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Solutions of stochastic partial differential equations considered as Dirichlet processes

LAURENT DENIS

Département de Mathématiques, Laboratoire de Statistiques et Processus E.A. 3263, Université du Maine, Avenue O. Messiaen, 72085 Le Mans Cedex 9, France. E-mail: ldenis@univ-lemans.fr

We consider the parabolic stochastic partial differential equation

$$u(t, x) = \Phi(x) + \int_0^t Lu(s, x) + f(s, x, u(s, x), Du(s, x))\mathrm{d}s$$

$$+ \int_0^t g_i(s, x, u(s, x), Du(s, x))\,\mathrm{d}B_s^i,$$

where f and g are supposed to be Lipschitzian and L is a self-adjoint operator associated with a Dirichlet form defined on a finite- or infinite-dimensional space. We prove that it admits a unique solution which is a Dirichlet process and, thanks to Itô formula for Dirichlet processes, we prove a comparison theorem.

Keywords: comparison theorem; Dirichlet processes; stochastic partial differential equation

1. Introduction

To illustrate the results which are proved in this paper, consider first the standard case. Let $O \subset \mathbb{R}^k$ be a bounded domain with boundary ∂O. We consider the nonlinear stochastic partial differential equation (SPDE)

$$\frac{\partial}{\partial t} u(t, x) = \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j} u(t, x)\right) + f(t, x, u(t, x), \Delta u(t, x)\sigma(x))$$

$$+ g_i(t, x, u(t, x), \nabla u(t, x)\sigma(x))\frac{\mathrm{d}B_t^i}{\mathrm{d}t}, \tag{1.1}$$

with Dirichlet boundary conditions $u(t, x) = 0$ for all $t > 0$, $x \in \partial O$ – as will be explained later, one can consider any other boundary conditions (von Neumann, mixed, etc.). The initial condition is

$$u(0, x) = \Phi(x) \in L^2(O).$$

B is a d-dimensional Brownian motion, $a = \sigma\sigma^*$ is a symmetric measurable matrix such that the bilinear form

$$\forall u, v \in C_0^\infty(O), \qquad e(u, v) = \int_O a_{ij}(x) \frac{\partial}{\partial x_j} u(x) \frac{\partial}{\partial x_i} v(x)\, \mathrm{d}x$$

is closable, where $C_0^\infty(O)$ denotes the set of infinitely derivable functions with compact support in O. This is the case if, for example, a is assumed to be strictly elliptic or if,

$$\forall i, j, l \in \{1, \ldots, k\}^2, \qquad \frac{\partial a_{i,j}}{\partial x_l} \in L^2_{\mathrm{loc}}(O);$$

see Fukushima *et al.* (1994). The coefficients $f(t, x, y, z)$, $g_i(t, x, y, z)$ are assumed to be Lipschitz in y and z; moreover, we suppose that $g(t, x, y, z) = (g_1(t, x, y, z), \ldots, g_d(t, x, y, z))$ is a contraction with respect to the variable z.

We prove existence and uniqueness in the weak sense of equation (1.1) and we prove that the solution $t \to u(t, \cdot)$, considered as an $L^2(O)$-valued process, is a (continuous) Dirichlet process, i.e. it may be decomposed as

$$u_t = M_t + A_t,$$

where M is a martingale and A a zero quadratic variation process. Thus as is now well known (see Bertoin 1986; 1987; Föllmer 1981a), the process u satisfies an Itô formula that we use to establish a comparison theorem. In this particular case, we have the following theorem:

Theorem 1.1. *Let $\tilde{f}$ be a coefficient which satisfies the same hypotheses as f, and $\tilde{\Phi} \in L^2(O)$. Let $\tilde{u}$ be the solution of*

$$\frac{\partial}{\partial t} \tilde{u}(t, x) = \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \tilde{u}(t, x) \right) + \tilde{f}(t, x, \tilde{u}(t, x), \Delta \tilde{u}(t, x) \sigma(x))$$

$$+ g_i(t, x, \tilde{u}(t, x), \nabla \tilde{u}(t, x) \sigma(x)) \frac{\mathrm{d}B_t^i}{\mathrm{d}t},$$

with initial condition $\tilde{u}_0 = \tilde{\Phi}$. Assume that $\Phi \leqslant \tilde{\Phi}$ almost everywhere, and that

$$f(t, x, u_t(x), \nabla u(t, x) \sigma(x)) \leqslant \tilde{f}(t, x, u_t(x), \nabla u(t, x) \sigma(x)) \qquad \mathrm{d}t \otimes \mathrm{d}x \otimes \mathrm{d}P\text{-}a.e.;$$

then

$$\forall t \geqslant 0, \qquad u_t \leqslant \tilde{u}_t \quad \mathrm{d}x \otimes \mathrm{d}P\text{-}a.e.$$

Since we use analytical methods, especially the theory of semigroups, we deal in fact with a much more general class of SPDEs. More precisely, we solve

$$u(t, x) = \Phi(x) + \int_0^t Lu(s, x) + f(s, x, u(s, x), Du(s, x)) \mathrm{d}s$$

$$+ \int_0^t g_i(s, x, u(s, x), Du(s, x))\, \mathrm{d}B_s^i,$$

where L is a non-positive (i.e. for all $f \in \mathrm{Dom}(L)$, $(Lf, f) \leqslant 0$) self-adjoint sub-Markovian operator associated with a symmetric Dirichlet form defined on some space $L^2(W, m(\mathrm{d}x))$ and which admits a gradient D. The previous standard case corresponds to the case where $W = \mathrm{O},\ m(\mathrm{d}x) = \mathrm{d}x$,

$$Lu(x) = \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}u(x)\right),$$

and $Du(x) = \nabla u(x)\sigma(x)$.

Note that there are many other examples: for instance, one can consider the same second-order differential operator with von Neumann conditions. For simplicity, one can assume that a is strictly elliptic; then it is well known that it defines a Dirichlet form (see Fukushima *et al*. 1994), and so all our results are valid in this case.

If instead of the Lebesgue measure we consider a measure m which admits a density $m(\mathrm{d}x) = p(x)\,\mathrm{d}x$, this allows us to consider the case of second-order differential operators of the form

$$Lu(x) = \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}u(x)\right) - b_i(x)\frac{\partial}{\partial x_i}u(x).$$

We only have to assume that L is a self-adjoint operator on $L^2(O, m)$. For example, the Ornstein–Uhlenbeck operator on $\mathbb{R}^k$ corresponds to the case where $W = \mathbb{R}^k$, $m(\mathrm{d}x) = \mathrm{e} - |x|^2/2\,\mathrm{d}x,\ Du = \nabla u$ and

$$Lu = \frac{1}{2}\Delta u - x\cdot\nabla u.$$

One can also consider the infinite-dimensional Ornstein–Uhlenbeck structure: $W = C_0([0, +\infty[; \mathbb{R}^k)$ is the Wiener space endowed with the Wiener measure m, L is the infinite-dimensional Ornstein–Uhlenbeck operator and D is the Malliavin operator.

SPDEs have been intensively studied in the recent past, and the literature is extensive. Semigroup methods are developed in Da Prato (1998), Da Prato and Zabczyk (1992) and Rozovski (1990). Comparison theorems for SPDEs driven by white noise may be found in Donati-Martin and Pardoux (1993); in Berge (2001), a comparison theorem is established in the case where L is a second-order operator with von Neumann-type conditions; and Gyöngy and Rovira (2000) obtained comparison theorems for SPDEs whose coefficients have polynomial growth. We emphasize the fact that in all these works, the coefficient in front of the noise does not depend on the term Du. We must also mention that some comparison theorems can be established in relation to the theory of backward stochastic differential equations (see Pardoux 1998, for example). Denis and Stoïca (2003) obtain the existence and uniqueness of solutions of SPDEs in a more general context.

2. Preliminaries

2.1. Hypotheses and definitions

Let $(W, \mathcal{G}, m)$ be a measurable space. We assume that a (symmetric) Dirichlet form (F, e) is defined on $L^2(W, \mathcal{G}, m)$. For the notion and definition of Dirichlet forms, we refer to

Fukushima *et al*. (1994) or Bouleau and Hirsch (1991). Moreover, we assume that this Dirichlet form admits a *carré du champ* operator, Γ, and a gradient operator, D. This means that Γ is a symmetric bilinear map from $F \times F$ into $L^1(W, \mathcal{G}, m)$ such that,

$$\forall (u, v) \in F \times F, \qquad e(u, v) = \int_W \Gamma(u, v)(x) m(\mathrm{d}x),$$

and that there exists a Hilbert space, K, such that D is a map from F into $L^2(W, \mathcal{G}, m; K)$ with,

$$\forall (u, v) \in F \times F, \qquad \Gamma(u, v)(x) = (Du(x), Dv(x))_K \quad m\text{-a.e.},$$

where $(\cdot, \cdot)_K$ denotes the inner product in K.

We recall that, by definition, F is a Hilbert space with respect to the norm,

$$\forall u \in F, \qquad \|u\|_F^2 = e(u, u) + \|u\|_{L^2(W,m)}^2,$$

where in a natural way, we set,

$$\forall u \in L^2(W, m), \qquad \|u\|_{L^2(W,m)}^2 = (u, u)_{L^2(W,m)} = \int_W u^2(x), m(\mathrm{d}x).$$

In the example we gave in Section 1, it is clear that F is the closure of $C_0^\infty(O)$ with respect to $\|\cdot\|_F$, so $F = H_0^1(O)$ if L is strictly elliptic,

$$\forall u, v \in H_0^1(O), \qquad e(u, v) = \sum_{i,j} \int_O a_{i,j}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_i}(x)\, \mathrm{d}x,$$

$K = \mathbb{R}^{n'}$, where n' is the number of rows of the matrix σ, $Du = \nabla u \sigma$ and,

$$\forall u, v \in F, \qquad \Gamma(u, v)(x) = a_{i,j}(x) \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_j}(x).$$

We denote by $(\Omega, (\mathcal{F}_t)_{t \geqslant 0}, P)$ a filtered probability space (satisfying the usual conditions) on which a d-dimensional Brownian motion $B = (B^i)_{i \in \{1,\dots,d\}}$ is defined.

Finally, a time $T > 0$ is fixed.

2.2. Associated operators

Still following Fukushima *et al*. (1994) or Bouleau and Hirsch (1991), we know that (F, e) is associated with an m-symmetric sub-Markovian semigroup denoted by $(P_t)_{t \geqslant 0}$; L is its generator with domain $\mathrm{Dom}(L)$, it is a non-positive sub-Markovian operator, and $(E_\lambda)_{\lambda \geqslant 0}$ is the resolution of identity associated with L. All these (bounded or unbounded) operators are defined on $L^2(W, m)$ and are self-adjoint. We have

$$-L = \int_0^{+\infty} \lambda \, \mathrm{d}E_\lambda,$$

and

$$\forall f \in F, \qquad e(f, f) = \int_0^{+\infty} \lambda \, \mathrm{d}(E_\lambda f, f)_{L^2(W,m)}.$$

We also have

$$F = \mathrm{Dom}((-L)^{1/2}).$$

We recall that $\mathrm{Dom}(L) \subset F$ and that,

$$\forall u \in \mathrm{Dom}(L), \forall v \in F, \qquad e(u, v) = -\int_W Lu(x)v(x)\mathrm{m}(\mathrm{d}x) = (-Lu, v)_{L^2(W,m)}.$$

2.3. Stochastic integral for Hilbert space-valued processes

In the sections to follow, we will consider stochastic integrals for $L^2(W, m)$-valued processes, so we need to define these precisely. A general theory of stochastic integration for Banach space-valued processes is developed in Da Prato and Zabczyk (1992) and in our setting in Berge (2001). Nevertheless, we briefly recall some results and proofs in order to set the notation.

Let H be a separable Hilbert space equipped with the norm $\|\cdot\|_H$. We continue to consider the filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, P)$ on which a d-dimensional Brownian motion B is defined up to time T. Naturally, the norm on the product space H^d that we will consider is

$$\forall x = (x_1, \ldots, x_d) \in H^d, \qquad \|x\|^2_{H^d} = \sum_{i=1}^{d} \|x_i\|^2_H.$$

Let X and Y be two H-valued processes. We shall say that X is a *modification* of Y if,

$$\forall t \in [0, T], \qquad X_t = Y_t \quad P\text{-a.e.}$$

We shall say that X and Y are *equivalent* if

$$X_t(w) = Y_t(w) \qquad \text{for } \mathrm{d}t \otimes P\text{-almost all } (t, w).$$

These two notions define equivalence relations and, as usual, we will not worry about the distinction between equivalence classes and processes which are members of these classes. We remark that the first notion is stronger than the second but that they coincide if the trajectories of X and Y are almost surely H-continuous.

Definition 2.1. *An H-valued process $(X_t)_{t\in[0,T]}$ is said to be progressively measurable if, for each $t \in [0, T]$, the map*

$$([0, t] \times \Omega, \mathcal{B}([0, t]) \otimes \mathcal{F}_t) \to (H, \mathcal{B}(H))$$

$$(s, w) \mapsto X_s(w)$$

is measurable; here $\mathcal{B}([0, t])$ is the Borel σ-field on $[0, t]$ and $\mathcal{B}(H)$ the Borel σ-field on H.

We denote by $\mathcal{P}(H)$ the class (with respect to the notion of equivalent processes) of H-valued progressively measurable processes X such that

$$\|X\|^2_{\mathcal{P}(H)} = \mathrm{E}\left(\int_0^T \|X_t\|^2_H \,\mathrm{d}t\right)$$

is finite.

To construct the stochastic integral, we start with simple processes and then go on to consider square-integrable progressively measurable processes.

Definition 2.2. *An H-valued process $(X_t)_{t\in[0,T]}$ is said to be simple if there exist $n \in \mathbb{N}^*$, $0 = t_0 < t_1 < \ldots < t_n = T$ and n square-integrable H-valued variables $X_0, X_1, \ldots, X_{n-1}$ such that, for all $i \in \{0, \ldots, n-1\}$, X_i is $\mathcal{F}_{t_i}$-measurable and,*

$$\forall t \in [0, T], \qquad X_t = X_0 \mathbf{1}_{\{0\}}(t) + \sum_{i=0}^{n-1} X_i \mathbf{1}_{]t_i, t_{i+1}]}(t).$$

We denote by $\mathcal{P}_0(H)$ the class (with respect to the notion of equivalent processes) of simple processes.

Following Karatzas and Shreve (1991, Chapter 3), by adapting the proofs of their Lemma 2.2 and Proposition 2.6, we have:

Proposition 2.1. $(\mathcal{P}(H), \|\cdot\|_{\mathcal{P}(H)})$ *is a Banach space and $\mathcal{P}_0(H)$ is dense in it.*

We can now construct stochastic integrals in an easy way: assume first that $X = (X^1, \ldots, X^d)$ belongs to $(\mathcal{P}_0(H))^d$. Thus, there exist $n \in \mathbb{N}^*$, $t_0 = 0 < t_1 < \ldots < t_n = T$ and, for all $(i, j) \in \{0, \ldots, n-1\} \times \{1, \ldots, d\}$, a square-integrable variable (X_i^j) which is $\mathcal{F}_{t_i}$-measurable, such that

$$\forall j \in \{1, \ldots, d\}, \forall t \in [0, T], \qquad X_t^j = X_0^j \mathbf{1}_{\{0\}}(t) + \sum_{i=0}^{n-1} X_i^j \mathbf{1}_{]t_i, t_{i+1}]}(t).$$

We set,

$$\forall t \in [0, T], \qquad I_t^X = \int_0^t X_s \,\mathrm{d}B_s = \sum_{j=1}^{d} \sum_{i=0}^{n-1} X_i^j (B^j_{t_{i+1}\wedge t} - B^j_{t_i \wedge t}).$$

We have:

Propostion 2.2. *In our previous notation, if $X \in (\mathcal{P}_0(H))^d$, the process I^X satisfies the following properties:*

(i) *I^X is a square-integrable H-valued martingale.*
(ii) *For all $t \in [0, T]$,* $\mathrm{E}[\|I_t^X\|^2_H] = \mathrm{E}[\int_0^t \|X_s\|^2_{H^d} \,\mathrm{d}s]$.
(iii) *For P-almost all w, the map*

$$
\begin{aligned}
[0, T] &\to H \\
t &\mapsto I_t^X(w)
\end{aligned}
$$

is continuous.

(iv) I^X *satisfies Doob's inequality:*

$$\mathrm{E}\left(\sup_{t\in[0,T]} \|I_t^X\|_H^2\right) \leqslant 4\mathrm{E}\left(\int_0^T \|X_t\|_{H^d}^2 \,\mathrm{d}t\right).$$

Proof. Assertions (i) and (iii) are obvious, and the proof of (ii) is straightforward. Let us prove Doob's inequality.

To this end, consider $(\mathrm{e}_k)_{k\in\mathbb{N}^*}$ an orthonormal basis in H (we consider the case where the dimension of H is infinite). Then, as X is simple, straightforward calculations yield

$$
\begin{aligned}
\mathrm{E}\left(\sup_{t\in[0,T]} \|I_t^X\|_H^2\right) &= \mathrm{E}\left(\sup_{t\in[0,T]}\left\{\sum_{k=1}^{+\infty}\left(\int_0^t X_s \,\mathrm{d}B_s,\, e_k\right)_H^2\right\}\right) \\
&\leqslant \mathrm{E}\left(\sum_{k=1}^{+\infty} \sup_{t\in[0,T]}\left\{\left(\int_0^t X_s \,\mathrm{d}B_s,\, e_k\right)_H^2\right\}\right) \\
&= \mathrm{E}\left(\sum_{k=1}^{+\infty} \sup_{t\in[0,T]}\left\{\sum_{i=1}^{d}\int_0^t (X_s^i,\, e_k)\,\mathrm{d}B_s^i\right\}^2\right) \\
&\leqslant 4\mathrm{E}\left(\sum_{k=1}^{+\infty}\left(\sum_{i=1}^{d}\int_0^T (X_s^i,\, e_k)\mathrm{d}B_s^i\right)^2\right) \\
&= 4\mathrm{E}\left(\sum_{k=1}^{+\infty}\sum_{i=1}^{d}\int_0^T (X_s^i,\, e_k)^2 \,\mathrm{d}s\right) \\
&= 4\mathrm{E}\left(\int_0^T \|X_s\|_{H^d}^2 \,\mathrm{d}s\right).
\end{aligned}
$$

□

Then, thanks to the density of $\mathcal{P}_0(H)$ in $\mathcal{P}(H)$ and Doob's inequality, we obtain:

Propostion 2.3. *Let X be in $(\mathcal{P}(H))^d$. There exists a unique H-valued square-integrable martingale that we denote as before $I_t^X = \int_0^t X_s \,\mathrm{d}B_s$ for all $t \in [0, T]$, such that, for any sequence $(X_n)_{n\in\mathbb{N}^*}$ in $(\mathcal{P}_0(H))^d$ which converges to X in $(\mathcal{P}(H))^d$,*

$$\lim_{n\to+\infty} \mathrm{E}\left(\sup_{t\in[0,T]} \|I_t^X - I_t^{X_n}\|_H^2\right) = 0.$$

Moreover, I^X satisfies the same properties as in Proposition 2.2.

More precisely, we have that I^X admits an H-continuous modification and satisfies Doob's inequality.

We end this subsection with the case we are interested in: $H = L^2(W, m)$. In this case, a third variable, $x \in W$, appears.

Definition 2.3. *Let X be an $L^2(W, m)$-valued process and $Y : [0, T] \times \Omega \times W \to \mathbb{R}$ be in $L^2([0, T] \times \Omega \times W)$. Y is said to be a version of X if*

$$\forall t \in [0, T], \qquad X_t(w, x) = Y_t(w, x) \quad \text{for } P \otimes m\text{-}all\ (w, x).$$

It is clear that in this case, Y may be viewed as an $L^2(W, m)$-valued process and is a modification of X.

Propostion 2.4. *Let $X \in (\mathcal{P}(L^2(W, m)))^d$. Then, the process I^X defined by,*

$$\forall t \in [0, T], \qquad I_t^X = \int_0^t X_s \, \mathrm{d}B_s,$$

admits a version that we denote as before by I^X such that, for $P \otimes m$-almost all $(w, x) \in \Omega \times W$, the map

$$t \in [0, T] \mapsto I_t^X(w, x)$$

is continuous. Moreover,

$$\mathrm{E}\left(\int_W \sup_{t\in[0,T]} |I_t^X(x)|^2 m(\mathrm{d}x)\right) \leqslant 4\mathrm{E}\left(\int_0^T \|X_s\|_{L^2(W,m)}^2 \, \mathrm{d}s\right).$$

Proof. Assume first that X is simple. Then, the proposition is clear and a density argument allows us to complete the proof. □

Remark. By the same proof, we also have that for m-almost all $x \in W$, $t \to \int_0^t X_s(x) \, \mathrm{d}B_s$ is a continuous martingale.

2.4. Dirichlet processes

We now introduce the notion of Dirichlet processes. For this purpose, we consider $\Theta = (\Delta_N)_{N\in\mathbb{N}^*}$, a sequence of subdivisions of $[0, T]$ such that

$$\lim_{N\to+\infty} |\Delta_N| = 0,$$

where $|\Delta|$ denotes the mesh of any subdivision Δ of $[0, T]$. As previously, we consider a separable Hilbert space $(H, \|\cdot\|_H)$, and, for any H-valued process $(A_t)_{t\in[0,T]}$, we put

$$\forall \Delta \in \Theta, \qquad V(A, \Delta) = \sum_{t_i \in \Delta\setminus\{T\}} \|A_{t_{i+1}} - A_{t_i}\|_H^2.$$

Definition 2.4. *Let $(A_t)_{t\in[0,T]}$ be an adapted and continuous H-valued stochastic process such that, for all $t \in [0, T]$, $A_t \in L^2(\Omega, P; H)$. We say that A is of zero quadratic variation throughout Θ if*

$$\lim_{N\to+\infty} \mathrm{E}(V(A, \Delta_N)) = 0.$$

Definition 2.5. *Let $(X_t)_{t\in[0,T]}$ be an adapted H-valued stochastic process. We say that X is a continuous Dirichlet process throughout Θ if and only if there exist*

- *a square-integrable continuous martingale $(M_t)_{t\in[0,T]}$ with values in H and satisfying $M_0 = 0$, and*
- *an H-valued process $(A_t)_{t\in[0,T]}$ of zero quadratic variation throughout Θ*

such that

$$\forall t \in [0, T], \qquad X_t = M_t + A_t.$$

Remarks.

1. Thus X is a continuous process.
2. There are many (non-equivalent) definitions for Dirichlet processes: one can consider all subdivisions, or stochastic subdivisions, etc.
3. The Dirichlet processes have been well studied, for example, by Bertoin (1986; 1987).
4. We refer to Föllmer (1981a; 1981b) who developed a stochastic calculus for Dirichlet processes, and to Fukushima *et al*. (1994) who proved that Dirichlet processes are naturally associated with Dirichlet forms.

We now construct a space of Dirichlet processes in which we are going to work.

Definition 2.6. *We denote by $\mathbb{D}_T^\Theta(H)$, the set of H-valued continuous Dirichlet processes, $t \in [0, T] \to X_t \in H$, such that*

$$\mathrm{E}\left[\sup_{t\in[0,T]} \|X_t\|_H^2\right] < +\infty.$$

For all $X \in \mathbb{D}_T^\Theta(H)$, we set

$$\|X\|_{\Theta,T,H} = \left(\mathrm{E}\left[\sup_{t\in[0,T]} \|X_t\|_H^2\right] + \sup_{\Delta\in\Theta} \mathrm{E}[V(X, \Delta)]\right)^{1/2}.$$

The following proposition is inspired by Bertoin (1986) and Denis (1994).

Propostion 2.5. $(\mathbb{D}_T^\Theta(H), \|\cdot\|_{,\Theta,T,H})$ *is a Banach space.*

Proof. First of all, we observe that if $X = M + A$ belongs to $\mathbb{D}_T^\Theta(H)$, then as M is a square-integrable martingale,

$$\mathrm{E}\left[\sup_{t\in[0,T]} \|M_t\|_H^2\right] < +\infty,$$

so

$$\mathrm{E}\left[\sup_{t\in[0,T]} \|A_t\|_H^2\right] < +\infty,$$

and as $\lim_{\Delta\in\Theta,|\Delta|\to 0}\mathrm{E}[V(A,\Delta)] = 0$, we have

$$\sup_{\Delta\in\Theta}\mathrm{E}[V(A,\Delta)] < +\infty,$$

which ensures that $\|\mathrm{X}\|_{\Theta,T,H}$ is finite. Moreover, it is easy to verify that $\|\cdot\|_{\Theta,T,H}$ is a norm.

Now consider $(X^n)_{n\in\mathbb{N}^*}$, a Cauchy sequence in $\mathbb{D}_T^\Theta(H)$. For each $n \in \mathbb{N}^*$, we write

$$X^n = M^n + A^n,$$

where M^n is a continuous square-integrable martingale which satisfies $M_0^n = 0$ and A^n is an adapted process of zero quadratic variation throughout Θ.

Thanks to Doob's inequality, and since, for all $k \in \mathbb{N}^*$, $\lim_{|\Delta|\to 0,\Delta\in\Theta}\mathrm{E}[V(A^k,\Delta)] = 0$, we have, for all $n, m \in \mathbb{N}^*$,

$$\begin{aligned}\mathrm{E}\left[\sup_{t\in[0,T]} \|M_t^n - M_t^m\|_H^2\right] &\leqslant 4 \lim_{|\Delta|\to 0,\Delta\in\Theta} \mathrm{E}[V(M^n - \mathrm{M}^m,\Delta)] \\ &= \lim_{|\Delta|\to 0,\Delta\in\Theta} \mathrm{E}[V(X^n - X^m,\Delta)] \\ &\leqslant \|X^n - X^m\|_{\Theta,T,H}^2.\end{aligned}$$

Thanks to the classical theory of martingales, we know that there exists a continuous square-integrable martingale M with $M_0 = 0$ such that

$$\lim_{n\to+\infty}\mathrm{E}\left[\sup_{t\in[0,T]} \|M_t - M_t^n\|_H^2\right] = 0.$$

Then we deduce that there exists a continuous adapted process A such that

$$\lim_{n\to+\infty}\mathrm{E}\left[\sup_{t\in[0,T]} \|A_t - A_t^n\|_H^2\right] = 0.$$

Let us prove now that A is of zero quadratic variation throughtout Θ.

Let $\Delta \in \Theta$, $p \in \mathbb{N}^*$; then, for all $n, m \in \mathbb{N}^*$, if we set

$$I_{n,m} = \mathrm{E}[|\mathrm{V}^{1/2}(A^n - A^p, \Delta) - V^{1/2}(A^m - A^p, \Delta)|^2],$$

we have

$$\begin{aligned} I_{n,m} &\leqslant \mathrm{E}[V(A^n - A^m, \Delta)] \\ &\leqslant \mathrm{E}[|V^{1/2}(X^n - X^m, \Delta) + V^{1/2}(M^n - M^m, \Delta)|^2] \\ &\leqslant 2\|X^n - X^m\|^2_{\Theta,T,H} + 2\mathrm{E}[\|M^n_T - M^m_T\|^2_H]. \end{aligned}$$

From this, it is clear that

$$\left[\mathrm{E}[V(A - A^p, \Delta)] = \lim_{n\to+\infty} \mathrm{E}[V(A^n - A^p, \Delta)\right],$$

uniformly with respect to $\Delta \in \Theta$. This yields

$$\lim_{n\to+\infty} \sup_{\Delta\in\Theta} \mathrm{E}[V(A - A^n, \Delta)] = 0.$$

Finally, as for each $\Delta \in \Theta$ and $n \in \mathbb{N}^*$,

$$\mathrm{E}[V(A, \Delta)] \leqslant 2(\mathrm{E}[V(A^n, \Delta)] + \mathrm{E}[V(A - A^n, \Delta)]),$$

it easy to conclude that A is of zero quadratic variation and that

$$\lim_{n\to+\infty} \|X - X^n\|_{\Theta,T,H} = 0,$$

where $X = M + A$, and the proof is complete. □

Note, finally, that if $H = \mathbb{R}$, we suppress it.

3. A stochastic partial differential equation

We consider the following SPDE:

$$\begin{aligned} u(t, x) = \Phi(x) &+ \int_0^t Lu(s, x) + f(s, x, u(s, x), Du(s, x))\mathrm{d}s \\ &+ \int_0^t g(s, x, u(s, x), Du(s, x))\,\mathrm{d}B_s. \end{aligned} \tag{3.1}$$

3.1. Hypotheses and notation

Hypothesis 3.1. $\Phi \in L^2(W, m)$.

Hypothesis 3.2. *f maps $\Omega \times [0, T] \times W \times \mathbb{R} \times K$ to $\mathbb{R}$ and*

(*a*) *f is $\mathcal{P} \times \mathcal{G} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(K)$-measurable, where $\mathcal{P}$ is the predictable σ-field on $\Omega \times [0, T]$;*
(b) $f(\cdot, \cdot, \cdot, 0, 0) \in L^2(\Omega \times [0, T] \times W, P \otimes \mathrm{d}t \otimes m)$;
(c) *there exists $C > 0$ such that, for all $w, t, x, y, z, y', z' \in \Omega \times [0, T] \times W \times \mathbb{R} \times K \times \mathbb{R} \times K$,*

$$|f(w, t, x, y, z) - f(w, t, x, y', z')|^2 \leqslant C(|y - y'|^2 + \|z - z'\|_K^2).$$

Hypothesis 3.3. *g maps $\Omega \times [0, T] \times W \times \mathbb{R} \times K$ to $\mathbb{R}^d$ and*

(*a*) *g is $\mathcal{P}r \times \mathcal{G} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(K)$-measurable where $\mathcal{P}r$ is the progressive σ-field on $\Omega \times [0, T]$;*
(b) $g(\cdot, \cdot, \cdot, 0, 0) \in L^2(\Omega \times [0, T] \times W; \mathbb{R}^d)$;
(c) *there exist $C > 0$ and $\alpha \in [0, 2[$ such that, for all $w, t, x, y, z, y', z' \in \Omega \times [0, T] \times W \times \mathbb{R} \times K \times \mathbb{R} \times K$,*

$$|g(w, t, x, y, z) - g(w, t, x, y', z')|^2 \leqslant C|y - y'|^2 + \alpha\|z - z'\|_K^2,$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$.

We observe that Hypothesis 3.3 is fulfilled if there exist $C' > 0$ and $\alpha' \in [0, 1[$ such that, for all w, t, x, y, z, y', z',

$$|g(w, t, x, y, z) - g(w, t, x, y', z')| \leqslant C'|y - y'| + \alpha'\|z - z'\|_K.$$

We will work in $\mathcal{P}(F)$. We recall that an F-valued process u belongs to $\mathcal{P}(F)$ if it is progressively measurable and

$$\|u\|_{\mathcal{P}(F)}^2 = \int_0^T \mathrm{E}(\|u(\cdot, t, \cdot)\|_F^2)\mathrm{d}t < +\infty,$$

and that $\mathcal{P}(F)$ is a Banach space.

3.2. Notion of weak solutions

Let $u \in \mathcal{P}(F)$. Following Bouleau and Hirsch (1991), the gradient of u,

$$(t, w, x) \in [0, T] \times \Omega \times W \to D(u(t, \cdot)(w))(x),$$

admits a version in $L^2([0, T] \times \Omega \times W; K)$ which is progressively measurable and so belongs to $\mathcal{P}(L^2(W; K))$ (it is easy to prove this if $u \in \mathcal{P}_0(F)$, and then using a density argument).

If f and g denote the coefficients of equation (3.1) which satisfy Hypotheses 3.1–3.3, the following lemma is easily proved:

Lemma 3.1. *Let $u \in \mathcal{P}(F)$. Then processes*

$$t \in [0, T] \to \int_0^t f(s, \cdot, u(s, \cdot), Du(s, \cdot))\mathrm{d}s$$

and

$$t \in [0, T] \to g(t, \cdot, u(t, \cdot), Du(t, \cdot))$$

admit a version in $\mathcal{P}(L^2(W))$ *and* $(\mathcal{P}(L^2(W)))^d$*, respectively.*

We denote by $\mathcal{D}$ the set of test functions, which is the tensor product of the two Dirichlet structures: $H^1([0, T])$, the standard Sobolev space on $[0, T]$, and F (for general settings on this notion, see Bouleau and Hirsch 1991). We recall that φ belongs to $\mathcal{D}$ if and only if:

- $\varphi \in L^2([0, T] \times W)$;
- for almost all $x \in W$, $\varphi(\cdot, x) \in H^1([0, T])$;
- for almost all $t \in [0, T]$, $\varphi(t, \cdot) \in F$;
- $\int_0^T \int_W \|D\varphi(t, x)\|_K^2 + |\partial_t \varphi(t, x)|^2 m(\mathrm{d}x)\mathrm{d}t < +\infty$.

We observe that as a consequence, for m-almost all $x \in W$, $t \to \varphi(t, x)$ is continuous, so that we can set

$$\mathcal{D}_0 = \{\varphi \in \mathcal{D}, \varphi(T, \cdot) = 0 \text{ a.s.}\}.$$

Definition 3.1. *A function* $u \in \mathcal{P}(F)$ *is said to be a weak solution of* (3.1) if, for all $\varphi \in \mathcal{D}_0$,

$$\int_0^T (u(t, \cdot), \partial_t \varphi(t, \cdot))_{L^2(W,m)} - e(u(t, \cdot), \varphi(t, \cdot)) + (f(t, \cdot, u(t, \cdot), Du(t, \cdot)), \varphi(t, \cdot))_{L^2(W,m)}\mathrm{d}t$$
$$+ \int_0^T (g(t, \cdot, u(t, \cdot), Du(t, \cdot)), \varphi(t, \cdot))_{L^2(W,m)}\, \mathrm{d}B_t + (\Phi, \varphi(0, \cdot))_{L^2(W,m)} = 0 \text{ a.s.}$$

In an abuse of notation, albeit a natural one,

$$t \in [0, T] \mapsto g(t, \cdot, u(t, \cdot), Du(t, \cdot)) = (g_t^1(\cdot), \ldots, g_t^d(\cdot)),$$

the process $t \in [0, T] \mapsto (g(t, \cdot, u(t, \cdot), Du(t, \cdot)), \varphi(t, \cdot))_{L^2(W,m)}$ is nothing but the $\mathbb{R}^d$-valued process

$$t \in [0, T] \mapsto ((g_t^1(\cdot), \varphi(t, \cdot))_{L^2(W,m)}, \ldots, (g_g^d(\cdot), \varphi(t, \cdot))_{L^2(W,m)})$$

and so

$$\int_0^T (g(t, \cdot, u(t, \cdot), Du(t, \cdot)), \varphi(t, \cdot))_{L^2(W,m)}\, \mathrm{d}B_t = \sum_{j=1}^d \int_0^T (g_t^j(\cdot), \varphi(t, \cdot))_{L^2(W,m)}\, \mathrm{d}B_t^j.$$

4. Study of (3.1): existence, uniqueness and probabilistic behaviour

4.1. The 'mild' equation

We wish to study (3.1) in terms of mild solution. This means that, formally for the moment, we will prove that the solution of (3.1) satisfies,

$$\forall t \in [0, T], \qquad u_t = P_t\Phi + \int_0^t P_{t-s} f(s, \cdot, u_s, Du_s)\mathrm{d}s + \int_0^t P_{t-s} g(s, \cdot, u_s, Du_s)\,\mathrm{d}B_s.$$

To make sense of this equation, we first study each term on the right, paying particular attention to its probabilistic behaviour.

Propostion 4.1. *Let Φ be in $L^2(W, m)$. Then:*

(i) *the process*

$$\gamma : t \in [0, T] \to P_t\Phi$$

admits a version in $L^2([0, T]; F)$;

(ii) *for all $\varphi \in \mathcal{D}_0$*

$$\int_0^T (P_t\Phi, \partial_t\varphi_t)_{L^2(W,m)}\mathrm{d}t = -(\Phi, \varphi(0, \cdot))_{L^2(W,m)} + \int_0^T e(P_t\Phi, \varphi_t)\mathrm{d}t;$$

(iii) *for all $0 \leqslant s < t \leqslant T$, $\int_s^t \gamma_u \,\mathrm{d}u$ belongs to $\mathrm{Dom}(L)$ and*

$$\gamma_t - \gamma_s = L\left(\int_s^t \gamma_u \,\mathrm{d}u\right).$$

Proof. It is well known that for all $t \in]0, T]$, $P_t\Phi \in F$. Moreover,

$$\forall t \in [0, T], \qquad P_t\Phi = \int_0^{+\infty} \mathrm{e}^{-\lambda t}\,\mathrm{d}E_\lambda\Phi,$$

so that

$$\forall t \in [0, T], \qquad \|P_t\Phi\|_F^2 = \int_0^{+\infty} (1+\lambda)\mathrm{e}^{-2\lambda t}\,\mathrm{d}(E_\lambda\Phi, \Phi)_{L^2(W,m)},$$

which yields

$$\begin{aligned}\int_0^T \|P_t\Phi\|_F^2\,\mathrm{d}t &= \int_0^{+\infty}(1+\lambda)\frac{1-\mathrm{e}^{-2\lambda T}}{2\lambda}\,\mathrm{d}(E_\lambda\Phi,\,\Phi)_{L^2(W,m)}\\ &= \int_0^{+\infty}\frac{1-\mathrm{e}^{-2\lambda T}}{2\lambda}\,\mathrm{d}(E_\lambda\Phi,\,\Phi)_{L^2(W,m)} + \int_0^{+\infty}\frac{1-\mathrm{e}^{-2\lambda T}}{2}\,\mathrm{d}(E_\lambda\Phi,\,\Phi)_{L^2(W,m)}\\ &\leqslant \int_0^{+\infty} T\,\mathrm{d}(E_\lambda\Phi,\,\Phi) + \int_0^{+\infty}\frac{1}{2}\,\mathrm{d}(E_\lambda\Phi,\,\Phi)\\ &= \left(\mathrm{T}+\frac{1}{2}\right)\|\Phi\|_{L^2(W,m)},\end{aligned}$$

which proves (i).

For assertions (ii) and (iii), assume first that $\Phi \in \mathrm{Dom}(L)$. Then, it is clear that for all $t \in [0,\,T]$, $\int_0^t P_u\Phi\,\mathrm{d}u$ belongs to $\mathrm{Dom}(L)$, that the map $t \in [0,\,T] \to P_t\Phi$ is $L^2(W,\,m)$-differentiable, $\partial P_t\Phi/\partial t = L(P_t\Phi)$, and so

$$\forall t \in [0,\,T], \qquad P_t\Phi - \Phi = \int_0^t L(P_u\Phi)\mathrm{d}u = L\left(\int_0^t P_u\Phi\,\mathrm{d}u\right),$$

which is assertion (iii).

Moreover, for all $\varphi \in \mathcal{D}_0$,

$$\begin{aligned}\partial_t(P_t\Phi,\,\varphi_t)_{L^2(W,m)} &= (LP_t\Phi,\,\varphi_t)_{L^2(W,m)} + (P_t\Phi,\,\partial_t\varphi_t)_{L^2(W,m)}\\ &= -\mathrm{e}(P_t\Phi,\,\varphi_t) + (P_t\Phi,\,\partial_t\varphi_t)_{L^2(W,m)};\end{aligned}$$

by integrating this relation, we get (ii).

Consider now $\Phi \in L^2(W,\,m)$ and $(\Phi^n)_{n\in\mathbb{N}^*}$ a sequence in $\mathrm{Dom}(L)$ which converges to Φ in $L^2(W,\,m)$. Thanks to the proof of (i), we know that $(P_t\Phi^n)$ tends to $P_t\Phi$ in $L^2([0,\,T];\,F)$, which yields (ii) by density.

Moreover, if $t \in [0,\,T]$, we have, for all $n,\,m \in \mathbb{N}^*$,

$$L\left(\int_0^t P_u\Phi^n\,\mathrm{d}u - \int_0^t P_u\Phi^m\,\mathrm{d}u\right) = P_t(\Phi^n - \Phi^m) - (\Phi^n - \Phi^m).$$

As P_t is continuous on $L^2(W,\,m)$, it is clear that $(L(\int_0^t P_u\Phi^n\,\mathrm{d}u)_{n\in\mathbb{N}^*})$ is a Cauchy sequence in $L^2(W,\,m)$ and so converges. As L is a closed operator, we conclude that $\int_0^t P_u\Phi\,\mathrm{d}u$ belongs to $\mathrm{Dom}(L)$ and that

$$L\left(\int_0^t P_u\Phi\,\mathrm{d}u\right) = \lim_{n\to+\infty} L\left(\int_0^t P_u\Phi^n\,\mathrm{d}u\right).$$

The proof is complete. □

Remark. The same proof gives that the (deterministic) process γ belongs to $\mathcal{P}(F)$. One has to note that $\gamma_0 = \Phi$ does not necessarily belong to F but, for all $t > 0$, $\gamma_t \in F$ and it is also well known that the map $t \in]0,\,T] \to \gamma_t \in F$ is continuous.

Proposition 4.2. *Let $\Theta = (\Delta_N)_{N\in\mathbb{N}^*}$ be a sequence of subdivisions of $[0, T]$ such that*

$$\lim_{N\to+\infty} |\Delta_N| = 0,$$

and let Φ be in $L^2(W, m)$. Then the process

$$\gamma : t \in [0, T] \to P_t\Phi$$

is a (deterministic) process of zero quadratic variation and hence belongs to $\mathbb{D}_T^\Theta(L^2(W, m))$. As a consequence, there exists a subsequence in Θ, γ, such that for m-almost all $x \in W$, the map $t \in [0, T] \to P_t\Phi(x)$ belongs to $\mathbb{D}_T^\gamma$ and is of zero quadratic variation throughout γ.

Proof. Let Δ be in Θ. We have

$$V(\gamma, \Delta) = \sum_{i=0}^{N-1} \int_0^{+\infty} (\mathrm{e}^{-\lambda t_{i+1}} - \mathrm{e}^{-\lambda t_i})^2 \, \mathrm{d}(E_\lambda\Phi, \Phi)_{L^2(W,m)}$$

$$= \int_0^{+\infty} \sum_{i=0}^{N-1} (\mathrm{e}^{-\lambda t_{i+1}} - \mathrm{e}^{-\lambda t_i})^2 \, \mathrm{d}(E_\lambda\Phi, \Phi)_{L^2(W,m)}.$$

Then, if $\lambda \geqslant 0$, since for all $i \in \{0, \ldots, N-1\}$, $|\mathrm{e}^{-\lambda t_{i+1}} - \mathrm{e}^{-\lambda t_i}| \leqslant 1$,

$$0 \leqslant \sum_{i=0}^{N-1} (\mathrm{e}^{-\lambda t_{i+1}} - \mathrm{e}^{-\lambda t_i})^2 \leqslant \sum_{i=0}^{N-1} (\mathrm{e}^{-\lambda t_i} - \mathrm{e}^{-\lambda t_{i+1}}) = 1 - \mathrm{e}^{-\lambda T} \leqslant 1,$$

and since

$$\sum_{i=0}^{N-1} (\mathrm{e}^{-\lambda t_{i+1}} - \mathrm{e}^{-\lambda t_i})^2 \leqslant \sup_i |\mathrm{e}^{-\lambda t_{i+1}} - \mathrm{e}^{-\lambda t_i}| \times \sum_{i=0}^{N-1} (\mathrm{e}^{-\lambda t_i} - \mathrm{e}^{-\lambda t_{i+1}}),$$

$$\leqslant \lambda |\Delta|,$$

we have that, for all $\lambda \geqslant 0$,

$$\lim_{N\to+\infty} \sum_{t_i \in \Delta_N \setminus \{T\}} (\mathrm{e}^{-\lambda t_{i+1}} - \mathrm{e}^{-\lambda t_i})^2 = 0,$$

and we conclude by the dominated convergence theorem with respect to the measure $\mathrm{d}(\mathrm{E}_\lambda\Phi, \Phi)$ that

$$\lim_{N\to+\infty} V(\gamma, \Delta_N) = 0.$$

But it is well known that $t \to P_t\Phi$ is $L^2(W, m)$-continuous, so we have the first part of the proposition.

Then as,

$$\forall N \in \mathbb{N}^*, \qquad V(\gamma, \Delta_N) = \int_W V(\gamma(x), \Delta_N) m(\mathrm{d}x),$$

there exists a subsequence $(N_i)_{i\in\mathbb{N}^*}$ such that, for m-almost all $x \in W$,

$$\lim_{i \to +\infty} V(\gamma(x), \Delta_{N_i}) = 0.$$

Moreover, thanks to results due to Stein (1970), we know that there exists a version of γ such that, for m-almost all $x \in W$,

$$t \in [0, T] \to P_t \Phi(x)$$

is continuous and there exists a constant, C, such that

$$\int_W \sup_{t \in [0,T]} P_t |\Phi(x)|^2 m(\mathrm{d}x) \leqslant C \|\Phi\|^2_{L^2(W,m)} < +\infty,$$

ensuring that, for almost all $x \in W$,

$$\sup_{t \in [0,T]} \gamma_t(x) < +\infty,$$

which concludes the proof. □

Propostion 4.3. *Let h be in $L^2([0, T] \times \Omega \times W)$ and adapted. Then:*

(i) *the process $\alpha : t \in [0, T] \to \int_0^t P_{t-s} h_s \, \mathrm{d}s$ admits a version in $\mathcal{P}(F)$;*
(ii) *for all $\varphi \in \mathcal{D}_0$,*

$$\int_0^T (\alpha_t, \partial_t \varphi_t)_{L^2(W,m)} \, \mathrm{d}t = -\int_0^T (h_t, \varphi_t)_{L^2(W,m)} \, \mathrm{d}t + \int_0^T e(\alpha_t, \varphi_t) \mathrm{d}t \qquad P\text{-}a.e.$$

(iii) *for all $0 \leqslant s < t \leqslant T$, $\int_s^t \alpha_u \, \mathrm{d}u$ belongs to* Dom(L) *P-a.e. and*

$$\alpha_t - \alpha_s = L\left(\int_s^t \alpha_u \, \mathrm{d}u \right) + \int_s^t h_u \, \mathrm{d}u \qquad P\text{-}a.e.$$

Proof. Assume first that h belongs to the algebraic tensor product $C^1([0, T]) \otimes L^2(\Omega, P) \otimes \mathrm{Dom}(L)$ and is adapted. Then, by the same kinds of arguments as in Proposition 4.1, it is clear that α belongs to $\mathcal{P}(F)$, that if we fix $w \in \Omega$, then for all $t \in [0, T]$, $\alpha_t(w)$ belongs to Dom(L), and that $t \to \alpha_t(w)$ is $L^2(W, m)$-differientiable and satisfies

$$\forall t \in [0, T], \qquad \frac{\mathrm{d}\alpha_t}{\mathrm{d}t}(w) = h_t(w) + L\alpha_t(w).$$

Henceforth, we suppress w from the notation.

So, integrating by parts, we obtain that, for all $\varphi \in \mathcal{D}_0$,

$$\int_0^T (\alpha_s, \partial_s \varphi_s)_{L^2(W,m)} \, \mathrm{d}s = -\int_0^T (h_s, \varphi_s)_{L^2(W,m)} \, \mathrm{d}s + \int_0^T e(\alpha_s, \varphi_s) \mathrm{d}s,$$

which is relation (ii).

Still integrating by parts, we have that, for all $t \in [0, T]$,

$$\begin{aligned}\|\alpha_t\|^2_{L^2(W,m)} &= 2\int_0^t (\partial_s \alpha_s, \alpha_s)_{L^2(W,m)}\, \mathrm{d}s \\ &= 2\int_0^t (h_s + L\alpha_s, \alpha_s)_{L^2(W,m)}\, \mathrm{d}s \\ &= 2\left(\int_0^t (h_s, \alpha_s)_{L^2(W,m)}\, \mathrm{d}s - \int_0^t e(\alpha_s)\mathrm{d}s\right)\end{aligned}$$

This yields

$$\begin{aligned}\|\alpha_t\|^2_{L^2(W,m)} + 2\int_0^t e(\alpha_s)\mathrm{d}s &= 2\int_0^t (h_s, \alpha_s)_{L^2(W,m)}\, \mathrm{d}s \\ &\leqslant \int_0^t [\|h_s\|^2_{L^2(W,m)} + \|\alpha_s\|^2_{L^2(W,m)}]\mathrm{d}s. \end{aligned} \tag{4.1}$$

Thanks to Gronwall's lemma, we conclude that

$$\sup_{t\in[0,T]} \|\alpha_t\|_{L^2(W,m)} \leqslant \mathrm{e}^T \int_0^T \|h_t\|^2_{L^2(W,m)}\, \mathrm{d}t \tag{4.2}$$

Then, using equation (4.1), we obtain

$$\int_0^T e(\alpha_t)\mathrm{d}t \leqslant \frac{1+T\mathrm{e}^T}{2}\int_0^T \|h_t\|^2_{L^2(W,m)}\, \mathrm{d}t.$$

By a density argument, we obtain (i) and (ii).

Consider now $0 \leqslant s < t \leqslant T$, $h \in L^2([0, T] \times \Omega \times W)$ and a sequence $(h^n)_{n\in\mathbb{N}}$ of elements in $C^1([0, T]) \otimes L^2(\Omega, P) \otimes \mathrm{Dom}(L)$ which converges to h in $L^2([0, T] \times \Omega \times W)$. We put,

$$\forall n \in \mathbb{N}^*, \forall_u \in [0, T], \qquad \alpha^n_u = \int_0^u P_{u-v} h^n_v\, \mathrm{d}v.$$

It is clear that, for all $n \in \mathbb{N}^*$ and P-almost all $w \in \Omega$, $\int_s^t \alpha^n_u\, \mathrm{d}u \in \mathrm{Dom}(L)$ and

$$L\left(\int_s^t \alpha^n_u\, \mathrm{d}u\right) = \alpha^n_t - \alpha^n_s - \int_s^t h^n_u\, \mathrm{d}u.$$

Thanks to the relations we have established at the beginning of this proof, we conclude that $\int_s^t \alpha^n_u\, \mathrm{d}u$ converges to $\int_s^t \alpha_u\, \mathrm{d}u$ in $L^2(W, m)$ and that, moreover, $L(\int_s^t \alpha^n_u\, \mathrm{d}u)$ converges in $L^2(W, m)$ to $\alpha_t - \alpha_s - \int_s^t h_u\, \mathrm{d}u$, for P-almost all $w \in W$. This ensures that $\int_s^t \alpha_u\, \mathrm{d}u$ belongs to $\mathrm{Dom}(L)$ and that

$$\alpha_t - \alpha_s = L\left(\int_s^t \alpha_u\, \mathrm{d}u\right) + \int_s^t h_u\, \mathrm{d}u.$$

$\square$

We now wish to prove that α is a zero quadratic variation process. We start with:

Lemma 4.4. *Let Δ be a subdivision of* $[0, T]$. *In the notation of Proposition 4.3,*

$$V(\alpha, \Delta) \leqslant 2(|\Delta| + T)\int_0^T \|h_t\|^2_{L^2(W,m)}\, \mathrm{d}t \qquad P\text{-}a.e.$$

Proof. As previously, we prove the inequality for each $w \in \Omega$. We have, for all $0 \leqslant s < t \leqslant T$,

$$\begin{aligned}\alpha_t - \alpha_s &= \int_0^t P_{t-u}h_u\, \mathrm{d}u - \int_0^s P_{s-u}h_u\, \mathrm{d}u \\ &= \int_s^t P_{t-u}h_u\, \mathrm{d}u + \int_0^s (P_{t-u}h_u - P_{s-u}h_u)\mathrm{d}u.\end{aligned}$$

Clearly,

$$\begin{aligned}\left\| \int_s^t P_{t-u}h_u\, \mathrm{d}u \right\|_{L^2(W,m)} &\leqslant \int_s^t \|P_{t-u}h_u\|_{L^2(W,m)}\, \mathrm{d}u \\ &\leqslant \int_s^t \|h_u\|_{L^2(W,m)}\, \mathrm{d}u \\ &\leqslant (t-s)^{1/2}\left(\int_s^t \|h_u\|^2_{L^2(W,m)}\, \mathrm{d}u\right)^{1/2}.\end{aligned}$$

We now fix $\Delta = \{t_0 = 0 < t_1 < \ldots < t_N = T\}$, with $N \geqslant 2$. Using the trivial inequality $(a+b)^2 \leqslant 2(a^2 + b^2)$, we have that

$$\begin{aligned}V(\alpha, \Delta) &\leqslant 2\left(\sum_{i=0}^{N-1}\left\| \int_{t_i}^{t_{i+1}} P_{t_{i+1}-u}h_u\, \mathrm{d}u\right\|^2_{L^2(W,m)} + \sum_{i=1}^{N-1}\left\| \int_0^{t_i}\big(P_{t_{i+1}-u}h_u - P_{t_i-u}h_u\big)\mathrm{d}u\right\|^2_{L^2(W,m)}\right) \\ &\leqslant 2\left(\sum_{i=0}^{N-1}|\Delta|\int_{t_i}^{t_{i+1}}\|h_u\|^2_{L^2(W,m)}\, \mathrm{d}u + \sum_{i=1}^{N-1} t_i\int_0^{t_i}\|P_{t_{i+1}-u}h_u - P_{t_i-u}h_u\|^2_{L^2(W,m)}\, \mathrm{d}u\right) \\ &\leqslant 2|\Delta|\int_0^T \|h_u\|^2_{L^2(W,m)}\, \mathrm{d}u + 2T\sum_{i=1}^{N-1}\int_0^{t_i}\|P_{t_{i+1}-u}h_u - P_{t_i-u}h_u\|^2_{L^2(W,m)}\, \mathrm{d}u.\end{aligned}$$

We now estimate

$$A = \sum_{i=1}^{N-1} \int_0^{t_i} \|P_{t_{i+1}-u} h_u - P_{t_i-u} h_u\|^2_{L^2(W,m)} \, \mathrm{d}u;$$

for this we use the spectral representation of the semigroup $(P_t)_{t\geqslant 0}$.

$$A = \sum_{i=1}^{N-1} \int_0^{t_i} \int_0^{+\infty} (\mathrm{e}^{-\lambda(t_{i+1}-u)} - \mathrm{e}^{-\lambda(t_i-u)})^2 \mathrm{d}(\mathrm{E}_\lambda h_u,\, h_u)_{L^2(W,m)} \, \mathrm{d}u$$

$$= \sum_{k=0}^{N-2} \int_{t_k}^{t_{k+1}} \int_0^{+\infty} \sum_{i=k_1}^{N-1} (\mathrm{e}^{-\lambda(t_{i+1}-u)} - \mathrm{e}^{-\lambda(t_i-u)})^2 \, \mathrm{d}(\mathrm{E}_\lambda h_u,\, h_u)_{L^2(W,m)} \, \mathrm{d}u$$

Let us fix k and $u \in [t_k,\, t_{k+1}]$, and set

$$B = \sum_{i=k+1}^{N-1} (\mathrm{e}^{-\lambda(t_{i+1}-u)} - \mathrm{e}^{-\lambda(t_i-u)})^2.$$

Since, for all $i \in \{k+1, \ldots, N-1\}$,

$$0 \leqslant (\mathrm{e}^{-\lambda(t_i-u)} - \mathrm{e}^{-\lambda(t_{i+1}-u)}) \leqslant 1,$$

$$B \leqslant \sum_{i=k+1}^{N-1} (\mathrm{e}^{-\lambda(t_i-u)} - \mathrm{e}^{-\lambda(t_{i+1}-u)})$$

$$= \mathrm{e}^{-\lambda(t_{k+1}-u)} - \mathrm{e}^{-\lambda(T-u)} \leqslant 1.$$

This yields

$$A \leqslant \sum_{k=0}^{N-2} \int_{t_k}^{t_{k+1}} \int_0^{+\infty} \mathrm{d}(\mathrm{E}_\lambda h_u,\, h_u)_{L^2(W,m)} \, \mathrm{d}u$$

$$\leqslant \int_0^T \|h_t\|^2_{L^2(W,m)} \, \mathrm{d}t,$$

and we are done. □

We are now able to prove:

Propostion 4.5. *Let* $\Theta = (\Delta_N)_{N\in\mathbb{N}^*}$ *a sequence of subdivisions of* $[0,\, T]$ *such that*

$$\lim_{N\to+\infty} |\Delta_N| = 0,$$

and h *be in* $L^2([0,\, T] \times \Omega \times W)$ *and adapted. Define*

$$\alpha : t \in [0,\, T] \to \int_0^t P_{t-s} h_s \, \mathrm{d}s.$$

Then, the process α admits a version in $\mathbb{D}_T^{\Theta}(L^2(W, m))$ which is of zero quadratic variation. Moreover, there exists a subsequence in Θ, γ, such that, for m-almost all $x \in W$, the map $t \in [0, T] \to \alpha_t(x)$ belongs to $\mathbb{D}_T^{\gamma}$ and is of zero quadratic variation throughout γ.

Proof. Consider a sequence $(h^n)_{n\in\mathbb{N}^*}$ in $C^1([0, T]) \otimes L^2(\omega, P) \otimes \mathrm{Dom}(L)$ which converges to h in $L^2([0, T] \times \Omega \times W)$, and define

$$\forall n \in \mathbb{N}^*, \forall t \in [0, T], \qquad \alpha_t^n = \int_0^t P_{t-s} h_s^n \, \mathrm{d}s.$$

For all $n \in \mathbb{N}^*$, the process α^n belongs to $\mathbb{D}_T^{\Theta}(L^2(W, m))$, because one has the decomposition

$$\forall t \in [0, T], \qquad \alpha_t^n = \int_0^t h_s^n \, \mathrm{d}s + \int_0^t L(\alpha_s^n) \mathrm{d}s. \tag{4.3}$$

Moreover, relation (4.2) obtained in the proof of Proposition 4.3 and Lemma 4.4 yield that there exists a constant $C > 0$ such that

$$\forall n, m \in \mathbb{N}^*, \qquad \|\alpha^n - \alpha^m\|_{\Theta, T, L^2(W,m)} \leqslant C\mathrm{E}\left(\int_0^T \|h_t^n - h_t^m\|_{L^2(W,m)} \, \mathrm{d}t \right).$$

We deduce from this that $(\alpha^n)_{n\in\mathbb{N}^*}$ is a Cauchy sequence in $\mathbb{D}_T^{\Theta}(L^2(W, m))$ and so converges. It is obvious that α is the limit.

To prove the second part, we remark that, for all $N \in \mathbb{N}^*$,

$$\begin{aligned} \mathrm{E}(V(\alpha, \Delta_N)) &= \mathrm{E}\left[\sum_i \|\alpha_{t_{i+1}^N} - \alpha_{t_i^N}\|_{L^2(W,m)}^2 \right] \\ &= \int_W \mathrm{E}\left[\sum_i |\alpha_{t_{i+1}^N}(x) - \alpha_{t_i^N}(x)|^2 \right] m(\mathrm{d}x) \\ &= \int_W \mathrm{E}[V(\alpha(x), \Delta_N)] m(\mathrm{d}x), \end{aligned}$$

and as $\lim_{N\to+\infty} \mathrm{E}(V(\alpha, \Delta_N)) = 0$ and m is σ-finite, we conclude that there exists a subsequence $\gamma = (\Delta_{N_i})_{i\in\mathbb{N}^*}$ such that, for m-almost all $x \in W$,

$$\lim_{i\to+\infty} \mathrm{E}[V(\alpha(x), \Delta_{N_i})] = 0.$$

All that remains is to prove that for m-almost all $x \in W$, $t \to \alpha_t(x)$ is continuous P-a.e. and that $\mathrm{E}[\sup_{t\in[0,T]} |\alpha_t(x)|^2]$ is finite. We have

$$\begin{aligned} \sup_{t\in[0,T]} |\alpha_t(x)| &= \sup_{t\in[0,T]} \left| \int_0^t P_{t-u} h_u(x) \, \mathrm{d}u \right| \\ &\leqslant \int_0^T \sup_{s\in[0,T]} |P_s h_u(x)| \, \mathrm{d}u. \end{aligned}$$

Results due to Stein (1970) ensure that there exists a constant C such that, for all

$f \in L^2(W, m)$, the map $t \in [0, T] \to P_t f$ admits a version such that, for m-almost all $x \in W$, $t \to P_t f(x)$ is continuous and, moreover,

$$\left\| \sup_{t\in[0,T]} |P_t f| \right\|_{L^2(W,m)} \leqslant C\|f\|_{L^2(W,m)}.$$

This yields

$$\begin{aligned}\int_W \mathrm{E}\left[\sup_{t\in[0,T]} |\alpha_t(x)|^2\right] m(\mathrm{d}x) &\leqslant T\int_W \mathrm{E}\left[\int_0^T \sup_{s\in[0,T]} |P_s h_u(x)|^2 \,\mathrm{d}u\right] m(\mathrm{d}x)\\ &= \mathrm{E}\left[\int_0^T \int_W \sup_{s\in[0,T]} |P_s h_u(x)|^2 m(\mathrm{d}x)\mathrm{d}u\right]\\ &\leqslant C^2 T\,\mathrm{E}\left[\int_0^T \|h_u\|^2_{L^2(W,m)}\,\mathrm{d}u\right].\end{aligned}$$

Since, if h belongs to $C^1([0, T]) \otimes L^2(\omega, P) \otimes \mathrm{Dom}(L)$, we have decomposition (4.3), it is easy to conclude, using a density argument, that for m-almost all $x \in W$, the process $t \in [0, T] \to \alpha_t(x)$ is continuous and that, moreover, $\mathrm{E}[\sup_{t\in[0,T]}|\alpha_t(x)|^2]$ is finite, and so belongs to $\mathbb{D}_T^\gamma$ and is of zero-quadratic variation throughout γ. □

We now study the stochastic part in the mild equation.

Propostion 4.6. *Let h be in $(\mathcal{P}(L^2(W, m)))^d$. Then:*

(i) *the process $t \in [0, T] \to \beta_t = \int_0^t P_{t-s} h_s \,\mathrm{d}B_s$ admits a version in $\mathcal{P}(F)$;*
(ii) *for all $\varphi \in \mathcal{D}_0$,*

$$\int_0^T (\beta_t, \partial\varphi_t)_{L^2(W,m)}\,\mathrm{d}t = -\int_0^T (h_t, \varphi_t)_{L^2(W,m)}\,\mathrm{d}B_t + \int_0^T e(\beta_t, \varphi_t)\mathrm{d}t \qquad P\text{-a.e.};$$

(iii) *for all $0 \leqslant s < t \leqslant T$, $\int_s^t \beta_u \,\mathrm{d}u$ belongs to* $\mathrm{Dom}(L)$ *and*

$$\beta_t - \beta_s = L\left(\int_s^t \beta_u \,\mathrm{d}u\right) + \int_s^t h_u \,\mathrm{d}B_u \qquad P\text{-a.e.}$$

Remark. Here again, if $h = (h^1, \ldots, h^d)$, then we take

- $P_{t-s}h_s$ to mean $(P_{t-s}h_s^1, \ldots, P_{t-s}h_s^d)$, and
- $(h_t, \varphi_t)_{L^2(W,m)}$ to mean $((h_t^1, \varphi_t)_{L^2(W,m)}, \ldots, (h_t^d, \varphi_t)_{L^2(W,m)})$.

Proof. We denote by $\mathcal{S}$ the set of processes h such that:

$$\forall (t, x, w) \in [0, T] \times W \times \Omega, \qquad h(t, x, w) = \sum_{i=0}^{n-1} \mathbf{1}_{]t_i,t_{i+1}]}(t) h_i(x, w),$$

where $n \in \mathbb{N}^*$, $0 \leqslant t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n \leqslant T$ and, for all $i \in \{0, 1, \ldots, n-1\}$,

$$\forall (x, w) \in W \times \Omega, \qquad h_i(x, w) = \sum_{j=1}^{n_i} \mathbf{1}_{A_i^j}(w) h_i^j(x),$$

where $n_i \in \mathbb{N}^*$ and, for all $j \in \{1, \ldots, n_i\}$, $A_i^j \in \mathcal{F}_{t_i}$ and $h_i^j \in \mathrm{Dom}(L)$.

As $\mathrm{Dom}(L)$ is dense in $L^2(W, m)$, we can easily prove that $\mathcal{S}$ is a dense subspace in $\mathcal{P}_0(L^2(W, m))$, hence in $\mathcal{P}(L^2(W, m))$.

Assume first that $h \in \mathcal{S}^d$. It is clear that the process

$$\forall t \in [0, T], \qquad \beta_t = \int_0^t P_{t-s} h_s \, \mathrm{d}B_s$$

admits a version both in $L^2([0; T] \times \Omega; \mathrm{Dom}(L))$ and $\mathcal{P}(F)$. A direct calculation or a generalized Itô formula (see, for example, Protter 1985, Theorem 3.2) yields

$$\begin{aligned} \beta_t &= \int_0^t h_s \, \mathrm{d}B_s + L \int_0^t \beta_u \, \mathrm{d}u \qquad (4.4) \\ &= \int_0^t h_s \, \mathrm{d}B_s + \int_0^t L\beta_u \, \mathrm{d}u. \end{aligned}$$

Let $\varphi \in \mathcal{D}_0$; thanks to the Itô's formula, we have

$$\begin{aligned} 0 &= \int_0^T (\beta_t, \partial_t \varphi_t)_{L^2(W,m)} \, \mathrm{d}t + \int_0^T (h_t, \varphi_t)_{L^2(W,m)} \, \mathrm{d}B_t + \int_0^T (L\beta_t, \varphi_t)_{L^2(W,m)} \, \mathrm{d}t \\ &= \int_0^T (\beta_t, \partial_t \varphi_t)_{L^2(W,m)} \, \mathrm{d}t + \int_0^T (h_t, \varphi_t)_{L^2(W,m)} \, \mathrm{d}B_t - \int_0^T e(\beta_t, \varphi_t) \mathrm{d}t, \end{aligned}$$

which corresponds to assertion (ii).

To conclude using a density argument, we estimate $\mathrm{E}[\sup_{t \in [0,T]} \|\beta_t\|^2_{L^2(W,m)}]$ and $\mathrm{E}[\int_0^T \mathrm{e}(\beta_t, \beta_t) \mathrm{d}t]$. Once again we apply Itô's formula, which yields that, for all $t \in [0, T]$,

$$\|\beta_t\|^2_{L^2(W,m)} + 2\int_0^t e(\beta_u, \beta_u) \mathrm{d}s = 2\int_0^t (\beta_u, h_u)_{L^2(W,m)} \, \mathrm{d}B_u + \int_0^t \|h_u\|^2_{L^2(W,m)} \, \mathrm{d}u \qquad (4.5)$$

so that

$$\mathrm{E}\left[\sup_{t \in [0,T]} \|\beta_t\|^2_{L^2(W,m)}\right] \leqslant 2\mathrm{E}\left[\sup_{t \in [0,T]} \left| \int_0^t (\beta_u, h_u)_{L^2(W,m)} \, \mathrm{d}B_u \right| \right] + 2\mathrm{E}\left[\int_0^T \|h_u\|^2_{L^2(W,m)} \, \mathrm{d}u\right].$$

Using the Burkholder–Davies–Gundy inequality, we obtain that there exists a constant C such that

$$\mathrm{E}\left[\sup_{t\in[0,T]}\|\beta_t\|^2_{L^2(W,m)}\right]\leqslant 2C\mathrm{E}\left[\left(\int_0^T(\beta_u,\,h_u)^2_{L^2(W,m)}\,\mathrm{d}u\right)^{1/2}\right]+2\mathrm{E}\left[\int_0^T\|h_u\|^2_{L^2(W,m)}\,\mathrm{d}u\right]$$

$$\leqslant 2C\mathrm{E}\left[\sup_{t\in[0,T]}\|\beta_t\|_{L^2(W,m)}\times\left(\int_0^T\|h_u\|^2_{L^2(W,m)}\,\mathrm{d}u\right)^{1/2}\right]$$

$$+\,2\mathrm{E}\left[\int_0^T\|h_u\|^2_{L^2(W,m)}\,\mathrm{d}u\right],$$

and so, for any $\varepsilon>0$, we have that

$$\mathrm{E}\left[\sup_{t\in[0,T]}\|\beta_t\|^2_{L^2(W,m)}\right]\leqslant C\varepsilon\mathrm{E}\left[\sup_{t\in[0,T]}\|\beta_t\|^2_{L^2(W,m)}\right]+\left(\frac{C}{\varepsilon}+2\right)\mathrm{E}\left[\int_0^T\|h_u\|^2_{L^2(W,m)}\,\mathrm{d}u\right].$$

For this inequality, we have used the trivial inequality

$$ab\leqslant\frac{1}{2}\left(\frac{a^2}{\varepsilon}+\varepsilon b^2\right).$$

Then, taking ε small enough, we obtain that there exists another constant, still denoted by C, such that

$$\mathrm{E}\left[\sup_{t\in[0,T]}\|\beta_t\|^2_{L^2(W,m)}\right]\leqslant C\mathrm{E}\left[\int_0^T\|h_u\|^2_{L^2(W,m)}\,\mathrm{d}u\right].$$

Then, relation (4.5) yields

$$\mathrm{E}\left[\int_0^T e(\beta_u,\,\beta_u)\mathrm{d}s\right]\leqslant\mathrm{E}\left[\int_0^T(\beta_u,\,h_u)_{L^2(W,m)}\,\mathrm{d}B_u\right]+\frac{1}{2}\mathrm{E}\left[\int_0^t\|h_u\|^2_{L^2(W,m)}\,\mathrm{d}u\right],$$

and so we have

$$\mathrm{E}\left[\int_0^T e(\beta_u,\,\beta_u)\mathrm{d}s\right]\leqslant\frac{1}{2}\mathrm{E}\left[\int_0^T\|h_u\|^2_{L^2(W,m)}\,\mathrm{d}u\right].$$

By density, we get assertions (i) and (ii).

For (iii), we remark that if $h\in\mathcal{S}$, it is given by relation (4.4). The general case is obtained by density as in the proof of Proposition 4.3. □

We now turn our attention to the probabilistic behaviour of the process β. This is given by the following propostion.

Propostion 4.7. *Let $\Theta=(\Delta_N)_{N\in\mathbb{N}^*}$ be a sequence of subdivisions of $[0,\,T]$ such that*

$$\lim_{N\to+\infty}|\Delta_N|=0,$$

and h be in $(\mathcal{P}(L^2(W,\,m)))^d$. Then the process $t\in[0,\,T]\to\beta_t=\int_0^t P_{t-s}h_s\,\mathrm{d}B_s$ admits a version in $\mathbb{D}^{\Theta}_T(L^2(W,\,m))$ whose martingale part is

$$t \in [0,\ T] \to \int_0^t h_s \, \mathrm{d}B_s,$$

and whose zero quadratic variation part is

$$t \in [0,\ T] \to L\left(\int_0^t \beta_s \, \mathrm{d}s\right).$$

Proof. Let $N \in \mathbb{N}^*$ and $\Delta = \{0 = t_0 < t_1 < \ldots < t_N = T\}$ be in Θ. Then

$$\begin{aligned}
\mathrm{E}[\mathrm{V}(\beta,\ \Delta)] &= \mathrm{E}\left[\sum_{i=0}^{N-1}\left\|\int_{t_i}^{t_{i+1}} P_{t_{i+1}-s} h_s \, \mathrm{d}B_s + \int_0^{t_i} (P_{t_{i+1}-s} h_s - P_{t_i-s} h_s) \mathrm{d}B_s\right\|_{L^2(W,m)}^2\right] \\
&\leqslant 2\sum_{i=0}^{N-1} \mathrm{E}\left[\left\|\int_{t_i}^{t_{i+1}} P_{t_{i+1}-s} h_s \, \mathrm{d}B_s\right\|_{L^2(W,m)}^2\right] \\
&\quad + 2\sum_{i=1}^{N-1} \mathrm{E}\left[\left\|\int_0^{t_i} (P_{t_{i+1}-\mathrm{s}} h_s - P_{t_i-s} h_s) \mathrm{d}B_s\right\|_{L^2(W,m)}^2\right] \\
&= 2\sum_{i=0}^{N-1} \mathrm{E}\left[\int_{t_i}^{t_{i+1}} \|P_{t_{i+1}-s} h_s\|_{L^2(W,m)}^2 \, \mathrm{d}s\right] \\
&\quad + 2\sum_{i=1}^{N-1} \mathrm{E}\left[\int_0^{t_i} \|P_{t_{i+1}-s} h_s - P_{t_i-s} h_s\|_{L^2(W,m)}^2 \, \mathrm{d}s\right] \\
&\leqslant 2\mathrm{E}\left[\int_0^T \|h_s\|_{L^2(W,m)}^2 \, \mathrm{d}s\right] + 2\sum_{i=1}^{N-1} \mathrm{E}\left[\int_0^{t_i} \|P_{t_{i+1}-s} h_s - P_{t_i-s} h_s\|_{L^2(W,m)}^2 \, \mathrm{d}s\right].
\end{aligned}$$

So, the same quantity A appears as in the proof of Lemma 4.4; this yields

$$\mathrm{E}(V(\beta,\ \Delta)) \leqslant 4\mathrm{E}\left(\int_0^T \|h_s\|_{L^2(W,m)}^2 \, \mathrm{d}s\right)$$

Moreover, as $h \in \mathcal{S}$, we have the decomposition,

$$\forall t \in [0,\ T], \qquad \beta_t = \int_0^T h_s \, \mathrm{d}B_s + \int_0^T L\beta_s \, \mathrm{d}s,$$

which ensures that $\beta \in \mathbb{D}_T^{\Theta}(L^2(W,\ m))$. Thanks to the previous inequalities, we have

$$\|\beta\|_{\Theta,T,L^2(W,m)} \leqslant C\mathrm{E}\left[\int_0^T \|h_s\|_{L^2(W,m)}^2 \, \mathrm{d}s\right],$$

where C is a constant.

Consider now the general case. Let $h \in \mathcal{P}(L^2(W, m))$ and $(h^n)_{n\in\mathbb{N}^*}$ a sequence in $\mathcal{S}$ which converges to h in $\mathcal{P}(L^2(W, m))$. We put

$$\forall n \in \mathbb{N}^*, \forall t \in [0, T], \qquad \beta_t^n = \int_0^t P_{t-s} h_s^n \, \mathrm{d}B_s.$$

We have that

$$\forall n, m \in \mathbb{N}^*, \qquad \|\beta^n - \beta^m\|^2_{\Theta,T,L^2(W,m)} \leqslant C\mathrm{E}\left(\int_0^T \|h_u^n - h_u^m\|^2_{L^2(W,m)} \, \mathrm{d}u\right).$$

So $(\beta^n)_{n\in\mathbb{N}^*}$ is a Cauchy sequence in $\mathbb{D}_T^\Theta(L^2(W, m))$ which converges to β. Moreover, it is clear that the martingale part of β^n converges to that of β, which allows us to conclude using Proposition 4.1. □

The question which now arises, is whether or not the process $t \to \beta_t(x)$ is a Dirichlet process for m-almost all x. Unfortunately, we do not know how to prove this. Nevertheless, we have the following propostion.

Propostion 4.8. *Let* $\Theta = (\Delta_N)_{N\in\mathbb{N}^*}$ *be a sequence of subdivisions of* $[0, T]$ *such that*

$$\lim_{N\to+\infty} |\Delta_N| = 0,$$

and h *be in* $(\mathcal{P}(F))^d$. *Then for* m*-almost all* $x \in W$, *the process* $t \in [0, T] \to \beta_t(x) = \int_0^t P_{t-s} h_s(x) \, \mathrm{d}B_s$ *is a semi-martingale which satisfies*

$$\mathrm{E}\left[\sup_{t\in[0,T]} |\beta_t(x)|^2\right] < +\infty,$$

so it admits a version in $\mathbb{D}_T^\Theta$.

Proof. Assume first that h belongs to $\mathcal{S}$; this clearly ensures that β is in $L^2([0, T] \times \Omega; \mathrm{Dom}(L))$ and we estimate $\mathrm{E}(\int_0^T \|L\beta_t\|^2 \, \mathrm{d}t)$, for which purpose we use the spectral representation. For all $t \in [0, T]$, one has

$$\begin{aligned} L\beta_t &= \int_0^t LP_{t-s}\beta_s \, \mathrm{dB}_s \\ &= \int_0^t \int_0^{+\infty} \lambda \mathrm{e}^{-\lambda(t-s)} \, \mathrm{d}E_\lambda h_s \, \mathrm{d}B_s, \end{aligned}$$

so

$$\mathrm{E}(\|L\beta_t\|^2_{L^2(W,m)}) = \mathrm{E}\left(\int_0^t \int_0^{+\infty} \lambda^2 \mathrm{e}^{-2\lambda(t-s)} (\mathrm{d}E_\lambda h_s, \, h_s)_{L^2(W,m)} \, \mathrm{d}s\right),$$

this yields

$$\begin{aligned}
\mathrm{E}\bigg(\int_0^T \|L\beta_t\|^2\,\mathrm{d}t\bigg) &= \mathrm{E}\bigg(\int_0^T\int_0^t\int_0^{+\infty}\lambda^2 e^{-2\lambda(t-s)}(\mathrm{d}E_\lambda h_s,\ h_s)_{L^2(W,m)}\,\mathrm{d}s\,\mathrm{d}t\bigg)\\
&= \mathrm{E}\bigg(\int_0^T\int_0^{+\infty}\int_s^T\lambda^2 e^{-2\lambda(t-s)}\,\mathrm{d}t(\mathrm{d}E_\lambda h_s,\ h_s)_{L^2(W,m)}\,\mathrm{d}s\bigg)\\
&\leqslant \mathrm{E}\bigg(\int_0^T\int_0^{+\infty}\frac{\lambda}{2}(\mathrm{d}E_\lambda h_s,\ h_s)_{L^2(W,m)}\,\mathrm{d}s\bigg)\\
&\leqslant \frac{1}{2}\mathrm{E}\bigg(\int_0^T\|h_s\|_F^2\,\mathrm{d}s\bigg).
\end{aligned}$$

As $\mathcal{S}$ is dense in $\mathcal{P}(F)$, we conclude that if $h \in \mathcal{P}(F)$ then $\beta \in L^2([0, T] \times \Omega; \mathrm{Dom}(L))$. Moreover, thanks to the previous proposition, we know that β admits the decomposition

$$\begin{aligned}
\beta_t &= \int_0^t h_s\,\mathrm{d}B_s + L\bigg(\int_0^t \beta_s\,\mathrm{d}s\bigg)\\
&= \int_0^t h_s\,\mathrm{d}B_s + \int_0^t L\beta_s\,\mathrm{d}s
\end{aligned}$$

for all $t \in [0, T]$; it is now easy to conclude using the remark at the end of Section 2.3. □

4.2. Equivalence between weak and mild solutions

We now consider the mild equation

$$\begin{aligned}
u(t, x) = P_t\Phi(x) &+ \int_0^t P_{t-s}f(s, \cdot, u(s, \cdot), Du(s, \cdot))(x)\,\mathrm{d}s\\
&+ \int_0^t P_{t-s}g(s, \cdot, u(s, \cdot), Du(s, \cdot))(x)\,\mathrm{d}B_s. \qquad (4.6)
\end{aligned}$$

Let us remark that thanks to previous results and Lemma 3.1, this equation makes sense in $\mathcal{P}(F)$. Moreover, we have:

Propostion 4.9. $u \in \mathcal{P}(F)$ *is a weak solution of* (3.1) *if and only if it satisfies* (4.6).

Proof. Let u be in $\mathcal{P}(F)$. We put:

$$\forall t \in [0, T], \qquad \alpha_t = \int_0^t P_{t-s}f(s, \cdot, u(s, \cdot), Du(s, \cdot))(x)\,\mathrm{d}s$$

and

$$\beta_t = \int_0^t P_{t-s} g(s, \cdot, u(s, \cdot), Du(s, \cdot))(x)\, \mathrm{d}B_s.$$

Let φ be in $\mathcal{D}_0$. Then, thanks to Proposition 4.1,

$$\int_0^T (P_t \Phi, \partial_t \varphi_t)_{L^2(W,m)}\, \mathrm{d}t = -(\Phi, \varphi(0, \cdot))_{L^2(W,m)} + \int_0^T e(P_t \Phi, \varphi_t).$$

Proposition 4.3 gives

$$\int_0^T (\alpha_t(\cdot), \partial_t \varphi_t(\cdot))_{L^2(W,m)}\, \mathrm{d}t = -\int_0^T (f(t, \cdot, u(t, \cdot), Du(t, \cdot)), \varphi_t)_{L^2(W,m)}\, \mathrm{d}t + \int_0^T e(\alpha_t, \varphi_t)\mathrm{d}t,$$

and from proposition 4.6,

$$\int_0^T (\beta_t, \partial \varphi_t)_{L^2(W,m)}\, \mathrm{d}t = -\int_0^T (g(t, \cdot, u(t, \cdot), Du(t, \cdot)), \varphi_t)_{L^2(W,m)}\, \mathrm{d}B_t + \int_0^T e(\beta_t, \varphi_t)\mathrm{d}t.$$

From this, we deduce easily that if $u \in \mathcal{P}(F)$ satisfies (4.6), then u satisfies (3.1).

Conversely, if $u \in \mathcal{P}(F)$ is a solution of (3.1), then we define the process

$$\begin{aligned}\hat{u}(t, x) = P_t \Phi(x) &+ \int_0^t P_{t-s} f(s, \cdot, u(s, \cdot), Du(s, \cdot))(x)\mathrm{d}s \\ &+ \int_0^t P_{t-s} g(s, \cdot, u(s, \cdot), Du(s, \cdot))(x)\, \mathrm{d}B_s.\end{aligned}$$

Using the previous calculus and notation, we have

$$\begin{aligned}\int_0^T (\hat{u}(t, \cdot), \partial_t \varphi(t, \cdot))_{L^2(W,m)}\, \mathrm{d}t = -(\Phi, \varphi(0, \cdot))_{L^2(W,m)} &+ \int_0^T e(P_t \Phi, \varphi(t, \cdot))\mathrm{d}t \\ &- \int_0^T (f(t, \cdot, u(t, \cdot), Du(t, \cdot)), \varphi(t, \cdot))_{L^2(W,m)}\, \mathrm{d}t \\ &+ \int_0^T e(\alpha(t, \cdot), \varphi(t, \cdot))\mathrm{d}t \\ &- \int_0^T (g(t, \cdot, u(t, \cdot), Du(t, \cdot)), \varphi(t, \cdot))_{L^2(W,m)}\, \mathrm{d}B_t \\ &+ \int_0^T \mathrm{e}(\beta(t, \cdot), \varphi(t, \cdot))\mathrm{d}t \text{ a.s.}\end{aligned}$$

So, we have that, for all $\varphi \in \mathcal{D}_0$,

$$\int_0^T (\hat{u}(t,\cdot), \partial_t\varphi(t,\cdot))_{L^2(W,m)}\,\mathrm{d}t = -(\Phi, \varphi(0,\cdot))_{L^2(W,m)} + \int_0^T e(\hat{u}(t,\cdot), \varphi(t,\cdot))\mathrm{d}t$$
$$- \int_0^T (g(t,\cdot, u(t,\cdot), Du(t,\cdot)), \varphi(t,\cdot))_{L^2(W,m)}\,\mathrm{d}B_t$$
$$- \int_0^T (f(t,\cdot, u(t,\cdot), Du(t,\cdot)), \varphi(t,\cdot))_{L^2(W,m)}\,\mathrm{d}t \text{ a.s.}$$

But, as u is a solution of (3.1), for all $\varphi \in \mathcal{D}_0$,

$$\int_0^T (u(t,\cdot), \partial_t\varphi(t,\cdot))_{L^2(W,m)}\,\mathrm{d}t = -(\Phi, \varphi(0,\cdot))_{L^2(W,m)} + \int_0^T e(u(t,\cdot), \varphi(t,\cdot))\mathrm{d}t$$
$$- \int_0^T (g(t,\cdot, u(t,\cdot), Du(t,\cdot)), \varphi(t,\cdot))_{L^2(W,m)}\,\mathrm{d}B_t$$
$$- \int_0^T (f(t,\cdot, u(t,\cdot), Du(t,\cdot)), \varphi(t,\cdot))_{L^2(W,m)}\,\mathrm{d}t \text{ a.s.}$$

We now put $v(t, x) = u(t, x) - \hat{u}(t, x)$; it is clear that v belongs to $\mathcal{P}(F)$ and that, for all $\varphi \in \mathcal{D}_0$,

$$\int_0^T (v(t,\cdot), \partial_t\varphi(t,\cdot))_{L^2(W,m)}\,\mathrm{d}t = \int_0^T e(v(t,\cdot), \varphi(t,\cdot))\mathrm{d}t \text{ a.s.}$$

So, v is solution in the weak sense of the equation $\partial_t v_t - Lv_t = 0$ with initial condition $v(0,\cdot) = 0$. Thanks to Lemma 4.10, the proposition is proved. □

Lemma 4.10. 0 *is the unique weak solution in* $\mathcal{P}(F)$ *of the equation:*

$$\partial_t v_t - Lv_t = 0,$$

with initial condition $v_0 = 0$.

Proof. We remark that this equation is deterministic. Let $v \in \mathcal{P}(F)$ be a solution in the weak sense of this equation.

Consider ψ, an element of $L^2([0, T]) \otimes \mathrm{Dom}(L)$, and define

$$\forall t \in [0, T], \qquad \varphi_t = \int_t^T P_{s-t}\psi_s\,\mathrm{d}s.$$

It is now standard to prove that $\varphi \in \mathcal{D}_0$ and

$$\partial_t\varphi_t = -\psi_t - L\varphi_t.$$

As v is a solution in the weak sense, we have

$$\int_0^T (v(t,\cdot), \partial_t\varphi(t,\cdot))_{L^2(W,m)}\,\mathrm{d}t = \int_0^T e(v(t,\cdot), \varphi(t,\cdot))\mathrm{d}t;$$

this yields

$$\int_0^T (v(t,\cdot),\, \psi(t,\cdot))_{L^2(W,m)}\,\mathrm{d}t = 0,$$

because

$$(v(t,\cdot),\, -L\varphi(t,\cdot))_{L^2(W,m)} = e(v(t,\cdot),\, \varphi(t,\cdot)).$$

We now conclude by density that for any $\psi \in L^2([0,T]\times W,\, \mathrm{d}t \otimes m(\mathrm{d}x))$,

$$\int_0^T \int_W v(t,x)\psi(t,x)m(\mathrm{d}x)\mathrm{d}t = 0,$$

and so $v = 0\,\mathrm{d}t \otimes m(\mathrm{d}x)$-everywhere.

4.3. Existence and uniqueness of the solution of (3.1)

4.3.1. Preliminaries on Dirichlet forms

In this short section, we recall a few properties satisfied by Dirichlet forms. All the proofs of the following properties may be found in Bouleau and Hirsch (1991). We denote by Ξ_1^0 the set of normal contraction from $\mathbb{R}$ into $\mathbb{R}$. More precisely, a function $G : \mathbb{R} \to \mathbb{R}$ belongs to Ξ_1^0 if and only if $G(0) = 0$ and

$$\forall (x,y) \in \mathbb{R}^2, \qquad |G(x) - G(y)| \leqslant |x - \mathrm{y}|.$$

One fundamental property of Dirichlet forms is the following:

Propostion 4.11. *Let $u \in F$; then for all $G \in \Xi_1^0$, $G(u)$ belongs to F and*

$$e(G(u), G(u)) \leqslant e(u,u).$$

Corollary 4.12. *For all $u \in F$, u^+ belongs to F and*

$$e(u^+, u^+) \leqslant e(u,u).$$

Definition 4.1. *(F, e) is said to be local if and only if, for all $u, v \in F$, for all $a \in \mathbb{R}$,*

$$(u + a)v = 0 \Rightarrow e(u,v) = 0.$$

We remark that if (F, e) is local then, for all $u \in F$, $e(u^+, u) = e(u^+, u^+)$.

Moreover, if λ denotes the Lebesgue measure on $\mathbb{R}$, it is well known that any function $G \in \Xi_1^0$ is λ-everywhere differentiable and so G' is defined (λ-a.e.). Still following Bouleau and Hirsch (1991, Section I.5.2), we have:

Propostion 4.13. *Assume that (F, e) is local. Let $u \in F$ and $(G_n)_{n\in\mathbb{N}^*}$ be a sequence in Ξ_1^0 and $G \in \Xi_1^0$. Assume that*

$$\lim_{n\to+\infty} G'_n = G' \qquad \lambda\text{-}a.e.$$

Then

$$\lim_{n\to+\infty} G_n(u) = G(u) \text{ in } F.$$

Proposition 4.14. *Assume that* (F, e) *is local. Let* $u \in F$ *and* $G \in \Xi_1^0$. *Then*

$$\Gamma(G(u), G(u)) = G'(u)^2\Gamma(u, u).$$

We remark that as G' is defined λ-a.e., $G'(u)^2\Gamma(u, u)$ is defined m-a.e. because we have the following fundamental property, which Bouleau and Hirsch (1991, Section I.7) refer to as the 'absolute continuity property of image measure':

Theorem 4.15. *Assume* (F, e) *is local. Let* u *be in* F; *then*

$$u * (\Gamma(u, u) \cdot m) \ll \lambda.$$

Corollary 4.16. *For all* $u \in F$,

$$\Gamma(u^+, u^+) = \mathbf{1}_{\{u>0\}}\Gamma(u, u), \qquad m\text{-}a.e.$$

In other words,

$$\|\mathrm{D}u^+\|_K^2 = \mathbf{1}_{\{u>0\}}\|Du\|_K^2, \qquad m\text{-}a.e.$$

4.3.2. An Itô formula for Dirichlet processes

An Itô formula for ($\mathbb{R}^k$-valued) Dirichlet processes was first proved in Föllmer (1981a) and also studied by Bertoin (1986; 1987). Our goal here is to establish an Itô formula for $L^2(W, m)$-valued Dirichlet processes. For this purpose, we restrict ourselves to a subset of $\mathbb{D}_T^\Theta(L^2(W, m))$.

Definition 4.2. *Let* $\Theta = (\Delta_N)_{N\in\mathbb{N}^*}$ *be a sequence of subdivisions of* $[0, T]$ *such that*

$$\lim_{N\to+\infty} |\Delta_N| = 0.$$

We denote by $\tilde{\mathbb{D}}_T^\Theta(L^2(W, m))$ *the set of processes* $X \in \mathbb{D}_T^\Theta(L^2(W, m))$ *such that there exist* $h \in (\mathcal{P}(L^2(W, m)))^d$ *and a zero-quadratic variation process* A *which satisfy*

$$\forall t \in [0, T], \qquad X_t = \int_0^t h_s \,\mathrm{d}B_s + A_t.$$

Definition 4.3. *Let* $k \in \mathbb{N}^*$ *and* $X = (X^1, \ldots, X^k)$ *be in* $(\tilde{\mathbb{D}}_T^\Theta(L^2(W, m)))^k$; *then for all* $i, j \in \{1, \ldots, k\}$ *the* $L^1(W, m)$*-valued finite variation process* $\langle X^i, X^j\rangle$ *is well defined:*

$$\langle X^i, X^j\rangle = \int_0^t h_s^i h_s^j \,\mathrm{d}s,$$

where $\int_0^\cdot h_s^i \,\mathrm{d}B_s(\int_0^\cdot h_s^j \,\mathrm{d}B_s)$ *is the martingale part of* X^i (X^j).

It is easy to verify that in this case,

$$\langle X^i, X^j\rangle_T = \lim_{|\Delta|\to 0} \sum_{t_l\in\Delta} (X^i_{t_{l+1}} - X^i_{t_l})\cdot(X^j_{t_{l+1}} - X^j_{t_l}),$$

in $L^1(\Omega\times W)$.

We remark also that if α is an adapted process in $L^\infty([0, T]\times\Omega\times W)$, then the process

$$t\in[0, T]\to \int_0^t \alpha_s \,\mathrm{d}\langle X^i, X^j\rangle_s = \int_0^t \alpha_s h_s^i h_s^j \,\mathrm{d}s$$

is well defined and may be viewed as a continuous $L^1(W)$-valued process.

Theorem 4.17. *Let* $k\in\mathbb{N}^*$, $\Theta = (\Delta_N)_{N\in\mathbb{N}^*}$ *be a sequence of subdivisions of* $[0, T]$ *such that*

$$\lim_{N\to+\infty} |\Delta_N| = 0,$$

and $(X_t)_{t\in[0,T]}$ *be in* $(\tilde{\mathbb{D}}_T^\Theta(L^2(W, m)))^k$. *Let* $f : \mathbb{R}^k\to\mathbb{R}$ *a* C^2 *function such that* $f(X_T)$ *is in* $L^1(\Omega\times W)$ *and, for all* $i, j\in\{1, \ldots, k\}$, $\partial^2 f/\partial x_i\partial x_j$ *is bounded-continuous. Then*

$$f(X_T) = f(X_0) + \sum_{i=1}^k \int_0^T \frac{\partial}{\partial x_i} f(X_t)\mathrm{d}X_t^i + \frac{1}{2}\sum_{i,j}\int_0^T \frac{\partial^2}{\partial x_i\partial x_j} f(X_s)\mathrm{d}\langle X^i, X^j\rangle_t,$$

where, for all $i\in\{1, \ldots, k\}$,

$$\int_0^T \frac{\partial}{\partial x_i} f(X_t)\mathrm{d}X_t^i = \lim_{\Delta\in\Theta, |\Delta|\to 0} \sum_{t_l\in\Delta\setminus\{T\}} \frac{\partial}{\partial x_i} f(X_{t_l})(X^i_{t_{l+1}} - X^i_{t_l})$$

in $L^1(\Omega\times W, P\otimes m)$.

Proof. For simplicity, we only give the proof in the case $k = 1$. Let N be in $\mathbb{N}^*$. By Taylor's formula we have, for m-almost all $x\in W$,

$$\begin{aligned} f(X_T(x)) - f(X_0(x)) &= \sum_{t_l<T\in\Delta_N} [f(X_{t_{l+1}}(x)) - f(X_{t_l}(x))] \\ &= \sum_{t_l<T\in\Delta_N} f'(X_{t_l}(x))(X_{t_{l+1}}(x) - X_{t_l}(x)) \\ &\quad + \frac{1}{2} f''(X_{t_l}(x))(X_{t_{l+1}}(x) - X_{t_l}(x))^2 + \mathrm{R}(X_{t_l}(x), X_{t_{l+1}}(x)), \end{aligned}$$

where, for all $a, h\in\mathbb{R}$,

$$R(a, a+h) = h^2 \int_0^1 [f''(a+th) - f''(a)](1-t)\mathrm{d}t.$$

So

$$\left| \sum_{t_l < T \in \Delta_N} R(X_{t_l}(x), X_{t_{l+1}}(x)) \right| \leqslant \sum_{t_l < T \in \Delta_N} \left\{ \left| \int_0^1 [f''(X_{t_l}(x) + t(X_{t_{l+1}}(x) - X_{t_l}(x))) \right.\right.$$

$$\left.\left. - f''(X_{t_l}(x))](1-t)\mathrm{d}t \right| \right\} \times \sum_{t_l < T \in \Delta_N} |X_{t_{l+1}}(x) - X_{t_l}(x)|^2.$$

We put

$$\forall N \in \mathbb{N}^*, \qquad R_N = \sup_{t_l < T \in \Delta N} \left| \int_0^1 [f''(X_{t_l} + t(X_{t_{l+1}} - x_{t_l})) - f''(X_{t_l})](1-t)\mathrm{d}t \right|$$

and we now prove that $R_N \times \sum_{t_l < T \in \Delta_N} |X_{t_{l+1}} - X_{t_l}|^2$ goes to 0 as N tends to $+\infty$ in $L^1(\Omega \times W)$. For this purpose, we decompose X as

$$\forall t \in [0, T], \qquad X_t = \int_0^t h_s \, \mathrm{d}B_s + A_t,$$

where A is a zero quadratic variation process.

Consider $(\Delta_{N_m})_{m \in \mathbb{N}^*}$ a subsequence of $(\Delta_N)_{N \in \mathbb{N}^*}$. Then as

$$\sup_{t_l < T \in \Delta_{N_m}} (A_{t_{l+1}} - A_{t_l})^2 \leqslant \sum_{t_l < T \in \Delta_{N_m}} (A_{t_{l+1}} - A_{t_l})^2,$$

we have that $\sup_{t_l < T \in \Delta_{N_m}} (A_{t_{l+1}} - A_{t_l})^2$ goes to 0 as m goes to $+\infty$ in $L^1(\Omega \times W)$. One can therefore extract another subsequence $(N_{m_q})_{q \in \mathbb{N}^*}$ such that, for m-almost all $x \in W$,

$$\lim_{q \to +\infty} \sup_{t_l < T \in \Delta_{N_{m_q}}} |A_{t_{l+1}}(x) - A_{t_l}(x)| = 0 \qquad P\text{-a.e.}$$

As the martingale part of X admits a continuous version for almost all $(w, x) \in \Omega \times W$, we have that, for almost all $x \in W$,

$$\lim_{q \to +\infty} \sup_{t_l < T \in \Delta_{N_{m_q}}} |X_{t_{l+1}}(x) - X_{t_l}(x)| = 0 \qquad P\text{-a.e.}$$

By the dominated convergence theorem one deduces that, for m-almost all $x \in W$,

$$\lim_{q \to +\infty} R_{N_{m_q}}(x) = 0 \qquad P\text{-a.e.}$$

As $\sum_{t_i < T \in \Delta_N} |X_{t_{i+1}} - X_{t_i}|^2$ converges in $L^1(\Omega \times W)$ and $(R_N)_{N \in \mathbb{N}^*}$ is uniformly bounded, we obtain that

$$\lim_{q \to +\infty} R_{N_{m_q}} \times \sum_{t_i < T \in \Delta_{N_{m_q}}} |X_{t_{i+1}} - X_{t_i}|^2 = 0,$$

in $L^1(\Omega \times W)$. From this, we deduce that $(R_N \times \sum_{t_i < T \in \Delta_N} |X_{t_{i+1}} - X_{t_i}|^2)_{N \in \mathbb{N}^*}$ goes to 0 in

$L^1(\Omega \times W)$. To see this, assume that it is not true, then by the dominated convergence theorem we easily get a contradiction.

This yields

$$\lim_{|\Delta|\to 0, \Delta\in\Theta} \mathrm{E}\left[\int_W \left| \sum_{t_l<T\in\Theta} \mathrm{R}(X_{t_1}(x), X_{t_{l+1}}(x)) \right| m(\mathrm{d}x)\right] = 0.$$

Moreover, standard arguments allow us to conclude that

$$\lim_{N\to+\infty} \sum_{t_l<T\in\Delta_N} f''(X_{t_l})(X_{t_{l+1}} - X_{t_l})^2 = \lim_{N\to+\infty} \sum_{t_l<T\in\Delta_N} f''(X_{t_l})\left(\int_{t_l}^{t_{l+1}} h_s \,\mathrm{d}B_s\right)^2$$
$$= \int_0^T f''(X_s)\mathrm{d}\langle X, X\rangle_s,$$

in $L^1(\Omega \times W, P \otimes m)$.

Now it is clear that $\sum_{t_l<T\in\Delta} f'(X_{t_l})(X_{t_{l+1}} - X_{t_l})$, converges in $L^1(\Omega \times W, P \otimes m)$, and we put

$$\int_0^T f'(X_t)\mathrm{d}X_t = \lim_{|\Delta|\to 0, \Delta\in\Theta} \sum_{t_l<T\in\Delta} f'(\mathrm{X}_{t_l})(X_{t_{l+1}} - X_{t_l}),$$

which ends the proof. □

Remark. We note that $\int_0^T f'(X_t)\mathrm{d}X_t$ is not a true stochastic integral (see Bertoin 1987).

Corollary 4.18. *Let $X \in \tilde{\mathbb{D}}_T^{\Theta}(L^2(W, m))$, φ a C^2 function defined from $[0, T]$ to $\mathbb{R}$ and G a C^2 function defined from $\mathbb{R}$ to $\mathbb{R}$ with bounded second derivative and such that $G(X_T)$ belongs to $L^1(\Omega \times W)$. Then*

$$\varphi(T)G(X_T) = \varphi(0)G(X_0) + \int_0^T \varphi'(s)G(X_s)\,\mathrm{d}s + \int_0^T \varphi(s)G'(X_s)\mathrm{d}X_s$$
$$+ \frac{1}{2}\int_0^T \varphi(s)G''(X_s)\mathrm{d}\langle X, X\rangle_s,$$

$P \otimes m$-a.e., where

$$\int_0^T \varphi(s)G'(X_s)\mathrm{d}X_s = \lim_{|\Delta|\to 0, \Delta\in\Theta} \sum_{t_{l\in\Delta\setminus\{T\}}} \varphi(t_l)G'(X_{t_l})(X_{t_{l+1}} - X_{t_l}),$$

in $L^1(\Omega \times W)$.

Proof. If $m(W) < +\infty$, this is a special case of the previous theorem. Otherwise, it is easy to verify that the same proof works. □

So, this weak stochastic integration gives a sense to the term ‘$\mathrm{d}u$’ in the equation. More precisely, we have

Propostion 4.19. *Let* $\bar{f}$ *be in* $L^2([0, T] \times \Omega \times W)$ *adapted and* $\bar{g}$ be in $(\mathcal{P}(L^2(W, m)))^d$. *We define*

$$\forall t \in [0, T], \qquad u_t = \int_0^t P_{t-s}\bar{f}_s \, \mathrm{d}s + \int_0^t P_{t-s}\bar{g}_s \, \mathrm{d}B_s.$$

Let φ *be a* C^2 *function defined from* $[0, T]$ *to* $\mathbb{R}$, $G : \mathbb{R} \to \mathbb{R}$ *be a* C^2 *function with bounded second-order derivative such that* $G(0) = 0$. *Then,*

$$\mathrm{E}\left[\int_W \int_0^T \varphi(t) G'(u_t) \mathrm{d}u_t(x) \, \mathrm{d}x\right] = -\mathrm{E}\left[\int_0^T \varphi(t) e(G'(u_t), u_t) \mathrm{d}t\right]$$
$$+ \mathrm{E}\left[\int_0^T \varphi(t) (G'(u_t), \bar{f}_t)_{L^2(W,m)} \, \mathrm{d}t\right].$$

Proof. Assume first that $\bar{f}$ belongs to $C^1([0, T]) \otimes \mathrm{Dom}(L) \otimes L^2(\Omega)$ and $\bar{g}$ to $(C^1([0, T]) \otimes \mathrm{Dom}(L) \otimes L^2(\Omega))^d$. We have already shown that u is a semi-martingale and that one has the decomposition,

$$\forall t \in [0, T], \qquad u_t = \int_0^t L u_s \, \mathrm{d}s + \int_0^t \bar{f}_s \, \mathrm{d}s + \int_0^t \bar{g}_s \, \mathrm{d}B_s,$$

see Propositions 4.3 and 4.6. Then the result is easy to prove.

One has to note that in all cases, the quantity $\int_0^T \varphi(t) e(G'(u_t), u_t) \mathrm{d}t$ is well defined (P-a.e.) because G'' is bounded and so, using the properties of Dirichlet forms recalled in Section 4.3.1, there exists a constant C such that,

$$\forall t \in [0, T], \forall v \in F, \qquad e(G'(v), G'(v)) \leqslant Ce(v, v).$$

For the general case, consider a sequence $(\bar{f}^{\,n})_{n \in \mathbb{N}}$ in $C^1([0, T]) \otimes \mathrm{Dom}(L) \otimes L^2(\Omega)$ which converges to $\bar{f}$ in $L^2([0, T] \times \Omega \times W)$ and a sequence $(\bar{g}^n)_{n \in \mathbb{N}}$ in $(C^1([0, T]) \otimes \mathrm{Dom}(L) \otimes L^2(\Omega))^d$ which converges to $\bar{g}$ in $(\mathcal{P}(L^2(W, m)))^d$. We define, for all $n \in \mathbb{N}$,

$$\forall t \in [0, T], \qquad u_t^n = \int_0^t P_{t-s}\bar{f}_s^n \, \mathrm{d}s + \int_0^t P_{t-s}\bar{g}_s^n \mathrm{d}B_s.$$

As $G(0) = 0$ and G'' is bounded, it is clear that quantities $G(u_T^n)$ and $G(u_T)$ are in $L^1(\Omega \times W)$, so thanks to Itô’s formula (see Corollary 4.18), we have that, for all $n \in \mathbb{N}$,

$$\int_0^T \varphi(t) G'(u_t^n) \mathrm{d}u_t^n = G(u_T^n) - \int_0^T \varphi'(t) G(u_t^n) \mathrm{d}t$$
$$- \frac{1}{2} \int_0^T \varphi(t) G''(u_t^n) |\bar{g}_t^n|^2 \, \mathrm{d}t.$$

In the proofs of Propositions 4.3 and 4.6, we established that u^n converges to u in $L^2([0, T] \times \Omega; F)$ and that

$$\lim_{n\to+\infty} \mathrm{E}\left(\sup_{t\in[0,T]} \|u_t^n - u_t\|^2_{L^2(W,m)}\right) = 0.$$

So $G(u_T^n)$ converges to $G(u_T)$ in $L^1(\Omega \times W)$ and $\int_0^T \varphi'(t)G(u_t^n)\mathrm{d}t$ to $\int_0^T \varphi'(t)G(u_t)\mathrm{d}t$. Moreover, as G'' is bounded and continuous, we obtain that $\varphi(t)G''(u_t^n)|\bar{g}_t^n|^2$ tends to $\varphi(t)G''(u_t)|\bar{g}_t|^2$ in $L^1([0, T] \times \Omega \times W)$ (once again, to see this, assume it is not true and use a double extraction procedure to yield a contradiction). As a consequence,

$$\int_0^T \varphi(t)G'(u_t^n)\mathrm{d}u_t^n$$

converges to

$$\int_0^T \varphi(t)G'(u_t)\mathrm{d}u_t$$

in $L^1(\Omega \times W)$. Since, for all $n \in \mathbb{N}$,

$$\mathrm{E}\left[\int_W \int_0^T \varphi(t)G'(u_t^n)\mathrm{d}u_t^n(x)\mathrm{d}x\right] = -\mathrm{E}\left[\int_0^T \varphi(t)e(G'(u_t^n), u_t^n)\mathrm{d}t\right]$$
$$+\,\mathrm{E}\left[\int_0^T \varphi(t)(G'(u_t^n), \bar{f}_t^n)_{L^2(W,m)}\mathrm{d}t\right],$$

we just have to let n tend to $+\infty$ to obtain the desired equality. Once again, we use the fact that as (F, e) is a Dirichlet form, there exists a positive constant C such that

$$\forall n \in \mathbb{N}, \forall t \in [0, T], \qquad e(G'(u_t^n), G'(u_t^n)) \leqslant Ce(u_t^n, u_t^n),$$

which ensures, for example, that

$$\mathrm{E}\left[\int_0^T \varphi(t)e(G'(u_t^n), u_t^n)\mathrm{d}t\right]$$

converges to

$$\mathrm{E}\left[\int_0^T \varphi(t)e(G'(u_t), u_t)\mathrm{d}t\right].$$

□

Remark. The hypothesis $G(0) = 0$ is not necessary if $m(W) < +\infty$.

4.3.3. The theorem of existence and uniqueness

Theorem 4.20. *Under Hypotheses* 3.1–3.3*, equation* (3.1) *admits a unique solution in* $\mathcal{P}(F)$.

Proof. We will prove existence and uniqueness of the solution of the mild equation. For this purpose, let γ, δ be positive constants. We consider the following norm on $\mathcal{P}(F)$:

$$\forall u \in \mathcal{P}(F), \qquad \|u\|_{\gamma,\delta} = \mathrm{E}\left(\int_0^T e^{-\gamma t}(\delta\|u_t\|^2_{L^2(W,m)} + e(u_t, u_t))\mathrm{d}t\right).$$

It is clear that $\|\cdot\|_{\gamma,\delta}$ is equivalent to $\|\cdot\|_{\mathcal{P}(F)}$. We consider the map, Λ, from $\mathcal{P}(F)$ into $\mathcal{P}(F)$ defined, for all $u \in \mathcal{P}(F)$, for all $(t, x) \in [0, T[\times W$, by

$$\begin{aligned}\Lambda u(t, x) = P_t\Phi(x) &+ \int_0^t P_{t-s}f(s, \cdot, u(s, \cdot), Du(s, \cdot))(x)\,\mathrm{d}s \\ &+ \int_0^t P_{t-s}g(s, \cdot, u(s, \cdot), Du(s, \cdot))(x)\,\mathrm{d}B_s.\end{aligned}$$

Let u and v be in $\mathcal{P}(F)$. We put:

$$\begin{aligned}\forall s \in [0, T], &\qquad \bar{f}_s = f(s, \cdot, u_s, Du_s) - f(s, \cdot, v_s, \mathrm{D}v_s), \\ \forall s \in [0, T], &\qquad \bar{g}_s = g(s, \cdot, u_s, \mathrm{D}u_s) - g(s, \cdot, v_s, Dv_s), \\ \forall t \in [0, T], &\qquad \bar{u}_t = \Lambda(u)_t - \Lambda(v)_t = \int_0^t P_{t-s}\bar{f}_s\,\mathrm{d}s + \int_0^t P_{t-s}\bar{g}_s\,\mathrm{d}B_s.\end{aligned}$$

Henceforth, we fix Θ, a sequence of subdivisions of $[0, T]$ whose mesh tends to 0. By Itô's formula, we obtain

$$\begin{aligned}\mathrm{e}^{-\gamma T}\bar{u}_T^2 &= -\gamma\int_0^T \mathrm{e}^{-\gamma s}\bar{u}_s^2\,\mathrm{d}s + 2\int_0^T \mathrm{e}^{-\gamma s}\bar{u}_s\,\mathrm{d}\bar{u}_s(x) + \int_0^T \mathrm{e}^{-\gamma s}\,\mathrm{d}\langle\bar{u}, \bar{u}\rangle_s \\ &= -\gamma\int_0^T \mathrm{e}^{-\gamma s}\bar{u}_s^2\,\mathrm{d}s + 2\int_0^T \mathrm{e}^{-\gamma s}\bar{u}_s\,\mathrm{d}\bar{u}_s + \int_0^T \mathrm{e}^{-\gamma s}|\bar{g}_s|^2\,\mathrm{d}s.\end{aligned}$$

Now, thanks to Proposition 4.19, we have that

$$\begin{aligned}\mathrm{E}\left(\int_W\int_0^T \mathrm{e}^{-\gamma s}\bar{u}_s(x)\,\mathrm{d}\bar{u}_s(x)m(\mathrm{d}x)\right) &= -\mathrm{E}\left(\int_0^T \mathrm{e}^{-\gamma s}e(\bar{u}_s, \bar{u}_s)\mathrm{d}s\right) \\ &\quad + \mathrm{E}\left(\int_0^T \mathrm{e}^{-\gamma s}(\bar{u}_s, \bar{f}_s)_{L^2(W,m)}\,\mathrm{d}s\right).\end{aligned}$$

This yields

$$\begin{aligned}\mathrm{E}(\mathrm{e}^{-\gamma T}\|\bar{u}_T\|^2_{L^2(W,m)}) = -\gamma\mathrm{E}\left(\int_0^T \mathrm{e}^{-\gamma s}\|\bar{u}_s\|^2_{L^2(W,m)}\,\mathrm{d}s\right)\\ -2\mathrm{E}\left(\int_0^T \mathrm{e}^{-\gamma s}e(\bar{u}_s,\,\bar{u}_s)\mathrm{d}s\right)\\ +2\mathrm{E}\left(\int_0^T \mathrm{e}^{-\gamma s}(\bar{u}_s,\,\bar{f}_s)_{L^2(W,m)}\,\mathrm{d}s\right)\\ +\mathrm{E}\left(\int_0^T \mathrm{e}^{-\gamma s}|||\bar{g}_s|||^2_{L^2(W,m)}\,\mathrm{d}s\right).\end{aligned}$$

Then, using the hypotheses satisfied by g, we have

$$\begin{aligned}\gamma\mathrm{E}\left(\int_0^T \mathrm{e}^{-\gamma s}\|\bar{u}_s\|^2_{L^2(W,m)}\,\mathrm{d}s\right)+2\mathrm{E}\left(\int_0^T \mathrm{e}^{-\gamma s}e(\bar{u}_s,\,\bar{u}_s)\mathrm{d}s\right)\leqslant 2\mathrm{E}\left(\int_0^T \mathrm{e}^{-\gamma s}(\bar{u}_s,\,\bar{f}_s)_{L^2(W,m)}\,\mathrm{d}s\right)\\ +C\mathrm{E}\left(\int_0^T \mathrm{e}^{-\gamma s}\|u_s-v_s\|^2_{L^2(W,m)}\,\mathrm{d}s\right)+\alpha\mathrm{E}\left(\int_0^T \mathrm{e}^{-\gamma s}e(u_s-v_s,\,u_s-v_s)\mathrm{d}s\right),\end{aligned}$$

where C and $\alpha\in[0,\,2[$ are the constants which appear in Hypotheses 3.1–3.3. Moreover, using hypotheses on f, we have, for all $\varepsilon>0$,

$$\begin{aligned}2\mathrm{E}\left(\int_0^T \mathrm{e}^{-\gamma s}(\bar{u}_s,\,\bar{f}_s)_{L^2(W,m)}\,\mathrm{d}s\right)\leqslant\frac{1}{\varepsilon}\mathrm{E}\left(\int_0^T \mathrm{e}^{-\gamma s}\|\bar{u}_s\|^2_{L^2(W,m)}\,\mathrm{d}s\right)\\ +\varepsilon\mathrm{E}\left(\int_0^T \mathrm{e}^{-\gamma s}\|\bar{f}_s\|^2_{L^2(W,m)}\,\mathrm{d}s\right)\\ \leqslant\frac{1}{\varepsilon}\mathrm{E}\left(\int_0^T \mathrm{e}^{-\gamma s}\|\bar{u}_s\|^2_{L^2(W,m)}\,\mathrm{d}s\right)\\ +C\varepsilon\mathrm{E}\left(\int_0^T \mathrm{e}^{-\gamma s}\|u_s-v_s\|^2_{L^2(W,m)}\,\mathrm{d}s\right)\\ +C\varepsilon\mathrm{E}\left(\int_0^T \mathrm{e}^{-\gamma s}e(u_s-v_s,\,u_s-v_s)\mathrm{d}s\right).\end{aligned}$$

Finally, we have that

$$\begin{aligned}(\gamma-1/\varepsilon)\mathrm{E}\left(\int_0^T \mathrm{e}^{-\gamma s}\|\bar{u}_s\|^2_{L^2(W,m)}\,\mathrm{d}s\right)+2\mathrm{E}\left(\int_0^T \mathrm{e}^{-\gamma s}e(\bar{u}_s,\,\bar{u}_s)\mathrm{d}s\right)\\ \leqslant C(1+\varepsilon)\mathrm{E}\left(\int_0^T \mathrm{e}^{-\gamma s}\|u_s-v_s\|^2_{L^2(W,m)}\,\mathrm{d}s\right)+(C\varepsilon+\alpha)\mathrm{E}\left(\int_0^T \mathrm{e}^{-\gamma s}e(u_s-v_s,\,u_s-v_s)\mathrm{d}s\right).\end{aligned}$$

Now, we choose ε small enough and then γ such that

$$C\varepsilon + \alpha < 2 \quad \text{and} \quad \frac{\gamma - 1/\varepsilon}{2} \times (C\varepsilon + \alpha) = C(1 + \varepsilon).$$

If we set $\delta = (\gamma - 1/\varepsilon)/2$, then

$$\forall u, v \in \mathcal{P}(F)^2, \qquad \|\Lambda(u) - \Lambda(v)\|_{\gamma,\delta} \leqslant \frac{(C\varepsilon + \alpha)}{2} \|u - v\|_{\gamma,\delta}.$$

We conclude thanks to the fixed point theorem. □

Theorem 4.21. *Assume Hypotheses* 3.1–3.3. *Let u be the unique solution of* (3.1) *in* $\mathcal{P}(F)$ *and* $\Theta = (\Delta_N)_{N \in \mathbb{N}^*}$ *a sequence of subdivisions of* $[0, T]$ *such that*

$$\lim_{N \to +\infty} |\Delta_N| = 0.$$

Then u admits a version that we still denote by u, which belongs to $\tilde{\mathbb{D}}_T^{\Theta}(L^2(W, m))$. *Moreover, its martingale part is*

$$t \to \int_0^t g(s, \cdot, u_s, Du_s)\, \mathrm{d}B_s,$$

and the zero quadratic variation part is

$$t \to \Phi + L\left(\int_0^t u_s\, \mathrm{d}s\right) + \int_0^t f(s, \cdot, u_s, Du_s)\mathrm{d}s.$$

Proof. As u satisfies the mild equation, this is a simple consequence of Propositions 4.2, 4.5 and 4.7. □

Remark. As a consequence, we have that $t \to u_t$ is $L^2(W, m)$-continuous P-a.e.

Theorem 4.22. *Let* $\Theta = (\Delta_N)_{N \in \mathbb{N}^*}$ *a sequence of subdivisions of* $[0, T]$ *such that*

$$\lim_{N \to +\infty} |\Delta_N| = 0.$$

Assume that f and Φ satisfy hypotheses 3.1–3.3 *and that g is defined from* $[0, T] \times \mathbb{R}$ *with values in* $\mathbb{R}^d$, *measurable and satisfies:*

(i) $g(\cdot, 0) \in L^2([0, T]; \mathbb{R}^d)$;
(ii) *there exists* $C > 0$ *such that, for all* t, y, y',

$$|g(t, y) - g(t, y')| \leqslant C|y - y'|.$$

Let u be the unique solution in $\mathcal{P}(F)$ of the SPDE

$$u(t, x) = \Phi(x) + \int_0^t Lu(s, x) + f(s, x, u(s, x), Du(s, x))\mathrm{d}s + \int_0^t g(s, u(s, x))\mathrm{d}B_s. \tag{4.7}$$

Then there exists a subsequence, γ, in Θ such that, for m-almost all $x \in W$, the map

$$t \in [0, T] \to u_t(x)$$

belongs to $\mathbb{D}_T^\gamma$.

Proof. First of all, we know that equation (4.7) admits a unique solution in $\mathcal{P}(F)$. Then, as g is uniformly Lipschitz, this yields that the map $t \in [0, T] \to g(t, u_t)$ belongs to $\mathcal{P}(F)$ (see Section 4.3.1). So the result is a consequence of Propositions 4.5, 4.8 and 4.2.

The fact that we can choose the same subsequence, γ, which appears in those propositions is easy to prove, either by a double extraction argument or by extracting an m-almost everywhere subsequence simultaneously in the proofs of Propostions 4.5 and 4.8. $\square$

Remark. As a consequence we have that, in this case, for m-almost all $x \in W$, $t \to u_t(x)$ is continuous P-a.e.

5. Application: a comparison theorem for parabolic SPDEs

One way to prove comparison theorems for partial differential equations is to use the probabilistic interpretation and thus the Itô calculus; for example, for PDEs with terminal condition, one can use the theory of backward differential equations (see Pardoux 1998) and for SPDEs one can use the doubly stochastic interpretation (see Bally and Matoussi 2001). We give here a direct proof based on stochastic calculus associated with Dirichlet processes which allows us to deal with a more general case and with SPDEs whose coefficient in front of the noise depends on the gradient of u.

We continue to consider equation (3.1), now rewritten as

$$\begin{aligned} \frac{\partial u_t}{\partial t} &= Lu_t + f(t, \cdot, u_t, Du_t) + g(t, \cdot, u_t, Du_t)\frac{\mathrm{d}B_t}{\mathrm{d}t}, \\ u_0 &= \Phi, \end{aligned}$$

with unique solution u.

Consider $\tilde{\Phi} \in L^2(W, m)$ and

$$\tilde{f} : \Omega \times [0, T] \times W \times \mathbb{R} \times K \to \mathbb{R}$$

which satisfies the same hypotheses as f (see Section 3.1). We denote by $\tilde{u}$ the unique solution of

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial t} &= L\tilde{u}_t + \tilde{f}(t, \cdot, \tilde{u}_t, D\tilde{u}_t) + g(t, \cdot, \tilde{u}_t, D\tilde{u}_t)\frac{\mathrm{d}B_t}{\mathrm{d}t}, \\ \tilde{u}_0 &= \tilde{\Phi}. \end{aligned}$$

Finally, we define, for all $t \in [0, T]$,

$$\hat{u}_t = u_t - \tilde{u}_t,$$

$$\hat{f}_t = f(t, \cdot, u_t, Du_t) - \tilde{f}(t, \cdot, \tilde{u}_t, D\tilde{u}_t),$$

$$\hat{g}_t = g(t, \cdot, u_t, Du_t) - g(t, \cdot, \tilde{u}_t, D\tilde{u}_t).$$

As in the case of (forward or backward) stochastic differential equations, the main idea is to evaluate $\mathrm{E}(\|\hat{u}_t^+\|^2_{L^2(W,m)})$ using Itôs formula and then to apply Gronwall's lemma.

Lemma 5.1. *Assume (F, e) is local. Then, for all $t \in [0, T]$,*

$$\begin{aligned}\mathrm{E}(\|\hat{u}_t^+\|^2_{L^2(W,m)}) = \mathrm{E}(((\Phi - \tilde{\Phi})^+)^2) - 2\mathrm{E}(\int_0^t \mathrm{e}(\hat{u}_s^+, \hat{u}_s^+)\mathrm{d}s) \\ + 2\mathrm{E}\left(\int_0^t (\hat{u}_s^+, \hat{f}_s)_{L^2(W,m)}\,\mathrm{d}s\right) \\ + \mathrm{E}\left(\int_0^t \|\mathbf{1}_{\{\hat{u}_s>0\}}|\hat{g}_s|\|^2_{L^2(W,m)}\,\mathrm{d}s\right).\end{aligned}$$

Proof. First of all, we fix Θ a sequence of subdivisions of $[0, T]$ whose mesh tends to 0. Thanks to the previous results, we know that $\hat{u}$ belongs to $\mathbb{D}_T^\Theta(L^2(W, m))$. To prove the lemma, we approximate the function $\psi : y \in \mathbb{R} \to (y^+)^2$ by a sequence $(\psi_n)_{n\in\mathbb{N}^*}$ of regular functions. Throughout this proof, the constant C may change from line to line.

Let φ be a C^∞ increasing function such that

$$\forall y \in]-\infty, 1], \quad \varphi(y) = 0, \qquad \forall y \in [2, +\infty[, \quad \varphi(y) = 1.$$

We set, for all $n \in \mathbb{N}^*$,

$$\forall y \in \mathbb{R}, \qquad \psi_n(y) = y^2\varphi(ny).$$

It is easy to verify that $(\psi_n)_{n\in\mathbb{N}^*}$ converges uniformly to the function ψ and that, moreover, we have the estimates

$$\forall y \in \mathbb{R}^+, \forall n \in \mathbb{N}^*, \qquad 0 \leqslant \psi_n(y) \leqslant \psi(y),\ 0 \leqslant \psi_n'(y) \leqslant Cy,\ |\psi_n''(y)| \leqslant C.$$

Thanks to Itôs formula, for all $n \in \mathbb{N}^*$ and $t \in [0, T]$, we have $m \otimes P$-almost everywhere

$$\psi_n(\hat{u}_t) = \psi_n(\Phi - \tilde{\Phi}) + \int_0^t \psi_n'(\hat{u}_s)\mathrm{d}\hat{u}_s + \frac{1}{2}\int_0^t \psi_n''(\hat{u}_s)\mathrm{d}\langle \hat{u}, \hat{u}\rangle_s.$$

As the martingale part of $\hat{u}$ is $\int_0^\cdot \hat{g}_s\,\mathrm{d}B_s$,

$$\int_0^t \psi_n''(\hat{u}_s)\mathrm{d}\langle \hat{u}, \hat{u}\rangle_s = \int_0^t \psi_n''(\hat{u}_s)|\hat{g}|_s^2\,\mathrm{d}s.$$

Proposition 4.19 yields

$$\mathrm{E}\left[\int_W \int_0^t \psi_n'(\hat{u}_s(x))\mathrm{d}\hat{u}_s(x)m(\mathrm{d}x)\right] = -\mathrm{E}\left[\int_0^t e(\psi_n'(\hat{u}_s),\, \hat{u}_s)\mathrm{d}s\right]$$
$$+\,\mathrm{E}\left[\int_0^t (\psi_n'(\hat{u}_s),\, \hat{f}_s)_{L^2(W,m)}\,\mathrm{d}s\right].$$

Thanks to the properties of Dirichlet forms we recalled in Section 4.3.1 and as ψ_n' is differientiable with (uniformly) bounded derivative, we know that, for all $v \in F$, $\psi_n'(v)$ belongs to F and, moreover,

$$e(\psi_n'(v),\, \psi_n'(v)) \leqslant Ce(v,\, v),$$

and so, for all $s \in [0,\, t]$ and $n \in \mathbb{N}^*$,

$$|e(\psi_n'(\hat{u}_s),\, \hat{u}_s)| \leqslant Ce(\hat{u}_s,\, \hat{u}_s).$$

At this stage, we have proved that, for all $n \in \mathbb{N}^*$ and all $t \in [0,\, T]$,

$$\mathrm{E}\left(\int_W \psi_n(\hat{u}_t(x))m(\mathrm{d}x)\right) = \mathrm{E}(\psi_n(\Phi - \tilde{\Phi})) - \mathrm{E}\left(\int_0^t e(\psi_n'(\hat{u}_s),\, \hat{u}_s)\mathrm{d}s\right)$$
$$+\,\mathrm{E}\left(\int_0^t (\psi_n'(\hat{u}_s),\, \hat{f}_s)_{L^2(W,m)}\,\mathrm{d}s\right)$$
$$+\,\frac{1}{2}\mathrm{E}\left(\int_W \int_0^t \psi_n''(\hat{u}_s(x))|\hat{g}|_s^2(x)\mathrm{d}s\,m(\mathrm{d}x)\right).$$

The dominated convergence theorem yields that $\psi_n''(\hat{u}_s(x))|\hat{g}|_s^2$ converges to $2\mathbf{1}_{\{\hat{u}_s>0\}}|\hat{g}_s|^2$ in $L^1(\Omega \times W \times [0,\, T],\, P \otimes m \otimes \mathrm{d}t)$ as n tends to $+\infty$ and it is clear that

$$\lim_{n\to+\infty} \mathrm{E}(\psi_n(\Phi - \tilde{\Phi})) = \mathrm{E}(((\Phi - \tilde{\Phi})^+)^2).$$

Let $s \in [0,\, T]$; as $(\psi_n'')_n$ converges λ-a.s. to the function $(t \in \mathbb{R} \to 2\mathbf{1}_{\{t>0\}})$), Proposition 4.13 ensures that $\psi_n'(\hat{u}_s)$ converges to $2\hat{u}_s^+$ in F. So, for all $s \in [0,\, T]$,

$$\lim_{n\to+\infty} e(\psi_n'(\hat{u}_s),\, \hat{u}_s) = 2e(\hat{u}_s^+,\, \hat{u}_s) = 2e(\hat{u}_s^+,\, \hat{u}_s^+),$$

because we assume that $(F,\, e)$ is local. This easily yields

$$\lim_{n\to+\infty} \mathrm{E}\left(\int_0^t e(\psi_n'(\hat{u}_s),\, \hat{u}_s)\mathrm{d}s\right) = 2\mathrm{E}\left(\int_0^t e(\hat{u}_s^+,\, \hat{u}_s^+)\mathrm{d}s\right).$$

In the same way, one has

$$\lim_{n\to+\infty} \mathrm{E}\left(\int_0^t (\psi_n'(\hat{u}_s),\, \hat{f}_s)_{L^2(W,m)}\mathrm{d}s\right) = 2\mathrm{E}\left(\int_0^t (\hat{u}_s^+,\, \hat{f}_s)_{L^2(W,m)}\,\mathrm{d}s\right),$$

and the proof is complete. □

We are now able to prove the comparison theorem:

Theorem 5.2. *Assume* (F, e) *is local,* $\Phi \leqslant \tilde{\Phi}$ *m-a.e. and*

$$f(t, x, u_t(x), Du_t(x)) \leqslant \tilde{f}(t, x, u_t(x), Du_t(x)) \qquad \mathrm{d}t \otimes m \otimes P\text{-a.e.}$$

Then

$$\forall t \in [0, T], \qquad u_t \leqslant \tilde{u}_t \quad m \otimes P\text{-a.e.},$$

Proof. Thanks to our hypotheses on g, we have that, for $\mathrm{d}t \otimes m$-almost all (s, x),

$$|\hat{g}_s(x)|^2 \leqslant C|\hat{u}_s(x)|^2 + \alpha \|D\hat{u}_s(x)\|_K^2 \qquad P\text{-a.e.},$$

where α is a constant in $[0, 2[$.

Using Corollary 4.16, we obtain that, for $\mathrm{d}t \otimes m$-almost all (s, x),

$$\begin{aligned}\mathbf{1}_{\{\hat{u}_s(x)>0\}}|\hat{g}_s(x)|^2 &\leqslant C\mathbf{1}_{\{\hat{u}_s(x)>0\}}|\hat{u}_s(x)|^2 + \alpha \mathbf{1}_{\{\hat{u}_s(x)>0\}}\|D\hat{u}_s(x)\|_K^2 \\ &= C|\hat{u}_s^+(x)|^2 + \alpha\|D\hat{u}_s^+(x)\|_K^2 \qquad P\text{-a.e.},\end{aligned}$$

and so

$$\begin{aligned}\mathrm{E}\left(\int_0^t \|\mathbf{1}_{\{\hat{u}_s(x)>0\}}\hat{g}_s(x)\|_{L^2(W,m)}^2\,\mathrm{d}s\right) &\leqslant C\mathrm{E}\left(\int_0^t \|\hat{u}_s^+(x)\|_{L^2(W,m)}^2\,\mathrm{d}s\right) \\ &\quad + \alpha\mathrm{E}\left(\int_0^t e(\hat{u}_s^+, \hat{u}_s^+)\mathrm{d}s\right).\end{aligned}$$

This yields, thanks to the previous lemma,

$$\begin{aligned}\mathrm{E}\left(\|\hat{u}_t^+(x)\|_{L^2(W,m)}^2\right) + 2(1-\alpha)\mathrm{E}\left(\int_0^t e(\hat{u}_s^+, \hat{u}_s^+)\mathrm{d}s\right) &\leqslant C\mathrm{E}\left(\int_0^t \|\hat{u}_s^+(x)\|_{L^2(W,m)}^2\,\mathrm{d}s\right) \\ &\quad + 2\mathrm{E}\left(\int_0^t (\hat{u}_s^+, \hat{f}_s)_{L^2(W,m)}\,\mathrm{d}s\right).\end{aligned}$$

We now decompose $\hat{f}$ in the following way

$$\hat{f}_s = \{f(s, \cdot, u_s, Du_s) - \tilde{f}(s, \cdot, u_s, Du_s)\} + \{\tilde{f}(s, \cdot, u_s, Du_s) - \tilde{f}(s, \cdot, \tilde{u}_s, D\tilde{u}_s)\}.$$

As we assumed that $f \leqslant \tilde{f}$,

$$\begin{aligned}\hat{u}_s^+ \cdot \hat{f}_s &= \hat{u}_s^+ \cdot \{f(s, \cdot, u_s, Du_s) - \tilde{f}(s, \cdot, u_s, Du_s)\} + \hat{u}_s^+ \cdot \{\tilde{f}(s, \cdot, u_s, Du_s) - \tilde{f}(s, \cdot, \tilde{u}_s, D\tilde{u}_s) \\ &\leqslant \hat{u}_s^+ \cdot \{\tilde{f}(s, \cdot, u_s, Du_s) - \tilde{f}(s, \cdot, \tilde{u}_s, D\tilde{u}_s)\}\end{aligned}$$

and so, thanks to our assumptions on $\tilde{f}$, for m-almost all $x \in W$,

$$\begin{aligned}|\hat{u}_s^+(x) \cdot \hat{f}_s(x)| &\leqslant C\hat{u}_s^+(x) \cdot (|\hat{u}_s(x)| + \|D\hat{u}_s(x)\|_K) \\ &= C|\hat{u}_s^+(x)|^2 + C\hat{u}_s^+(x) \times \mathbf{1}_{\{u_s(x)>0\}}\|D\hat{u}_s(x)\|_K \\ &= C|\hat{u}_s^+(x)|^2 + C\hat{u}_s^+(x)\|D\hat{u}_s^+(x)\|_K.\end{aligned}$$

Once again, we use the inequality $2ab \leqslant \varepsilon a^2 + \frac{b^2}{\varepsilon}$ to conclude that, for any $\varepsilon > 0$, there exists a constant $C > 0$ such that, for almost all $s \in [0, T]$ and almost all $x \in W$,

$$|\hat{u}_s^+(x)\hat{f}_s(x)| \leqslant C(\hat{u}_s^+(x))^2 + \varepsilon\|\mathrm{D}\hat{u}_s(x)\|_K^2.$$

This yields

$$\mathrm{E}(\|\hat{u}_t^+(x)\|^2_{L^2(W,m)}) + (2 - \alpha - \varepsilon)\mathrm{E}\left(\int_0^t e(\hat{u}_s, \hat{u}_s)\mathrm{d}s\right) \leqslant C\mathrm{E}\left(\int_0^t \|\hat{u}_s^+(x)\|^2_{L^2(W,m)}\,\mathrm{d}s\right).$$

Taking ε small enough, we obtain that, for all $t \in [0, T]$,

$$\mathrm{E}(\|\hat{u}_t^+(x)\|^2_{L^2(W,m)}) \leqslant C\mathrm{E}\left(\int_0^t \|\hat{u}_s^+(x)\|^2_{L^2(W,m)}\,\mathrm{d}s\right),$$

and we conclude thanks to Gronwall's lemma.

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