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Large-noise asymptotics for one-dimensional diffusions

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We establish a law of large numbers and a central limit theorem for a class of additive functionals related to the solution of a one-dimensional stochastic differential equation perturbed by a large noise.

Keywords: additive functional; central limit theorem; large-noise; law of large numbers; stochastic differential equations

1. Introduction

There is a vast literature on small-noise perturbation of dynamical systems. This includes topics such as the Freidlin–Wentzell large-deviation estimates and the Varadhan estimates on law densities with connections to the Malliavin calculus. On the other hand, dynamical systems with large noise have been, to our knowledge, much less considered. It is worth noting that the influence of small noise on solutions to stochastic differential equations can be closely approached by the small-time asymptotic behaviour. However, the influence of large noise generally cannot be reduced to the large-time behaviour.

We consider the one-dimensional stochastic differential equation

$$\mathrm{d}X_t = (b(X_t) + vX_t)\mathrm{d}t + \eta\, \mathrm{d}B_t, \qquad X_0 = 0, \tag{1.1}$$

where b is a real Borel bounded function, η is a large real constant, and B is a one-dimensional Brownian motion. This equation has a unique strong solution; see, for example, Le Gall (1983), Nakao (1972), Perkins (1982), Revuz and Yor (1999, Theorem 3.8, Chapter IX) for $v = 0$, and Flandoli and Russo (2002) for $v \neq 0$.

One can ask under what conditions on b the solution $X = X^{v,\eta}$ completely ignores at macroscopic level the nonlinear part $b(X)$. In other words, denoting by ηB^v the Ornstein–Uhlenbeck process being the solution to (1.1) with $b = 0$, one would like to know whether

$$\sup_{t\in[0,T]} |X_t^{v,\eta} - \eta B_t^v| \overset{(\mathbb{P})}{\to} 0. \tag{1.2}$$

If (1.2) holds, then one says that the *triviality phenomenon* occurs.

In examining the convergence of high-dimensional stochastic partial differential equations driven by white noise, Russo and Oberguggenberger (1999) introduced the concept of a

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massless-at-zero Schwartz tempered distribution: a Borel real function b has a Fourier transform massless at zero, or simply is *Fourier massless at zero*, if

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}} b\left(\frac{y}{\varepsilon}\right)\mathrm{e}^{-y^2/2}\,\mathrm{d}y = 0. \tag{1.3}$$

Notice that p-integrable functions with an arbitrary $p \in [1, \infty)$ and bounded measurable functions vanishing at ∞ are Fourier massless at zero. Notice also that if $|b|$ is Fourier massless at zero, then the triviality phenomenon occurs. In fact we have

$$\mathbb{E}\left(\int_0^T |b(\eta B_s^{\upsilon})|\,\mathrm{d}s\right)^p \to 0, \qquad p \geqslant 1,$$

which yields (1.2) by means of the Girsanov transformation.

By analogy with the case of high-dimensional stochastic partial differential equations, one might expect that the triviality phenomenon takes place for any Fourier massless-at-zero function. This, however, does not happen in the present framework of large-noise analysis. Taking for simplicity $\upsilon = 0$, consider a Lipschitz bounded function satisfying

$$\lim_{x\to+\infty} b(x) = \ell = -\lim_{x\to-\infty} b(x),$$

for some $\ell > 0$. Then (1.3) is fulfilled but the triviality phenomenon does not occur since

$$|X_t^{0,\eta} - \eta B_t| \to l\left|\int_0^t \operatorname{sgn}(B_s)\mathrm{d}s\right|,$$

which is not even a Gaussian process.

One objective of this paper is to study precise asymptotics for a class of functions for which triviality occurs. This will include laws of large numbers and central limit theorems of some kind. We focus on the class $\mathcal{C}$ of functions b having a bounded primitive. Integrating by parts, it is easy to see that $b \in \mathcal{C}$ satisfies (1.3), and so b is Fourier massless at zero. Moreover, as illustrated by Proposition 2.1, the triviality phenomenon occurs by means of the inverse Itô formula applied to

$$X_t^{\upsilon,\eta} - \eta B_t^{\upsilon} = \int_0^t b(X_s^{\upsilon,\eta})\mathrm{d}s. \tag{1.4}$$

The class $\mathcal{C}$ includes

(a) the class of integrable functions,
(b) trigonometric polynomials.

Clearly, if $b \in L^1$ then $|b|$ is Fourier massless at zero. This, however, generally fails for a trigonometric polynomial b: take $b(x) = \cos(x)$, $x \in \mathbb{R}$.

In fact we consider (1.1) for any bounded measurable function b and study precise asymptotics of additive functionals

$$A_t^{\upsilon,\eta}(\rho) = \int_0^t \rho(\eta B_s^{\upsilon})\mathrm{d}s \quad \text{and } \mathcal{A}_t^{\upsilon,\eta}(\rho) = \int_0^t \rho(X_s^{\upsilon,\eta})\mathrm{d}s \tag{1.5}$$

for any Schwartz distribution ρ belonging to the union of

(a′) the class of finite signed measures,
(b′) the class of the Schwartz distribution $\rho = H''$ where $H \in W^{1,\infty}$ and $(H')^2 - c_\rho$ has a bounded primitive for a certain c_ρ.

Note that the first class is a generalization of the space L^1, whereas the second class contains the space of trigonometric polynomials.

The triviality phenomenon (1.2) also occurs for $b \in L^2$, and for additive functionals (1.5) driven by $\rho \in L^2$. We will return to this, however, in a future paper. We note that in the case $\upsilon = 0$, that is, for Brownian motion, Yamada (1986; 1996) proved that uniformly in t on bounded intervals,

$$\frac{1}{\lambda}\int_0^{\lambda t} \rho(B_s)\mathrm{d}s \overset{(\mathbb{P})}{\to} \frac{1}{\pi}\int_{\mathbb{R}} (\mathcal{H}^{-1}\rho)(x)\mathrm{d}x\, C_t^0, \qquad \text{as } \lambda \to +\infty, \tag{1.6}$$

where C^0 is a given process depending on the Brownian motion and $\mathcal{H}^{-1}\rho$ is the inverse Hilbert transform of $\rho \in L^2$. Clearly, through an obvious change of time variable $\eta = \sqrt{\lambda}$, (1.6) can be formulated in terms of asymptotic behaviour of additive functionals, for which we are interested in obtaining a non-zero limit, see Proposition 2.2 for pathwise convergence when ρ is a finite measure and Theorem 2.2 for convergence in law when ρ belongs to class (b′).

The next step in the analysis of large noise will be the study of large deviations. For this purpose, natural tools will be Remillard (2000) and Takeda (1998; 2003). That investigation will be the subject of a subsequent paper.

The paper is organized as follows. In Section 2, we state the basic limit results for $\eta A^{\upsilon,\eta}(\rho)$ and $\eta \mathcal{A}^{\upsilon,\eta}(\rho)$. Proposition 2.2 states a law of large numbers (pathwise convergence) when ρ is a finite measure. If, moreover, ρ has finite first moment, then Theorem 2.1 gives a central limit theorem. The class (b′) is treated in Theorem 2.2. Sections 3–5 are devoted to the proof of the limit results.

2. Formulation of the results

Let $(L^\infty, \|\cdot\|_\infty)$ be the space of classes (with respect to Lebesgue measure) of bounded measurable functions $f : \mathbb{R} \to \mathbb{R}$ equipped with the essential supremum norm. Let $W^{-1,\infty}$ be the space of all distributions F' where $F \in L^\infty$. We equip $W^{-1,\infty}$ with the norm $\|\rho\|_{W^{-1,\infty}} = \|F\|_\infty$, where $F \in L^\infty$ is such that $F' = \rho$ and $F(0) = 0$. Let $C_0(\mathbb{R})$ be the class of all continuous functions with a compact support. Clearly, $C_0(\mathbb{R})$ is dense in $W^{-1,\infty}$.

In this paper, all the initial conditions are assumed to be equal to zero and all the equations will be taken on a compact interval $[0, T]$.

Let $X^{\upsilon,\eta}$ be the solution to the stochastic differential equation (1.1), where B is a one-dimensional Brownian motion defined on a probability space $\mathscr{A} = (\Omega, \mathscr{F}, \mathbb{P})$, $b : \mathbb{R} \to \mathbb{R}$ is a bounded measurable function, and the parameter $\upsilon \in \mathbb{R}$. Let $B_t^\upsilon := \int_0^t \mathrm{e}^{\upsilon(t-s)}\,\mathrm{d}B_s$ be the

Ornstein–Uhlenbeck process being the solution to the equation obtained from (1.1) by putting $b = 0$ and $\eta = 1$.

Given $\rho \in C_0(\mathbb{R})$ and $\eta > 0$, we define additive functionals $A^{v,\eta}(\rho)$ and $\mathcal{A}^{v,\eta}(\rho)$ by (1.5). The lemma below enables us to extend the functionals $A^{v,\eta}$ and $\mathcal{A}^{v,\eta}$ from the space $C_0(\mathbb{R})$ to the space $W^{-1,\infty}$. In the paper we use $\|\cdot\|_T$ to denote the supremum norm on $C([0, T]; \mathbb{R})$, that is,

$$\|\psi\|_T = \sup_{0\leqslant t\leqslant T} |\psi(t)|, \qquad \psi \in C([0, T]; \mathbb{R}).$$

Lemma 2.1. *Let $T \in (0, \infty)$ and $p \in [1, \infty)$.*

(i) *The linear operators $\rho \to A^{v,\eta}(\rho)$ and $\rho \to \mathcal{A}^{v,\eta}(\rho)$ are continuous from the space $(C_0(\mathbb{R}), \|\cdot\|_{W^{-1,\infty}})$ into $\mathcal{L}_T^p := L^p(\Omega, \mathscr{F}, \mathbb{P}; C([0, T]; \mathbb{R}))$, and hence $A^{v,\eta}$ and $\mathcal{A}^{v,\eta}$ can be uniquely extended to the continuous linear operators (denoted also by $A^{v,\eta}$ and $\mathcal{A}^{v,\eta}$) acting from $W^{-1,\infty}$ to $\mathcal{L}_T^p$.*

(ii) *For any $\rho \in W^{-1,\infty}$, the following inverse Itô formulae hold:*

$$A_t^{v,\eta}(\rho) = \frac{2H(\eta B_t^v) - 2H(0)}{\eta^2} - \frac{2}{\eta}\int_0^t F\left(\eta B_s^v\right) \mathrm{d}B_s^v \tag{2.1}$$

and

$$\mathcal{A}_t^{v,\eta}(\rho) = \frac{2H(X_t^{v,\eta}) - 2H(0)}{\eta^2} - \frac{2}{\eta^2}\int_0^t F\left(X_s^{v,\eta}\right) \mathrm{d}X_s^{v,\eta}, \tag{2.2}$$

where $\rho = F'$, $F \in L^\infty$ and H is a primitive of F.

Proof. By Itô's formula, (2.1) and (2.2) hold for $\rho \in C_0(\mathbb{R})$. Since any primitive H of a bounded function F has a linear growth one can obtain (i). Then by standard approximation arguments one obtains (ii). □

As a consequence of Lemma 2.1(ii) and Girsanov's theorem, we have the following asymptotic result on $A^{v,\eta}$ and $\mathcal{A}^{v,\eta}$.

Proposition 2.1. *Let $0 < r < 1$. For all $\rho \in W^{-1,\infty}$, $T \in (0, \infty)$ and $p \in [1, \infty)$,*

$$\lim_{\eta\to\infty} \mathbb{E}\|\eta^r A^{v,\eta}(\rho)\|_T^p = 0 = \lim_{\eta\to\infty} \mathbb{E}\|\eta^r \mathcal{A}^{v,\eta}(\rho)\|_T^p.$$

Let us denote by $\mathcal{M}_{\mathrm{fin}}(\mathbb{R})$ the collection of all finite Radon signed measures on $\mathbb{R}$. We denote by $\|\rho\|_{\mathrm{Var}}$ the total variation of a $\rho \in \mathcal{M}_{\mathrm{fin}}(\mathbb{R})$. Let $\mathcal{M}_{1,\mathrm{fin}}(\mathbb{R})$ be the subspace of $\mathcal{M}_{\mathrm{fin}}(\mathbb{R})$ consisting of all measures ρ with finite first moment, that is,

$$\int_{\mathbb{R}} |x| \, \|\rho\|_{\mathrm{Var}}(\mathrm{d}x) < \infty.$$

Clearly, $\mathcal{M}_{1,\mathrm{fin}}(\mathbb{R}) \subset \mathcal{M}_{\mathrm{fin}}(\mathbb{R}) \subset W^{-1,\infty}$.

Let L^B be the local time of B, let $f \in L^1(\mathbb{R})$, and let

$$M_t^{\eta}(f) := \eta A_t^{0,\eta}(f) - \int_{\mathbb{R}} f(x)\mathrm{d}x\, L_t^B(0).$$

Then $\mathbb{E}\,\|M^{\eta}(f)\|_T^p \to 0$, for all $p \in [1, \infty)$ and $T < \infty$ (see Revuz and Yor 1999, Proposition 2.1, Chapter XIII). The theorem below provides similar results on the asymptotic behavior of

$$M^{v,\eta}(\rho) := \eta A^{v,\eta}(\rho) - \rho(\mathbb{R})\, L^{B^v}(0) \quad \text{and} \mathcal{M}^{v,\eta}(\rho) := \eta \mathcal{A}^{v,\eta}(\rho) - \rho(\mathbb{R})\, L^{B^v}(0),$$

where $\rho \in \mathcal{M}_{\mathrm{fin}}(\mathbb{R})$ and L^{B^v} is the local time of B^v.

Proposition 2.2. *For all* $\rho \in \mathcal{M}_{\mathrm{fin}}(\mathbb{R})$, $T \in (0, \infty)$, *and* $p \in [1, \infty)$,

$$\lim_{\eta\to\infty} \mathbb{E}\|M^{v,\eta}(\rho)\|_T^p = 0 = \lim_{\eta\to\infty} \mathbb{E}\|\mathcal{M}^{v,\eta}(\rho)\|_T^p.$$

Remark 2.1. If $\rho \in L^1(\mathbb{R})$, then, using Girsanov's transformation, one can easily derive from Revuz and Yor (1999, Proposition 2.1, Chapter XIII), that for any finite T, $\|M^{v,\eta}(\rho)\|_T \to 0$, $\mathbb{P}$-almost surely.

Our next result provides a weak convergence of $\sqrt{\eta}M^{v,\eta}(\rho)$ and $\sqrt{\eta}\mathcal{M}^{v,\eta}(\rho)$ under the assumption $\rho \in \mathcal{M}_{1,\mathrm{fin}}(\mathbb{R})$. Let $F_\rho(x) = \rho((-\infty, x))$ and let

$$\alpha_\rho = \sqrt{\int_{\mathbb{R}} \big(F_\rho - \rho(\mathbb{R})\,\chi_{(0,+\infty)}\big)^2(x)\mathrm{d}x}.$$

Clearly $F_\rho \in L^\infty$ and $\rho = F'_\rho$.

Remark 2.2. Note that if $\rho \in \mathcal{M}_{1,\mathrm{fin}}(\mathbb{R})$, then $\alpha_\rho < \infty$. In fact

$$\begin{aligned}\alpha_\rho^2 &= \int_{-\infty}^0 (\rho((-\infty, x)))^2\,\mathrm{d}x + \int_0^\infty (\rho([x, \infty)))^2\,\mathrm{d}x \\ &\leqslant \|\rho\|_{\mathrm{Var}}(\mathbb{R})\left(\int_{-\infty}^0 \|\rho\|_{\mathrm{Var}}((-\infty, x])\mathrm{d}x + \int_0^\infty \|\rho\|_{\mathrm{Var}}([x, \infty))\mathrm{d}x\right) \\ &\leqslant \|\rho\|_{\mathrm{Var}}(\mathbb{R})\int_{\mathbb{R}} |x|\, \|\rho\|_{\mathrm{Var}}(\mathrm{d}x).\end{aligned}$$

Remark 2.3. Note that $\alpha_\rho = 0$ if and only if $\rho = c\delta_0$ for some $c \in \mathbb{R}$. Then, obviously, $M^{v,\eta}(\rho) = 0$. This shows that next result is in some sense optimal and it is not possible to get a non-trivial limit, renormalizing further by a bigger power of η, when $\rho \in \mathcal{M}_{1,\mathrm{fin}}(\mathbb{R})$.

Theorem 2.1. *Assume that* $\rho \in \mathcal{M}_{1,\mathrm{fin}}(\mathbb{R})$. *Then, uniformly in* t *on compact intervals,*

$$(B_t, \sqrt{\eta}\, M_t^{v,\eta}(\rho)) \Rightarrow \left(\beta_t, 2\alpha_\rho \gamma_{L_t^{\beta^v}(0)}\right), \qquad \text{as } \eta \to \infty, \tag{2.3}$$

and

$$(B_t, \sqrt{\eta}\,\mathcal{M}_t^{v,\eta}(\rho)) \Rightarrow \left(\beta_t, 2\alpha_\rho \gamma_{L_t^{\beta^v}(0)}\right), \qquad as\ \eta \to \infty, \tag{2.4}$$

where β and γ are independent standard Brownian motions, and L^{β^v} is the local time of the Ornstein–Uhlenbeck process β^v.

Let $W^{1,\infty}(\mathbb{R})$ be the space of all bounded absolutely continuous functions $H : \mathbb{R} \to \mathbb{R}$ such that $H' \in L^\infty$, and let $D_b^{-1}(\mathbb{R})$ be the space of all Schwartz distributions ρ such that $\rho = H''$ for some $H \in W^{1,\infty}(\mathbb{R})$. Clearly $D_b^{-1}(\mathbb{R}) \subset W^{-1,\infty}$.

Note that for any $\rho \in D_b^{-1}(\mathbb{R})$ there is a unique $H_\rho \in W^{1,\infty}(\mathbb{R})$ such that $\rho = H_\rho''$. Let $\mathcal{D}$ be the class of all $\rho \in D_b^{-1}(\mathbb{R})$ for which there is a (unique) constant c_ρ such that $(H_\rho')^2 - c_\rho \in W^{-1,\infty}$. Note that $c_\rho \geqslant 0$.

The last result of this section provides a generalized central limit theorem for additive functionals $A^{v,\eta}(\rho)$ and $\mathcal{A}^{v,\eta}(\rho)$, where $\rho \in \mathcal{M}_{\text{fin}} + \mathcal{D}$. We note that the class $\mathcal{D}$ contains trigonometrical polynomials (see Example 2.2).

Theorem 2.2. *Assume that $\rho = \rho_{\mathcal{M}} + \rho_{\mathcal{D}}$, where $\rho_{\mathcal{M}} \in \mathcal{M}_{\text{fin}}$ and $\rho_{\mathcal{D}} \in \mathcal{D}$. Then, uniformly in t on compact intervals,*

$$\left(B_t, \eta A_t^{v,\eta}(\rho)\right) \Rightarrow \left(\beta_t, \rho_{\mathcal{M}}(\mathbb{R}) L_t^{\beta^v}(0) + 2\sqrt{c_{\rho_{\mathcal{D}}}}\,\gamma_t\right)$$

and

$$\left(B_t, \eta \mathcal{A}_t^{v,\eta}(\rho)\right) \Rightarrow \left(\beta_t, \rho_{\mathcal{M}}(\mathbb{R}) L_t^{\beta^v}(0) + 2\sqrt{c_{\rho_{\mathcal{D}}}}\,\gamma_t\right),$$

where β and γ are independent standard Brownian motions.

Example 2.1. Suppose that $\rho = F'$ where $F(x) = \int_{\mathbb{R}} e^{ixy} \mu(dy)$ is the Fourier transform of a complex measure μ on $\mathbb{R}$. To ensure that F takes real values we assume that $\mu(A) = \overline{\mu(-A)}$ for any Borel set A. Clearly, $\rho \in D_b^{-1}(\mathbb{R})$ if μ and $x^{-1}\mu(dx)$ are finite. Assuming this we obtain:

(i) If $\rho \in \mathcal{D}$, then $c_\rho = \mu * \bar{\mu}(\{0\}) = \int_{\mathbb{R}} \bar{\mu}(\{x\})\mu(dx)$. Thus, in particular, $c_\rho = 0$ if μ is atomless.

(ii) A sufficient condition for $\rho \in \mathcal{D}$ is

$$\iint_{\{|y-x| \neq 0\}} \frac{\|\mu\|_{\text{Var}}(dy)\, \|\mu\|_{\text{Var}}(dx)}{|y-x|} < \infty.$$

For $F^2 = F\overline{F}$ is the Fourier transform of $\mu * \bar{\mu}$, which again is a measure. Then we derive (i) from the fact that any constant function C is the Fourier transform of $C\delta_0$, and the following observation: if a Fourier transform of a measure ν has a bounded primitive, say h, then $\nu(\{0\}) = 0$, and $h(x) = -i \int_{\mathbb{R}} e^{ixy} y^{-1} \nu(dy)$, $x \in \mathbb{R}$. Set

$$\xi(dy) = -iy^{-1}(\mu * \bar{\mu} - \mu * \bar{\mu}(\{0\})\delta_0)(dy).$$

We infer that if there is a bounded primitive H of $F^2 - \mu * \bar{\mu}(\{0\})$, then $H(x) = \int_{\mathbb{R}} e^{ixy} \xi(dy)$. Finally, H given by the formula above is bounded if ξ is a finite measure, which is guaranteed by (ii).

Taking in the example above any purely atomic spectral measure μ with a finite number of atoms we obtain the following.

Example 2.2. Any trigonometrical polynomial $\rho(x) = \sum_{j=-m}^{m} a_j \mathrm{e}^{\mathrm{i} b_j x}$, where $0 < m < \infty$, $a_j = \overline{a_{-j}}$, and $b_j = -b_{-j} \in \mathbb{R}\backslash\{0\}$, belongs to $\mathcal{D}$ with

$$c_\rho = \sum_{j=-m}^{m} |a_j|^2 |b_j|^{-2}.$$

We can now formulate a result concerning the triviality phenomenon for $X^{\upsilon,\eta}$. To do this, note that $X^{\upsilon,\eta}$ is the solution to

$$X_t^{\upsilon,\eta} = \int_0^t \mathrm{e}^{\upsilon(t-s)} b\big(X_s^{\upsilon,\eta}\big)\, \mathrm{d}t + \eta B_t^\upsilon.$$

Thus

$$\begin{aligned} \mathcal{D}_t^{\upsilon,\eta}(b) := X_t^{\upsilon,\eta} - \eta B_t^\upsilon &= \int_0^t \mathrm{e}^{\upsilon(t-s)} b\big(X_s^{\upsilon,\eta}\big)\, \mathrm{d}t = \int_0^t \mathrm{e}^{\upsilon(t-s)} \frac{\mathrm{d}}{\mathrm{d}s} \mathcal{A}_s^{\upsilon,\eta}(b) \mathrm{d}s \\ &= \mathcal{A}_t^{\upsilon,\eta}(b) + \upsilon \int_0^t \mathrm{e}^{\upsilon(t-s)} \mathcal{A}_s^{\upsilon,\eta}(b) \mathrm{d}s = \mathcal{J}^{\mathrm{v}}\big(\mathcal{A}^{\mathrm{v},\eta}(\mathrm{b})\big)_{\mathrm{t}}, \end{aligned} \tag{2.5}$$

where $\mathcal{J}^\upsilon$ is a bounded linear operator on $C([0, \infty); \mathbb{R})$ given by

$$\mathcal{J}^\upsilon(\psi)_t = \psi_t + \upsilon \int_0^{\mathrm{t}} \mathrm{e}^{\upsilon(t-s)} \psi_s \, \mathrm{d}s, \qquad \psi \in C([0, \infty); \mathbb{R}). \tag{2.6}$$

Consequently, we have the following Corollary to Propositions 2.1 and 2.2 and Theorems 2.1 and 2.2. Recall that in the present paper b is a bounded measurable function. If $b \in L^1$ then we set

$$\mathscr{M}_t^{\upsilon,\eta} := \eta \mathcal{D}_t^{\upsilon,\eta}(b) - \int_{\mathbb{R}} b(x) \mathrm{d}x \, \mathcal{J}^\upsilon\big(L^{B^\upsilon}(0)\big)_t.$$

Corollary 2.1. (*i*) *If* $b \in W^{-1,\infty}$ *then, for all* $T \in [0, \infty)$, $r \in (0, 1)$ and $p \in [1, \infty)$, *one has* $\mathbb{E}\,\|\eta^r \mathcal{D}^{\upsilon,\eta}(b)\|_T^p \to 0$ *as* $\eta \to \infty$.

(ii) *If* $b \in L^1$ *then, for all* $T \in [0, \infty)$ *and* $p \in [1, \infty)$, *one has* $\mathbb{E}\|\mathscr{M}^{\upsilon,\eta}\|_T^p \to 0$ *as* $\eta \to \infty$.

(iii) *If* b *and* $x \to xb(x)$ *belong to* L^1 *then, uniformly in* t *on compact intervals,*

$$(B_t, \sqrt{\eta}\mathscr{M}_t^{\upsilon,\eta}(b)) \Rightarrow \Big(\beta_t, 2\alpha_b\, \mathcal{J}^{\mathrm{v}}\Big(\gamma_{L^{\beta^\upsilon}(0)}\Big)_t\Big), \qquad \text{as } \eta \to \infty,$$

where β *and* γ *are independent standard Brownian motions, and* L^{β^υ} *is the local time of the Ornstein–Uhlenbeck process* β^υ.

(iv) *If* $b = b_{\mathcal{M}} + b_{\mathcal{D}}$, *where* $b_{\mathcal{M}} \in L^1$ *and* $b_{\mathcal{D}} \in \mathcal{D}$, *then, uniformly in* t *on compact intervals,*

$$\left(B_t,\, \eta\mathcal{D}_t^{v,\eta}(b)\right) \Rightarrow \left(\beta_t,\, \int_{\mathbb{R}} b(x)\mathrm{d}x\, \mathcal{J}^v\left(L^{\beta^v}(0)\right)_t + 2\sqrt{c_{b_D}}\, \mathcal{J}^v(\gamma)_t\right),$$

where β and γ are independent standard Brownian motions.

3. Proof of Proposition 2.2

Let us fix T, p, and $\rho \in \mathcal{M}_{\mathrm{fin}}(\mathbb{R})$. Let $t \in [0, T]$. By the occupation density times formula (see, for example, Revuz and Yor 1999, Corollary 1.6, Chapter VI), for any $f \in L^1(\mathbb{R})$ we have

$$\eta A_t^{v,\eta}(f) = \eta\int_{\mathbb{R}} f(\eta x) L_t^{B^v}(x)\mathrm{d}x = \int_{\mathbb{R}} L_t^{B^v}\left(\eta^{-1}x\right) f(x)\mathrm{d}x.$$

Thus by a standard approximation argument we obtain

$$\eta A_t^{v,\eta}(\rho) = \int_{\mathbb{R}} L_t^{B^v}\left(\eta^{-1}x\right)\rho(\mathrm{d}x). \tag{3.1}$$

Consequently,

$$M_t^{v,\eta}(\rho) = \int_{\mathbb{R}} \left\{L_t^{B^v}\left(\eta^{-1}x\right) - L_t^{B^v}(0)\right\}\rho(\mathrm{d}x)$$

and by Jensen's inequality,

$$\mathbb{E}\|M^{v,\eta}(\rho)\|_T^p \leqslant \left(\|\rho\|_{\mathrm{Var}}(\mathbb{R})\right)^{p-1}\int_{\mathbb{R}} \mathbb{E}\|L^{B^v}\left(\eta^{-1}x\right) - L^{B^v}(0)\|_T^p \|\rho\|_{\mathrm{Var}}(\mathrm{d}x).$$

Next, by Tanaka's formula (see, for example, Revuz and Yor 1999, Theorem 1.2, Chapter VI), we have

$$L_t^{B^v}\left(\eta^{-1}x\right) - L_t^{B^v}(0) = I_t^\eta(x) + J_t^\eta(x),$$

where

$$I_t^\eta(x) := |B_t^v - \eta^{-1}x| - |\eta^{-1}x| - |B_t^v|$$

and

$$J_t^\eta(x) := -\int_0^t \left\{\mathrm{sgn}\left(B_s^v - \eta^{-1}x\right) - \mathrm{sgn}\left(B_s^v\right)\right\}\mathrm{d}B_s^v.$$

Note that $|I_t^\eta(x)| \leqslant 2|B_t^v|$. Hence, it is easy to see that

$$\sup_{\eta>0}\sup_{x\in\mathbb{R}} \mathbb{E}\|L^{B^v}\left(\eta^{-1}x\right) - L^{B^v}(0)\|_T^p < \infty.$$

Moreover, since $\mathbb{P}\left(B_s^v = 0\right) = 0$, $s > 0$, one has, for any x,

$$\lim_{\eta\to\infty} \mathbb{E}\|L^{B^v}\left(\eta^{-1}x\right) - L^{B^v}(0)\|_T^p = 0.$$

Therefore one obtains $\mathbb{E}\|M^{v,\eta}(\rho)\|_T^p \to 0$ by means of Lebesgue's dominated convergence theorem. Clearly, to prove $\mathbb{E}\|\mathcal{M}^{v,\eta}(\rho)\|_T^p \to 0$ it is enough to show that $\mathbb{E}\|\eta(A^{v,\eta}(\rho) - \mathcal{A}^{v,\eta}(\rho))\|_T^p \to 0$. To do this, note that $Y^{v,\eta} := \eta^{-1}X^{v,\eta}$ is a continuous semimartingale and $Y^{v,\eta} = B^v + \eta^{-1}\mathcal{B}^{v,\eta}$, where

$$\mathcal{B}_t^{v,\eta} = \int_0^t \mathrm{e}^{v(t-s)} b\big(X_s^{v,\eta}\big)\mathrm{d}s.$$

Let us denote by L^η the local time of $Y^{v,\eta}$. Then using the arguments used in the proof of (3.1) we obtain

$$\eta\big(\mathcal{A}_t^{v,\eta}(\rho) - A_t^{v,\eta}(\rho)\big) = \int_{\mathbb{R}} \big\{L_t^\eta\big(\eta^{-1}x\big) - L_t^{B^v}\big(\eta^{-1}x\big)\big\}\rho(\mathrm{d}x).$$

Applying Tanaka's formula again, we obtain

$$L_t^\eta\big(\eta^{-1}x\big) - L_t^{B^v}\big(\eta^{-1}x\big) = I_t^\eta(x) + J_t^\eta(x),$$

where

$$I_t^\eta(x) := |Y_t^{v,\eta} - \eta^{-1}x| - |B_t^v - \eta^{-1}x| - \eta^{-1}\int_0^t \operatorname{sgn}\big(Y_s^{v,\eta} - \eta^{-1}x\big)\,\mathrm{d}\mathcal{B}_s^{v,\eta},$$

and

$$J_t^\eta(x) := \int_0^t \big\{\operatorname{sgn}\big(B_s^v - \eta^{-1}x\big) - \operatorname{sgn}\big(Y_s^{v,\eta} - \eta^{-1}x\big)\big\}\,\mathrm{d}B_s^v.$$

Since b is bounded it is easy to see that there is a constant c depending on v and T such that

$$\|Y^{v,\eta} - B^v\|_T + \sup_{x\in\mathbb{R}}\|I^\eta(x)\|_T \leqslant c\eta^{-1}, \qquad \eta > 0. \tag{3.2}$$

Thus the proof will be complete as soon as we show that

$$\sup_{\eta>0}\sup_{x\in\mathbb{R}} \mathbb{E}\|J^\eta(x)\|_T^p < \infty \qquad \text{and} \lim_{\eta\to\infty}\mathbb{E}\|J^\eta(x)\|_T^p = 0, \qquad \forall x\in\mathbb{R}. \tag{3.3}$$

Since

$$|B_t^u| = |B_t + u\int_0^t \mathrm{e}^{u(t-s)}B_s\,\mathrm{d}s| \leqslant \Big(1 + |u|\mathrm{e}^{|u|t}\Big)\|B\|_t,$$

we have $|J_t^\eta(x)| \leqslant C_1\Big(J^{\eta,1}(x) + J_t^{\eta,2}(x)\Big)$, where

$$J^{\eta,1}(x) := \|B\|_T\int_0^T |\operatorname{sgn}\big(B_t^v - \eta^{-1}x\big) - \operatorname{sgn}\big(Y_t^{v,\eta} - \eta^{-1}x\big)|\,\mathrm{d}t$$

and

$$J_t^{\eta,2}(x) := \left|\int_0^t \big\{\operatorname{sgn}\big(B_s^v - \eta^{-1}x\big) - \operatorname{sgn}\big(Y_s^{v,\eta} - \eta^{-1}x\big)\big\}\,\mathrm{d}B_s\right|.$$

Then (3.3) follows easily from (3.2) and Burkholder's inequality.

4. Proof of Theorem 2.1

Let us fix $T < \infty$ and $\rho \in \mathcal{M}_{1,\mathrm{fin}}(\mathbb{R})$. Recall that $\rho = F_\rho'$, where $F_\rho(x) = \rho((-\infty, x))$. Let $g = F_\rho - \rho(\mathbb{R})\chi_{(0,\infty)}$. Then g is bounded and square integrable (see Remark 2.2). Clearly $\|g\|_{L^2(\mathbb{R})} = \alpha_\rho$. Define

$$G_t^\eta := \int_0^t g\left(\eta B_s^\upsilon\right) \mathrm{d}B_s^\upsilon \qquad \text{and} \qquad \mathcal{G}_t^\eta := \eta^{-1}\int_0^t g\left(X_s^{\upsilon,\eta}\right) \mathrm{d}X_s^{\upsilon,\eta}.$$

We have

$$G_t^\eta = \int_0^t F_\rho\left(\eta B_s^\upsilon\right) \mathrm{d}B_s^\upsilon - \rho(\mathbb{R})\int_0^t \chi_{(0,\infty)}\left(\eta B_s^\upsilon\right) \mathrm{d}B_s^\upsilon.$$

By Tanaka's formula,

$$\int_0^t \chi_{(0,\infty)}\left(\eta B_s^\upsilon\right) \mathrm{d}B_s^\upsilon = (B_t^\upsilon)^+ - \frac{1}{2} L_t^{B^\upsilon}(0).$$

Thus

$$-2\int_0^t F_\rho\left(\eta B_s^\upsilon\right) \mathrm{d}B_s^\upsilon = -2G_t^\eta - 2\rho(\mathbb{R})(B_t^\upsilon)^+ + \rho(\mathbb{R}) L_t^{B^\upsilon}(0).$$

Let H be a primitive of F. Then using the inverse Itô formula (2.1), we obtain

$$M^{\upsilon,\eta}(\rho) = -2G^\eta + I^\eta, \tag{4.1}$$

where

$$I_t^\eta = \frac{2}{\eta}\left(H\left(\eta B_t^\upsilon\right) - H(0)\right) - 2\rho(\mathbb{R})(B_t^\upsilon)^+.$$

First, we will show that, uniformly in $t \in [0, T]$,

$$(B_t, \sqrt{\eta} G_t^\eta) \Rightarrow \left(\beta_t, \alpha_\rho \gamma_{L_t^{\beta^\upsilon}(0)}\right), \tag{4.2}$$

where β and γ are independent Brownian motions. To do this, note that $G^\eta = \upsilon J^{\eta,1} + J^{\eta,2}$, where

$$J_t^{\eta,1} := \int_0^t g\left(\eta B_s^\upsilon\right) B_s^\upsilon \,\mathrm{d}s \qquad \text{and } J_t^{\eta,2} := \int_0^t g\left(\eta B_s^\upsilon\right) \mathrm{d}B_s.$$

Note that $g \in L^1(\mathbb{R})$. In fact

$$\begin{aligned}\int_{\mathbb{R}} |g(x)| \,\mathrm{d}x &\leqslant \int_{-\infty}^0 \|\rho\|_{\mathrm{Var}}((-\infty, x])\mathrm{d}x + \int_0^\infty \|\rho\|_{\mathrm{Var}}([x, \infty))\mathrm{d}x \\ &\leqslant \int_{\mathbb{R}} |x| \, \|\rho\|_{\mathrm{Var}}(\mathrm{d}x) < \infty.\end{aligned}$$

We have

$$\|J^{\eta,1}\|_T \leqslant \|B^v\|_T \int_0^T |g(\eta B_s^v)|\,\mathrm{d}s.$$

Hence, by the Hölder inequality,

$$\eta^{p/2}\,\mathbb{E}\|J^{\eta,1}\|_T^p \leqslant \left(\mathbb{E}\,\|B^v\|_T^{2p}\right)^{1/2}\left(\eta^p\,\mathbb{E}\left(\int_0^T |g(\eta B_s^v)|\,\mathrm{d}s\right)^{2p}\right)^{1/2}.$$

Since $g \in L^1(\mathbb{R})$, it has a bounded primitive, and Proposition 2.1 yields

$$\lim_{\eta\to\infty} \eta^p\,\mathbb{E}\left(\int_0^T |g(\eta B_s^v)|\,\mathrm{d}s\right)^{2p} = 0.$$

Therefore $\eta^{p/2}\,\mathbb{E}\|J^{\eta,1}\|_T^p \to 0$, and it remains to show that, uniformly in $t \in [0, T]$,

$$\left(B_t, \sqrt{\eta}J_t^{\eta,2}\right) \Rightarrow \left(\beta_t, \alpha_\rho \gamma_{L_t^{\beta v}}\right). \tag{4.3}$$

Let W^η be the Dambis–Dubins–Schwartz (DDS) Brownian motion of $\sqrt{\eta}J^{\eta,2}$ (see, for example, Revuz and Yor 1999, Theorem 1.6, Chapter V). Then $\sqrt{\eta}J_t^{\eta,2} = W^\eta_{\psi^\eta(t)}$, where $\psi^\eta(t) = \eta\int_0^t g^2\left(\eta B_s^v\right)\mathrm{d}s$. Since g has a bounded primitive, Proposition 2.1 yields

$$\langle B, \sqrt{\eta}J^{\eta,2}\rangle_t = \sqrt{\eta}\int_0^t g\left(\eta B_s^v\right)\mathrm{d}s \to 0,$$

and by Proposition 2.2, $\langle\sqrt{\eta}J^{\eta,2}, \sqrt{\eta}J^{\eta,2}\rangle_t = \psi_t^\eta \to \alpha_\rho^2 L_t^{B^v}(0)$. Thus (4.2) follows from Revuz and Yor (1999, Theorem 2.3, Chapter XIII); see also the proof of Theorem 2.6, Chapter XIII, p. 526 from Revuz and Yor (1999).

Having shown (4.1) and (4.2), the proof of (2.3) will be complete as soon as we show that

$$\sqrt{\eta}I_t^\eta \to 0, \qquad \mathbb{P}\text{-a.s. uniformaly in } t \in [0.T]. \tag{4.4}$$

To see this, set

$$h_\eta(x) = \sqrt{\eta}\left(\eta^{-1}H(\eta x) - \rho(\mathbb{R})(x)^+\right). \tag{4.5}$$

Then, for $x \leqslant 0$,

$$\begin{aligned}
|h_\eta(x)| &\leqslant \eta^{-1/2}|H(\eta x)| \leqslant \eta^{-1/2}\left(|H(0)| + \int_{\eta x}^0 |F(z)|\,\mathrm{d}z\right) \\
&\leqslant \eta^{-1/2}\left(|H(0)| + \int_{-\infty}^0 \|\rho\|_{\mathrm{Var}}((-\infty, y])\mathrm{d}y\right) \\
&\leqslant \eta^{-1/2}\left(|H(0)| + \int_{-\infty}^0 |y|\|\rho\|_{\mathrm{Var}}(\mathrm{d}y)\right).
\end{aligned}$$

Note that, by Fubini's theorem,

$$\int_0^{\eta x} F(z)\mathrm{d}z = \eta x\rho((-\infty, 0]) + \int_0^{\eta x} \rho((0, z))\mathrm{d}z = \eta x\rho(-\infty, \eta x) - \int_{(0,\eta x)} y\rho(\mathrm{d}y).$$

Hence, for $x > 0$,

$$\begin{aligned} |h_\eta(x)| &= \sqrt{\eta}\left|\eta^{-1}\left(H(0) + \int_0^{\eta x} F(z)\mathrm{d}z\right) - x\rho(\mathbb{R})\right| \\ &= \sqrt{\eta}\left|\eta^{-1}H(0) - \eta^{-1}\int_{(0,\eta x)} y\rho(\mathrm{d}y) - x\rho([\eta x, \infty))\right| \\ &\leqslant \eta^{-1/2}|H(0)| + 2\eta^{-1/2}\int_{\mathbb{R}} |y|\, \|\rho\|_{\mathrm{Var}}(\mathrm{d}y), \end{aligned}$$

as

$$|\eta x\rho([\eta x, \infty))| \leqslant \int_{\mathbb{R}} |y|\, \|\rho\|_{\mathrm{Var}}(\mathrm{d}y).$$

Thus there is a constant $C < \infty$ such that

$$|h_\eta(x)| \leqslant C\eta^{-1/2}\left(1 + \int_{\mathbb{R}} |y|\, \|\rho\|_{\mathrm{Var}}(\mathrm{d}y)\right) \qquad \text{for all } \eta > 0,\ x \in \mathbb{R}, \tag{4.6}$$

and (4.4) follows from the identity $\sqrt{\eta}I_t^\eta = 2h_\eta\left(B_t^v\right) - 2\left(\sqrt{\eta}\right)^{-1}H(0)$.

We proceed to the proof of (2.4). Let L^η be the local time of $Y^{v,\eta} := \eta^{-1}X^{v,\eta}$. Since

$$\int_0^t \chi_{(0,\infty)}\left(X_s^{v,\eta}\right) \mathrm{d}Y_s^{v,\eta} = \left(Y_t^{v,\eta}\right)^+ - \frac{1}{2}L_t^\eta(0),$$

the inverse Itô formula (2.2) yields $\mathcal{M}^{v,\eta}(\rho) = -2\mathcal{G}^\eta + \mathcal{I}^\eta$, where

$$\mathcal{I}_t^\eta = \frac{2}{\eta}\left(H\left(X_t^{v,\eta}\right) - H(0)\right) - 2\rho(\mathbb{R})\left(Y_t^{v,\eta}\right)^+.$$

Thus the proof will be complete, as soon as we show that

$$(B_t, \sqrt{\eta}\mathcal{G}_t^\eta) \Rightarrow \left(\beta_t, \alpha_\rho\gamma_{L_t^{\beta^v}(0)}\right) \tag{4.7}$$

and

$$\sqrt{\eta}\mathcal{I}_t^\eta \to 0, \quad \mathbb{P}\text{-a.s. uniformly in } t \in [0.T]. \tag{4.8}$$

To show (4.7) we will use the ideas from the proof of (4.2). That is, first we note that $\mathcal{G}^\eta = \mathcal{J}^{\eta,1} + \mathcal{J}^{\eta,2} + \mathcal{J}^{\eta,3}$, where

$$\mathcal{J}_t^{\eta,1} := \eta^{-1}\int_0^t g\big(X_s^{v,\eta}\big)\, b\big(\mathrm{X}_s^{v,\eta}\big)\mathrm{d}s,$$

$$\mathcal{J}_t^{\eta,2} := \eta^{-1}\int_0^t g\big(X_s^{v,\eta}\big)\, v\, X_s^{v,\eta}\,\mathrm{d}s$$

$$\mathcal{J}_t^{\eta,3} := \int_0^t g\big(X_s^{v,\eta}\big)\,\mathrm{d}B_s.$$

Recall that the function g is integrable (see the proof of (4.2)). Hence, as b is bounded the function gb is integrable, and Proposition 2.1 yields

$$\eta^{p/2}\mathbb{E}\|\mathcal{J}^{\eta,1}\|_T^p \to 0.$$

As

$$\sup_{\eta>0}\mathbb{E}\,\|\eta^{-1}v\,X^{v,\eta}\|_T^{2p} < \infty,$$

the Hölder inequality and Proposition 2.1 yield

$$\eta^{p/2}\mathbb{E}\|\mathcal{J}^{\eta,2}\|_{\mathrm{T}}^p \to 0.$$

To show that, uniformly in $t\in[0, T]$,

$$\Big(B_t,\, \sqrt{\eta}\mathcal{J}_t^{\eta,3}\Big) \Rightarrow \Big(\beta_t,\, \alpha_f\gamma_{L_t^{\beta^v}}\Big),$$

we note that $\sqrt{\eta}\mathcal{J}_t^{\eta,3} = W^{\eta}_{\psi^\eta(t)}$, where $\psi^\eta(t) = \eta\int_0^t g^2\big(X_s^{v,\eta}\big)\,\mathrm{d}s$ and W is the DDS Brownian motion of $\sqrt{\eta}\mathcal{J}^{\eta,3}$. By Proposition 2.1,

$$\langle B, \sqrt{\eta}\mathcal{J}^{\eta,3}\rangle_t = \sqrt{\eta}\int_0^t g\big(X_s^{v,\eta}\big)\mathrm{d}s \to 0,$$

and by Proposition 2.2,

$$\langle\sqrt{\eta}\mathcal{J}^{\eta,3},\, \sqrt{\eta}\mathcal{J}^{\eta,2}\rangle_t = \psi_t^\eta \to \alpha_\rho^2 L_t^{B^v}(0).$$

Thus (4.7) follows from Revuz and Yor (1999, Theorem 2.3, Chapter XIII). To see (4.8), note that $\sqrt{\eta}\mathcal{I}_t^\eta = 2h_\eta\big(Y_t^{v,\eta}\big) - 2\big(\sqrt{\eta}\big)^{-1}H(0)$, where h_η is given by (4.5), and consequently (4.8) follows from (4.6).

5. Proof of Theorem 2.2

We will need the following lemma.

Lemma 5.1. *Let* (B^η) *be a family of two-sided Brownian motions, let* $\psi : [0, \infty) \to \mathbb{R}$ *be a deterministic function, and let* $(\psi^\eta(t),\, t \geqslant 0)$ *be a family of continuous processes such that, for any* $0 \leqslant T < \infty$, $\|\psi^\eta - \psi\|_T \to 0$ *in probability, as* $\eta \to \infty$. *Then there is a Brownian motion* β *such that* $B^\eta_{\psi^\eta(t)} \Rightarrow \beta_{\psi(t)}$ *uniformly in* t *on compact intervals.*

Proof. Clearly it is enough to show that for a fixed $T < \infty$, $\gamma_t^\eta := B^\eta_{\psi^\eta(t)} - B^\eta_{\psi(t)}$ converges in probability to 0, uniformly with respect to $t \in [0, T]$. This follows from the convergence of ψ^η to ψ, the uniform continuity of Brownian motion, and the fact that for all η, and ε, $\delta > 0$, one has

$$\mathbb{P}\{\|\gamma^\eta\|_T > \varepsilon\} \leqslant \mathbb{P}\{\|\gamma^\eta\|_T > \varepsilon;\ \|\psi^\eta - \psi\|_T \leqslant \delta\} + \mathbb{P}\{\|\psi^\eta - \psi\|_T > \delta\}$$

$$\leqslant \mathbb{P}\left\{\sup_{t,s:\ 0\leqslant t,s\leqslant T,\ |t-s|\leqslant\delta} |B_t^\eta - B_s^\eta| > \varepsilon\right\} + \mathbb{P}\{\|\psi^\eta - \psi\|_T > \delta\}.$$

□

Let L^η be the local time of $Y^\eta := \eta^{-1}X^{v,\eta}$. Write

$$V_t^\eta := \int_0^t \left(L_t^{B^v}\left(\eta^{-1}x\right) - L_t^{B^v}(0)\right)\rho_{\mathcal{M}}(\mathrm{d}x),$$

$$\mathcal{V}_t^\eta := \int_0^t \left(L_t^\eta\left(\eta^{-1}x\right) - L_t^\eta(0)\right)\rho_{\mathcal{M}}(\mathrm{d}x).$$

Then (see the proof of Proposition 2.2) we have

$$\eta A^{v,\eta}(\rho) = V_t^\eta + \rho_{\mathcal{M}}(\mathbb{R})L_t^{B^v}(0) + \eta A^{v,\eta}(\rho_{\mathcal{D}})$$

and

$$\eta \mathcal{A}^{v,\eta}(\rho) = V_t^\eta + \rho_{\mathcal{M}}(\mathbb{R})L_t^\eta(0) + \eta \mathcal{A}^{v,\eta}(\rho_{\mathcal{D}}).$$

Moreover (again see the proof of Proposition 2.2), we have $\mathbb{E}(\|V^\eta\|_T + \|\mathcal{V}^\eta\|_T) \to 0$.

Let $H \in W^{1,\infty}(\mathbb{R})$ be such that $\rho_{\mathcal{D}} = H''$. Let $F = H'$, and let

$$R_t^\eta = \int_0^t F\left(\eta B_s^v\right)\mathrm{d}B_s \quad \text{and } \mathcal{R}_t^\eta = \int_0^t F\left(X_s^{v,\eta}\right)\mathrm{d}B_s. \tag{5.1}$$

Let

$$N_t^{v,\eta} := \int_0^t F\left(\eta B_s^v\right)B_s^v\,\mathrm{d}s \qquad \text{and } \mathcal{N}_t^{v,\eta} := \eta^{-1}\int_0^t F\left(X_s^{v,\eta}\right)X_s^{v,\eta}\,\mathrm{d}s.$$

Finally, let $I_t^\eta = 2\eta^{-1}\left(H(\eta B_t^v) - H(0)\right)$, and let

$$\mathcal{I}^\eta = \frac{2}{\eta}\left(H(X_t^{v,\eta}) - H(0)\right) - \frac{2}{\eta}\int_0^t F\left(X_s^{v,\eta}\right)b\left(X_s^{v,\eta}\right)\mathrm{d}s.$$

By (2.1) and (2.2),

$$\eta A_t^{v,\eta}(\rho_{\mathcal{D}}) = I_t^\eta - 2vN_t^{v,\eta} - 2R_t^\eta$$

and

$$\eta \mathcal{A}_t^{v,\eta}(\rho_{\mathcal{D}}) = \mathcal{I}_t^\eta - 2\mathrm{v}\mathcal{N}_t^{v,\eta} - 2\mathcal{R}_t^\eta.$$

Obviously, as H, F and b are bounded, we have $\mathbb{E}(\|I^\eta\|_T + \|\mathcal{I}^\eta\|_T) \to 0$ for any $T < \infty$. Thus, as

$$L_t^{B^v}(0) = |B_t^v| - \int_0^t \operatorname{sgn} B_s^v \, \mathrm{d}B_s^v$$

and

$$L_t^\eta(0) = |Y^\eta| - \int_0^t \operatorname{sgn} Y_s^\eta \, \mathrm{d}Y_s^\eta,$$

the proof will be complete as soon as we show that

$$\mathbb{E}(\|N^{v,\eta}\|_T + \|\mathcal{N}^{v,\eta}\|_T) \to 0, \qquad \text{for any } T < \infty, \tag{5.2}$$

and that uniformly with respect to t on compact intervals,

$$\left(B_t^v, \int_0^t \operatorname{sgn} B_s^v \, \mathrm{d}B_s^v, R_t^\eta\right) \Rightarrow \left(\beta_t^v, \int_0^t \operatorname{sgn} \beta_s^v \, \mathrm{d}\beta_s^v, \sqrt{c_{\rho_\mathcal{D}}}\gamma_t\right) \tag{5.3}$$

and

$$\left(Y_t^\eta, \int_0^t \operatorname{sgn} Y_s^\eta \, \mathrm{d}Y_s^\eta, \mathcal{R}_t^\eta\right) \Rightarrow \left(\beta_t^v, \int_0^t \operatorname{sgn} \beta_s^v \, \mathrm{d}\beta_s^v, \sqrt{c_{\rho_\mathcal{D}}}\gamma_t\right), \tag{5.4}$$

where β and γ are independent Brownian motions.

To see (5.2), note that

$$N_t^{v,\eta} = A_t^{v,\eta}(F)B_t^v - \int_0^t A_s^{v,\eta}(F)\mathrm{d}B_s^v$$

and

$$\mathcal{N}_t^{v,\eta} = \eta^{-1}\mathcal{A}_t^{v,\eta}(F)X_t^{v,\eta} - \eta^{-1}\int_0^t \mathcal{A}_s^{v,\eta}(F)\mathrm{d}X_s^{v,\eta}.$$

Thus (5.2) follows easily from Proposition 2.1 as F has a bounded primitive.

It remains to check (5.3) and (5.4). Let W^η and $\mathcal{W}^\eta$ be DDS Brownian motions of R^η and $\mathcal{R}^\eta$, respectively. Then $R_t^\eta = W_{\psi^\eta(t)}^\eta$ and $\mathcal{R}_t^\eta = \mathcal{W}_{\varphi^\eta(t)}^\eta$, where

$$\psi^\eta(t) = \int_0^t F^2\big(\eta B_s^v\big)\,\mathrm{d}s \qquad \text{and } \varphi^\eta(t) = \int_0^t F^2\big(X_s^{v,\eta}\big)\,\mathrm{d}s.$$

Proposition 2.1 yields $\psi_t^\eta = \langle R^\eta\rangle_t \to c_{\rho_\mathcal{D}} t$ and $\varphi^\eta = \langle \mathcal{R}^\eta\rangle_\mathrm{t} \to \mathrm{c}_{\rho_\mathcal{D}}$ uniformly in t on bounded intervals. Thus, by Lemma 5.1, $R_t^\eta \Rightarrow \sqrt{c_{\rho_\mathcal{D}}}\gamma_t$ and $\mathcal{R}_\mathrm{t}^\eta \Rightarrow \sqrt{c_{\rho_\mathcal{D}}}\gamma_t$ uniformly in $t \in [0, T]$, where γ is a Brownian motion. Finally, by Proposition 2.1, we have

$$\langle B^v, R^\eta\rangle_t = \int_0^t F\big(\eta B_s^v\big)\,\mathrm{d}s \to 0 \quad \text{and} \quad \langle Y^\eta, \mathcal{R}^\eta\rangle_\mathrm{t} = \int_0^\mathrm{t} \mathrm{F}\big(\eta \mathrm{X}_\mathrm{s}^{\mathrm{v},\eta}\big)\,\mathrm{ds} \to 0,$$

and, again by Proposition 2.1, for $Z_t = \int_0^t \operatorname{sgn} B_s^v \, \mathrm{d}B_s^v$ and $\mathcal{Z}_t^\eta = \int_0^t \operatorname{sgn} Y_s^\eta \, \mathrm{d}Y_s^\eta$,

$$\langle Z, R^\eta\rangle_t = \int_0^t F\big(\eta B_s^v\big) \operatorname{sgn} \eta B_s^v \, \mathrm{d}s \to 0$$

and

$$\langle \mathcal{Z}^\eta, \mathcal{R}^\eta \rangle_t = \int_0^t F\left(\eta X_s^{v,\eta}\right) \operatorname{sgn} \eta X_s^{v,\eta} \, \mathrm{d}s \to 0,$$

because the function $\tilde{F}(x) = \operatorname{sgn} x \, F(x)$, $x \in \mathbb{R}$, has a bounded primitive. Thus the desired conclusion follows from Revuz and Yor (1999, Theorems 2.3 and 2.6, Chapter XIII).

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References

Flandoli, F. and Russo, F. (2002) Generalized calculus and SDEs with non-regular drift. *Stochastics Stochastics Rep.*, **72**, 11–54.

Le Gall, J.F. (1983) Applications du temps local aux équations différentielles stochastiques unidimensionnelles. In J. Azéma and M. Yor (eds), *Séminaire de Probabilités XVII*, Lecture Notes in Math. 986, pp. 15–31. Berlin: Springer-Verlag.

Nakao, S. (1972) On the pathwise uniqueness of one-dimensional stochastic differential equations. *Osaka J. Math.*, **9**, 513–518.

Perkins, E. (1982) Local time and pathwise uniqueness for stochastic differential equations. In J. Azéma and M. Yor (eds), *Séminaire de Probabilités XVI*, Lecture Notes in Math. 920, pp. 201–208. Berlin: Springer–Verlag.

Remillard, B. (2000) Large deviations for occupation time integrals of Brownian motion. In L.G. Gorostiza and G. Ivanoff (eds), *Stochastic Models, A Volume in Honor of Donald A. Dawson*, pp. 375–398. Providence, RI: American Mathematical Society.

Revuz, D. and Yor, M. (1999) *Continuous Martingales and Brownian Motion.* Berlin: Springer-Verlag.

Russo, F. and Oberguggenberger, M. (1999) White noise driven stochastic partial differential equations: triviality and non-triviality. In M. Grosser, G. Hörmann, M. Kunzinger and M. Oberguggenberger (eds), *Nonlinear Theory of Generalized Functions*, pp. 315–333, Research Notes in Math. Series. Boca Raton, FL.: Chapman & Hall/CRC.

Takeda, M. (1998) Asymptotic properties of generalized Feynman–Kac functionals. *Potential Anal.*, **5**, 261–291.

Takeda, M. (2003) Large deviation principle for additive functionals of Brownian motion corresponding to Kato measures. *Potential Anal.*, **19**, 51–67.

Yamada, T. (1986) On some limit theorems for occupation times of one dimensional Brownian motion and its continuous additive functionals locally of zero energy. *J. Math. Kyoto Univ.*, **26**, 309–322.

Yamada, T. (1996) Principal values of Brownian local times and their related topics. In N. Ikeda, S. Watanabe, M. Fukushima and H. Kunita (eds), *Itô's Stochastic Calculus and Probability Theory*, pp. 413–422. Tokyo: Springer–Verlag.

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