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On L^2-convergence of controlled branching processes with random control function

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The topic of this paper is Yanev's class of controlled branching processes with a random control function. We investigate the L^2-convergence of the suitably normed process to a non-degenerate limit. Necessary and sufficient conditions for this convergence are established under certain constraints on the means and variances of the control variables.

Keywords: controlled branching processes; L^2-convergence

1. Introduction

Consider, on the same probability space, two independent sets of non-negative integer-valued random variables, $\{X_{nj} : n = 0, 1, \ldots; j = 1, 2, \ldots\}$ and $\{\phi_n(k) : n = 0, 1, \ldots; k = 0, 1, \ldots,\}$ where the X_{nj} are independent and identically distributed, and the $\{\phi_n(k)\}_{k\geqslant 0}$ are independent stochastic processes with identical one-dimensional probability distributions.

The common probability law of the X_{nj} is $p_k := P(X_{01} = k)$, $k = 0, 1, \ldots,$ called the offspring distribution. The $\phi_n(k)$ are called control variables. The controlled branching process with random control function $\{Z_n\}_{n\geqslant 0}$ is then defined as

$$Z_0 = N, \qquad Z_{n+1} = \sum_{j=1}^{\phi_n(Z_n)} X_{nj}, \qquad n = 0, 1, \ldots,$$

where N is a positive integer and we adopt the convention that the empty sum is zero.

Intuitively, X_{nj} is the number of offspring of the jth individual in the nth generation and Z_n represents the population size in the nth generation. Each individual, independently of all others, gives rise to new individuals with the same probability distribution but, unlike the standard Galton–Watson process, the population size in the $(n+1)$th generation is determined by an additional random process. This branching model could describe the evolution of populations in which, for various reasons (environmental, social, etc.), a random mechanism establishes the number of progenitors who participate in each generation. In addition to its theoretical interest, this process therefore has a major practical dimension because of potential applications in such diverse fields as epidemiology, genetics, nuclear physics, demography, and actuarial mathematics.

This stochastic model, introduced by Yanev (1975), is a homogeneous Markov chain. Its

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extinction probability, asymptotic behaviour and inferential theory have been investigated by Bruss (1980), Nakagawa (1994), Dion and Essebbar (1995), and González *et al*. (2002; 2003). In the latter two papers, the present authors studied the process under the condition of asymptotically linear growth of the expectation of the control variables, providing some results on the extinction probability and limiting behaviour to a finite non-degenerate random variable. In particular, conditions for geometric growth of the population were established using almost sure or L^1-convergence. The present paper continues this research by focusing on L^2-convergence, as is usual in the methodology of branching processes. Section 2 presents the basic notation, working assumptions and some preliminary results. Sections 3 and 4 determine necessary and sufficient conditions, respectively, for the L^2-convergence of the suitably normed process to a non-degenerate limit. These conditions are obtained using a method analogous to that introduced by Klebaner (1984) for branching processes dependent on population size.

2. Preliminary

We assume that $P(\phi_0(0) = 0) = 1$ and that at least one of the following conditions holds:

(a) $p_0 > 0$;
(b) $P(\phi_0(k) = 0) > 0$, for $k = 1, 2, \ldots$.

Consequently, zero is an absorbing state and, as proved by Yanev (1975), the positive integers are transient states. Hence, from Markov chain theory,

$$P\Big(\lim_{n\to\infty} Z_n = 0\Big) + P\Big(\lim_{n\to\infty} Z_n = \infty\Big) = 1. \tag{2.1}$$

Let $m := \mathrm{E}[X_{01}]$ be the mean and $\sigma^2 := \mathrm{var}[X_{01}]$ the variance of the offspring distribution, and, for $k = 0, 1, \ldots,$ let $\varepsilon(k) := \mathrm{E}[\phi_0(k)]$ be the mean and $\nu(k) := \mathrm{var}[\phi_0(k)]$ the variance of the control variables. All these parameters are assumed finite. We assume that $\lim_{k\to\infty} \varepsilon(k)k^{-1} =: \tau < \infty$. We consider the sequence $\{\tau(k)\}_{k\geqslant 0}$, where $\tau(0) := 0$ and $\tau(k) := \tau - \varepsilon(k)k^{-1}$, $k = 1, 2, \ldots$. Finally, for $N = 1, 2, \ldots,$ let $q_N := P(Z_n \to 0 | Z_0 = N)$ be the extinction probability when initially there are N individuals in the population. Our purpose is to study the L^2-convergence of the sequence $\{W_n\}_{n\geqslant 0}$, where $W_n := Z_n(\tau m)^{-n}$, with $\tau m > 1$. To this end, it is necessary to use some previously known results. In brief, González *et al*. (2003) proved that if either (a) $\{\tau(k)\}_{k\geqslant 1}$ is non-increasing, or (b) $\{\tau(k)\}_{k\geqslant 1}$ is non-decreasing and $\{\mathrm{E}[W_n]\}_{n\geqslant 0}$ is bounded, then $\{W_n\}_{n\geqslant 0}$ converges almost surely to a non-negative and finite random variable W. Moreover, if $P(W > 0) > 0$, then

$$P(w \in [W > 0] : \sum_{k=0}^{\infty} |\tau(Z_k(w))| < \infty) = 1.$$

In order to extend the class of controlled processes such that $\{W_n\}_{n\geqslant 0}$ has an almost sure limit, González *et al*. (2003) assume that $\sum_{k=1}^{\infty} |\tau(k)|k^{-1} < \infty$ and, when $\{|\tau(k)|\}_{k\geqslant 1}$ is non-increasing, prove the existence of this almost sure limit. Furthermore, for all $N \geqslant 1$ such that $q_N < 1$,

$$0 < \lim_{n\to\infty} \mathrm{E}[W_n | Z_0 = N] < \infty. \tag{2.2}$$

In that paper, it is shown that, for processes such that $\{\tau(k)\}_{k\geqslant 1}$ is monotone, $\sum_{k=1}^{\infty}|\tau(k)|k^{-1} < \infty$ is a necessary condition for the L^1-convergence of $\{W_n\}_{n\geqslant 0}$ to a non-degenerate random variable. Notice that, in this context, the L^r-convergence of $\{W_n\}_{n\geqslant 0}$ ($r \geqslant 1$) implies almost sure convergence, and both limits are, almost surely, the same.

We now give some preliminary results.

Proposition 1. *For* $n = 0, 1, \ldots,$

$$\mathrm{E}[W_{n+1}^2] = Z_0^2 + \tau^{-2}\sum_{k=0}^{n}\mathrm{E}[W_k^2(\nu(Z_k)Z_k^{-2} + \tau(Z_k)^2 - 2\tau\tau(Z_k))]$$

$$+ \sigma^2\sum_{k=0}^{n}\mathrm{E}[\varepsilon(Z_k)](\tau m)^{-2(k+1)}.$$

Proof. Let $\mathcal{F}_n := \sigma(Z_0, \ldots, Z_n)$, $n = 0, 1, \ldots$. It follows that

$$\begin{aligned}\mathrm{E}[Z_{n+1}^2] &= \mathrm{E}[\mathrm{E}[Z_{n+1}^2|\mathcal{F}_n]] = \mathrm{E}[\varepsilon(Z_n)\sigma^2 + m^2\nu(Z_n) + \varepsilon(Z_n)^2m^2]\\ &= \mathrm{E}[\varepsilon(Z_n)\sigma^2 + m^2\nu(Z_n) + Z_n^2(\tau - \tau(Z_n))^2m^2]\\ &= \mathrm{E}\left[Z_n^2\tau^2m^2 + Z_n^2m^2\left(\frac{\nu(Z_n)}{Z_n^2} + \tau(Z_n)^2 - 2\tau\tau(Z_n)\right) + \sigma^2\varepsilon(Z_n)\right],\end{aligned}$$

and therefore

$$\mathrm{E}[W_{n+1}^2] = \mathrm{E}[W_n^2] + \frac{1}{\tau^2}\mathrm{E}\left[W_n^2\left(\frac{\nu(Z_n)}{Z_n^2} + \tau(Z_n)^2 - 2\tau\tau(Z_n)\right)\right] + \sigma^2\frac{\mathrm{E}[\varepsilon(Z_n)]}{(\tau m)^{2(n+1)}}. \tag{2.3}$$

By iteration, the proof is concluded. □

Proposition 2. *Assume that:*

(i) $\{|\tau(k)|\}_{k\geqslant 1}$ *is non-increasing;*
(ii) $\sum_{k=1}^{\infty}|\tau(k)|k^{-1} < \infty$.

Then, for all $N \geqslant 1$ *such that* $q_N < 1$,

$$\sum_{n=0}^{\infty}\mathrm{E}[\varepsilon(Z_n)](\tau m)^{-2(n+1)} < \infty.$$

Proof. Since $\{k^{-1}\varepsilon(k)\}_{k\geqslant 1}$ is convergent, it is bounded. Hence, there exists $M > 0$ such that $k^{-1}\varepsilon(k) \leqslant M$, $k = 1, 2, \ldots,$ and we deduce that $\mathrm{E}[\varepsilon(Z_n)] \leqslant M\mathrm{E}[Z_n]$, $n = 0, 1 \ldots$. By assumptions (i) and (ii), (2.2) holds, and consequently there exists $M' > 0$ such that $\mathrm{E}[W_n] \leqslant M'$, $n = 0, 1, \ldots$. Hence, using the fact that $\tau m > 1$, we obtain the inequality

$$\sum_{n=0}^{\infty} \mathrm{E}[\varepsilon(Z_n)](\tau m)^{-2(n+1)} \leqslant MM' \sum_{n=0}^{\infty} (\tau m)^{-(n+2)} < \infty.$$

□

3. Necessary conditions

We now find necessary conditions for the L^2-convergence of $\{W_n\}_{n\geqslant 0}$. For simplicity, let us introduce the sequence $\{\xi(k)\}_{k\geqslant 1}$ where

$$\xi(k) := \nu(k)k^{-2} + \tau(k)^2 - 2\tau\tau(k).$$

Theorem 1. *A necessary condition for the L^2-convergence of $\{W_n\}_{n\geqslant 0}$ is*

$$\inf_{k\geqslant 1} \xi(k) \leqslant 0.$$

Proof. Assume the existence of $\alpha > 0$ such that $\xi(n) \geqslant \alpha$, $n = 1, 2, \ldots$. By (2.3), it follows that, for $n = 0, 1, \ldots,$

$$\begin{aligned} \mathrm{E}[W_{n+1}^2] &\geqslant \mathrm{E}[W_n^2](1 + \alpha\tau^{-2}) + \sigma^2 \mathrm{E}[\varepsilon(Z_n)](\tau m)^{-2(n+1)} \\ &\geqslant \mathrm{E}[W_n^2](1 + \alpha\tau^{-2}) \geqslant Z_0^2(1 + \alpha\tau^{-2})^{n+1}, \end{aligned}$$

and hence $\lim_{n\to\infty} \mathrm{E}[W_n^2] = \infty$, thus contradicting the convergence of $\{W_n\}_{n\geqslant 0}$ in L^2. □

The following result provides a necessary condition for the L^2- convergence of $\{W_n\}_{n\geqslant 0}$ to a non-degenerate limit.

Theorem 2. *Assume that:*

(i) $\{\tau(k)\}_{k\geqslant 1}$ *is monotonic;*
(ii) *there exists a positive function* $\tilde{\nu}(x)$ *on* $[0, \infty)$ *such that* $\tilde{\nu}(k) = \nu(k)$*,* $k = 0, 1, \ldots,$ *and* $\tilde{\nu}(x)x^{-2}$ *is non-increasing;*
(iii) $\{\xi(k)\}_{k\geqslant 1}$ *is such that its elements have the same sign.*

Then, if $\{W_n\}_{n\geqslant 0}$ *converges in* L^2 *to a non-degenerate limit,*

$$\sum_{k=1}^{\infty} \nu(k)k^{-3} < \infty.$$

Proof. Assume that $\{W_n\}_{n\geqslant 0}$ converges in L^2 to W with $P(W > 0) > 0$. Taking the results of Section 2 into account, it follows that

$$P\left(\omega \in [W > 0] : \sum_{k=0}^{\infty} |\tau(Z_k(\omega))| < \infty\right) = 1 \quad \text{and} \quad \sum_{k=1}^{\infty} |\tau(k)|k^{-1} < \infty. \tag{3.1}$$

Moreover, since $\{E[W_n^2]\}_{n\geq 0}$ is bounded, by Propositions 1 and 2,

$$\sum_{k=0}^{\infty} E[W_k^2 \xi(Z_k)] \text{ converges.}$$

Therefore, from (iii), it follows that $\sum_{k=0}^{\infty} E[W_k^2 |\xi(Z_k)|] < \infty$.

From (3.1), using (i) and (ii), and applying similar reasoning to that used by Klebaner (1984), it can be shown that on $[W > 0]$,

$$\sum_{k=0}^{\infty} \tilde{\nu}((\tau m)^k \tilde{W})(\tau m)^{-2k} \tilde{W}^{-2} < \infty, \qquad \text{almost surely,} \tag{3.2}$$

where $\tilde{W} := \sup_{n\geq 0} W_n < \infty$. Finally, since $\tilde{\nu}(x)x^{-2}$ is a positive non-increasing function and $P(W > 0) > 0$, using the integral test, (3.2) is equivalent to $\sum_{k=1}^{\infty} \tilde{\nu}(n)k^{-3} = \sum_{k=1}^{\infty} \nu(k)k^{-3} < \infty$. □

Corollary 1. *Assume that:*

(i) $\{\tau(k)\}_{k\geq 1}$ *is monotonic;*
(ii) $\{\nu(k)k^{-2}\}_{k\geq 1}$ *is non-increasing,*
(iii) $\{\xi(k)\}_{k\geq 1}$ *is such that its elements have the same sign.*

Then a necessary condition for the L^2-convergence of $\{W_n\}_{n\geq 0}$ to a non-degenerate limit is

$$\sum_{k=1}^{\infty} \nu(k)k^{-3} < \infty.$$

Proof. From Theorem 2, it suffices to verify the existence of a positive function $\tilde{\nu}(x)$ on $[0, \infty)$ such that $\tilde{\nu}(n) = \nu(n)$, $n = 0, 1, \ldots,$ and $\tilde{\nu}(x)x^{-2}$ is non-increasing. Straightforward calculation shows that the function on $[0, \infty)$,

$$\tilde{\nu}(x) := x^2(\nu(1)\mathbf{1}_{[0\leq x<1]} + \nu(\lfloor x \rfloor)\lfloor x \rfloor^{-2}\mathbf{1}_{[x\geq 1]})$$

with $\lfloor x \rfloor$ denoting the integer part of x, satisfies these requirements. □

4. Sufficient conditions

In this section we investigate sufficient conditions for L^2-convergence of $\{W_n\}_{n\geq 0}$ to a non-degenerate limit. Throughout, we shall consider the sequence $\{|\tau(k)|\}_{k\geq 1}$ to be non-increasing and $P(Z_n \to \infty) > 0$.

Theorem 3. *If there exist positive functions $\tilde{\tau}(x)$ and $\tilde{\nu}(x)$ defined on $[0, \infty)$, with $\tilde{\tau}(x)$ non-increasing, such that*

(i) $\tilde{\tau}(k) \geq |\tau(k)|$, $\tilde{\nu}(k) \geq \nu(k)$, $k = 1, 2, \ldots,$
(ii) $g(x) := \tilde{\nu}(x)x^{-2} + \tilde{\tau}(x)^2 + 2\tau\tilde{\tau}(x)$, $x \in (0, \infty)$, *is non-increasing, and* $h(x) := xg(x^{1/2})$ *and* $x\tilde{\tau}^2(x^{1/2})$ *are concave on* $(0, \infty)$,

(iii) $\sum_{k=1}^{\infty}\tilde{\tau}(k)k^{-1} < \infty$, $\sum_{k=1}^{\infty}\tilde{\nu}(k)k^{-3} < \infty$,

then $\{W_n\}_{n\geqslant 0}$ *converges in* L^2 *to a finite non-degenerate variable.*

Proof. The proof is divided into two parts. First let us prove that $\{\mathrm{E}[W_n^2]\}_{n\geqslant 0}$ is a bounded sequence, and then that $\{W_n\}_{n\geqslant 0}$ converges in L^2.

By (2.3) and (i), we deduce that

$$\begin{aligned}\mathrm{E}[W_{n+1}^2] &\leqslant \mathrm{E}[W_n^2] + \frac{1}{\tau^2}\mathrm{E}\left[W_n^2\left(\frac{\tilde{\nu}(Z_n)}{Z_n^2} + 2\tau\tilde{\tau}(Z_n) + \tilde{\tau}^2(Z_n)\right)\right] + \sigma^2\frac{\mathrm{E}[\varepsilon(Z_n)]}{(\tau m)^{2(n+1)}}\\ &= \mathrm{E}[W_n^2] + \mathrm{E}[W_n^2 g(Z_n)]\tau^{-2} + \sigma^2\mathrm{E}[\varepsilon(Z_n)](\tau m)^{-2(n+1)}\\ &= \mathrm{E}[W_n^2] + \mathrm{E}[h(Z_n^2)](\tau m)^{-2n}\tau^{-2} + \sigma^2\mathrm{E}[\varepsilon(Z_n)](\tau m)^{-2(n+1)}.\end{aligned}$$

Now, since $h(x)$ is concave and $g(x)$ non-increasing, we obtain, by Jensen's inequality,

$$\mathrm{E}[h(Z_n^2)](\tau m)^{-2n} \leqslant \mathrm{E}[W_n^2]g(\mathrm{E}[Z_n]).$$

From (2.2), there exists $\delta > 0$ such that $\mathrm{E}[Z_n] \geqslant \delta(\tau m)^n$, $n = 0, 1, \ldots,$ and we have $g(\mathrm{E}[Z_n]) \leqslant g(\delta(\tau m)^n)$. Consequently,

$$\mathrm{E}[W_{n+1}^2] \leqslant \mathrm{E}[W_n^2]\big(1 + g(\delta(\tau m)^n)\tau^{-2}\big) + \sigma^2\mathrm{E}[\varepsilon(Z_n)](\tau m)^{-2(n+1)}$$

and, for $n = 1, 2, \ldots,$ it follows that,

$$\begin{aligned}\mathrm{E}[W_{n+1}^2] \leqslant\ & Z_0^2\prod_{k=0}^{n}\big(1 + g(\delta(\tau m)^k)\tau^{-2}\big)\\ &+ \sigma^2\sum_{k=0}^{n}\mathrm{E}[\varepsilon(Z_k)](\tau m)^{-2(k+1)}\prod_{s=k+1}^{n}\big(1 + g(\delta(\tau m)^s\tau^{-2})\big).\end{aligned}$$

Considering Proposition 1, to conclude that $\{\mathrm{E}[W_n^2]\}_{n\geqslant 0}$ is a bounded sequence it only remains to verify that

$$\prod_{k=0}^{\infty}(1 + m^2(\tau m)^{-2}g(\delta(\tau m)^k)) < \infty, \tag{4.1}$$

or equivalently $\sum_{k=0}^{\infty}g(\delta(\tau m)^k) < \infty$. Since $g(x)$ is non-increasing, (4.1) holds if

$$\sum_{k=1}^{\infty}g(k)k^{-1} = \sum_{k=1}^{\infty}\big(\tilde{\nu}(k)k^{-3} + \tilde{\tau}^2(k)k^{-1} + 2\tau\tilde{\tau}(k)k^{-1}\big) < \infty,$$

which is true by (iii) and the fact that $\tilde{\tau}(x)$ is non-increasing.

We now prove that $\{W_n\}_{n\geqslant 0}$ converges in L^2. Let us consider the variables $Y_n := W_n + T_n$, $n = 0, 1, \ldots,$ where $T_0 := 0$ and $T_n := \tau^{-1}\sum_{k=0}^{n-1}W_k\tau(Z_k)$, $n = 0, 1, \ldots.$ It can be proved that $\{Y_n\}_{n\geqslant 0}$ is a martingale with respect to $\{\mathcal{F}_n\}_{n\geqslant 0}$, with $\mathcal{F}_n = \sigma(Z_0, \ldots, Z_n)$.

Since $x\tilde{\tau}^2(x^{1/2})$ is concave, $\tilde{\tau}(x)$ is non-increasing, and $\{E[W_n^2]\}_{n\geqslant 0}$ is bounded, and using (2.2), we have, for some constant K,

$$\left\| \sum_{k=0}^{\infty} W_k \tau(Z_k) \right\|_2 \leqslant K \sum_{k=0}^{\infty} \tilde{\tau}(\delta(\tau m)^k),$$

where $\|X\|_2 := E[|X|^2]^{1/2}$. Since

$$\sum_{k=0}^{\infty} \tilde{\tau}(\delta(\tau m)^k) < \infty \text{ is equivalent to } \sum_{k=1}^{\infty} \tilde{\tau}(k)k^{-1} < \infty,$$

we deduce that $\{T_n\}_{n\geqslant 0}$ converges in L^2.

Since $\{W_n\}_{n\geqslant 0}$ is bounded in L^2, it follows that $\sup_{n\geqslant 0}\|Y_n\|_2 < \infty$. Hence, from the L^2-bounded martingale convergence theorem, we deduce that $\{Y_n\}_{n\geqslant 0}$ converges in L^2 to a finite variable, and therefore $\{W_n\}_{n\geqslant 0}$ is also L^2-convergent to a finite variable. Finally, taking into account that L^2-convergence implies L^1-convergence, and that both limits are the same almost surely, by (2.2) we obtain the non-degeneracy of such a limit. $\square$

Corollary 2. *Assume that:*

(i) $\{\nu(k)k^{-2}\}_{k\geqslant 1}$ *is non-increasing;*
(ii) $\sum_{k=1}^{\infty}|\tau(k)|k^{-1} < \infty$;
(iii) $\sum_{k=1}^{\infty}\nu(k)k^{-3} < \infty$.

Then $\{W_n\}_{n\geqslant 0}$ converges in L^2 to a finite non-degenerate random variable.

Proof. From Theorem 3, it is sufficient to show that there exist positive functions $\tilde{\tau}(x)$ and $\tilde{\nu}(x)$ on $[0, \infty)$ such that:

(a) $\tilde{\tau}(k) \geqslant |\tau(k)|$, $\tilde{\nu}(k) \geqslant \nu(k)$, $k = 1, 2, \ldots$;
(b) $\tilde{\tau}(x)$ and $\tilde{\nu}(x)x^{-2}$ are non-increasing, and $x\tilde{\tau}(x^{1/2})$, $x\tilde{\tau}^2(x^{1/2})$ and $\tilde{\nu}(x^{1/2})$ are concave on $(0, \infty)$;
(c) $\sum_{k=1}^{\infty}\tilde{\tau}(k)k^{-1} < \infty$, $\sum_{k=1}^{\infty}\tilde{\nu}(k)k^{-3} < \infty$.

Let us consider the functions

$$\tilde{\tau}(x) := |\tau(1)|\mathbf{1}_{[0\leqslant x<1]} + x^{-1}\left(|\tau(1)| + \int_1^x |\tau(\lfloor t \rfloor)|\mathrm{d}t\right)\mathbf{1}_{[x\geqslant 1]}$$

and

$$\tilde{\nu}(x) := x^2\nu(1)\mathbf{1}_{[0\leqslant x<1]} + x\left(\nu(1) + \int_1^x (\lfloor t \rfloor)^{-2}\nu(\lfloor t \rfloor)\mathrm{d}t\right)\mathbf{1}_{[x\geqslant 1]}$$

on $[0, \infty)$. It is clear that $\tilde{\tau}(k) \geqslant |\tau(k)|$, $\tilde{\nu}(k) \geqslant \nu(k)$, $k = 1, 2, \ldots$. Moreover, $\tilde{\tau}(x)$ and $\tilde{\nu}(x)x^{-2}$ are non-increasing. It is simple to verify that on $(1, \infty)$,

$$x\tilde{\tau}(x^{1/2}) = x^{1/2}\left(|\tau(1)| + \int_1^{x^{1/2}} |\tau(\lfloor t\rfloor)|\mathrm{d}t\right),$$

$$\tilde{\nu}(x^{1/2}) = x^{1/2}\left(\nu(1) + \int_1^{x^{1/2}} (\lfloor t\rfloor)^{-2}\nu(\lfloor t\rfloor)\mathrm{d}t\right).$$

Consequently, taking into account Lemma A.1 given in the Appendix, we deduce the concavity of $x\tilde{\tau}(x^{1/2})$, $x\tilde{\tau}^2(x^{1/2})$, and $\tilde{\nu}(x^{1/2})$ on $(1, \infty)$. Since, for $x \in (0, 1)$, $\tilde{\tau}(x^{1/2}) = |\tau(1)|$ and $\tilde{\nu}(x^{1/2}) = x\nu(1)$, (b) holds.

Finally, to prove (c), it is sufficient to verify that

$$\int_1^{\infty} \tilde{\tau}(x)x^{-1}\,\mathrm{d}x < \infty \quad \text{and} \quad \int_1^{\infty} \tilde{\nu}(x)x^{-3}\,\mathrm{d}x < \infty.$$

Since

$$\int_1^{\infty} \tilde{\tau}(x)x^{-1}\,\mathrm{d}x = |\tau(1)| + \int_1^{\infty} x^{-2}\int_1^{x} |\tau(\lfloor t\rfloor)|\,\mathrm{d}t\,\mathrm{d}x,$$

and using the fact that $\sum_{k=1}^{\infty} |\tau(k)|k^{-1} < \infty$, we obtain

$$\int_1^{\infty} x^{-2}\int_1^{x} |\tau(\lfloor t\rfloor)|\mathrm{d}t\,\mathrm{d}x = \int_1^{\infty} t^{-1}|\tau(\lfloor t\rfloor)|\mathrm{d}t < \infty.$$

Analogously,

$$\int_1^{\infty} \tilde{\nu}(x)x^{-3}\,\mathrm{d}x = \nu(1) + \int_1^{\infty} t^{-1}(\lfloor t\rfloor)^{-2}\nu(\lfloor t\rfloor)\mathrm{d}t,$$

and the last integral is finite since $\sum_{k=1}^{\infty} \nu(k)k^{-3} < \infty$. □

Remark 1. A condition which guarantees a positive probability for the non-extinction of $\{Z_n\}_{n\geqslant 0}$ is $\nu(k) = O(k)$ as $k \to \infty$ (see Theorem 2 of González *et al*. 2002). Under this hypothesis, condition (iii) in Corollary 2 is a trivial consequence and can be dropped.

Remark 2. A result concerning the L^{α}-convergence, $1 < \alpha \leqslant 2$, of a controlled branching process with random control was provided by Nakagawa (1994). For $\alpha = 2$, this result is improved by Corollary 2. Briefly, whereas Nakagawa's result requires the convergence of $\sum_{k=1}^{\infty} \nu(k)k^{-2}$, we consider the convergence of $\sum_{k=1}^{\infty} \nu(k)k^{-3}$. Also notice that Corollary 2 remains true if $\tau(n) = O(a_n)$ and $\nu(n) = O(b_n)$, with $\{a_n\}_{n\geqslant 0}$ and $\{b_n\}_{n\geqslant 0}$ being sequences of real numbers satisfying the conditions of growth and summability assumed for $\{\tau(n)\}_{n\geqslant 0}$ and $\{\nu(n)\}_{n\geqslant 0}$, respectively. An example is $\tau(n) = O(\log^{-\alpha} n)$ and $\nu(n) = O(n^2 \log^{-\beta} n)$, $\alpha, \beta > 1$.

Appendix: Lemma

Lemma A.1. *Let* $\{a_n\}_{n\geqslant 1}$ *be a non-increasing sequence of positive real numbers. Then the functions on* $(1, \infty)$ *given by*

$$g(x) := x^{1/2}\left(a_1 + \int_1^{x^{1/2}} a_{\lfloor t\rfloor}\,\mathrm{d}t\right) \quad \text{and} \quad h(x) := \left(a_1 + \int_1^{x^{1/2}} a_{\lfloor t\rfloor}\,\mathrm{d}t\right)^2$$

are concave on $(1, \infty)$.

Proof. Let us prove that $g(x)$ is concave. Note that

$$g(x) = xa_1\mathbf{1}_{[1<x<4]} + \sum_{n=2}^{\infty} x^{1/2}\left(a_1 + \sum_{k=1}^{n-1} a_k + a_n(x^{1/2} - n)\right)\mathbf{1}_{[n^2\leqslant x<(n+1)^2]}.$$

It is clear that $g(x)$ is concave on $(1, 4)$. For $x \in (n^2, (n+1)^2)$, $n = 2, 3, \ldots,$ taking into account that $a_n \leqslant a_k$, $k = 1, \ldots, n-1$, we deduce that

$$g''(x) = (4x^{3/2})^{-1}\left(na_n - \left(a_1 + \sum_{k=1}^{n-1} a_k\right)\right) \leqslant 0.$$

In order to conclude that $g(x)$ is concave let us show that $g'(n^2-) \geqslant g'(n^2+)$ for $n = 2, 3, \ldots,$ where $g'(n^2-)$ and $g'(n^2+)$ denote the left-hand and right-hand derivatives of $g(x)$ at $x = n^2$, respectively. In effect,

$$\begin{aligned} g'(n^2-) &= (2n)^{-1}\left(a_1 + \sum_{k=1}^{n-1} a_k\right) + 2^{-1}a_{n-1} \\ &\geqslant (2n)^{-1}\left(a_1 + \sum_{k=1}^{n-1} a_k\right) + 2^{-1}a_n = g'(n^2+). \end{aligned}$$

The proof that $h(x)$ is concave is similar. □

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