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Convergence results for conditional expectations

IRENE CRIMALDI[1] and LUCA PRATELLI[2]

[1]*Department of Mathematics, University of Bologna, Piazza di Porta San Donato 5, 40126 Bologna, Italy. E-mail: crimaldi@dm.unibo.it*
[2]*Accademia Navale, Viale Italia 72, 57100 Livorno, Italy. E-mail: pratel@mail.dm.unipi.it*

Let E, F be two Polish spaces and $[X_n, Y_n]$, $[X, Y]$ random variables with values in $E \times F$ (not necessarily defined on the same probability space). We show some conditions which are sufficient in order to assure that, for each bounded continuous function f on $E \times F$, the conditional expectation of $f(X_n, Y_n)$ given Y_n converges in distribution to the conditional expectation of $f(X, Y)$ given Y.

Keywords: conditional expectation; Skorohod's theorem; weak convergence of probability measures

1. Introduction

Let E, F be two Polish spaces. Let X, Y be two random variables defined on a probability space $(\Omega, \mathcal{A}, P)$ with values in E, F, respectively. Moreover, for each integer $n \geqslant 0$, on a probability space $(\Omega_n, \mathcal{A}_n, P_n)$, let X_n be a random variable with values in E and Y_n a random variable with values in F. The problem we consider here is to find conditions under which, for each bounded continuous function f on $E \times F$, we have the weak convergence of the distribution under P_n of the conditional expectation $\mathrm{E}^{P_n}[f(X_n, Y_n)|Y_n]$ to the distribution under P of the conditional expectation $\mathrm{E}^P[f(X, Y)|Y]$. Problems of this kind arise in the theory of filtering, which plays a fundamental role in various fields, such as mathematical finance, biology and telecommunications. Indeed, in the theory of filtering, it is known that the conditional expectation of the signal given the observation is the optimal estimate, in the sense of the minimum mean square error. Computation of this conditional expectation is, in general, extremely difficult and so it is natural to seek approximations. Thus the problem is to find conditions under which the approximation of the signal–observation pair leads to a conditional expectation that is close (in some sense) to the conditional expectation of the signal given the observation.

A first result in this direction may be found in Goggin (1994; 1997), where a change of probability measure is assumed, from P_n to a suitable Q_n and from P to a suitable Q, in such a way that, in particular, P_n is absolutely continuous with respect to Q_n on the σ-field $\sigma(X_n, Y_n)$, P is absolutely continuous with respect to Q on the σ-field $\sigma(X, Y)$, and, for each n, the random variables X_n, Y_n are independent under Q_n and the random variables X, Y are independent under Q. In this paper, we replace the assumption of independence by the less restrictive assumption that, for each bounded continuous function g on E, the distribution under Q_n of $\mathrm{E}^{Q_n}[g(X_n)|Y_n]$ converges weakly to the distribution under Q of

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$\mathrm{E}^Q[g(X)|Y]$. This condition becomes *necessary* if we assume that P is equivalent to Q on $\sigma(X, Y)$ (see Corollary 4.2). Moreover, we obtain a result for the convergence of the conditional expectations not only of the form $\mathrm{E}^{P_n}[f(X_n)|Y_n]$ (as in Goggin 1994; 1997), but also of the form $\mathrm{E}^{P_n}[f(X_n, Y_n)|Y_n]$. This allows us (see Corollary 4.1) to obtain that, if the distribution under P_n of $[X_n, Y_n]$ converges weakly to the distribution under P of $[X, Y]$, then the weak convergence of the distribution under P_n of $\mathrm{E}^{P_n}[f(X_n)|Y_n]$ to the distribution under P of $\mathrm{E}^P[f(X)|Y]$ for each bounded continuous function f on E is *equivalent* to the weak convergence of the distribution under P_n of $\mathrm{E}^{P_n}[f(X_n, Y_n)|Y_n]$ to the distribution under P of $\mathrm{E}^P[f(X, Y)|Y]$, for each bounded continuous function f on $E \times F$.

Finally, we would like to point out the simplicity of our proof compared to the one presented by Goggin.

The paper is structured as follows. We present our main result (Theorem 2.1) in Section 2 and prove it in Section 3. In Section 4 we find some characterizations for the convergence of conditional expectations and prove that, if the two probability measures P, Q are equivalent on $\sigma(X, Y)$, condition (b) in Theorem 2.1 is necessary. In Section 5, we show that the result given by Goggin is a particular case of our Theorem 2.1.

2. Main result

Let E, F be two Polish spaces, endowed with their Borel σ-fields. On a probability space $(\Omega, \mathcal{A}, P)$, let X be a random variable with values in E and Y a random variable with values in F. For each integer $n \geqslant 0$, on a probability space $(\Omega_n, \mathcal{A}_n, P_n)$, let X_n be a random variable with values in E and Y_n a random variable with values in F.

Let Q be a probability measure on $(\Omega, \mathcal{A})$ such that P is absolutely continuous with respect to Q on the σ-field $\sigma(X, Y)$ generated by $[X, Y]$, and let us denote by Z a version of the corresponding Radon–Nikodym derivative. Moreover, for each n, let Q_n be a probability measure on $(\Omega_n, \mathcal{A}_n)$ such that P_n is absolutely continuous with respect to Q_n on the σ-field $\sigma(X_n, Y_n)$ generated by $[X_n, Y_n]$, and let us denote by Z_n a version of the corresponding Radon–Nikodym derivative. Thus, we have

$$Z = l(X, Y), \qquad Z_n = l_n(X_n, Y_n),$$

where l, l_n are suitable positive real Borel functions on $E \times F$. Denote by $Z.Q$ the probability measure on $\mathcal{A}$ which has density Z with respect to Q. Similarly, for each $n \geqslant 0$, denote by $Z_n.Q_n$ the probability measure on $\mathcal{A}_n$ which has density Z_n with respect to Q_n.

We shall prove the following result:

Theorem 2.1. *In the above setting, let us assume the following conditions:*

(a) *The distribution ν_n of $[X_n, Y_n, Z_n]$ under Q_n converges weakly to the distribution ν of $[X, Y, Z]$ under Q.*
(b) *For each bounded continuous function g on E, the distribution under Q_n of the conditional expectation $\mathrm{E}^{Q_n}[g(X_n)|Y_n]$ converges weakly to the distribution under Q of the conditional expectation $\mathrm{E}^Q[g(X)|Y]$.*

Then, for each bounded continuous function f on $E \times F$, the distribution under P_n of the conditional expectation $\mathrm{E}^{P_n}[f(X_n, Y_n)|Y_n]$ converges weakly to the distribution under P of the conditional expectation $\mathrm{E}^{P}[f(X, Y)|Y]$.

3. Proof of Theorem 2.1

Let us start by observing that, since we have the relations

$$P_n = Z_n.Q_n \text{ on } \sigma(X_n, Y_n), \qquad P = Z.Q \text{ on } \sigma(X, Y),$$

in order to prove Theorem 2.1 we may replace the probability measures P, P_n by the probability measures $Z.Q$, $Z_n.Q_n$, respectively. Thus, we may work only with the probability spaces $(\Omega, \mathcal{A}, Q)$, $(\Omega_n, \mathcal{A}_n, Q_n)$ and the triplets of random variables (X, Y, Z), (X_n, Y_n, Z_n).

For each bounded continuous function g on E, the distribution under Q of the conditional expectation $\mathrm{E}^Q[g(X)|Y]$ depends only on the distribution of $[X, Y]$ under Q. Moreover, if f is a bounded continuous function on $E \times F$, and U, V are versions of the conditional expectations

$$\mathrm{E}^Q[Z|Y], \qquad \mathrm{E}^Q[f(X, Y)Z|Y],$$

then a version W of the conditional expectation

$$\mathrm{E}^{Z.Q}[f(X, Y)|Y]$$

is given by $W = \rho(U, V)$, where ρ is the real Borel function defined on $\mathbb{R}^2$ by

$$\rho(u, v) = \begin{cases} v/u, & \text{for } u \neq 0 \\ 0, & \text{for } u = 0. \end{cases} \tag{1}$$

Therefore, the distribution of W under $Z.Q$ depends only on the distribution ν of $[X, Y, Z]$ under Q. Thus, we see that, in order to prove Theorem 2.1, we may replace the triplet (X, Y, Z) by another one, say (X', Y', Z') (possibly, defined on a new probability space), provided that its joint distribution is ν. It is worthwhile to observe that, since we require that the joint distribution of the new triplet is the same as the old one, we have the equality $Z' = l(X', Y')$ almost everywhere. Similarly, for each n, we may replace the triplet (X_n, Y_n, Z_n) by another one, say (X'_n, Y'_n, Z'_n) (possibly, defined on a new probability space), provided that its joint distribution is ν_n. On the other hand, assumption (a) and Skorohod's theorem allow us to choose the new triplets (X', Y', Z') and (X'_n, Y'_n, Z'_n) in such a way that they are defined on a *common* probability space $(\Omega', \mathcal{A}', Q')$ and, on this space, the random variable $[X'_n, Y'_n, Z'_n]$ converges *almost surely* to $[X', Y', Z']$.

Summing up, what we have just observed allows us, *without loss of generality*, to consider only the particular case in which all the probability spaces $(\Omega_n, \mathcal{A}_n, Q_n)$ coincide with $(\Omega, \mathcal{A}, Q)$ and, on this space, the random variable $[X_n, Y_n, Z_n]$ converges almost surely to the random variable $[X, Y, Z]$. Assuming this is the case, let us observe that, by Scheffé's theorem, the sequence (Z_n) converges in $L^1(Q)$ to Z. Now, we divide the proof into two steps.

Step 1. Let us prove that, if g is a bounded continuous function on E, and T, T_n are versions of the conditional expectations

$$\mathrm{E}^Q[g(X)|Y], \qquad \mathrm{E}^Q[g(X_n)|Y_n],$$

then T_n converges in probability to T. To this end, let us observe that the random variables T_n are uniformly bounded and, by assumption, the sequence (T_n) converges in distribution under Q to T. Thus, since we have the equality

$$\int |T_n - T|^2 \,\mathrm{d}Q = \int T_n^2 \,\mathrm{d}Q - 2\int TT_n \,\mathrm{d}Q + \int T^2 \,\mathrm{d}Q,$$

it suffices to prove that we have $\int T^2 \,\mathrm{d}Q = \lim_n \int TT_n \,\mathrm{d}Q$, or, more generally,

$$\int RT \,\mathrm{d}Q = \lim_n \int RT_n \,\mathrm{d}Q \tag{2}$$

for each bounded random variable R which is measurable with respect to the σ-field $\sigma(Y)$ generated by Y, that is, of the form $h(Y)$, where h is a bounded Borel function on F. On the other hand, since the functions of this type for which the desired convergence holds form a monotone class, we can limit ourselves to taking into account only the case of a bounded *continuous* function h on F. In this case, the assertion immediately follows: indeed, by the convergence in distribution of $[X_n, Y_n]$ to $[X, Y]$ and the convergence in probability of Y_n to Y, we have

$$\int h(Y)T \,\mathrm{d}Q = \int h(Y)g(X) \,\mathrm{d}Q = \lim_n \int h(Y_n)g(X_n) \,\mathrm{d}Q$$

$$= \lim_n \int h(Y_n)T_n \,\mathrm{d}Q = \lim_n \int h(Y)T_n \,\mathrm{d}Q.$$

Step 2. Let f be a bounded continuous function on $E \times F$ and V, V_n versions of the conditional expectations

$$\mathrm{E}^Q[f(X, Y)Z|Y], \qquad \mathrm{E}^Q[f(X_n, Y_n)Z_n|Y_n].$$

Let us prove that V_n converges in $L^1(Q)$ to V.

To this end, recall that we have $Z = l(X, Y)$ and observe that, if we denote by μ the distribution of $[X, Y]$ under Q, for each $\epsilon > 0$, we can find an integer k and k pairs of functions $(g_1, h_1), \dots, (g_k, h_k)$, where g_k is a bounded continuous function on E and h_k is a bounded continuous function on F, such that

$$\left\| f(x, y)l(x, y) - \sum_{j=1}^{k} g_j(x)h_j(y) \right\|_{\mathrm{L}^1(\mu)} < \epsilon,$$

that is,

$$\left\| f(X, Y)Z - \sum_{j=1}^{k} g_j(X)h_j(Y) \right\|_{L^1(Q)} < \epsilon. \tag{3}$$

On the other hand, under Q, the sequence $(f(X_n, Y_n)Z_n - \sum_{j=1}^k g_j(X_n)h_j(Y_n))$ converges almost surely to the random variable

$$f(X, Y)Z - \sum_{j=1}^{k} g_j(X)h_j(Y).$$

Moreover, it is uniformly integrable: indeed, the functions f, g_j, h_j are bounded and, as we have already observed, the sequence (Z_n) converges in $L^1(Q)$ to Z. Therefore, the above convergence is also in $L^1(Q)$. Thus, by inequality (3), for n sufficiently large, we have

$$\left\| f(X_n, Y_n)Z_n - \sum_{j=1}^{k} g_j(X_n)h_j(Y_n) \right\|_{L^1(Q)} < \epsilon. \tag{4}$$

Let T_j, $T_{j,n}$ be versions of the conditional expectations

$$\mathrm{E}^Q[g_j(X)|Y], \qquad \mathrm{E}^Q[g_j(X_n)|Y_n].$$

Then, by Jensen's inequality and relations (3) and (4), we find

$$\|V - V_n\|_{L^1(Q)} < 2\epsilon + \sum_{j=1}^{k} \|T_j h_j(Y) - T_{j,n} h_j(Y_n)\|_{L^1(Q)}.$$

Hence, letting n go to $+\infty$ and using what we have proved in step 1 and the fact that Y_n converges in probability under Q to Y, we obtain

$$\limsup_n \|V - V_n\|_{L^1(Q)} \leqslant 2\epsilon.$$

Since ϵ is arbitrary, the convergence of V_n is proved. In particular (for $f = 1$), it follows that, if U, U_n are versions of the conditional expectations

$$\mathrm{E}^Q[Z|Y], \qquad \mathrm{E}^Q[Z_n|Y_n],$$

then U_n converges in $L^1(Q)$ to U. Thus, we have that the random variable $[U_n, V_n]$ converges to $[U, V]$ in probability under Q (and so under $Z.Q$). Moreover, since we have

$$\int_{\{U=0\}} Z \, \mathrm{d}Q = \int_{\{U=0\}} U \, \mathrm{d}Q = 0,$$

the set of the discontinuity points of the function ρ (defined by (1)) is negligible with respect to the distribution of $[U, V]$ under the probability measure $Z.Q$. Therefore, we can affirm that the random variable $W_n = \rho(U_n, V_n)$ converges to $W = \rho(U, V)$ in probability (and so in distribution) under $Z.Q$. Finally, remembering that Z_n converges in $L^1(Q)$ to Z, we find that the distribution of W_n under $Z_n.Q$ converges weakly to the distribution of W under $Z.Q$. This proves the theorem since the random variables W_n, W are versions of the conditional expectations

$$\mathrm{E}^{Z.Q}[f(X, Y)|Y], \qquad \mathrm{E}^{Z_n.Q}[f(X_n, Y_n)|Y_n].$$

4. Some complements

From Theorem 2.1 we obtain the following corollaries:

Corollary 4.1. *Let E, F be two Polish spaces, endowed with their Borel σ-fields. On a probability space $(\Omega, \mathcal{A}, P)$, let X be a random variable with values in E and Y a random variable with values in F. Moreover, for each integer $n \geqslant 0$, on a probability space $(\Omega_n, \mathcal{A}_n, P_n)$, let X_n be a random variable with values in E and Y_n a random variable with values in F. Then the following statements are equivalent:*

(i) *For each bounded continuous function f on $E \times F$, the distribution under P_n of the conditional expectation $\mathrm{E}^{P_n}[f(X_n, Y_n)|Y_n]$ converges weakly to the distribution under P of the conditional expectation $\mathrm{E}^P[f(X, Y)|Y]$.*

(ii) *The distribution under P_n of $[X_n, Y_n]$ converges weakly to the distribution under P of $[X, Y]$ and, for each bounded continuous function g on E, the distribution under P_n of $\mathrm{E}^{P_n}[g(X_n)|Y_n]$ converges weakly to the distribution under P of $\mathrm{E}^P[g(X)|Y]$.*

Proof. Implication (i) $\Rightarrow$ (ii) is obvious. Implication (ii) $\Rightarrow$ (i) is a particular case of Theorem 2.1; that is, the case in which we have $P = Q$ (and so $Z = 1$) and $P_n = Q_n$ (and so $Z_n = 1$) for each n. □

Corollary 4.2. *Under the notation of Theorem 2.1, let us assume condition (a) and $Q\{Z > 0\} = 1$ (that is, P equivalent to Q on $\sigma(X, Y)$). Moreover, let us assume that, for each bounded continuous function f on E, the distribution under P_n of the conditional expectation $\mathrm{E}^{P_n}[f(X_n)|Y_n]$ converges weakly to the distribution under P of the conditional expectation $\mathrm{E}^P[f(X)|Y]$. Then condition (b) of Theorem 2.1 holds.*

Proof. Since, by assumption, the distribution of Z_n under Q_n converges weakly to the distribution of Z under Q and we have $Q\{Z > 0\} = 1$, we obtain that

$$\lim_n Q_n\{Z_n > 0\} = Q\{Z > 0\} = 1. \tag{5}$$

Therefore, for n sufficiently large, the following random variables are well defined:

$$W_n = Q_n\{Z_n > 0\}^{-1} I_{\{Z_n > 0\}}.$$

Further, we find

$$\lim_n \mathrm{E}^{Q_n}\left[(W_n - 1)^2\right] = 0. \tag{6}$$

If we set $Z'_n = W_n/Z_n$ and $Z' = 1/Z$, the probability measure $Q'_n = W_n.Q_n$ is absolutely continuous with respect to P_n on $\sigma(X_n, Y_n)$ with Radon–Nikodym derivative Z'_n and $Q' = Q$ is absolutely continuous with respect to P on $\sigma(X, Y)$ with Radon–Nikodym derivative Z'. It is easy to see (using Skorohod's theorem and Scheffé's theorem) that, by condition (a) of Theorem 2.1 and equality (5), we have that the distribution under P_n of the random variable $[X_n, Y_n, Z'_n]$ converges weakly to the distribution under P of the random variable $[X, Y, Z']$.

Hence, applying Theorem 2.1, we obtain that, for each bounded continuous function f on $E \times F$, the distribution under Q'_n of $\mathrm{E}^{Q'_n}[f(X_n, Y_n)|Y_n]$ converges weakly to the distribution under Q' of $\mathrm{E}^{Q'}[f(X, Y)|Y]$. In particular, we obtain that, for each bounded continuous function g on E, the distribution under Q'_n of $\mathrm{E}^{Q_n{}'}[g(X_n)|Y_n]$ converges weakly to the distribution under Q' of $\mathrm{E}^{Q'}[g(X)|Y]$. Recalling that $Q'_n = W_n.Q_n$ and $Q' = Q$, since equality (6) holds, we may conclude that condition (b) of Theorem 2.1 is satisfied. $\square$

With an argument similar to the one used in the proof of step 1 of Theorem 2.1, we obtian the following proposition:

Proposition 4.3. *In the setting of Corollary 4.1, let us assume that, for each n, the probability space* $(\Omega_n, \mathcal{A}_n, P_n)$ *coincides with* $(\Omega, \mathcal{A}, P)$*. Then the following conditions are equivalent:*

(i) *For each bounded continuous function f on $E \times F$, the conditional expectation* $\mathrm{E}^P[f(X_n, Y_n)|Y_n]$ *converges in* $L^1(P)$ *to the conditional expectation* $\mathrm{E}^P[f(X, Y)|Y]$.

(ii) *The sequence* (Y_n) *converges in probability to* Y *and, for each bounded continuous function f on $E \times F$, the conditional expectation* $\mathrm{E}^P[f(X_n, Y_n)|Y_n]$ *converges in distribution to the conditional expectation* $\mathrm{E}^P[f(X, Y)|Y]$.

Proof. Regarding implication (i) $\Rightarrow$ (ii), we have only to prove that the convergence in probability of $h(Y_n)$ to $h(Y)$ for each bounded continuous function h on F is equivalent to the convergence in probability of Y_n to Y. To this end, let us fix a countable basis $\mathcal{U}$ of F and, for each open set U in $\mathcal{U}$, let us denote by h_U a bounded positive continuous function on F such that $\{h_U > 0\} = U$. Thus, if we start from a given subsequence of (Y_n), it is possible to extract (by a diagonal argument) a sub-subsequence, say (Y'_n), such that, for each U in $\mathcal{U}$, the sequence $(h_U(Y'_n))$ converges almost surely to $h_U(Y)$. Therefore, there exists a set H in $\mathcal{A}$ with $P(H) = 1$ and such that, for each ω in H and U in $\mathcal{U}$, the sequence $(h_U(Y'_n(\omega)))$ converges to $h_U(Y(\omega))$. Then it is easy to see that, if ω belongs to H, the sequence $(Y'_n(\omega))$ converges to $Y(\omega)$: indeed, for each U in $\mathcal{U}$ with $Y(\omega) \in U$, since $h_U(Y(\omega)) > 0$, we have $h_U(Y'_n(\omega)) > 0$, that is, $Y'_n(\omega) \in U$, for n sufficiently large.

Implication (ii) $\Rightarrow$ (i) follows from an argument similar to the one used in step 1 of the proof of Theorem 2.1. $\square$

From Corollary 4.1 and Proposition 4.3 we obtain the following corollary:

Corollary 4.4. *In the setting of Proposition 4.3, the following conditions are equivalent:*

(i) *For each bounded continuous function f on $E \times F$, the conditional expectation* $\mathrm{E}^P[f(X_n, Y_n)|Y_n]$ *converges in* $L^1(P)$ *to the conditional expectation* $\mathrm{E}^P[f(X, Y)|Y]$.

(ii) *The sequence* (Y_n) *converges in probability to* Y*, the random variable* $[X_n, Y_n]$ *converges in distribution to* $[X, Y]$ *and, for each bounded continuous function g on E, the conditional expectation* $\mathrm{E}^P[g(X_n)|Y_n]$ *converges in distribution to the conditional expectation* $\mathrm{E}^P[g(X)|Y]$.

5. Comparison with Goggin's result

Let us use the same notation as in the previous sections. The result proved in Goggin (1994) is the following:

Theorem 5.1. *For each n, let Q_n be a probability measure on $(\Omega_n, \mathcal{A}_n)$ such that the probability measure P_n is absolutely continuous with respect to Q_n on the σ-field $\sigma(X_n, Y_n)$, and let us denote by Z_n a version of the corresponding Radon–Nikodym derivative. Let us assume the following conditions:*

(A) *On the probability space $(\Omega, \mathcal{A})$ there exist a probability measure Q and a positive $\sigma(X, Y)$-measurable random variable Z with $\mathrm{E}^Q[Z] = 1$ such that the distribution of $[X_n, Y_n, Z_n]$ under Q_n converges weakly to the distribution of $[X, Y, Z]$ under Q.*
(B) *The distribution under P_n of $[X_n, Y_n]$ converges weakly to the distribution under P of $[X, Y]$.*
(C) *For each n, the two random variables X_n, Y_n are independent under Q_n and the two random variables X, Y are independent under Q.*

Then the following statements hold:

(i) *The probability measure P is absolutely continuous with respect to Q on the σ-field $\sigma(X, Y)$, and Z is the corresponding Radon–Nikodym derivative.*
(ii) *For each bounded continuous function f on E, we have that the distribution under P_n of the conditional expectation $\mathrm{E}^{P_n}[f(X_n)|Y_n]$ converges weakly to the distribution under P of the conditional expectation $\mathrm{E}^P[f(X)|Y]$.*

It is easy to see that we can obtain the above theorem as a corollary of Theorem 2.1. More precisely, we have the following corollary:

Corollary 5.2. *With the notation of Theorem 5.1, let us assume conditions (A), (B) and (C). Then, the probability measure P is absolutely continuous with respect to Q on $\sigma(X, Y)$ with Radon–Nikodym derivative Z and condition (b) – and so the assertion – of Theorem 2.1 holds.*

Proof. We observe that, by conditions (A) and (B), for each bounded continuous function f on $E \times F$, we have

$$\int f(X, Y)\,\mathrm{d}P = \lim_n \int f(X_n, Y_n)\,\mathrm{d}P_n = \lim_n \int f(X_n, Y_n) Z_n\,\mathrm{d}Q_n$$
$$= \int f(X, Y) Z\,\mathrm{d}Q$$

(where the last equality follows by Skorohod's theorem and Scheffé's theorem). Moreover, by condition (C), for each bounded continuous function g on E and for each n, we have

$$\mathrm{E}^{Q_n}\big[g(X_n)|Y_n\big] = \mathrm{E}^{Q_n}[g(X_n)] \quad \text{and} \quad \mathrm{E}^{Q}\big[g(X)|Y\big] = \mathrm{E}^{Q}[g(X)].$$

Thus, in order to arrive at the conclusion, it is sufficient to remember that, by assumption, the distribution under Q_n of X_n converges weakly to the distribution under Q of X. $\square$

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