. \tag{2.5}$$

Formula (2.2) also gives two remarkable particular cases, for $\lambda = \pm 1/2$. In these cases, the Laplace transforms are

$$L_{\mu_{-1/2,a,b}}(\theta) = \exp\{\sqrt{ab^*} - \sqrt{a(b+\theta)^*}\} \tag{2.6}$$
$$L_{\mu_{1/2,a,b}}(\theta) = (b^*)^{1/2}\{(b+\theta)^*\}^{-1/2} \exp\{\sqrt{ab^*} - \sqrt{a(b+\theta)^*}\}.$$

From (2.6) it is easy to see that

$$\mu_{1/2,a,b} * \mu_{-1/2,a',b} = \mu_{1/2,a'',b} \tag{2.7}$$

where $\sqrt{a''} = \sqrt{a} + \sqrt{a'}$ (we thank O.E. Barndorff-Nielsen for this remark). The natural exponential family $F_a = \{\mu_{-1/2,a,b};\ b \in P\}$ is especially interesting. Its variance function is $V_{F_a}(m_0, \vec{m}) = (m_0 m \oplus m/a) + \left[\begin{smallmatrix} 0 & 0 \\ 0 & m_0 I_V \end{smallmatrix}\right]$, as can be verified from (2.6). It is cubic. This is why Hassaïri (1992; 1993) has called any affine transformation of $\mu_{-1/2,a,b}$ an *inverse Gaussian distribution on* $\mathbb{R}^{d+1}$. He has shown that, as in the one-dimensional case, these distributions, together with the Gaussian ones, are the whole $G$-orbit (in Hassaïri's sense) of the Gaussian distributions.

For general $\lambda$, the marginal distribution of $\vec{x}$ under (1.3) is not elementary. For $\lambda = -1/2$ and $\vec{b} = 0$, (2.5) shows that $\vec{x}$ is Cauchy distributed in $V$. Furthermore, from (2.2), things are computable if $\lambda = (d \pm 1)/2$. For $\lambda = (d+1)/2$, (2.2) is

$$(2\pi)^{-d/2} K_{(d+1)/2}(\sqrt{ab^*}) a^{-(d+1)/4} (b^*)^{(d+1)/4} b_0^{-1/2} \exp(-\sqrt{a + \vec{x}^2}\sqrt{b_0} + \vec{b}\cdot\vec{x})\,d\vec{x}. \tag{2.8}$$

For $\lambda = (d-1)/2$, it is

$$(2\pi)^{-d/2} K_{(d-1)/2}(\sqrt{ab^*}) a^{-(d-1)/4} (b^*)^{(d-1)/4} \times (a + \vec{x}^2)^{-1/2} \exp(-\sqrt{a + \vec{x}^2}\sqrt{b_0} + \vec{b}\cdot\vec{x})\,d\vec{x}. \tag{2.9}$$

For fixed $a$, if $b$ varies in $P$, (2.8) and (2.9) are general exponential families on $V$, generated by the measures $d\vec{x}$ and $(a + \vec{x}^2)^{-1/2}\,d\vec{x}$, and the map $t : V \to \mathbb{R} \times V$, $t(\vec{x}) = ((a + \vec{x}^2)^{1/2}, \vec{x})$; from (2.7) and (2.8) the Laplace transforms of these measures can be computed, and can even be put in elementary form (because of (2.2) and the fact that $K_{n+1/2}(s)/K_{1/2}(s)$ is a polynomial in $1/s$ for all $n$ in $\mathbb{N}^*$: see Watson (1966, p. 80, formula (12)). These measures are concentrated not on a cone of revolution (as in (2.3) and (2.4)) but on a hyperboloid of revolution. The distributions (2.7) and (2.8) are called respectively *multivariate hyperbolic* and *multivariate hyperboloid* distributions in Barndorff-Nielsen and Blæsild (1983a) – see also Barndorff-Nielsen *et al.* (1989, p.100) and references therein.

5. Pierre Vallois (private communication) has a nice probabilistic proof of Proposition 1.1, simply by introducing four independent random variables $X_0$, $Y_0$, $\vec{N}$, $\vec{N}'$, such that if $\vec{X} = X_0\vec{b} + \sqrt{X_0}\vec{N}$ and $\vec{Y} = Y_0\vec{b} + \sqrt{Y_0}\vec{N}'$, then $\mathscr{L}(X_0, \vec{X}) = \mu_{-\lambda,a,b}$ and $\mathscr{L}(Y_0, \vec{Y}) = \gamma_{\lambda,b}$. Here $\vec{N}$ and $\vec{N}'$ are Gaussian with mean 0 and covariance $I_V$. (But a similar interpretation is also available for the extension of Proposition 1.1 mentioned in section 4, with a suitable distribution for $\vec{N}$ and $\vec{N}'$.) The idea is to note that $\sqrt{X_0}\vec{N} + \sqrt{Y_0}\vec{N}' = (\sqrt{X_0 + Y_0})\vec{N}''$ with $\mathscr{L}(\vec{N}'') = \mathscr{L}(\vec{N})$ and $\vec{N}''$ independent of $X_0 + Y_0$. Vallois then observes that this explains Proposition 1.2, since

$$h(X_0 + Y_0, \vec{X} + \vec{Y}; \vec{b}, \vec{c}) = \left(\frac{1}{X_0 + Y_0}, \frac{\vec{c}}{X_0 + Y_0} + \frac{\vec{N}''}{\sqrt{X_0 + Y_0}}\right).$$

6. The trick for proving that (1.16) or (1.17) actually characterizes $\mathscr{L}(X)$ relies on the following principle (see Letac 1986; Chamayou and Letac 1991; Goldie 1991). Although its proof is short, it will not be repeated here.

**Proposition 2.1** Let $E$ be a locally compact space with countable basis; $\mathscr{E}$ is its Borel field, and $C$ is the set of continuous functions $f : E \to E$. Furthermore, $\mathscr{C}$ is the smallest $\sigma$-field on $C$ such that for all $x$ in $E$, the maps $C \to E f \mapsto f(x)$ are measurable, and $(F_n)_{n=1}^{\infty}$ is an independently and identically distributed sequence of random variables in $C$. Suppose that, almost surely

$$Z = \lim_{n\to\infty} F_1 \circ F_2 \circ \cdots \circ F_n(x)$$

exists and does not depend on $x$. Then $\mathscr{L}(Z)$ is a stationary distribution of the Markov chain on $E$ given by

$$X_n = F_n \circ \cdots \circ F_1(X_0)$$

and this stationary distribution is unique.
This will be applied to $E = (0, +\infty) \times V$ and to the $F_n$ of (1.18).

## 3. A random walk on projectivities of $\mathbb{R}^{d+1}$

Instead of providing an algebraically correct definition of a projectivity on a projective space (which may be pedantic), we shall be content with an intuitive view of a projectivity on a finite-dimensional vector space $E$. Let $a$ be a linear endomorphism of $E$, $b$ in $E$, $c$ in the dual space $E^*$ and $d$ in $\mathbb{R}$ such that

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is in the group $GL(E \times \mathbb{R})$ of automorphisms of $E \times \mathbb{R}$. For $x$ in $E$ such that $c(x) + d \neq 0$, define $h_M(x)$ in $E$ by

$$h_M(x) = \{a(x) + b\}/\{c(x) + d\}. \tag{3.1}$$

The map $h_M$ is called the *projectivity* associated with $M$. Note that $h_{\lambda M} = h_M$ for $\lambda \neq 0$. (A complete definition of $h_M$ would involve a projective completion of $E$ by a hyperplane at infinity.) It is easy to see that if $M_1$ is in $GL(E\times\mathbb{R})$ then

$$h_M\circ h_{M_1}=h_{MM_1};\tag{3.2}$$

thus the set of projectivities is a group under composition.

Suppose now that $E=\mathbb{R}\times V$. For convenience, elements $x$ of $E$ are now denoted $\begin{pmatrix}\vec{x}\\x_0\end{pmatrix}$. The projectivity $h(\cdot;\vec{b},\vec{c})$ of (1.12) can be represented by

$$M=\begin{bmatrix}I_V & -\vec{b} & \vec{c}\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}.\tag{3.3}$$

Here $a$ in (3.1) is $\left(\begin{smallmatrix}I_V & -\vec{b}\\ 0 & 0\end{smallmatrix}\right)$, in $b$ in (3.1) is $\begin{pmatrix}\vec{c}\\1\end{pmatrix}$, $c(x)$ is the linear form $\begin{pmatrix}\vec{x}\\x_0\end{pmatrix}\mapsto x_0$ and $d=0$. With the $M$ of (3.3) we have $h_M=h(\cdot;\vec{b},\vec{c})$.

Similarly if $y\in(0,+\infty)\times V$, the projectivity $x\mapsto h(y+x;\vec{b},\vec{c})$ is represented by

$$M=\begin{bmatrix}I_V & -\vec{b} & \vec{c}+\vec{y}-y_0\vec{b}\\ 0 & 0 & 1\\ 0 & 1 & y_0\end{bmatrix}.\tag{3.4}$$

Clearly enough, the projectivities $h_M$, with $M$ of type (3.4), belong to the semigroup of all projectivities of $\mathbb{R}\times V$ which map $(0,+\infty)\times V$ into itself. They also belong to the group $T$ of projectivities $h_M$ such that $M$ has the form

$$M(W,C)=\begin{bmatrix}I_V & W\\ 0 & C\end{bmatrix}\tag{3.5}$$

where $C$ is a (2.2) invertible matrix, and $W=(\vec{w}_1,\vec{w}_2)$ is in $V^2$. Note that:

$$M(W,C)M(W',C')=M(W'+WC',CC').\tag{3.6}$$

For (3.4), for instance, $W=(-\vec{b},c+\vec{y}-y_0\vec{b})$ and $C=\left[\begin{smallmatrix}0 & 1\\ 1 & y_0\end{smallmatrix}\right]$.

We consider now a sequence $(y_n)_{n=1}^{\infty}$ of independent random variables valued in $(0,+\infty)\times V$ such that

$$\mathscr{L}(V_{2n-1})=\gamma_{\lambda,b},\qquad \mathscr{L}(Y_{2n})=\gamma_{\lambda,c},\tag{3.7}$$

where $\lambda>0$ and $(b,c)$ in $P^2$ are fixed. To avoid a multiple index, we choose to write

$$Y_n=(S_n,\vec{Y}_n)\text{ instead of }((Y_n)_0,\vec{Y}_n).$$

($S$ is not for 'sum', but for 'scalar'). We consider the random $2\times 2$ matrix

$$C_n=\begin{bmatrix}0 & 1\\ 1 & S_n\end{bmatrix}\tag{3.8}$$

and the random elements of $V^2$,

$$W_{2n-1}=(-\vec{c},\vec{b}+\vec{Y}_{2n-1}-S_{2n-1}\vec{c})$$
$$W_{2n}=(-\vec{b},\vec{c}+\vec{Y}_{2n}-S_{2n}\vec{b}).\tag{3.9}$$

Thus, with the notation (3.5),

$$x \mapsto h(Y_{2n-1} + x; \vec{c}, \vec{b}) = h_{M(W_{2n-1}, c_{2n-1})}(x)$$
$$x \mapsto h(Y_{2n} + x; \vec{b}, \vec{c}) = h_{M(W_{2n}, c_{2n})}(x),$$

and using (3.2), (3.6) and the definition (1.18) of $F_n$,

$$F_n = h_{M(W_{2n-1} + W_{2n}c_{2n-1}, c_{2n}c_{2n-1})}. \tag{3.10}$$

We now state the results of this paper.

**Theorem 3.1** Let $(F_n)_{n=1}^{\infty}$ be the independent random mappings of $(0, +\infty) \times V$ onto itself defined by (1.18). Then almost surely

$$Z = \lim_{n \to \infty} (F_1 \circ F_2 \circ \cdots \circ F_n)(x)$$

exists and does not depend on $x$. Furthermore, the law of $Z$ is $\mu_{-\lambda, c^*, b}$.

**Corollary 3.2** If $X$, $Y_1$, $Y_2$ are independent random variables on $(0, +\infty) \times V$ such that $\mathscr{L}(Y_1) = \gamma_{\lambda, b}$, $\mathscr{L}(Y_2) = \gamma_{\lambda, c}$ and (1.16) holds, then $\mathscr{L}(X) = \mu_{-\lambda, c^*, b}$.

*Proof of Corollary 3.2*
Equation (1.16) implies that $\mathscr{L}(X)$ is a stationary distribution of the Markov chain

$$X_n = F_n \circ \cdots \circ F_1(x)$$

Since from (1.15) $\mu_{-\lambda, c^*, b}$ is such a stationary distribution, and since Theorem 3.1 shows that the hypothesis of Proposition 2.1 is fulfilled, Proposition 2.1 guarantees the uniqueness of the stationary distribution. □

*Proof of Theorem 3.1*
Since we are considering $F_1 \circ \cdots \circ F_n$ rather than $F_n \circ \cdots \circ F_1$, products such as $C_2C_1C_4C_3 \cdots C_{2n}C_{2n-1}$ are going to appear (see (3.5)), (3.6) and (3.10)). To simplify the notation we denote

$$c_{2n} = C_{2n-1}, \qquad c_{2n-1} = C_{2n}, \qquad s_{2n} = S_{2n-1}, \qquad s_{2n-1} = S_{2n},$$
$$w_{2n} = W_{2n-1} \qquad \text{and} \qquad w_{2n-1} = W_{2n}.$$

Denote by $u_k$ the element of $V^2$ defined by

$$u_k = \sum_{j=1}^{k} w_j c_{j+1} c_{j+2} \cdots c_k, \tag{3.11}$$

with the convention that $w_j c_{j+1} c_{j+2} \cdots c_k = w_k$ if $j = k$. From (3.6), we easily see by induction on $k$ that

$$M(w_1, c_1)M(w_2, c_2) \cdots M(w_k, c_k) = M(u_k, c_1 \cdots c_k). \tag{3.12}$$

To express the product $c_{j+1}c_{j+2} \cdots c_k$ in (3.11) for fixed $j = 0, 1, \ldots$, we define, as in Olds (1963), the sequences $(p_k^{(j)})_{k=j}^{\infty}$ and $(q_k^{(j)})_{k=j}^{\infty}$ by

$$\begin{aligned} p_j^{(j)} &= 1, \qquad p_{j+1}^{(j)} = s_{j+1}, \qquad p_k^{(j)} = s_k p_{k-1}^{(j)} + p_{k-2}^{(j)}, \\ q_j^{(j)} &= 0, \qquad q_{j+1}^{(j)} = 1 \qquad \text{and} \qquad q_k^{(j)} = s_k q_{k-1}^{(j)} + q_{k-2}^{(j)}. \end{aligned} \tag{3.13}$$

One easily sees by induction on $k > j$ that

$$c_{j+1} \cdots c_k = \begin{bmatrix} 0 & 1 \\ 1 & s_{j+1} \end{bmatrix} \cdots \begin{bmatrix} 0 & 1 \\ 1 & s_k \end{bmatrix} = \begin{bmatrix} q_{k-1}^{(j)} & q_k^{(j)} \\ p_{k-1}^{(j)} & p_k^{(j)} \end{bmatrix}. \tag{3.14}$$

For simplicity, we denote

$$p_k = p_k^{(0)} \qquad \text{and} \qquad q_k = q_k^{(0)}.$$

In order to compute $F_1 \circ F_2 \circ \cdots \circ F_n(x)$, we observe that from (3.12) and (3.14)

$$M(u_k, c_1 \cdots c_k) \begin{pmatrix} \vec{x} \\ x_0 \\ 1 \end{pmatrix} = \begin{pmatrix} \vec{x} + u_k \begin{pmatrix} x_0 \\ 1 \end{pmatrix} \\ c_1 \cdots c_k \begin{pmatrix} x_0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \vec{v}_k \\ q_{k-1}x_0 + q_k \\ p_{k-1}x_0 + p_k \end{pmatrix},$$

and we have to show that

$$F_1 \circ \cdots \circ F_n(x) = \left( \frac{q_{2n-1}x_0 + q_{2n}}{p_{2n-1}x_0 + p_{2n}}, \frac{\vec{v}_{2n}}{p_{2n-1}x_0 + p_{2n}} \right) \tag{3.15}$$

has a limit independent of $x$. Clearly the scalar part of (3.15) is $h_{c_1 \cdots c_{2n}}(x_0)$, and it has already been proved in Letac and Seshadri (1983) that its limit exists and is independent of $x_0$. We only have to concentrate on the vector part of (3.15). For this, we make the basic observation from (3.13) that

$$p_k = (s_k s_{k-1} + 1)p_{k-2} + s_k p_{k-3} \geqslant (s_k s_{k-1} + 1)p_{k-2}$$

which implies

$$p_{2n} \geqslant \prod_{j=1}^{n} (s_{2j-1}s_{2j} + 1).$$

Denoting $K = \exp\left[\mathrm{E}\{\ln(s_1 s_2 + 1)\}\right] > 1$, from the law of large numbers we have

$$\liminf (p_{2n})^{1/n} \geqslant K, \tag{3.16}$$

which then implies $\vec{x}/(p_{2n-1}x_0 + p_{2n}) \to_{n\to\infty} \vec{o}$. We have to show that if

$$\vec{L}_k = u_k \begin{pmatrix} x_0 \\ 1 \end{pmatrix} \frac{1}{p_{k-1}x_0 + p_k}$$

then $\lim_{n\to\infty} \vec{L}_{2n}$ exists. To do this, we show that the series $\sum_k \|\vec{L}_{k+1} - \vec{L}_k\|$ converges almost surely.

A tedious computation starting from (3.11) shows that if $w_j = (\vec{w}_j^{(1)},\ \vec{w}_j^{(2)})$ and if $d_k = p_{k-1}x_0 + p_k$, then

$$\vec{L}_{k+1} - \vec{L}_k = \vec{A}_k + \sum_{j=1}^{k} \vec{B}_{k,j}, \tag{3.17}$$

with

$$\vec{A}_k = \frac{1}{d_{k+1}} (\vec{w}_{k+1}^{(1)} x_0 + \vec{w}_{k+1}^{(2)}),$$

$$\vec{B}_{k,j} = \frac{(-1)^{k+1}}{d_k d_{k+1}} (x_0^2 + x_0 s_{k+1} - 1) \times (-1)^j (\vec{w}_j^{(1)} p_j - \vec{w}_j^{(2)} p_{j-1}).$$

Details of this computation are as follows. We write, from the definition of $\vec{L}_k$ and the definition (3.11) of $u_k$:

$$\vec{L}_{k+1} - \vec{L}_k = \vec{A}_k + \sum_{j=1}^{k} w_j c_{j+1} \cdots c_k \left\{ \frac{1}{d_{k+1}} \begin{pmatrix} 0 & 1 \\ 1 & s_{k+1} \end{pmatrix} \begin{pmatrix} x_0 \\ 1 \end{pmatrix} - \frac{1}{d_k} \begin{pmatrix} x_0 \\ 1 \end{pmatrix} \right\}. \tag{3.18}$$

Using $p_{k+1} = s_{k+1}p_k + p_{k-1}$, the term in braces in (3.18) is

$$\left\{ \frac{1}{d_{k+1}} \begin{pmatrix} 0 & 1 \\ 1 & s_{k+1} \end{pmatrix} \begin{pmatrix} x_0 \\ 1 \end{pmatrix} - \frac{1}{d_k} \begin{pmatrix} x_0 \\ 1 \end{pmatrix} \right\} = \frac{x_0^2 + x_0 s_{k+1} - 1}{d_k d_{k+1}} \begin{pmatrix} -p_k \\ p_{k-1} \end{pmatrix}.$$

Since now

$$c_s \begin{pmatrix} -p_s \\ p_{s-1} \end{pmatrix} = \begin{pmatrix} p_{s-1} \\ -p_{s-2} \end{pmatrix} \qquad \text{for all } s \geqslant 2,$$

expression (3.18) is easily transformed into (3.17). Inequality (3.16) implies easily that $\sum_k \|\vec{A}_k\| < +\infty$. Furthermore, if $\vec{w}_j^{(3)} = \vec{w}_j^{(1)} + \vec{w}_{j+1}^{(2)}$, then $\|\sum_{j=1}^k \vec{B}_{k,j}\|$ can be rewritten as

$$\left\| \frac{(-1)^k}{d_k d_{k+1}} (x_0^2 + x_0 s_{k+1} - 1) \left\{ \vec{w}_1^{(2)} p_0 + (-1)^k \vec{w}_k^{(1)} p_k + \sum_{j=1}^{k-1} (-1)^j \vec{w}_j^{(3)} p_j \right\} \right\|$$

and is majorized by

$$\frac{x_0^2 + x_0 s_{k+1} + 1}{p_k p_{k+1}} \sum_{j=1}^{k} \|\vec{w}_j^{(3)}\| p_j.$$

Since $k \mapsto (1/k)\sum_{j=1}^{k}\|\vec{w}_j^{(3)}\|$ is bounded from the law of large numbers and since $p_j \leqslant p_k + p_{k+1}$ for all $j = 1,\ldots,k$, we just have to check that for positive constants $a$ and $b$

$$\sum_{k=1}^{\infty} k(as_{k+1}+b)\left(\frac{1}{p_k}+\frac{1}{p_{k+1}}\right)$$

converges almost surely. But this is clear from (3.16).

We prove in a similar way that actually the limit of $\vec{L}_k$ does not depend on $x_0$. For this, we write first $u_j = (\vec{u}_j^{(1)}, \vec{u}_j^{(2)})$, thus

$$\vec{L}_k = \frac{\vec{u}_k^{(1)}x_0 + \vec{u}_k^{(2)}}{p_{k-1}x_0 + p_k}.$$

We now imitate the classical formula:

$$\frac{ax+b}{cx+d} = \frac{a}{c} - \frac{ad-bc}{c^2}\cdot\frac{1}{x+\dfrac{d}{c}},$$

and write

$$\vec{L}_k = \frac{\vec{u}_k^{(1)}}{p_{k-1}} + \frac{1}{p_{k-1}^2}u_k\begin{pmatrix}-p_k\\ p_{k-1}\end{pmatrix}\frac{1}{x_0+\dfrac{p_k}{p_{k-1}}}.$$

Thus, we have only to prove that

$$\frac{1}{p_{k-1}^2}u_k\begin{pmatrix}-p_k\\ p_{k-1}\end{pmatrix}\underset{k\to\infty}{\longrightarrow} 0.$$

We have again

$$\frac{1}{p_{k-1}^2}u_k\begin{pmatrix}-p_k\\ p_{k-1}\end{pmatrix} = \frac{(-1)^{k+1}}{p_{k-1}^2}\left(w_1^{(2)}p_0 + (-1)^k w_k^{(1)}p_k + \sum_{j=1}^{k-1}(-1)^j\vec{w}_j^{(3)}p_j\right)$$

and we conclude with the law of large numbers and (3.16). □

## 4. Remarks on a generalization

Suppose that for all $x_0 > 0$ there exists a positive measure $Q_{x_0}(\mathrm{d}\vec{x})$ on $V$ such that there exists a non-void open convex set $\Theta$ and a function $f : \Theta \to \mathbb{R}$ with

$$\exp\left(-\frac{x_0}{2}f(\vec{\theta})\right) = \int_V \exp\,(\vec{\theta},\vec{x})Q_{x_0}(\mathrm{d}\vec{x}).$$

We denote $b^* = b_0 - f(\vec{b})$ if $b = (b_0,\vec{b}) \in (0,+\infty)\times\Theta$. We define for such $b$ two probabilities on

$(0, +\infty) \times V$ as follows: if $\lambda \in \mathbb{R}$ and $a > 0$,

$$\mu_{\lambda,a,b}(\mathrm{d}x) = K_\lambda(\sqrt{ab^*})a^{-\lambda/2}(b^*)^{\lambda/2}x_0^{\lambda-1}\exp\left(-\frac{a}{2x_0} - \frac{b_0x_0}{2} + \vec{b}\cdot\vec{x}\right)1_{(0,+\infty)}(x_0)\,\mathrm{d}x_0Q_{x_0}(\mathrm{d}\vec{x})$$

$$\gamma_{\lambda,b}(\mathrm{d}x) = \{\Gamma(\lambda)\}^{-1}(b^*)^\lambda y_0^{\lambda-1}\exp\left(-\frac{b_0y_0}{2} + \vec{b}\cdot\vec{y}\right)1_{(0,+\infty)}(y_0)\,\mathrm{d}y_0Q_{y_0}(\mathrm{d}\vec{y}).$$

Then it is easy to prove that Proposition 1.1 still holds with these more general $\mu$ and $\gamma$. However, it is not difficult to see that if for all $(\vec{b}, \vec{c})$ in $V^2$ we have the result of Proposition 2.2, then $Q_{x_0}(\mathrm{d}\vec{x})$ is nothing but $x_0^{-1/2}\exp(-\|\vec{x} - \vec{x}_1\|^2/2x_0)$ up to some constant.

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