Bernoulli **10**(2), 2004, 327–355

Stable limits of sums of bounded functions of long-memory moving averages with finite variance

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We discuss limit distributions of partial sums of bounded functions h of a long-memory moving-average process $X_t = \sum_{j=1}^{\infty} b_j \zeta_{t-j}$ with coefficients b_j decaying as $j^{-\beta}$, $1/2 < \beta < 1$, and independent and identically distributed innovations ζ_s whose probability tails decay as $x^{-\alpha}$, $2 < \alpha < 4$. The case of h having Appell rank $k_* = 2$ or 3 is discussed in detail. We show that in this case and in the parameter region $\alpha\beta < 2$, the partial sums process, normalized by $N^{1/\alpha\beta}$, weakly converges to an $\alpha\beta$-stable Lévy process, provided that the normalization dominates the corresponding k_*th-order Hermite process normalization, or that $1/\alpha\beta > 1 - (2\beta - 1)k_*/2$. A complete characterization of limit distributions of the partial sums process remains open.

Keywords: Appell rank; fractional derivative; Hermite process; long memory; moving-average process; partial sums process; stable Lévy process

1. Introduction

We discuss limit distributions of partial sums processes

$$S_{N,h}(\tau) := \sum_{t=1}^{[N\tau]} (h(X_t) - \mathrm{E}h(X_t)), \qquad \tau \geqslant 0, \tag{1.1}$$

where h is a (nonlinear) function, and X_t, $t \in \mathbb{Z}$, is a long-memory moving-average process

$$X_t = \sum_{j=1}^{\infty} b_j \zeta_{t-j} \tag{1.2}$$

in standardized (i.e. zero mean and unit variance) independent and identically distributed (i.i.d.) innovations ζ_t, $t \in \mathbb{Z}$, whose coefficients b_j, $j \geqslant 1$, are non-random and decay hyperbolically:

$$b_j \sim c_0 j^{-\beta} \qquad (j \to \infty) \tag{1.3}$$

for some constants $c_0 \neq 0$, $\beta \in (1/2, 1)$. We expect that our main results (Theorems 2.1 and 2.2 below) also hold, suitably modified, in the more general case $b_j = \ell(j) j^{-\beta}$, where $\ell(x)$, $x \in [1, \infty)$, is slowly varying at infinity. The stronger assumption (1.3) is chosen mainly

1350–7265

to avoid additional technicalities and to simplify notation. Assumption (1.3) is satisfied for some important parametric families of long-memory processes (1.1) such as FARIMA(p, d, q), $d = 1 - \beta \in (0, 1/2)$, defined by the autoregressive equation $\phi(B)(1-B)^d X_t = \psi(B)\zeta_t$, where B is the backward shift operator, $(1-B)^d$ is the fractional differencing operator, and $\phi(B)$, $\psi(B)$ are polynomials in B of degree p, q, respectively, $\phi(\cdot)$ satisfying the usual root requirement for stationarity of the process (see, for example, Brockwell and Davies 1991). In this case, the asymptotic constant in (1.3) equals $c_0 = |\psi(1)|/(|\phi(1)|\Gamma(d))$ (Hosking 1981).

Moving averages (1.2)–(1.3) constitute probably the most important class of long-memory processes. The problem of the limiting behaviour of partial sums processes (1.1) has been investigated by many authors; see, for example, Dobrushin and Major (1979), Taqqu (1975; 1979), Surgailis (1982; 2000), Breuer and Major (1983), Avram and Taqqu (1987), Dehling and Taqqu (1989), Giraitis *et al.* (1996), Ho and Hsing (1996; 1997), Koul and Surgailis (1997; 2002), Wu (2003) and the references therein. Set

$$h_\infty(x) := \mathrm{E}h(x + X_0), \qquad x \in \mathbb{R}. \tag{1.4}$$

Let $h_\infty^{(k)}(x) = \mathrm{d}^k h_\infty(x)/\mathrm{d}x^k$ denote the kth derivative of $h_\infty (k = 1, 2, \ldots)$, provided it exists. As shown in Ho and Hsing (1996; 1997) (see also Koul and Surgailis 1997; Wu 2003), the limit distribution of $S_{N,h}(\tau)$ is determined by the integer

$$k_* := \min\{k \geqslant 1\colon h_\infty^{(k)}(0) \neq 0\}, \tag{1.5}$$

called the *Appell rank of h* (Koul and Surgailis 1997; Surgailis 2000) or the *power rank of h* (Ho and Hsing 1996; 1997). More precisely, under some growth condition on h and some moment and regularity conditions on the cdf of the innovations, and for $k_*(2\beta - 1) < 1$, one has the finite-dimensional convergence

$$N^{-1+(k_*/2)(2\beta-1)} S_{N,h}(\tau) \Rightarrow \frac{c_0^{k_*}}{k_*!} h_\infty^{(k_*)}(0) J^{(k_*)}(\tau), \tag{1.6}$$

where $J^{(k)}(\tau)$ is a Hermite process of order k (see Section 2 for the definition). The last result is well known if the long-memory process X_t is Gaussian (Dobrushin and Major 1979; Taqqu 1979). On the other hand, if $k_*(2\beta - 1) > 1$, then $N^{-1/2} S_{N,h}(\tau)$ tends in distribution to a Brownian motion with variance depending on h (Ho and Hsing 1997; Koul and Surgailis 1997).

The notion of Appell rank generalizes that of *Hermite rank* which is defined, for any function $h \in L^2(\mathbb{R}, \mathrm{e}^{-x^2/2}\,\mathrm{d}x)$, as the smallest index $k \geqslant 1$ with $c_k \neq 0$ in the expansion $h(x) = \sum_{k=0}^\infty c_k H_k(x)/k!$ in Hermite polynomials with the generating series $\sum_{k=0}^\infty (\mathrm{i}z)^k H_k(x)/k! = \mathrm{e}^{\mathrm{i}zX}/\mathrm{E}\mathrm{e}^{\mathrm{i}zX}$, $X \sim N(0, 1)$. *Appell polynomials* $A_k(x)$, $k \geqslant 0$, are defined by an analogous generating function: $\sum_{k=0}^\infty (\mathrm{i}z)^k A_k(x)/k! = \mathrm{e}^{\mathrm{i}zX}/\mathrm{E}\mathrm{e}^{\mathrm{i}zX}$, where X is an arbitrary random variable having all finite moments (Avram and Taqqu 1987; Surgailis 2000). Unfortunately, Appell polynomials lack the orthogonality property of Hermite polynomials, and the formal Appell expansion

$$h(x) = \sum_{k=0}^\infty a_k A_k(x)/k! \tag{1.7}$$

has been justified for h satisfying some strong analyticity conditions only; see, for example, Kaz'min (1969). (To stress the formality of this expansion, note that even its terms starting with some $k > k_0$ need not be well defined, as Appell polynomials exist under corresponding moment conditions only.) Nevertheless, the coefficients of (1.7), $a_k = \mathrm{d}^k \mathrm{E}h(y+X)/\mathrm{d}y^k|_{y=0}$, as well as the notion of Appell rank, make sense under fairly mild regularity conditions on h and the distribution of X, in particular on the marginal distribution $X = X_0$ of the linear process of (1.2)–(1.3). Following the suggestion of a referee, in what follows we use the notation

$$a_k = h_\infty^{(k)}(0), \tag{1.8}$$

invoking the relationship between long-memory properties of partial sums process $S_{N,h}(\tau)$ and the formal Appell expansion (1.7). One may conclude that the (nonlinear) process $h(X_t)$ has long-memory if $k_*(2\beta - 1) < 1$ and short memory if $k_*(2\beta - 1) > 1$; moreover, in the long-memory case the limit law of the partial sums process is determined by the first non-zero coefficent a_{k_*} of the formal Appell expansion (1.7).

However, the above characterization of the limit behaviour of partial sums (1.1) holds under sufficiently high moment assumptions of the innovations only. In particular, the fourth moment condition $\mathrm{E}\zeta_0^4 < \infty$ seems crucial. More precisely, the 'first-order' convergence

$$N^{-1+(1/2)(2\beta-1)} S_{N,h}(\tau) \Rightarrow a_1 c_0 J^{(1)}(\tau) \tag{1.9}$$

to a fractional Brownian motion $J^{(1)}(\tau)$, holds for bounded h under mild conditions on the innovations, namely $\mathrm{E}|\zeta_0|^{2+\delta} < \infty (\exists \delta > 0)$ and condition (2.1) below; see Giraitis and Surgailis (1999). Of course, (1.9) solves the problem of the limit distribution of $S_{N,h}(\tau)$ for bounded h having Appell rank $k_* = 1$ only. If $\mathrm{E}\zeta_0^4 = \infty$ and $k_* > 1$, the problem becomes much harder and is still open. We show that in this case the class of the limit laws of partial sums (1.1) is richer and contains stable Lévy processes in particular.

In the the present paper we assume that distribution tails of the innovations decay regularly at infinity, that is,

$$P(\zeta_0 \leqslant -x) \sim q_- x^{-\alpha}, \qquad P(\zeta_0 > x) \sim q_+ x^{-\alpha}, \tag{1.10}$$

as $x \to \infty$, where $q_+, q_- \geqslant 0$ are constants such that $q_+ + q_- > 0$, and $2 < \alpha < 4$. Assumption (1.10) implies that $\mathrm{E}|\zeta_0|^\alpha = \infty$ and $\mathrm{E}|\zeta_0|^r < \infty$, for all $r < \alpha$. Our main results refer to the case of Appell rank $k_* = 2$ and $k_* = 3$. They are summarized in Tables 1 and 2.

Note that for $\alpha \geqslant 8/3$, the middle column in Table 1 vanishes, $\alpha = 8/3$ being a root of $(\alpha + \sqrt{\alpha^2 - 2\alpha})/2\alpha = 2/\alpha$. Similarly, for $\alpha \geqslant 3$, the middle column in Table 2 vanishes, $\alpha = 3$ being a root of $(5\alpha + \sqrt{25\alpha^2 - 48\alpha})/12\alpha = 2/\alpha$. The cases $k_* = 2, 8/3 \leqslant \alpha < 4$ and $k_* = 3, 3 \leqslant \alpha < 4$ are somewhat less interesting, in the sense that the limit laws of partial sums process are the same as in the case $\mathrm{E}\zeta_0^4 < \infty$ studied in Ho and Hsing (1997) and elsewhere.

Note that for both $k_* = 2$ and $k_* = 3$, the limit of $S_{N,h}(\tau)$ is determined by the dominant normalization of the three types of limiting behaviour:

(I) Hermite process of order k_*, normalization $N^{1-(2\beta-1)k_*/2}$;

Table 1. Summary of results for $k_* = 2, 2 < \alpha < 8/3$

	$\frac{1}{2} < \beta < \frac{\alpha + \sqrt{\alpha^2 - 2\alpha}}{2\alpha}$	$\frac{\alpha + \sqrt{\alpha^2 - 2\alpha}}{2\alpha} < \beta < \frac{2}{\alpha}$	$\frac{2}{\alpha} < \beta < 1$
Limit distribution of $S_{N,h}(\tau)$	Hermite process of order 2	$\alpha\beta$-stable Lévy motion	Brownian motion
Normalization	$N^{2-2\beta}$	$N^{1/\alpha\beta}$	$N^{1/2}$

Table 2. Summary of results for $k_* = 3, 2 < \alpha < 3$

	$\frac{1}{2} < \beta < \frac{5\alpha + \sqrt{25\alpha^2 - 48\alpha}}{12\alpha}$	$\frac{5\alpha + \sqrt{25\alpha^2 - 48\alpha}}{12\alpha} < \beta < \frac{2}{\alpha}$	$\frac{2}{\alpha} < \beta < 1$
Limit distribution of $S_{N,h}(\tau)$	Hermite process of order 3	$\alpha\beta$-stable Lévy motion	Brownian motion
Normalization	$N^{(5-6\beta)/2}$	$N^{1/\alpha\beta}$	$N^{1/2}$

(II) $\alpha\beta$-stable Lévy process, normalization $N^{1/(\alpha\beta)}$;
(III) Brownian motion, normalization $N^{1/2}$.

The boundaries between different normalizations in the 'rectangle' $\{(\alpha, \beta) : 2 < \alpha < 4, 1/2 < \beta < 1\}$ are shown in Figure 1. A similar limit behaviour can be expected in the general case $k_* > 1$. However, technical difficulties increase with k_* and our proofs in the case $k_* = 3$ are quite involved. Note that stable limits of empirical functionals of moving averages (1.1) with *infinite variance* were recently discussed in Koul and Surgailis (2001) and Surgailis (2002).

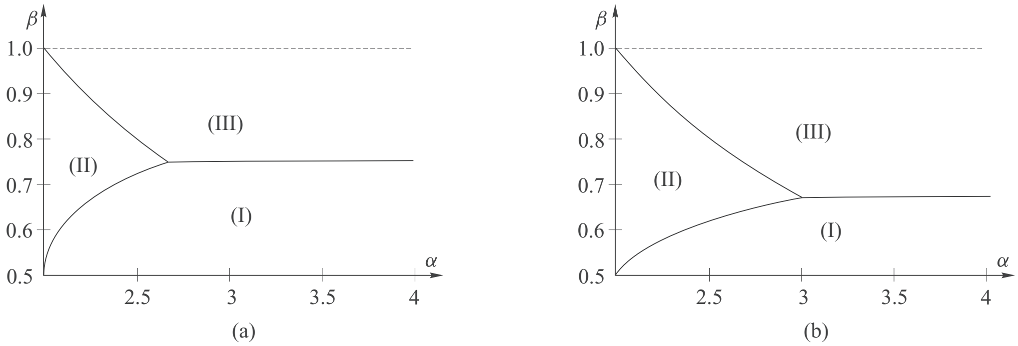

Figure 1. Limiting behaviours: (a) $k_* = 2$; (b) $k_* = 3$

While behaviours (I) and (III) are commonly referred to as long memory and short memory, respectively, (II) is less clear to classify. On the one hand, the limit process has independent increments, which is characteristic of short-memory processes. On the other hand, the variance of $S_{N,h}$ in region (II) grows faster than N (which is usually interpreted as an indication of long memory), because a linear growth of the variance would imply $S_{N,h} = O_P(N^{1/2})$, which contradicts the normalization $N^{1/(\alpha\beta)}$. The limiting $\alpha\beta$–stable distribution in region (II) depends on h via *fractional* derivatives of h_∞ of order $1+(1/\beta)$ (see Remark 2.1), and therefore behaviour (II) cannot be characterized in terms of Appell rank (or any finite set of coefficients a_k). We also note that convergence to stable limit law of sums of strongly dependent stationary random variables with finite variance has been proved for several time series models arising in econometrics and telecommunications; see Taqqu and Levy (1986), Mikosch *et al.* (2002), Davidson and Sibbertsen (2002), Leipus and Surgailis (2003), Leipus *et al.* (2003) and Pipiras *et al.* (2003).

This paper is organized as follows. Section 2 contains a rigorous formulation of our main results (Theorems 2.1 and 2.2). We also give a heuristic explanation of the $\alpha\beta$–stable limit, and discuss the interesting open question of what happens if this limit is zero. Examples of (bounded and unbounded) functions having Appell rank $k_* = 1, 2$ or 3 are discussed in Remarks 2.2–2.4. Section 3 is devoted to the proof of the convergence to Lévy process. Theorems 2.1 and 2.2 are proved in Sections 4 and 5, respectively. Section 5 also contains formulations of the main auxiliary Lemmas 5.1 and 5.2, whose proofs occupy the rest of the paper.

2. Main results

In the rest of the paper we assume that the innovations ζ_s, $s \in \mathbb{Z}$, are i.i.d., have zero mean and unit variance, and satisfy condition (1.10), for some $q_+, q_- \geqslant 0$, $q_+ + q_- > 0$ and some $2 < \alpha < 4$. The notation $\sim$ means that the ratio of both sides tends to 1. We shall denote by C a generic constant which may change from line to line. We also assume the following condition on the characteristic function of the innovations:

$$|\mathrm{E}e^{\mathrm{i}u\zeta_0}| \leqslant C(1+|u|)^{-\delta} \qquad (\forall u \in \mathbb{R}, \exists C, \delta > 0). \tag{2.1}$$

Conditions (2.1) and (1.3) guarantee infinite differentiability of the function h_∞ in (1.4); see Lemma 3.1 below.

We introduce a kth-order Hermite process, $J^{(k)}(\tau)$, as the k-tuple Itô–Wiener stochastic integral

$$J^{(k)}(\tau) := \int_{\mathbb{R}^k} \left\{ \int_0^\tau \prod_{i=1}^k (t-u_i)_+^{-\beta}\,\mathrm{d}t \right\} W(\mathrm{d}u_1)\ldots W(\mathrm{d}u_k) \tag{2.2}$$

with respect to a standard Gaussian white noise $W(\mathrm{d}u)$ with zero mean and variance $\mathrm{d}u$; $a_+^{-\beta}$ is defined as $a^{-\beta}$ if $a > 0$, and as 0 otherwise. It is well known (Taqqu 1979) that (2.2) is well defined for any integer $1 \leqslant k < 1/(2\beta - 1)$, $\beta \in (1/2, 1)$. The process $J^{(1)}(\tau)$ is a fractional Brownian motion, and $J^{(2)}(\tau)$ is called the Rosenblatt process. For simplicity, we shall assume below that $1/(2\beta-1)$ is not an integer.

For $\alpha\beta < 2$, let $L^{\pm}_{\alpha\beta}(\tau)$ denote independent copies of a totally skewed (i.e. with skewness parameter equal to 1) stable Lévy motion $L_{\alpha\beta}(\tau)$, with independent and stationary increments and the characteristic function

$$\mathrm{E}\mathrm{e}^{\mathrm{i}uL_{\alpha\beta}(\tau)} = \exp\{-\tau|u|^{\alpha\beta}(1 - \mathrm{i}\,\mathrm{sgn}(u)\tan(\pi\alpha\beta/2))\}, \qquad u \in \mathbb{R}.$$

Also, put

$$c_h^{\pm} := \sigma\int_0^{\infty}(h_{\infty}(\pm t) - a_0 \mp a_1 t)t^{-1-1/\beta}\,\mathrm{d}t, \tag{2.3}$$

where $a_0 = h_{\infty}(0) = \mathrm{E}h(X_0)$, a_k are given in (1.8), as usual, and where

$$\sigma := \beta^{-1}c_0^{1/\beta}\{\Gamma(2-\alpha\beta)|\cos(\pi\alpha\beta/2)|/(\alpha\beta-1)\}^{1/\alpha\beta}. \tag{2.4}$$

Write $\Rightarrow$ for weak convergence of finite-dimensional distributions.

Theorem 2.1. *Assume conditions* (1.3), (1.10) *and* (2.1). *Let h be a bounded measurable function with $a_1 = 0$, that is, having Appell rank $k_* \geqslant 2$.*

(i) *If $\beta < 3/4$ and $\beta < (\alpha + \sqrt{\alpha^2 - 2\alpha})/2\alpha$, then*

$$N^{2\beta-2}S_{N,h}(\tau) \Rightarrow \frac{c_0^2}{2}a_2 J^{(2)}(\tau).$$

(ii) *If $(\alpha + \sqrt{\alpha^2 - 2\alpha})/2\alpha < \beta < 2/\alpha$, then*

$$N^{-1/\alpha\beta}S_{N,h}(\tau) \Rightarrow c_h^+ q_+^{1/\alpha\beta}L^+_{\alpha\beta}(\tau) + c_h^- q_-^{1/\alpha\beta}L^-_{\alpha\beta}(\tau). \tag{2.5}$$

(iii) *If $\beta > 3/4$ and $\beta > 2/\alpha$, then*

$$N^{-1/2}S_{N,h}(\tau) \Rightarrow \sigma_h B(\tau), \tag{2.6}$$

where $B(\tau)$ is a standard Brownian motion and $\sigma_h^2 := \sum_{t\in\mathbb{Z}}\mathrm{cov}(h(X_0), h(X_t))$.

Theorem 2.2. *Assume the same conditions as in Theorem 2.1. Let h be a bounded measurable function with $a_1 = a_2 = 0$, that is having Appell rank $k_* \geqslant 3$.*

(i) *If $\beta < 2/3$ and $\beta < (5\alpha + \sqrt{25\alpha^2 - 48\alpha})/12\alpha$, then*

$$N^{(6\beta-5)/2}S_{N,h}(\tau) \Rightarrow \frac{c_0^3}{6}a_3 J^{(3)}(\tau).$$

(ii) *If $(5\alpha + \sqrt{25\alpha^2 - 48\alpha})/12\alpha < \beta < 2/\alpha$, then the convergence* (2.5) *holds.*
(iii) *If $\beta > 2/3$ and $\beta > 2/\alpha$, then the convergence* (2.6) *holds.*

We note that statements (i) of Theorems 2.1 and 2.2 also hold in the cases $a_2 = 0$ and $a_3 = 0$ respectively, in which cases the corresponding limits are trivial.

Let us explain the main idea of the proof of Theorem 2.1. Put

$$R_t^{(k,0)} := h(X_t) - \mathrm{E}h(X_t) - \sum_{\ell=1}^{k} a_\ell X_t^{(\ell)} I_{\{\ell(2\beta-1)<1\}}, \tag{2.7}$$

$$R_t^{(k,1)} := R_t^{(k,0)} - Z_t^{(1)} I_{\{\alpha\beta<2\}}, \tag{2.8}$$

where

$$X_t^{(k)} := \sum_{s_k<\ldots<s_1<t} b_{t-s_1} \ldots b_{t-s_k} \zeta_{s_1} \ldots \zeta_{s_k}, \tag{2.9}$$

$$Z_t^{(1)} := \sum_{j=1}^{\infty} \mathrm{E}[h(X_t) - \mathrm{E}h(X_t) - a_1 X_t | \zeta_{t-j}] = \sum_{j=1}^{\infty} \mathrm{E}[R_t^{(1,0)} | \zeta_{t-j}]. \tag{2.10}$$

Then

$$S_{N,h}(\tau) = a_2 \sum_{t=1}^{[N\tau]} X_t^{(2)} + \sum_{t=1}^{[N\tau]} Z_t^{(1)} + \sum_{t=1}^{[N\tau]} R_t^{(2,1)}. \tag{2.11}$$

It is easy to show (see, for example, Surgailis 1982; Avram and Taqqu 1987) that, for any $1 \leqslant k < 1/(2\beta - 1)$,

$$N^{-1+(k/2)(2\beta-1)} \sum_{t=1}^{[N\tau]} X_t^{(k)} \Rightarrow \frac{c_0^k}{k!} J^{(k)}(\tau). \tag{2.12}$$

According to Lemma 3.2 below, for $1 < \alpha\beta < 2$,

$$N^{-1/\alpha\beta} \sum_{t=1}^{[N\tau]} Z_t^{(1)} \Rightarrow c_h^+ q_+^{1/\alpha\beta} L_{\alpha\beta}^+(\tau) + c_h^- q_-^{1/\alpha\beta} L_{\alpha\beta}^-(\tau). \tag{2.13}$$

With (2.11)–(2.13) in mind, statements (i) and (ii) of Theorem 2.1 follow from the relevant estimate of sums of the 'remainder' $R_t^{(2,1)}$ proved in Lemma 4.2. The proof of Theorem 2.2 uses a similar rearrangement of $S_{N,h}(\tau)$ involving an additional centring $Z_t^{(2)}$ of (2.14), with the crucial estimates proved in Lemmas 5.1 and 5.2.

Heuristically, the $\alpha\beta$-stable limit in Theorems 2.1 and 2.2 can be explained as a 'joint effect of heavy tails in the innovations and moving-average coefficients'. In the parameter region $\alpha\beta < 2$, a single large fluctuation $|\zeta_s| = O(N^{1/\alpha})$ occurring at some $1 \leqslant s \leqslant N$ is 'remembered by $h(X_t)$' at typical distances $t - s = O(N^{1/(\alpha\beta)})$, $s < t \leqslant N$, and 'dominates the contributions of the remaining innovations' in the sense that, at such distances, $h(X_t) \approx \mathrm{E}[h(X_t)|\zeta_s]$ is 'approximately constant'. A similar heuristic explanation in the infinite-variance case is given in Surgailis (2002).

Remark 2.1. Note that the asymptotic constants $c_h^\pm$ of (2.3), up to a multiplicative constant, are the Marchaud (right and left) fractional derivatives of h_∞ of order $1 + 1/\beta$ at $x = 0$. If $c_h^+ = c_h^- = 0$, the $\alpha\beta$-stable limits in Theorems 2.1 and 2.2 are trivial. This raises the natural question of the existence of a non-trivial limit distribution of $S_{N,h}(\tau)$ under a different normalization. As suggested by Lemma 5.2, in such a case the role of $Z_t^{(1)}$ could be played by a similar 'second-order' centring term

$$Z_t^{(2)} := \sum_{0<j_1<j_2} \mathrm{E}[R_t^{(2,1)}|\zeta_{t-j_1}, \zeta_{t-j_2}], \tag{2.14}$$

where $R_t^{(2,1)}$ is given in (2.8). According to Lemma 5.1, $\sum_{t=1}^N Z_t^{(2)} = O_P(N^{(3-2\beta)/2\alpha\beta})$ provided $2\alpha\beta/(3-2\beta) < 2$, or $\beta < 3/(2+\alpha)$, holds. We conjecture that for such α, β, the partial sums process of the stationary sequence $Z_t^{(2)}$ of (2.14), normalized by $N^{(3-2\beta)/2\alpha\beta}$, weakly converges to a $(2\alpha\beta/(3-2\beta))$-stable Lévy process. See also (6.1)–(6.3) and Lemma 6.1 for further approximation of this partial sums process. As $2\alpha\beta/(3-2\beta) > \alpha\beta$ for $\beta > 1/2$, one could expect in the case $c_h^+ = c_h^- = 0$ a similar competition between Hermite, Lévy and Brownian asymptotic behaviours of $S_{N,h}(\tau)$, with $\alpha\beta$ replaced by $2\alpha\beta/(3-2\beta)$. The analysis of $Z_t^{(2)}$ is more involved than that of $Z_t^{(1)}$, and plays an important role in the proof of Theorem 2.2.

Remark 2.2. The simplest example of functions h with a given Appell rank $k_* \geqslant 1$ are the Appell polynomials $h = A_{k_*}$, relative to the marginal distribution X_0 of the linear process in (1.2). In particular, $A_1(x) = x$, $A_2(x) = x^2 - \mu_2$, $A_3(x) = x^3 - 3\mu_2 x - \mu_3$, etc., where $\mu_k := \mathrm{E}X_0^k$, $\mu_1 = \mathrm{E}X_0 = 0$. However, Appell polynomials are unbounded (and require the existence of corresponding moments of X_0), so they are excluded from our discussion. The limit distribution of partial sums $S_{N,h}$ for polynomial h was studied in Surgailis (1982), Giraitis (1985), Avram and Taqqu (1987), Vaičiulis (2003) and elsewhere. We note that Theorems 2.1 and 2.2 essentially use the boundedness of h; the extension of these theorems to unbounded functions (satisfying, for example, some growth condition at infinity) is an open problem.

Remark 2.3. An important class of bounded functions form indicator functions $h_y(x) = I_{\{x\leqslant y\}}$, $y \in \mathbb{R}$, in which case $S_{N,h_y} = N(\hat{F}_N(y) - F(y))$ is the (normalized) empirical distribution function; $F(y) := P(X_0 \leqslant y)$, $\hat{F}_N(y) := N^{-1}\sum_{t=1}^N I_{\{X_t\leqslant y\}}$. The formal Appell expansion of such indicator functions is very explicit:

$$h_y(x) = \sum_{k=0}^{\infty} (-1)^k F^{(k)}(y) A_k(x)/k!;$$

see Koul and Surgailis (1997). In particular, the Appell rank of h_y equals 1 for each $y \in \mathbb{R}$ with $f(y) \neq 0$, where $f(y) = F^{(1)}(y)$ is the probability density. An interesting open problem is to extend Theorems 2.1 and 2.2 to obtain an asymptotic expansion of the empirical distribution function in the spirit of Ho and Hsing (1996), including the functional convergence.

Remark 2.4. To give an example of bounded functions h with Appell rank $k_* \geqslant 2$, assume that h and the probability density f of X_0 are *symmetric*: $h(-x) = h(x)$ and $f(-x) = f(x)$, for almost every $x \in \mathbb{R}$. (The symmetry of f holds, for example, if the distribution of ζ_0 is symmetric.) Then the derivative $f^{(1)}$ is antisymmetric: $f^{(1)}(-x) = -f^{(1)}(x)$ and therefore $a_1 = 0$, where $a_k = (-1)^k \int_{\mathbb{R}} h(x) f^{(k)}(x)\,\mathrm{d}x\,(k \geqslant 0)$; see (1.4), (1.8) (the existence and integrability of all derivatives of f under conditions (1.3) and (2.1) follow from Koul and Surgailis 2002, Lemma 5.1). Accordingly, for such h, f, the Appell rank $k_* = 2$ or $k_* \geqslant 3$ depending on whether $a_2 \neq 0$ or $a_2 = 0$ holds. For example, let $h_\lambda(x) := \cos(\lambda x)\,(\lambda \in \mathbb{R})$; then S_{N,h_λ} is the (normalized) empirical characteristic function and the Appell rank of h_λ equals 2 for any $\lambda \neq 0$ such that $\int_{\mathbb{R}} \cos(\lambda x) f(x)\mathrm{d}x = \mathrm{E}\mathrm{e}^{\mathrm{i}\lambda X_0} \neq 0$. Next, let h_1, h_2 be any linearly independent *antisymmetric* bounded and measurable functions with $a_{1i} \neq 0$, $i = 1, 2$, where $a_{ki} := (-1)^k \int_{\mathbb{R}} h_i(x) f^{(k)}(x)\mathrm{d}x$; then a linear combination $h = \sum_{i=1}^2 c_i h_i$ is a bounded function whose Appell rank $k_* = 3$, provided constants c_1, c_2 satisfy $a_1 = \sum_{i=1}^2 c_i a_{1i} = 0$, $a_3 = \sum_{i=1}^2 c_i a_{3i} \neq 0$ (the relation $a_2 = 0$ for any c_1, c_2 follows by antisymmetry of h_i, $i = 1, 2$). The above examples are based on general symmetry properties of h and f only; they suggest that there is no intrinsic relationship between Appell rank k_* and parameters α, β in Theorems 2.1 and 2.2.

3. Convergence to stable Lévy process

We introduce the following notation. For any integers $j > 0$, $j_2 > j_1 > 0$, put

$$X_{t,j} := \sum_{0<i\leqslant j} b_i \zeta_{t-i}, \qquad \tilde{X}_{t,j} := \sum_{i>j} b_i \zeta_{t-i},$$

$$\hat{X}_{t,j} := \sum_{i>0, i\neq j} b_i \zeta_{t-i}, \quad \hat{X}_{t,j_1,j_2} := \sum_{i>0, i\neq j_1,j_2} b_i \zeta_{t-i}, \quad (3.1)$$

where $X_{t,0} := 0$ and $\tilde{X}_{t,0} := X_t$. Note that

$$X_{t,j} + \tilde{X}_{t,j} = X_t, \qquad \hat{X}_{t,j} = X_t - b_j \zeta_{t-j}, \qquad \hat{X}_{t,j_1,j_2} = X_t - b_{j_1}\zeta_{t-j_1} - b_{j_2}\zeta_{t-j_2}.$$

Also let

$$h_j(x) := \mathrm{E}h(x + X_{0,j}),\ \hat{h}_j(x) := \mathrm{E}h(x + \hat{X}_{0,j}),\ \hat{h}_{j_1,j_2}(x) := \mathrm{E}h(x + \hat{X}_{0,j_1,j_2}), \qquad x \in \mathbb{R}. \quad (3.2)$$

Lemma 3.1 is an easy corollary of the bounds of marginal distribution functions and their derivatives of stationary processes (3.1) given in Koul and Surgailis (2002, Lemma 5.1).

Lemma 3.1. *Assume conditions* (2.1) *and* (1.3) *only. For any bounded measurable function h and any $p = 0, 1, \ldots,$ there exist $j_0 \geqslant 1$ and a constant $C < \infty$ such that, for any $j > j_0$ and any $j_2 > j_1 > 0$, the functions h_j, h_∞, $\hat{h}_{j_1}$, $\hat{h}_{j_1,j_2}$ are p times differentiable and*

$$|h_j^p(x)| + |h_\infty^{(p)}(x)| + |\hat{h}_{j_1}^{(p)}(x)| + |\hat{h}_{j_1,j_2}^{(p)}(x)| \leqslant C. \quad (3.3)$$

Moreover,

$$|\hat{h}_j^{(p)}(x) - h_\infty^{(p)}(x)| \leqslant C b_j^2. \quad (3.4)$$

Lemma 3.2. *Let $\alpha\beta < 2$. Then the convergence* (2.13) *holds.*

Proof. We follow the argument in Surgailis (2002). We have

$$Z_t^{(1)} = \sum_{j=1}^{\infty} \mathrm{E}[h(X_t) - a_0 - a_1 X_t | \zeta_{t-j}] = \sum_{j=1}^{\infty} (\hat{h}_j(b_j\zeta_{t-j}) - a_0 - a_1 b_j \zeta_{t-j}), \tag{3.5}$$

where the series converges in mean and almost surely; see (3.9) and (3.10) below. Let $\mathcal{Z}_N^{(1)}(\tau) := \sum_{t=1}^{[N\tau]} Z_t^{(1)}$,

$$Q_N(\tau) := \sum_{t=1}^{[N\tau]} \eta(\zeta_{t-1}), \qquad \eta(z) := \sum_{j=1}^{\infty} (h_\infty(b_j z) - \mathrm{E}h_\infty(b_j\zeta_0) - a_1 b_j z).$$

Clearly the lemma follows from

$$\mathcal{Z}_N^{(1)}(\tau) - Q_N(\tau) = o_P(N^{1/(\alpha\beta)}), \tag{3.6}$$

$$N^{-1/\alpha\beta} Q_N(\tau) \Rightarrow c_h^+ q_+^{1/\alpha\beta} L_{\alpha\beta}^+(\tau) + c_h^- q_-^{1/\alpha\beta} L_{\alpha\beta}^-(\tau). \tag{3.7}$$

We prove (3.6) for $\tau = 1$ only. Let $V_N := \mathcal{Z}_N^{(1)}(1) - Q_N(1)$. Then $V_N = V_{N1} - V_{N2} + V_{N3}$, where $V_{N1} := \sum_{s<0} \phi_{N1,s}(\zeta_s)$, $V_{Ni} := \sum_{0 \leqslant s < N} \phi_{Ni,s}(\zeta_s)$, $i = 2, 3$, and where

$$\phi_{N1,s}(z) := \sum_{j=1-s}^{N-s} (\hat{h}_j(b_j z) - a_0 - a_1 b_j z),$$

$$\phi_{N2,s}(z) := \sum_{j>N-s} (h_\infty(b_j z) - \mathrm{E}h_\infty(b_j\zeta_0) - a_1 b_j z),$$

$$\phi_{N3,s}(z) := \sum_{j=1}^{N-s} \{(\hat{h}_j(b_j z) - h_\infty(b_j z)) - (\mathrm{E}\hat{h}_j(b_j\zeta_0) - \mathrm{E}h_\infty(b_j\zeta_0))\}.$$

Then (3.6) follows from

$$\mathrm{E}|V_{Ni}|^r \leqslant CN^{1+r-\alpha\beta}, \qquad i = 1, 2 (\exists 1 < r < \alpha\beta),$$

$$\mathrm{E}V_{N3}^2 \leqslant CN, \tag{3.8}$$

and the inequalities $(1 + r - \alpha\beta)/r < 1/\alpha\beta (1 < r < \alpha\beta)$, $1/2 < 1/\alpha\beta (\alpha\beta < 2)$. From (3.3), it is easy to show that

$$|h_\infty(b_j z) - \mathrm{E}h_\infty(b_j\zeta_0) - a_1 b_j z| \leqslant C \min(|b_j|(|z| + 1), b_j^2(|z| + 1)^2), \tag{3.9}$$

$$|\hat{h}_j(b_j z) - a_0 - a_1 b_j z| \leqslant C \min(|b_j|(|z| + 1), b_j^2(|z| + 1)^2). \tag{3.10}$$

Indeed, the left-hand side of (3.9) can be written as $|\mathrm{E}\int_{b_j\zeta}^{b_j z}(h'_\infty(u) - h'_\infty(0))\mathrm{d}u|$ and so (3.9) is obvious by (3.3) and the mean value theorem, while (3.10) additionally uses $a_p = \mathrm{E}\hat{h}_j^{(p)}(b_j\zeta_0)$, $\mathrm{E}\zeta_0 = 0$ and $\hat{h}_j(b_j z) - a_0 - a_1 b_j z = \mathrm{E}\{\hat{h}_j(b_j z) - \hat{h}_j(b_j\zeta_0) - b_j(z - \zeta_0)\hat{h}_j^{(1)}$

$(b_j\zeta_0)\} - b_j\mathrm{E}\{\zeta_0(\hat{h}_j^{(1)}(b_j\zeta_0) - \hat{h}_j^{(1)}(0))\}$. Using condition (1.10) and integration by parts, for any $\alpha/2 < r < \alpha$ one has

$$\mathrm{E}|\min(|b_j|(|\zeta_0|+1),\, b_j^2(|\zeta_0|+1)^2)|^r \leqslant C|b_j|^\alpha. \tag{3.11}$$

In particular, (3.11) holds for $r = \alpha\beta - \epsilon$, where $\epsilon > 0$ is small enough. Note that the random variables $\phi_{Ni,s}(\zeta_s)$, $s \in \mathbb{Z}$, are independent and have zero mean, $i = 1, 2, 3$. Then, for any $1 \leqslant r \leqslant 2$,

$$\begin{aligned}
\mathrm{E}|V_{N1}|^r &\leqslant 2\sum_{s<0}\mathrm{E}|\phi_{N1,s}(\zeta_s)|^r \\
&\leqslant C\sum_{s>0}\mathrm{E}\left|\sum_{j=1+s}^{N+s}\min(|b_j|(|\zeta_0|+1),\, b_j^2(|\zeta_0|+1)^2)\right|^r \\
&\leqslant C\sum_{s>0}\left(\sum_{j=1+s}^{N+s}\mathrm{E}^{1/r}\left|\min(|b_j|(|\zeta_0|+1),\, b_j^2(|\zeta_0|+1)^2)\right|^r\right)^r \\
&\leqslant C\int_1^\infty \mathrm{d}s\left(\int_s^{N+s} j^{-\alpha\beta/r}\mathrm{d}j\right)^r.
\end{aligned}$$

Decompose the last integral as $\int_1^\infty \mathrm{d}s\{\ldots\} = \int_1^N \mathrm{d}s\{\ldots\} + \int_N^\infty \mathrm{d}s\{\ldots\} =: I_{1N} + I_{2N}$. Then $I_{1N} = C\int_1^N s^{r-\alpha\beta}\mathrm{d}s(\int_1^{1+(N/s)} x^{-\alpha\beta/r}\,\mathrm{d}x)^r \leqslant CN^{1+r-\alpha\beta}$, as $\alpha\beta/r > 1$ and the inner integral is bounded by a constant. On the other hand, $I_{2N} \leqslant C\int_N^\infty s^{r-\alpha\beta}(N/s)^r\,\mathrm{d}s = CN^r\int_N^\infty s^{-\alpha\beta}\,\mathrm{d}s \leqslant CN^{1+r-\alpha\beta}$. This proves (3.8) for $i = 1$. Similarly,

$$\begin{aligned}
\mathrm{E}|V_{N2}|^r &\leqslant C\int_0^N \mathrm{d}s\left(\int_{N-s}^\infty j^{-\alpha\beta/r}\,\mathrm{d}j\right)^r \\
&\leqslant C\int_0^N s^{r-\alpha\beta}\,\mathrm{d}s\left(\int_1^\infty x^{-\alpha\beta/r}\,\mathrm{d}x\right)^r \leqslant C\int_0^N s^{r-\alpha\beta}\,\mathrm{d}s = CN^{1+r-\alpha\beta}.
\end{aligned}$$

Finally, using (3.4) and the fact that $\phi_{N3,s}(\zeta_s)$, $s \in \mathbb{Z}$, are independent and have zero mean, we obtain

$$\mathrm{E}V_{N3}^2 \leqslant C\sum_{0\leqslant s<N}\left(\sum_{j=1}^{N-s} b_j^2\right)^2 \leqslant CN,$$

thereby proving (3.6).

Relation (3.7) follows by the classical central limit theorem (see Ibragimov and Linnik 1971, Theorem 2.6.7), as $\eta(\zeta_s)$, $s \in \mathbb{Z}$, are i.i.d. Thus it suffices to show that $\eta(\zeta)$ belongs to the domain of attraction of $\alpha\beta$-stable law, namely,

$$\lim_{x\to\infty} x^{\alpha\beta}P(\eta(\zeta) > x) = \gamma_+, \qquad \lim_{x\to-\infty}|x|^{\alpha\beta}P(\eta(\zeta) < x) = \gamma_-, \tag{3.12}$$

where the constants $\gamma_\pm$ are given by

$$\gamma_+ := \nu\Big(q_+|c_h^+|^{\alpha\beta} I_{\{c_h^+>0\}} + q_-|c_h^-|^{\alpha\beta} I_{\{c_h^->0\}}\Big),$$

$$\gamma_- := \nu\Big(q_+|c_h^+|^{\alpha\beta} I_{\{c_h^+<0\}} + q_-|c_h^-|^{\alpha\beta} I_{\{c_h^-<0\}}\Big),$$

$\nu := c_0^\alpha/(\beta\sigma)^{\alpha\beta}$. Using (3.9), it is easy to check that the series in the definition of $\eta(z)$ converges absolutely for each $z \in \mathbb{R}$, and defines a locally bounded function on $\mathbb{R}$. The limits (3.12) follow by (1.10) and the existence of the limits

$$\lim_{z\to\pm\infty} |z|^{-1/\beta}\eta(z) = \nu^{1/\alpha\beta} c_h^\pm. \tag{3.13}$$

Note $\eta(z) = \tilde{\eta}(z) - \mathrm{E}\tilde{\eta}(\zeta)$, where $\tilde{\eta}(z) = \sum_{j=1}^\infty (h_\infty(b_j z) - h_\infty(0) - b_j z h_\infty^{(1)}(0))$ and $\mathrm{E}|\tilde{\eta}(\zeta)| < C < \infty$. Let $z > 0$. Then

$$\begin{aligned} z^{-1/\beta}\eta(z) &= z^{-1/\beta}\tilde{\eta}(z) + O(z^{-1/\beta}) \\ &= z^{-1/\beta}\int_0^\infty (h_\infty(c_0 z t^{-\beta}) - h_\infty(0) - c_0 z t^{-\beta} h_\infty^{(1)}(0))\mathrm{d}t + \rho(z) + O(z^{-1/\beta}) \\ &= \nu^{1/\alpha\beta} c_h^+ + \rho(z) + O(z^{-1/\beta}), \end{aligned}$$

where

$$\rho(z) := z^{-1/\beta}\int_0^\infty \big\{h_\infty(b_{1+[t]}z) - h_\infty(c_0 z t^{-\beta}) - z(b_{1+[t]} - c_0 t^{-\beta}) h_\infty^{(1)}(0)\big\}\mathrm{d}t = o(1)$$

by the dominated convergence theorem. The limit $z \to -\infty$ in (3.13) is analogous. Further details can be found in Surgailis (2002, proof of Lemma 3.1). □

Remark 3.1. If $c_h^\pm = 0$, the limit in (2.13) is trivial. In such a case, we expect $\sum_{t=1}^N Z_t^{(1)} = o_P(N^{(3-2\beta)/2\alpha\beta})$, in agreement with the conjecture in Remark 2.1 on the existence of a second stable limit.

4. Proof of Theorem 2.1

The proof uses bounds of covariances of $R_t^{(k,0)}$ and $R_t^{(k,1)}$ given in Lemmas 4.1 and 4.2 below. We asume everywhere below that conditions (1.3), (1.10) and (2.1) are satisfied, with $1/2 < \beta < 1$, $\alpha > 2$, as well as the boundedness and measurability of h.

Lemma 4.1 (Koul and Surgailis 2002, Lemma 6.1). *Let $\alpha\beta > 2$. Then for any integer $1 \leqslant k < 1/(2\beta - 1)$ one can find $C, \kappa > 0$ such that, for all $t \geqslant 1$,*

$$|\mathrm{cov}(R_t^{(k,0)}, R_0^{(k,0)})| \leqslant C\begin{cases} t^{-(k+1)(2\beta-1)}, & \text{if } (k+1)(2\beta-1) < 1, \\ t^{-1-\kappa}, & \text{if } (k+1)(2\beta-1) > 1. \end{cases}$$

Lemma 4.2. *Let $\alpha\beta < 2$. Then for any sufficiently small $\kappa > 0$ there exists a constant $C > 0$ such that, for all $t \geqslant 1$,*

$$|\mathrm{cov}(R_t^{(2,1)}, R_0^{(2,1)})| \leqslant C \begin{cases} t^{-1-\kappa}, & \text{if } \beta > 3/(2+\alpha) \text{ and } \beta > 2/3, \\ t^{-(2\beta+\alpha\beta-2-\kappa)\wedge(6\beta-3)}, & \text{otherwise}. \end{cases}$$

Proof. Note that $\mathrm{E}[h(X_t)|\zeta_s, s \leqslant t-j] = h_{j-1}(\tilde{X}_{t,j-1})$ for any integer $j \geqslant 1$, where $h_j, \tilde{X}_{t,j}$ are defined in (3.1), (3.2). Following Ho and Hsing (1996; 1997), one can write the telescoping identity

$$R_t^{(1,0)} = h(X_t) - a_0 - a_1 X_t = \sum_{j=1}^{\infty} U_{t,j}, \tag{4.1}$$

where

$$U_{t,j} := h_{j-1}(\tilde{X}_{t,j-1}) - h_j(\tilde{X}_{t,j}) - a_1 b_j \zeta_{t-j},$$

$h_0(\tilde{X}_{t,0}) = h(X_t)$. The series (4.1) is orthogonal and converges in L^2. From (4.1) and (3.5) we obtain

$$R_t^{(2,1)} = \sum_{j=1}^{\infty} V_{t,j}, \tag{4.2}$$

where

$$V_{t,j} := h_{j-1}(\tilde{X}_{t,j-1}) - h_j(\tilde{X}_{t,j}) - \hat{h}_j(b_j\zeta_{t-j}) + a_0 - a_2 b_j \zeta_{t-j} \tilde{X}_{t,j} I_{\{\beta<3/4\}}. \tag{4.3}$$

Note that $V_{t,j}$ is measurable with respect to ζ_s, $s \leqslant t-j$, and that $\mathrm{E}[V_{t,j}|\zeta_s, s \leqslant t-j-1] = 0$; in particular, $\mathrm{E}[V_{t,j}V_{t',j'}] = 0$ unless $t-j = t'-j'$, $t, t \in \mathbb{Z}$, $j, j' \geqslant 1$. Using this fact and the Cauchy–Schwarz inequality,

$$|\mathrm{cov}(R_0^{(2,1)}, R_t^{(2,1)})| = \left| \sum_{j=1}^{\infty} \mathrm{E} V_{0,j} V_{t,t+j} \right| \leqslant \sum_{j=1}^{\infty} \mathrm{E}^{1/2}(V_{0,j})^2 \mathrm{E}^{1/2}(V_{t,t+j})^2. \tag{4.4}$$

We claim that for any $\kappa' > 0$ there is a constant $C < \infty$ such that, for all $t, j \geqslant 1$,

$$\mathrm{E}(V_{t,j})^2 \leqslant C \begin{cases} j^{-(2\beta+\alpha\beta-1-\kappa')\wedge(6\beta-2)}, & \text{if } 1/2 < \beta < 3/4, \\ j^{-(4\beta-1)}, & \text{if } 3/4 < \beta < 1. \end{cases} \tag{4.5}$$

The lemma follows from (4.4), (4.5) and the following elementary inequality: for any $\mu > 0$, $\nu > 0$, $\mu + \nu > 1$ and any $t > 0$,

$$\sum_{j=1}^{\infty} j^{-\mu}(t+j)^{-\nu} \leqslant C \begin{cases} t^{1-\mu-\nu}, & \text{if } 0 < \mu < 1, \\ t^{-\nu}, & \text{if } \mu > 1. \end{cases} \tag{4.6}$$

Indeed, let

$$\mu = \nu := \begin{cases} \{(2\beta + \alpha\beta - 1 - \kappa') \wedge (6\beta - 2)\}/2, & \text{if } 1/2 < \beta < 3/4, \\ (4\beta - 1)/2, & \text{if } 3/4 < \beta < 1. \end{cases}$$

First, let $\beta < 3/(2+\alpha)$ $(< 3/4)$, $\beta < 2/3$. Then $1 < 2\beta + \alpha\beta - 1 < 2$ (the lower bound follows from $\alpha\beta > 1$, $\beta > 1/2$) and $1 < 6\beta - 2 < 2$, so that $\mu \in (1/2, 1)$ provided $\kappa' > 0$ is chosen small enough. By (4.4)–(4.6) we obtain

$$|\mathrm{cov}(R_0^{(2,1)}, R_t^{(2,1)})| \leqslant Ct^{1-(2\beta+\alpha\beta-1-\kappa')\wedge(6\beta-2)} = Ct^{-(2\beta+\alpha\beta-2-\kappa')\wedge(6\beta-3)}.$$

Next, let $\beta < 3/(2+\alpha)$, $\beta > 2/3$. Then $6\beta - 2 > 2$ and we again have $\mu = (2\beta + \alpha\beta - 1 - \kappa')/2 \in (1/2, 1)$ and the lemma follows similarly.

In the case $\beta > 3/(2+\alpha)$, $\beta < 2/3$, one has $\mu = (6\beta - 2)/2 \in (1/2, 1)$ for $\kappa' > 0$ small enough and again the conclusion follows. Next, let $\beta > 3/(2+\alpha)$, $2/3 < \beta < 3/4$, then $\mu > 1$, implying the statement of the lemma with $\kappa := \mu - 1 > 0$. Finally, in the case $\beta > 3/4$, we again have $\mu > 1$ and the statement of the lemma follows similarly, with $\kappa := \mu - 1 > 0$.

It remains to prove the claim (4.5). To that end, note that

$$\mathrm{E}[V_{t,j}|\zeta_{t-j}] = 0. \tag{4.7}$$

Fix t, j, and write the following telescoping identity similar to (4.1):

$$\begin{aligned} V_{t,j} &= \sum_{\ell>j}\big(\mathrm{E}[V_{t,j}|\zeta_{t-j}, \zeta_u, u \leqslant t-\ell] - \mathrm{E}[V_{t,j}|\zeta_{t-j}, \zeta_u, u \leqslant t-\ell-1]\big) \\ &=: \sum_{\ell>j} W_{t,j,\ell}, \end{aligned} \tag{4.8}$$

which converges by orthogonality with respect to *conditional* probability $P[\cdot|\zeta_{t-j}]$. With (4.3) in mind,

$$\begin{aligned} W_{t,j,\ell} := {} & h_{\neq j,\ell-1}(b_j\zeta_{t-j} + \tilde{X}_{t,\ell-1}) - h_{\ell-1}(\tilde{X}_{t,\ell-1}) \\ & - h_{\neq j,\ell}(b_j\zeta_{t-j} + \tilde{X}_{t,\ell}) + h_\ell(\tilde{X}_{t,\ell}) - a_2 b_j\zeta_{t-j}b_\ell\zeta_{t-\ell}, \end{aligned} \tag{4.9}$$

where

$$h_{\neq j,\ell}(x) := \mathrm{E}h\left(x + \sum_{1\leqslant i\leqslant \ell, i\neq j} b_i\zeta_i\right), \qquad 1 \leqslant j < \ell.$$

By orthogonality of the decomposition (4.8), we have

$$\mathrm{E}V_{t,j}^2 = \sum_{\ell>j}\mathrm{E}W_{t,j,\ell}^2. \tag{4.10}$$

To evaluate the last expectation, we need a convenient representation of $W_{t,j,\ell}$. We introduce the following notation, as t, j, ℓ will be fixed temporarily:

$$H(x) := h_{\neq j,\ell-1}(x), \qquad \zeta := \zeta_{t-j}, \qquad \eta := \zeta_{t-\ell}, \qquad \tilde{X} := \tilde{X}_{t,\ell}, \tag{4.11}$$

$H^{(k)}(x) = \mathrm{d}^k H(x)/\mathrm{d}x^k$. We also introduce an independent copy $(\zeta^0, \eta^0, \tilde{X}^0)$ of $(\zeta, \eta, \tilde{X})$, and let E^0 denote the expectation with respect to $(\zeta^0, \eta^0, \tilde{X}^0)$ only. Then $W_{t,j,\ell}$ can be rewritten as

$$W_{t,j,\ell} = \mathrm{E}^0\{H(b_j\zeta + b_\ell\eta + \tilde{X}) - H(b_j\zeta^0 + b_\ell\eta + \tilde{X}) - H(b_j\zeta + b_\ell\eta^0 + \tilde{X}) + H(b_j\zeta^0 + b_\ell\eta^0 + \tilde{X}) - b_jb_\ell\zeta\eta H^{(2)}(b_j\zeta^0 + b_\ell\eta^0 + \tilde{X}^0)\}. \tag{4.12}$$

To evaluate this expectation, write $W_{t,j,\ell} = \sum_{i=1}^3 W_{t,j,\ell}^{(i)}$, where

$$W_{t,j,\ell}^{(1)} := \mathrm{E}^0\{H(b_j\zeta + b_\ell\eta + \tilde{X}) - H(b_j\zeta^0 + b_\ell\eta + \tilde{X}) - H(b_j\zeta + b_\ell\eta^0 + \tilde{X}) + H(b_j\zeta^0 + b_\ell\eta^0 + \tilde{X}) - b_jb_\ell(\zeta - \zeta^0)(\eta - \eta^0)H^{(2)}(b_j\zeta^0 + b_\ell\eta^0 + \tilde{X})\},$$

$$W_{t,j,\ell}^{(2)} := b_jb_\ell\zeta\eta\mathrm{E}^0\{H^{(2)}(b_j\zeta^0 + b_j\eta^0 + \tilde{X}) - H^{(2)}(b_j\zeta^0 + b_\ell\eta^0 + \tilde{X}^0)\},$$

$$W_{t,j,\ell}^{(3)} := b_jb_\ell\mathrm{E}^0\{H^{(2)}(b_j\zeta^0 + b_\ell\eta^0 + \tilde{X})(\zeta^0\eta^0 - \zeta^0\eta - \zeta\eta^0)\}.$$

Let $0 < \gamma < 1$, which will be specified later. By using the mean value theorem together with the boundedness of $H^{(3)}$ and the inequality $\min(1, u_1 + u_2) \leqslant (u_1 + u_2)^\gamma \leqslant u_1^\gamma + u_2^\gamma$, $u_1, u_2 \geqslant 0$, we obtain

$$\begin{aligned}
|W_{t,j,\ell}^{(1)}| &= \left|\mathrm{E}^0\int_{b_j\zeta^0}^{b_j\zeta}\int_{b_\ell\eta^0}^{b_\ell\eta}(H^{(2)}(u_1 + u_2 + \tilde{X}) - H^{(2)}(b_j\zeta^0 + b_\ell\eta^0 + \tilde{X}))\mathrm{d}u_1\,\mathrm{d}u_2\right| \\
&\leqslant C\mathrm{E}^0\int_0^{|b_j(\zeta-\zeta_0)|}\int_0^{|b_\ell(\eta-\eta^0)|}\min(1, (u_1 + u_2))\mathrm{d}u_1\,\mathrm{d}u_2 \\
&\leqslant C\mathrm{E}^0\int_0^{|b_j(\zeta-\zeta_0)|}\int_0^{|b_\ell(\eta-\eta^0)|}(u_1^\gamma + u_2^\gamma)\mathrm{d}u_1\,\mathrm{d}u_2 \\
&\leqslant C\mathrm{E}^0(|b_j(\zeta - \zeta^0)|^{1+\gamma}|b_\ell(\eta - \eta^0)| + |b_j(\zeta - \zeta^0)||b_\ell(\eta - \eta^0)|^{1+\gamma}) \\
&\leqslant C\big(|b_j|^{1+\gamma}|b_\ell|(|\zeta|^{1+\gamma} + 1)(|\eta| + 1) + |b_j||b_\ell|^{1+\gamma}(|\zeta| + 1)(|\eta|^{1+\gamma} + 1)\big),
\end{aligned}$$

almost surely. Choose γ so that $2(1 + \gamma) = \alpha - \kappa < \alpha$; then

$$\mathrm{E}|W_{t,j,\ell}^{(1)}|^2 \leqslant C(|b_j|^{\alpha-\kappa}b_\ell^2 + b_j^2|b_\ell|^{\alpha-\kappa}). \tag{4.13}$$

Next, consider $W_{t,j,\ell}^{(2)}$. By the uniform Lipschitz condition of $H^{(2)}$ (see (3.3)), for any $\ell > j_0$ large enough,

$$\mathrm{E}|W_{t,j,\ell}^{(2)}|^2 \leqslant Cb_j^2b_\ell^2(\mathrm{E}\tilde{X}^2 + (\mathrm{E}^0|\tilde{X}^0|)^2) \leqslant Cb_j^2b_\ell^2\ell^{1-2\beta}. \tag{4.14}$$

Finally, $W_{t,j,\ell}^{(3)} = b_jb_\ell\mathrm{E}^0\{(H^{(2)}(b_j\zeta^0 + b_\ell\eta^0 + \tilde{X}) - H^{(2)}(\tilde{X}))(\zeta^0\eta^0 - \zeta^0\eta - \zeta\eta^0)\}$ by $\mathrm{E}^0\zeta^0 = \mathrm{E}^0\eta^0 = \mathrm{E}^0\zeta^0\eta^0 = 0$. Hence, using the uniform Lipschitz condition of $H^{(2)}$, one obtains $|W_{t,j,\ell}^{(3)}| \leqslant C|b_jb_\ell|(|b_j| + |b_\ell|)$ almost surely, or

$$\mathrm{E}|W_{t,j,\ell}^{(3)}|^2 \leqslant Cb_j^2b_\ell^2(b_j^2 + b_\ell^2). \tag{4.15}$$

The claim (4.5) for $\beta < 3/4$ now follows from (4.13), (4.14), (4.15) and (4.10).

It remains to check (4.5) for $\beta > 3/4$. In this case, the last term on the right-hand side of (4.3) vanishes, and $V_{t,j}$ can be represented as

$$\mathrm{E}^0\{(h_{j-1}(b_j\zeta_{t-j}+\tilde{X}_{t,j})-h_{j-1}(b_j\zeta^0+\tilde{X}_{t,j})-h_{j-1}(b_j\zeta_{t-j}+\tilde{X}^0)+h_{j-1}(b_j\zeta^0+\tilde{X}^0))\}$$

$$=\mathrm{E}^0\left\{\int_{b_j\zeta^0}^{b_j\zeta_{t-j}}\int_{\tilde{X}^0}^{\tilde{X}_{t,j}} h_{j-1}^{(2)}(u_1+u_2)\mathrm{d}u_1\,\mathrm{d}u_2\right\}.$$

Hence, $|V_{t,j}|\leqslant C|b_j|\mathrm{E}^0|\zeta_{t-j}-\zeta^0||\tilde{X}_{t,j}-\tilde{X}^0|$ almost surely, and we obtain $\mathrm{E}V_{t,j}^2\leqslant Cb_j^2\mathrm{E}\tilde{X}_{t,j}^2\leqslant Cj^{1-4\beta}$, thereby proving (4.5) and thus the lemma. □

Proof of Theorem 2.1. (i) Consider the decomposition (2.11). Let $\alpha\beta>2$. Then $S_{N,h}(\tau)=h_\infty^{(2)}(0)\sum_{t=1}^{[N\tau]}X_t^{(2)}+\sum_{t=1}^{[N\tau]}R_t^{(2,0)}$ and the statement follows from Lemma 4.1 and (2.12). Let $\alpha\beta<2$. Note $\beta<(\alpha+\sqrt{\alpha^2-2\alpha})/2\alpha$ is equivalent to $2-2\beta>1/\alpha\beta$. Hence, the first term on the right-hand side of (2.11) dominates the second one, in probability, in view of relations (2.12) and (2.13). By Lemma 4.2, the third term on the right-hand side of (2.11) is $O_P(N^{1-(\beta-1/2)(1+\alpha/2+\kappa)})=o_P(N^{2-2\beta})$ for $\kappa>0$ small enough, which easily follows by $\alpha>2$. This proves (i).

(ii) This follows similarly from (2.11), where now the second term on the right-hand side dominates the other two terms.

(iii) Note that $\mathrm{cov}(h(X_0),\,h(X_t))=\mathrm{cov}(R_0^{(1,0)},\,R_t^{(1,0)})$ is absolutely summable, according to Lemma 4.1; in particular, σ_h^2 is well defined. The proof of the asymptotic normality is similar to Koul and Surgailis (1997) or Ho and Hsing (1997). □

5. Proof of Theorem 2.2

Note that the order of the 'remainder' $\sum_{t=1}^N R_t^{(2,1)}$ given in Lemma 4.2 is greater than the order of $\sum_{t=1}^N X_t^{(3)}$ unless $\alpha>4$. Therefore, to prove Theorem 2.2, we need to modify the remainder term by introducing the additional centring term $Z_t^{(2)}$ (2.14). More explicitly, from (2.8) and (3.5),

$$Z_t^{(2)}=$$

$$\sum_{j_2>j_1\geqslant 1}\Big(\hat{h}_{j_1,j_2}(b_{j_1}\zeta_{t-j_1}+b_{j_2}\zeta_{t-j_2})-\hat{h}_{j_1}(b_{j_1}\zeta_{t-j_1})-\hat{h}_{j_2}(b_{j_2}\zeta_{t-j_2})+a_0-a_2b_{j_1}b_{j_2}\zeta_{t-j_1}\zeta_{t-j_2}\Big),\tag{5.1}$$

where

$$\hat{h}_{j_1,j_2}(x):=\mathrm{E}h\left(x+\sum_{i\neq j_1,j_2}b_i\zeta_i\right),\qquad 1\leqslant j_1<j_2,\,x\in\mathbb{R}.\tag{5.2}$$

Accordingly, put

$$R_t^{(3,2)}:=R_t^{(3,0)}-Z_t^{(1)}I_{\{\alpha\beta<2\}}-Z_t^{(2)}I_{\{(2+\alpha)\beta<3\}}=R_t^{(3,1)}-Z_t^{(2)}I_{\{(2+\alpha)\beta<3\}}.\tag{5.3}$$

Lemma 5.1. *Let* $2\alpha\beta/(3-2\beta) < 2$, *or* $(2+\alpha)\beta < 3$. *Then*

$$\sum_{t=1}^{N} Z_t^{(2)} = O_P(N^{(3-2\beta)/2\alpha\beta}). \tag{5.4}$$

Lemma 5.2. *For any sufficiently small* $\kappa > 0$, *there exists* $C < \infty$ *such that, for all* $t \geqslant 1$,

$$|\mathrm{cov}(R_t^{(3,2)}, R_0^{(3,2)})| \leqslant C \begin{cases} t^{-(2\beta-1)(2+\alpha/2)+\kappa}, & \textit{if } \beta < \min\left(\dfrac{3}{2+\alpha}, \dfrac{2}{3}\right), \\ \left(t^{-(2\beta-1)(2+\alpha/2)+\kappa}\right) + \left(t^{-(2\beta-1)-(\alpha\beta-1)+\kappa}\right), & \textit{if } \dfrac{3}{2+\alpha} < \beta < \dfrac{2}{3}, \\ t^{-1-\kappa}, & \textit{if } \dfrac{2}{3} < \beta < \dfrac{3}{2+\alpha}. \end{cases}$$

The proofs of the above lemmas are given in Sections 6 and 7 below. With their help, we shall prove Theorem 2.2.

Proof of Theorem 2.2. (i) Similarly to (2.11), write

$$S_{N,h}(\tau) = a_3 \sum_{t=1}^{[N\tau]} X_t^{(3)} I_{\{\beta<2/3\}} + \sum_{t=1}^{[N\tau]} Z_t^{(1)} I_{\{\alpha\beta<2\}} + \sum_{t=1}^{[N\tau]} Z_t^{(2)} I_{\{(2+\alpha)\beta<3\}} + \sum_{t=1}^{[N\tau]} R_t^{(3,2)}. \tag{5.5}$$

Note that $\beta < (5\alpha + \sqrt{25\alpha^2 - 48\alpha})/12\alpha$ is equivalent to $1/\alpha\beta < (5-6\beta)/2$. Accordingly, the first term on the right-hand side of (5.5) dominates the second one, in probability; see (2.12) and (2.13). As $(3-2\beta)/2\alpha\beta < 1/\alpha\beta$ for $\beta > \frac{1}{2}$, by Lemma 5.1 the first term dominates the third one as well. Hence (i) follows from

$$\sum_{t=1}^{N} R_t^{(3,2)} = o_P(N^{(5-6\beta)/2}).$$

By Lemma 5.2,

$$\sum_{t=1}^{N} R_t^{(3,2)} = \begin{cases} O_P(N^{1-(\beta-1/2)(2+\alpha/2)+\kappa}), & \text{if } \beta < \min\left(\dfrac{3}{2+\alpha}, \dfrac{2}{3}\right), \\ O_P\left(N^{1-(\beta-1/2)(2+\alpha/2)+\kappa} + N^{1-(\beta-1/2)-(\alpha\beta-1)/2+\kappa}\right), & \text{if } \dfrac{3}{2+\alpha} < \beta < \dfrac{2}{3}, \\ O_P(N^{1/2}), & \text{if } \dfrac{2}{3} < \beta < \dfrac{3}{2+\alpha}. \end{cases} \tag{5.6}$$

Let $\beta < 3/(2+\alpha)$. Then $1 - (\beta - 1/2)(2+\alpha/2) < (5-6\beta)/2 = 1 - 3(\beta - 1/2)$ is obvious by $\alpha > 2$. Next, assume $3/(2+\alpha) < \beta < 2/3$. Note that this assumption implies $3/(2+\alpha) < 2/3$, or $\alpha > 5/2$. Then we need to check $1 - (\beta - 1/2) - (\alpha\beta - 1)/2 < 1 - 3(\beta - 1/2)$ only. The last inequality is equivalent to $\beta(4-\alpha) < 1$. As $\beta < 2/3$, this follows from $4 - \alpha < 3/2$, or $\alpha > 5/2$. This proves (i).

(ii) In this case, the second term on the right-hand side of (5.5) dominates the first and the third, as $1/\alpha\beta > (5-6\beta)/2 = 1 - 3(\beta - \frac{1}{2})$. Hence (ii) follows from

$$\sum_{t=1}^{N} R_t^{(3,2)} = o_P(N^{1/\alpha\beta}). \tag{5.7}$$

We need to examine the case $2/3 < \beta < 2/\alpha$ only, as (5.7) in the case $\beta < 2/3$, $\beta < 2/\alpha$ follows from the argument in (i). We have two possibilities: (a) $2/3 < \beta$, $3/(2+\alpha) < \beta < 2/\alpha$; (b) $2/3 < \beta < 3/(2+\alpha)$. In case (a), we have $R_t^{(3,2)} = R_t^{(2,1)}$ and (5.7) follows from the proof of Theorem 2.1. In case (b), (5.7) is immediate by (5.6). Theorem 2.2 is proved. □

6. Proof of Lemma 5.1

The proof uses the decomposition

$$Z_t^{(2)} = F_t + D_t, \tag{6.1}$$

with F_t, the remainder term and D_t, the main term, defined by

$$F_t := \sum_{j_2 > j_1 > 0} f_{j_1,j_2}(b_{j_1}\zeta_{t-j_1},\, b_{j2}\zeta_{t-j_2}), \tag{6.2}$$

$$D_t := \sum_{j>0} \Big(\hat{h}_j^{(1)}(b_j\zeta_{t-j}) - a_1 - a_2 b_j \zeta_{t-j} \Big) \hat{X}_{t,j}, \tag{6.3}$$

where

$$\hat{X}_{t,j} := \sum_{i>0: i\neq j} b_i \zeta_{t-i} = X_t - b_j\zeta_{t-j}$$

and where the function $f_{j_1,j_2}(z_1, z_2)$, $z_1, z_2 \in \mathbb{R}$, is given by

$$\begin{aligned} f_{j_1,j_2}(z_1, z_2) := \hat{h}_{j_1,j_2}(z_1+z_2) - \hat{h}_{j_1}(z_1) - \hat{h}_{j_2}(z_2) + a_0 - \Big(\hat{h}_{j_1}^{(1)}(z_1) - a_1 - a_2 z_1 \Big) z_2 \\ - \Big(\hat{h}_{j_2}^{(1)}(z_2) - a_1 - a_2 z_2 \Big) z_1 - a_2 z_1 z_2. \end{aligned} \tag{6.4}$$

Lemma 6.1.

$$\sum_{t=1}^{N} F_t = O_P(N^{((3-2\beta)/2\alpha\beta)-\kappa}) \qquad (\exists \kappa > 0). \tag{6.5}$$

Proof. Write $F_t = F_t^+ + F_t^-$, where $F_t^+ := \sum_{0<j_1<j_2: |j_2-j_1| > N^\lambda} \cdots$, $F_t^- := \sum_{0<j_1<j_2: |j_2-j_1| \leqslant N^\lambda} \cdots$, and where $0 < \lambda < 1$ is specified below. Then

$$\sum_{t=1}^{N} F_t^+ = \sum_{k>N^\lambda}\sum_{s<N}\left\{\sum_{t=1\vee(s+1)}^{N} f_{t-s,t-s+k}(b_{t-s}\zeta_s,\, b_{t-s+k}\zeta_{s-k})\right\}.$$

By definition (6.4), the random variable in the braces has zero conditional expectation given either ζ_s or ζ_{s-k}. By applying (twice) the von Bahr–Esseen inequality (von Bahr and Esseen 1965) for any $1 \leqslant r \leqslant 2$, one obtains

$$\begin{aligned} \mathrm{E}\left|\sum_{t=1}^{N} F_t^+\right|^r &\leqslant 4\sum_{s<N}\sum_{k>N^\lambda} \mathrm{E}\left|\sum_{t=1\vee(s+1)}^{N} f_{t-s,t-s+k}(b_{t-s}\zeta_s,\, b_{t-s+k}\zeta_{s-k})\right|^r \\ &\leqslant 4\sum_{s<N}\sum_{k>N^\lambda}\left(\sum_{t=1\vee(s+1)}^{N} \mathrm{E}^{1/r}|f_{t-s,t-s+k}(b_{t-s}\zeta_s,\, b_{t-s+k}\zeta_{s-k})|^r\right)^r, \end{aligned} \tag{6.6}$$

where the last line follows by the Minkowski inequality. In a similar way,

$$\mathrm{E}\left|\sum_{t=1}^{N} F_t^-\right|^2 \leqslant \sum_{s<N}\sum_{1\leqslant k\leqslant N^\lambda}\left(\sum_{t=1\vee(s+1)}^{N} \mathrm{E}^{1/2}|f_{t-s,t-s+k}(b_{t-s}\zeta_s,\, b_{t-s+k}\zeta_{s-k})|^2\right)^2. \tag{6.7}$$

Put $\bar r := 2\alpha\beta/(3-2\beta)$. Clearly, Lemma 6.1 follows if we show that there exist $1 < r < 2$, $\kappa, \lambda > 0$ such that, for all $N \geqslant 1$,

$$\mathrm{E}\left|\sum_{t=1}^{N} F_t^+\right|^r \leqslant CN^{(r/\bar r)-\kappa}, \tag{6.8}$$

$$\mathrm{E}\left|\sum_{t=1}^{N} F_t^-\right|^2 \leqslant CN^{(2/\bar r)-\kappa}. \tag{6.9}$$

To proceed, we need the following lemma, whose proof will be postponed till later.

Lemma 6.2. *For any* $1 \leqslant r \leqslant 2$, $r > \alpha/2$ *and any* $0 < j_1 < j_2$,

$$\mathrm{E}|f_{j_1,j_2}(b_{j_1}\zeta_{j_1},\, b_{j_2}\zeta_{j_2})|^r \leqslant C|b_{j_1}|^\alpha|b_{j_2}|^\alpha. \tag{6.10}$$

Let us prove (6.8). Below we choose $r = \bar r - \delta$, with $\delta > 0$ small enough. Note that $2 > \bar r > \alpha\beta$ and $\bar r > \alpha/2$, so that $r > \alpha\beta$, $r > \alpha/2$, for suitable δ, and (6.10) applies. We shall also need the inequality

$$1 + \bar r - 2\alpha\beta < 0, \tag{6.11}$$

which will be verified below. Then by (6.6) and (6.10),

$$\mathrm{E}\left|\sum_{t=1}^{N} F_t^+\right|^r \leqslant C \sum_{s<N} \sum_{k>N^\lambda} \left(\sum_{t=1\vee(s+1)}^{N} |b_{t-s}|^{\alpha/r} |b_{t-s+k}|^{\alpha/r} \right)^r$$

$$= C\left(\sum_{|s|<N} \ldots + \sum_{s<-N} \ldots \right) =: C(\Sigma_{1N}^+ + \Sigma_{2N}^+).$$

Here

$$\Sigma_{1N}^+ \leqslant CN \int_{N^\lambda}^{\infty} \mathrm{d}k \left(\int_0^N \tau^{-\alpha\beta/r} (\tau + k)^{-\alpha\beta/r} \, \mathrm{d}\tau \right)^r = CN^{2+r-2\alpha\beta} I_N,$$

where $I_N := \int_{N^{\lambda-1}}^\infty \mathrm{d}k (\int_0^1 \tau^{-\alpha\beta/r} (\tau + k)^{-\alpha\beta/r} \mathrm{d}\tau)^r$. Write $I_N = \int_{N^{\lambda-1}}^1 \mathrm{d}k(\ldots)^r + \int_1^\infty \mathrm{d}k(\ldots)^r =: I_{1N} + I_{2N}$. Here $I_{2N} \leqslant C \int_1^\infty k^{-\alpha\beta} \, \mathrm{d}k (\int_0^1 \tau^{-\alpha\beta/r} \mathrm{d}\tau)^r \leqslant C$, while $I_{1N} \leqslant C \int_{N^{\lambda-1}}^1 (k^{-(2\alpha\beta/r)+1})^r \, \mathrm{d}k = O(N^{(\lambda-1)(1+r-2\alpha\beta)})$ because of (6.11). Thus,

$$\Sigma_{1N}^+ \leqslant CN^{1+\lambda(1+r-2\alpha\beta)}. \tag{6.12}$$

Next,

$$\Sigma_{2N}^+ \leqslant C \int_N^\infty \mathrm{d}s \int_1^\infty \mathrm{d}k \left(\int_0^N (t+s)^{-\alpha\beta/r} (t+s+k)^{-\alpha\beta/r} \, \mathrm{d}t \right)^r$$

$$= CN^{1+(1+r-2\alpha\beta)} I = CN^{1+(1+r-2\alpha\beta)}, \tag{6.13}$$

as the integral $I := \int_1^\infty \mathrm{d}s \int_0^\infty \mathrm{d}k (\int_0^1 (t+s)^{-\alpha\beta/r} (t+s+k)^{-\alpha\beta/r} \, \mathrm{d}t)^r \leqslant C \int_1^\infty s^{-\alpha\beta} \, \mathrm{d}s \int_0^\infty (s+k)^{-\alpha\beta} \, \mathrm{d}k \leqslant C \int_1^\infty s^{-\alpha\beta} \, \mathrm{d}s \int_0^\infty (1+k)^{-\alpha\beta} \, \mathrm{d}k < \infty$, by $\alpha\beta > 1$.

The bounds (6.11)–(6.13) imply (6.8), with arbitrarily small $\lambda > 0$ and $r = \bar{r} - \delta$, where $\delta = \delta(\lambda) > 0$ is sufficiently small. Indeed, the desired inequality $1 + \lambda(1 + r - 2\alpha\beta) < r/\bar{r}$ follows by $\lambda(2\alpha\beta - 1 - r) + r/\bar{r} > \lambda(2\alpha\beta - 1 - \bar{r}) + r/\bar{r}$, where $2\alpha\beta - 1 - \bar{r} > 0$ by (6.11) and $\bar{r}/r$ is arbitrary close to 1 by taking $\delta > 0$ small enough.

Let us prove (6.11), or

$$3 + 4\alpha\beta^2 < 4\alpha\beta + 2\beta. \tag{6.14}$$

Indeed, for $\alpha = 2$, (6.14) becomes $(\beta - 1/2)(8\beta - 6) < 0$, which is true for all $1/2 < \beta < 3/4$ ($> 3/(2+\alpha)$). Then (6.14) holds for $1/2 < \beta < 3/4$ and $\alpha \geqslant 2$ as well, which follows by taking the derivative of both sides of (6.14) with respect to α.

Now let us prove (6.9). Similarly to the proof of (6.8), $\mathrm{E}|\sum_{t=1}^N F_t^-|^2 \leqslant C(\Sigma_{1N}^- + \Sigma_{2N}^-)$, where

$$\begin{aligned}\Sigma_{1N}^{-} &\leqslant CN\int_1^{N^\lambda} \mathrm{d}k\left(\int_0^N \tau^{-\alpha\beta/2}(\tau+k)^{-\alpha\beta/2}\,\mathrm{d}\tau\right)^2\\ &\leqslant CN\int_1^{N^\lambda} \mathrm{d}k(k/k^{\alpha\beta})^2\left(\int_0^\infty \tau^{-\alpha\beta/2}(\tau+1)^{-\alpha\beta/2}\,\mathrm{d}\tau\right)^2\\ &\leqslant CN\int_1^{N^\lambda} k^{2-2\alpha\beta}\,\mathrm{d}k \leqslant C\begin{cases} N^{1+\lambda(3-2\alpha\beta)}, & \text{if } \alpha\beta < 3/2,\\ N\log N, & \text{otherwise.}\end{cases}\end{aligned}$$

Finally,

$$\Sigma_{2N}^{-} \leqslant CN^\lambda\int_N^\infty \mathrm{d}s\left(\int_0^N (\tau+s)^{-\alpha\beta}\,\mathrm{d}\tau\right)^2 \leqslant CN^{\lambda+3-2\alpha\beta},$$

whence (6.9) is immediate if $\alpha\beta \geqslant 3/2$, while in the case $\alpha\beta < 3/2$, λ has to be chosen so that the inequalities $1+\lambda(3-2\alpha\beta) < 2/\bar{r}$ and $\lambda+3-2\alpha\beta < 2/\bar{r}$ are satisfied. Since $\bar{r} < 2$, $3-2\alpha\beta < 1$, these inequalities are clearly satisfied for $\lambda > 0$ small enough. Lemma 6.1 is proved. □

Proof of Lemma 6.2. Recall the definition (6.4) of f_{j_1,j_2}. Let us show the bound

$$|f_{j_1,j_2}(z_1, z_2)| \leqslant C\begin{cases}(z_1^2+b_{j_1}^2)(z_2^2+b_{j_2}^2), & \text{if } |z_1|\leqslant 1, |z_2|\leqslant 1,\\ |z_1|(z_2^2+b_{j_2}^2), & \text{if } |z_1| > 1, |z_2|\leqslant 1,\\ |z_2|(z_1^2+b_{j_1}^2), & \text{if } |z_1|\leqslant 1, |z_2| > 1,\\ (|z_1|+|z_2|), & \text{if } |z_1| > 1, |z_2| > 1.\end{cases} \tag{6.15}$$

Let $U_1 := b_{j_1}\zeta_{t-j_1}$, $U_2 := b_{j_2}\zeta_{t-j_2}$, $H(x) := \hat{h}_{j_1,j_2}(x)$, $H^{(k)}(x) = \mathrm{d}^k H(x)/\mathrm{d}x^k$; then f_{j_1,j_2} can be rewritten as

$$\begin{aligned} f_{j_1,j_2}(z_1, z_2) = \mathrm{E}\{&H(z_1+z_2) - H(z_1+U_2) - H(U_1+z_2) + H(U_1+U_2)\\ &- (H^{(1)}(z_1+U_2) - H^{(1)}(U_1+U_2) - H^{(2)}(U_1+U_2)z_1)z_2\\ &- (H^{(1)}(U_1+z_2) - H^{(1)}(U_1+U_2) - H^{(2)}(U_1+U_2)z_2)z_1\\ &- H^{(2)}(U_1+U_2)z_1z_2\}.\end{aligned}$$

The expression inside the expectation can be rearranged as $\sum_{i=0}^{4} q_i$, where

$$q_0 := U_1(H^{(1)}(z_2) - H^{(1)}(U_2) - z_2 H^{(2)}(U_2)) + U_2(H^{(1)}(z_1) - H^{(1)}(U_1) - z_1 H^{(2)}(U_1)) + U_1 U_2 H^{(2)}(0),$$

$$q_1 := \int_{U_1}^{z_1}\int_{U_2}^{z_2}\left\{\int_{U_1}^{u_1}\int_{U_2}^{u_2} H^{(4)}(t_1+t_2)\mathrm{d}t_1\,\mathrm{d}t_2\right\}\mathrm{d}u_1\,\mathrm{d}u_2,$$

$$q_2 := U_1 U_2 \int_0^{U_1}\int_0^{U_2} H^{(4)}(t_1+t_2)\mathrm{d}t_1\,\mathrm{d}t_2,$$

$$q_3 := -U_2\int_{U_1}^{z_1}\left\{\int_0^{U_2}\int_{U_1}^{u_1} H^{(4)}(t_1+t_2)\mathrm{d}t_1\,\mathrm{d}t_2\right\}\mathrm{d}u_1,$$

$$q_4 := -U_1\int_{U_2}^{z_2}\left\{\int_0^{U_1}\int_{U_2}^{u_2} H^{(4)}(t_1+t_2)\mathrm{d}t_1\,\mathrm{d}t_2\right\}\mathrm{d}u_2.$$

Note that $\mathrm{E}q_0 = 0$ and therefore $f_{j_1,j_2}(z_1, z_2) = \mathrm{E}\left\{\sum_{i=1}^{4} q_i\right\}$. Next, by boundedness of $H^{(4)}$, almost surely,

$$|q_1| \leqslant C(z_1 - U_1)^2(z_2 - U_2)^2,$$

$$|q_2| \leqslant C U_1^2 U_2^2,$$

$$|q_3| \leqslant C(z_1 - U_1)^2 U_2^2,$$

$$|q_4| \leqslant C(z_2 - U_2)^2 U_1^2,$$

yielding $|f_{j_1,j_2}(z_1, z_2)| \leqslant C\mathrm{E}(z_1^2 + U_1^2)(z_2^2 + U_2^2) \leqslant C(z_1^2 + b_{j_1}^2)(z_2^2 + b_{j_2}^2)$, or the first inequality of (6.15); the remaining inequalities follow similarly. With (6.15) in mind, the lemma follows from the inequality $\mathrm{E}\min(|b_j\zeta|^{2r}, |b_j\zeta|^r) \leqslant C|b_j|^\alpha$ which is an easy consequence of (1.10). Lemma 6.2 is proved. □

Lemma 6.3. $\sum_{t=1}^{N} D_t = O_P(N^{(3-2\beta)/2\alpha\beta})$.

Proof. Write $D_t = D_t^+ + D_t^-$, where

$$D_t^+ := \sum_{j>0}\left(\hat{h}_j^{(1)}(b_j\zeta_{t-j}) - a_1 - a_2 b_j\zeta_{t-j}\right)\sum_{i>j} b_i\zeta_{t-i},$$

$$D_t^- := \sum_{j>0}\left(\hat{h}_j^{(1)}(b_j\zeta_{t-j}) - a_1 - a_2 b_j\zeta_{t-j}\right)\sum_{j>i>0} b_i\zeta_{t-i}.$$

We shall prove the lemma for $\sum_{t=1}^{N} D_t^+$ only, as the bound for $\sum_{t=1}^{N} D_t^-$ follows similarly. We have $\sum_{t=1}^{N} D_t^+ = \mathcal{D}_1 + \mathcal{D}_2$, where

$$\mathcal{D}_1 := \sum_{s<N} \sum_{k>N^\lambda} \sum_{t=(s+1)\vee 1}^{N} \Delta h_{t-s}(b_{t-s}\zeta_s) b_{t-s+k}\zeta_{s-k},$$

$$\mathcal{D}_2 := \sum_{s<N} \sum_{1\leqslant k\leqslant N^\lambda} \sum_{t=(s+1)\vee 1}^{N} \Delta h_{t-s}(b_{t-s}\zeta_s) b_{t-s+k}\zeta_{s-k},$$

where $\Delta h_{t-s}(z) := \hat{h}^{(1)}_{t-s}(z) - a_1 - a_2 z$, and where $\lambda := 1/\alpha\beta < 1$. It suffices to show that there exists $1 < r < \bar{r} := 2\alpha\beta/(3-2\beta)$ such that

$$\mathrm{E}|\mathcal{D}_2|^2 \leqslant CN^{2/\bar{r}}, \qquad \mathrm{E}|\mathcal{D}_1|^r \leqslant CN^{r/\bar{r}}. \tag{6.16}$$

Put $q_{N,k,s}(z) := \sum_{t=(s+1)\vee 1}^{N} \Delta h_{t-s}(z) b_{t-s+k}$. Using $a_1 = h^{(1)}_\infty(0) = \mathrm{E}\hat{h}^{(1)}_{t-s}(b_{t-s}\zeta_s)$, we obtain by orthogonality

$$\mathrm{E}\mathcal{D}_2^2 = \sum_{s<N} \sum_{1\leqslant k\leqslant N^\lambda} \mathrm{E}(q_{N,k,s}(\zeta_s))^2. \tag{6.17}$$

Next, using the von Bahr-Esseen inequality, for any $1 \leqslant r \leqslant 2$,

$$\begin{aligned}
\mathrm{E}|\mathcal{D}_1|^r &\leqslant C \sum_{s<N} \mathrm{E}\left| \sum_{k>N^\lambda} \sum_{t=(s+1)\vee 1}^{N} \Delta h_{t-s}(b_{t-s}\zeta_s) b_{t-s+k}\zeta_{s-k} \right|^r \\
&= C \sum_{s<N} \mathrm{E}\mathrm{E}\left[\left| \sum_{k>N^\lambda} \sum_{t=(s+1)\vee 1}^{N} \Delta h_{t-s}(b_{t-s}\zeta_s) b_{t-s+k}\zeta_{s-k} \right|^r \middle| \zeta_s \right] \\
&\leqslant C \sum_{s<N} \mathrm{E}\left\{ \mathrm{E}\left[\left(\sum_{k>N^\lambda} \sum_{t=(s+1)\vee 1}^{N} \Delta h_{t-s}(b_{t-s}\zeta_s) b_{t-s+k}\zeta_{s-k} \right)^2 \middle| \zeta_s \right] \right\}^{r/2} \\
&= C \sum_{s<N} \mathrm{E}\left\{ \sum_{k>N^\lambda} (q_{N,s,k}(\zeta_s))^2 \right\}^{r/2}.
\end{aligned} \tag{6.18}$$

Exactly as in (3.9), (3.10) and (3.11), one has $\mathrm{E}|\Delta h_{t-s}(b_{t-s}\zeta_s)|^2 \leqslant C|b_{t-s}|^\alpha$ and therefore

$$\mathrm{E}(q_{N,s,k}(\zeta_s))^2 \leqslant \left(\sum_{t=(s+1)\vee 1}^{N} |b_{t-s+k}||b_{t-s}|^{\alpha/2} \right)^2. \tag{6.19}$$

With (6.17) and (6.19) in mind, the first bound (6.16) follows by integral approximation of corresponding sums. Indeed, $\mathrm{E}\mathcal{D}_2^2 \leqslant C(\sum_{|s|<N} \ldots + \sum_{s<-N} \ldots) =: C(I_1 + I_2)$. Here

$$I_1 \leqslant CN\int_0^{N^\lambda} \mathrm{d}k\left(\int_0^\infty \tau^{-\alpha\beta/2}(\tau+k)^{-\beta}\,\mathrm{d}\tau\right)^2$$
$$= CN\int_0^{N^\lambda} k^{2-2\beta-\alpha\beta}\mathrm{d}k\left(\int_0^\infty \tau^{-\alpha\beta/2}(\tau+1)^{-\beta}\,\mathrm{d}\tau\right)^2$$
$$\leqslant CN\int_0^{N^\lambda} k^{2-2\beta-\alpha\beta}\,\mathrm{d}k = CN^{2/\bar r},$$

according to the definition of λ, $\bar r$. Next,

$$I_2 \leqslant C\int_N^\infty \mathrm{d}s\int_0^{N^\lambda} \mathrm{d}k\left(\int_0^N (t+s)^{-\alpha\beta/2}(t+s+k)^{-\beta}\,\mathrm{d}t\right)^2$$
$$= CN^{4-\alpha\beta-2\beta}\int_1^\infty \mathrm{d}s\int_0^{N^\lambda/N} \mathrm{d}k\left(\int_0^1 (t+s)^{-\alpha\beta/2}(t+s+k)^{-\beta}\,\mathrm{d}t\right)^2$$
$$\leqslant CN^{4-\alpha\beta-2\beta}(N^\lambda/N)\int_1^\infty s^{-\alpha\beta-2\beta}\,\mathrm{d}s = CN^{3-\alpha\beta-2\beta+\lambda} \leqslant CN^{2/\bar r},$$

where the last inequality follows from $\alpha\beta(3-\alpha\beta-2\beta) \leqslant 2-2\beta$ which is true for any $\beta \geqslant 1/2$, $\alpha \geqslant 2$. Indeed, for $\alpha = 2$ it becomes $\beta(3-4\beta) \leqslant 1-\beta$, or $(2\beta-1)^2 \geqslant 0$. Now, for any $\beta \geqslant 1/2$, the function $u(\alpha) := 2-2\beta-\alpha\beta(3-\alpha\beta-2\beta)$ increases in α, which follows from $u'(\alpha) = \beta(2\alpha\beta+2\beta-3) > 0$, due to $\alpha\beta > 1$, $\beta > 1/2$. This proves the first bound of (6.16).

The proof of the second bound of (6.16) is more delicate. We shall need the following bound:

$$\sum_{k>N^\lambda}(q_{N,k,s}(z))^2 \leqslant C\begin{cases} N^{-\lambda(2\beta-1)}|z|_1^{2/\beta}, & \text{if } |s|<N,\ |z|<N^{1/\alpha},\\ |z|^{(3-2\beta)/\beta}, & \text{if } |s|<N,\ |z|\geqslant N^{1/\alpha},\\ N^2|s|^{1-2\beta}\min(|z|_1^2|s|^{-2\beta},\ |z|_1^4|s|^{-4\beta}), & \text{if } s\leqslant -N,\end{cases} \tag{6.20}$$

where $|z|_1 := |z| \vee 1$. To show (6.20), note that $|\Delta h_{t-s}(b_{t-s}z)| \leqslant C\min(|b_{t-s}||z|_1,\ b_{t-s}^2|z|_1^2)$ exactly as in (3.9) and (3.10), implying that

$$|q_{N,s,k}(z)| \leqslant C\sum_{t=(s+1)\vee 1}^{N}|b_{t-s+k}|\min(|b_{t-s}||z|_1,\ b_{t-s}^2|z|_1^2).$$

Let $|s| < N$ and $z \geqslant 1$. Then

$$\sum_{k>N^\lambda}(q_{N,k,s}(z))^2 \leqslant C\int_{N^\lambda}^\infty \mathrm{d}k\left(\int_0^\infty \min(z\tau^{-\beta},\ z^2\tau^{-2\beta})(\tau+k)^{-\beta}\,\mathrm{d}\tau\right)^2 \leqslant C(J_1+J_2),$$

where

$$J_1 := z^2\int_{N^\lambda}^\infty \mathrm{d}k\left(\int_0^{z^{1/\beta}} \tau^{-\beta}(\tau+k)^{-\beta}\,\mathrm{d}\tau\right)^2, \qquad J_2 := z^4\int_{N^\lambda}^\infty \mathrm{d}k\left(\int_{z^{1/\beta}}^\infty \tau^{-2\beta}(\tau+k)^{-\beta}\,\mathrm{d}\tau\right)^2.$$

Here

$$J_1 = z^{(3-2\beta)/\beta}\int_{N^\lambda/z^{1/\beta}}^{\infty} \mathrm{d}k\left(\int_0^1 \tau^{-\beta}(\tau+k)^{-\beta}\,\mathrm{d}\tau\right)^2$$

$$\leqslant z^{(3-2\beta)/\beta}\int_0^{\infty} \mathrm{d}k\left(\int_0^1 \tau^{-\beta}(\tau+k)^{-\beta}\,\mathrm{d}\tau\right)^2 = Cz^{(3-2\beta)/\beta}$$

if $N^\lambda < z^{1/\beta}$ or $z > N^{1/\alpha}$ holds; the convergence of the last integral follows from

$$\int_0^1 \tau^{-\beta}(\tau+k)^{-\beta}\,\mathrm{d}\tau \leqslant \begin{cases} Ck^{-\beta}, & \text{if } k \geqslant 1, \\ k^{-(2\beta-1)}, & \text{if } 0 < k < 1, \end{cases}$$

and the inequality $2(2\beta-1) < 1$ which follows from $\beta < 3/(2+\alpha) < 3/4$. On the other hand, if $N^\lambda \geqslant z^{1/\beta}$, then

$$J_1 \leqslant z^{(3-2\beta)/\beta}\int_{N^\lambda/z^{1/\beta}} k^{-2\beta}\,\mathrm{d}k = CN^{-\lambda(2\beta-1)}z^{2/\beta}.$$

As J_2 can be similarly estimated, this proves (6.20) in the case $|s| < N$. Finally, for $s \leqslant -N$,

$$\sum_{k>N^\lambda}(q_{N,k,s}(z))^2 \leqslant C\int_0^\infty \mathrm{d}k\left(\int_0^N \min(z(\tau-s)^{-\beta}, z^2(\tau-s)^{-2\beta})(\tau+k-s)^{-\beta}\,\mathrm{d}\tau\right)^2$$

$$= C|s|^{3-2\beta}\int_0^\infty \mathrm{d}k\left(\int_0^{N/|s|}\min\left(\frac{z|s|^{-\beta}}{(1+\tau)^\beta}, \frac{z^2|s|^{-2\beta}}{(1+\tau)^{2\beta}}\right)\frac{\mathrm{d}\tau}{(\tau+k+1)^\beta}\right)^2$$

$$\leqslant C|s|^{3-2\beta}(N/|s|)^2\min(z^2|s|^{-2\beta}, z^4|s|^{-4\beta})\int_0^\infty (k+1)^{-2\beta}\,\mathrm{d}k$$

$$\leqslant CN^2|s|^{1-2\beta}\min(z^2|s|^{-2\beta}, z^4|s|^{-4\beta}),$$

proving (6.20). Hence by (6.18), $\mathrm{E}|\mathcal{D}_1|^r \leqslant C(\Sigma_1+\Sigma_2+\Sigma_3)$, where

$$\Sigma_1 := \sum_{|s|<N}\mathrm{E}\left\{\sum_{k>N^\lambda}(q_{N,k,s}(\zeta_s))^2\right\}^{r/2} I(|\zeta_s| \leqslant N^{1/\alpha}),$$

$$\Sigma_2 := \sum_{|s|<N}\mathrm{E}\left\{\sum_{k>N^\lambda}(q_{N,k,s}(\zeta_s))^2\right\}^{r/2} I(|\zeta_s| < N^{1/\alpha}),$$

$$\Sigma_3 := \sum_{s\leqslant -N}\mathrm{E}\left\{\sum_{k>N^\lambda}(q_{N,k,s}(\zeta_s))^2\right\}^{r/2}.$$

Here $\Sigma_1 \leqslant CN^{1-\lambda(2\beta-1)r/2}\mathrm{E}|\zeta|^{r/\beta}I(|\zeta| \leqslant N^{1/\alpha}) \leqslant CN^{1-\lambda(2\beta-1)r/2+(r/\beta-\alpha)/\alpha} =: CN^\nu$, provided $r/\beta > \alpha$, or $r > \alpha\beta$, holds. Recall $\lambda = 1/\alpha\beta$ and hence the exponent $\nu =$

$-(r/2\alpha\beta)(2\beta-1)+r/\alpha\beta=(r/2\alpha\beta)(3-2\beta)=r/\bar{r}$, so that the desired bound $\Sigma_1 \leqslant CN^{r/\bar{r}}$ (cf. (6.16)), for Σ_1 is satisfied. Next,

$$\Sigma_2 \leqslant CN\mathrm{E}|\zeta|^{(r/2\beta)(3-2\beta)}I(|\zeta|>N^{1/\alpha}) \leqslant CN^{1+((r/2\beta)(3-2\beta)-\alpha)/\alpha} = CN^{r/\bar{r}}$$

provided $(r/2\beta)(3-2\beta)<\alpha$, or $r<2\alpha\beta/(3-2\beta)=\bar{r}$, holds. Finally, for $s \leqslant -N$,

$$\begin{aligned}\Sigma_3 &\leqslant CN^r \sum_{s\geqslant N} s^{(1-2\beta)r/2}\mathrm{E}\min(|\zeta|^r s^{-r\beta}, |\zeta|^{2r}s^{-2r\beta}) \\ &\leqslant CN^r\int_N^\infty s^{(1-2\beta)r/2}\left(\frac{1}{s^{2r\beta}}\int_0^{s^\beta}\frac{z^{2r}\,\mathrm{d}z}{z^{1+\alpha}}+\frac{1}{s^{r\beta}}\int_{s^\beta}^\infty\frac{z^r\,\mathrm{d}z}{z^{1+\alpha}}\right)\mathrm{d}s \\ &\leqslant CN^r\int_N^\infty s^{(1-2\beta)r/2-\alpha\beta}\,\mathrm{d}s = CN^\nu,\end{aligned}$$

where $\nu := r+(1-2\beta)r/2-\alpha\beta+1 \leqslant r/\bar{r}$ is equivalent to $r \leqslant \bar{r}$. The above estimate also uses $\alpha/2<r<\alpha$, in addition to the previous $\alpha\beta<r<\bar{r}=2\alpha\beta/(3-2\beta)$ and $1\leqslant r\leqslant 2$. Note all these inequalities are satisfied by choosing $r=\alpha\beta+\epsilon$, where $\epsilon>0$ is small enough. This proves (6.16). Lemma 6.3 is proved. □

7. Proof of Lemma 5.2

The proof is similar to that of Lemma 4.2. First, let $\beta<2/3$, $\beta<3/(2+\alpha)$. From (4.2), (4.3), (4.9), (4.8), (5.3) and (5.1) we obtain

$$R_t^{(3,2)} = \sum_{0<j_1<j_2} U_{t,j_1,j_2}, \tag{7.1}$$

where $R_t^{(3,2)}$ is defined in (5.3) and

$$\begin{aligned}U_{t,j_1,j_2} := {}& h_{\neq j_1,j_2-1}(b_{j_1}\zeta_{t-j_1}+b_{j_2}\zeta_{t-j_2}+\tilde{X}_{t,j_2}) - h_{j_2-1}(b_{j_2}\zeta_{t-j_2}+\tilde{X}_{t,j_2}) \\ & - h_{\neq j_1,j_2}(b_{j_1}\zeta_{t-j_1}+\tilde{X}_{t,j_2}) + h_{j_2}(\tilde{X}_{t,j_2}) - \hat{h}_{j_1,j_2}(b_{j_1}\zeta_{t-j_1}+b_{j_2}\zeta_{t-j_2}) \\ & + \hat{h}_{j_1}(b_{j_1}\zeta_{t-j_1}) + \hat{h}_{j_2}(b_{j_2}\zeta_{t-j_2}) - a_0 - a_3 b_{j_1}b_{j_2}\zeta_{t-j_1}\zeta_{t-j_2}\tilde{X}_{t,j_2}.\end{aligned} \tag{7.2}$$

From the definition of $h_{\neq j_1,j_2}$, h_j, $\hat{h}_{j_1,j_2}$ and $\hat{h}_j$, we have $\mathrm{E}[U_{t,j_1,j_2}|\zeta_s, s\neq t-j_i]=0 (i=1,2)$ and hence the orthogonality property: $\mathrm{E}U_{t,j_1,j_2}U_{t',j_1',j_2'}=0$ if either $t-j_1\neq t'-j_1'$ or $t-j_2\neq t'-j_2'$. Using this property, we can write

$$\begin{aligned}|\mathrm{E}R_0^{(3,2)}R_t^{(3,2)}| &= \left|\sum_{0<j_1<j_2}\mathrm{E}U_{0,j_1,j_2}U_{t,t+j_1,t+j_2}\right| \\ &\leqslant \sum_{0<j_1<j_2}\left(\mathrm{E}U^2_{t,j_1,j_2}\right)^{1/2}\left(\mathrm{E}U^2_{t,t+j_1,t+j_2}\right)^{1/2}.\end{aligned} \tag{7.3}$$

We now use Lemma 7.1 below. It is easy to note that the main contribution to (7.3) is provided by the second term on the right-hand side of (7.5), and we obtain

$$|\mathrm{E}R_0^{(3,2)}R_t^{(3,2)}| \leqslant C\sum_{j_1,j_2>0}(j_1(t+j_1))^{-\beta}(j_2(t+j_2))^{-\beta-(2\beta-1)\alpha/4+\kappa/2}$$
$$\leqslant Ct^{-(2\beta-1)(2+\alpha/2)+\kappa}, \tag{7.4}$$

thereby proving the lemma in the case $\beta < 2/3$, $\beta < 3/(2+\alpha)$.

Next, let $3/(2+\alpha) < \beta < 2/3$. In this case, (7.1) and (7.3) hold with U_{t,j_1,j_2} replaced by $U'_{t,j_1,j_2} := U_{t,j_1,j_2} + \Lambda_{t,j_1,j_2}$, where

$$\Lambda_{t,j_1,j_2} := \hat{h}_{j_1,j_2}(b_{j_1}\zeta_{t-j_1} + b_{j_2}\zeta_{t-j_2}) - \hat{h}_{j_1}(b_{j_1}\zeta_{t-j_1}) - \hat{h}_{j_2}(b_{j_2}\zeta_{t-j_2}) + a_0 - a_2 b_{j_1} b_{j_2}\zeta_{t-j_1}\zeta_{t-j_2},$$

so that $Z_t^{(2)} = \sum_{0<j_1<j_2}\Lambda_{t,j_1,j_2}$. Then $\mathrm{E}(U'_{t,j_1,j_2})^2 \leqslant 2(\mathrm{E}(U_{t,j_1,j_2})^2 + \mathrm{E}(\Lambda_{t,j_1,j_2})^2)$ and the statement of Lemma 5.2 follows similarly from Lemma 7.1.

Finally, in the case $2/3 < \beta < 3/(2+\alpha)$, (7.1) and (7.3) hold with U_{t,j_1,j_2} replaced by $U''_{t,j_1,j_2} := U_{t,j_1,j_2} + a_3 b_{t-j_1} b_{t-j_2}\zeta_{t-j_1}\zeta_{t-j_2}\tilde{X}_{t,j_2}$ leading to $\mathrm{E}(U''_{t,j_1,j_2})^2 \leqslant C(\mathrm{E}U^2_{t,j_1,j_2} + b^2_{j_1} b^2_{j_2}\mathrm{E}\zeta^2_{t-j_1}\mathrm{E}\zeta^2_{t-j_2}\mathrm{E}\tilde{X}^2_{t,j_2})$, where $b^2_{j_1} b^2_{j_2}\mathrm{E}\tilde{X}^2_{t,j_2} \leqslant Cj_1^{-2\beta}j_2^{1-4\beta}$. The remaining computations are similar. Lemma 5.2 is proved. □

Lemma 7.1. *For any $\kappa > 0$, there exists $C < \infty$ such that*

$$\mathrm{E}(U_{t,j_1,j_2})^2 \leqslant C(j_1^{-\alpha\beta+\kappa}j_2^{1-4\beta} + j_1^{-2\beta}j_2^{-2\beta-(2\beta-1)\alpha/2+\kappa}), \tag{7.5}$$

$$\mathrm{E}(\Lambda_{t,j_1,j_2})^2 \leqslant C(j_1^{-\alpha\beta+\kappa}j_2^{-2\beta} + j_1^{-2\beta}j_2^{-\alpha\beta+\kappa}). \tag{7.6}$$

Proof. The proof is similar to that of (4.13), (4.14), (4.15) and Lemma 6.2, so we confine ourselves to giving an outline. To obtain a convenient expression for U_{t,j_1,j_2}, fix t, j_1, j_2, $0 < j_1 < j_2$, and let $H(x) := h_{\neq j_1,j_2-1}(x)$, $W_1 := b_{j_1}\zeta_{t-j_1}$, $W_2 := b_{j_2}\zeta_{t-j_2}$, $\tilde{X} := \tilde{X}_{t,j_2}$. Let $(W_1^0, W_2^0, \tilde{X}^0)$ be an independent copy of $(W_1, W_2, \tilde{X})$. Then $U_{t,j_1,j_2} = U^{(1)}_{t,j_1,j_2} + U^{(2)}_{t,j_1,j_2}$, where

$$U^{(1)}_{t,j_1,j_2} := \mathrm{E}^0\left\{\int_{W_1^0}^{W_1}\int_{W_2^0}^{W_2}\int_{\tilde{X}^0}^{\tilde{X}}(H^{(3)}(u_1+u_2+u_3) - H^{(3)}(W_1^0+W_2^0+\tilde{X}^0))\mathrm{d}u_1\mathrm{d}u_2\mathrm{d}u_3\right\},$$

$$U^{(2)}_{t,j_1,j_2} := \mathrm{E}^0\{(H^{(3)}(0) - H^{(3)}(W_1^0+W_2^0+\tilde{X}^0))(W_1W_2\tilde{X}^0 + W_1W_2^0\tilde{X}$$
$$+ W_1^0W_2\tilde{X} - W_1W_2^0\tilde{X}^0 - W_1^0W_2\tilde{X}^0 - W_1^0W_2^0\tilde{X} + W_1^0W_2^0\tilde{X}^0)\}.$$

Using the boundedness of $H^{(i)}$, $i = 3, 4$ (see lemma 3.1), and the inequality $|\int_{y^0}^{y}\min(1, |u-y^0|)\mathrm{d}u| \leqslant \min(|y-y^0|, |y-y^0|^2) \leqslant |y-y^0|^{1+\gamma}$ which is valid for any $y, y^0 \in \mathbb{R}$ and any $0 \leqslant \gamma \leqslant 1$, we obtain

$$|U^{(1)}_{t,j_1,j_2}| \leqslant C|b_{j_1}|^{1+\gamma}(|\zeta_{t-j_1}|^{1+\gamma}+1)|b_{j_2}|(|\zeta_{t-j_2}|+1)(|\tilde{X}_{t,j_2}|+\mathrm{E}|\tilde{X}_{t,j_2}|)$$
$$+ C|b_{j_1}|(|\zeta_{t-j_1}|+1)|b_{j_2}|^{1+\gamma}(|\zeta_{t-j_2}|^{1+\gamma}+1)(|\tilde{X}_{t,j_2}|+\mathrm{E}|\tilde{X}_{t,j_2}|)$$
$$+ C|b_{j_1}|(|\zeta_{t-j_1}|+1)|b_{j_2}|(|\zeta_{t-j_2}|+1)(|\tilde{X}_{t,j_2}|^{1+\gamma}+\mathrm{E}|\tilde{X}_{t,j_2}|^{1+\gamma}),$$

almost surely. By taking $2 < 2(1+\gamma) < \alpha$, we obtain

$$\mathrm{E}(U^{(1)}_{t,j_1,j_2})^2 \leqslant C\Big\{|b_{j_1}|^{2(1+\gamma)}b^2_{j_2}\mathrm{E}\tilde{X}^2_{t,j_2} + b^2_{j_1}|b_{j_2}|^{2(1+\gamma)}\mathrm{E}\tilde{X}^2_{t,j_2} + b^2_{j_1}b^2_{j_2}\mathrm{E}|\tilde{X}_{t,j_2}|^{2(1+\gamma)}\Big\}$$
$$\leqslant C\Big\{j_1^{-2(1+\gamma)\beta}j_2^{1-4\beta} + j_1^{-2\beta}j_2^{-2(1+\gamma)\beta+1-2\beta} + j_1^{-2\beta}j_2^{-2\beta+(1-2\beta)(1+\gamma)}\Big\}.$$

Here, we have used $\mathrm{E}|\tilde{X}_{t,j_2}|^{2(1+\gamma)} \leqslant Cj_2^{-(2\beta-1)(1+\gamma)}$ which follows by the Rosenthal inequality (see, for example, Petrov 1975). The rest of the proof of (7.5), including the estimation of $\mathrm{E}(U^{(2)}_{t,j_1,j_2})^2$, is similar to (4.13) and (4.14). The proof of (7.6) follows similarly. □

Acknowledgement

The author thanks the referees and associate editor for many helpful comments.

References

Avram, F. and Taqqu, M.S. (1987) Noncentral limit theorems and Appell polynomials. *Ann. Probab.*, **15**, 767–775.

Breuer, P. and Major, P. (1983) Central limit theorems for non-linear functionals of Gaussian random fields. *J. Multivariate Anal.*, **13**, 425–441.

Brockwell, P.J. and Davis, R.A. (1991) *Time Series: Theory and Methods.* New York: Springer-Verlag.

Davidson, J. and Sibbertsen, Ph. (2002) Generating schemes for long memory processes: regimes, aggregation and linearity. Preprint.

Dehling, H. and Taqqu, M.S. (1989) The empirical process of some long-range dependent sequences with an application to U-statistics. *Ann. Statist.*, **17**, 1767–1783.

Dobrushin, R.L. and Major, P. (1979) Non-central limit theorems for non-linear functions of Gaussian fields. *Z. Wahrscheinlichkeitstheorie Verw. Geb.*, **50**, 27–52.

Giraitis, L. (1985) Central limit theorem for functionals of linear processes. *Lithuanian Math. J.*, **25**, 43–57.

Giraitis, L. and Surgailis, D. (1999) Central limit theorem for the empirical process of a linear sequence with long memory. *J. Statist. Plann. Inference*, **80**, 81–93.

Giraitis, L., Koul, H.L. and Surgailis, D. (1996) Asymptotic normality of regression estimators with long memory errors. *Statist. Probab. Lett.*, **29**, 317–335.

Ho, H.-C. and Hsing, T. (1996) On the asymptotic expansion of the empirical process of long memory moving averages. *Ann. Statist.*, **24**, 992–1024.

Ho, H.-C. and Hsing, T. (1997) Limit theorems for functionals of moving averages. *Ann. Probab.*, **25**, 1636–1669.

Hosking, J.R.M. (1981) Fractional differencing. *Biometrika*, **68**, 165–176.

Ibragimov, I.A. and Linnik, Yu. V. (1971) *Independent and Stationary Sequences of Random Variables.* Groningen: Wolters-Noordhoff.

Kaz'min, J.A. (1969) Appell polynomial series expansions. *Mat. Zametki*, **5**, 509–520.

Koul, H.L. and Surgailis, D. (1997) Asymptotic expansion of M-estimators with long memory errors. *Ann. Statist.*, **25**, 818–850.

Koul, H.L. and Surgailis, D. (2001) Asymptotics of the empirical processes of long memory moving averages with infinite variance. *Stochastic Process. Appl.*, **91**, 309–336.

Koul, H.L. and Surgailis, D. (2002) Asymptotic expansion of the empirical processes of long memory moving averages. In H. Dehling, T. Mikosch and M. Sørensen (eds), *Empirical Process Techniques for Dependent Data*. Boston: Birkhäuser.

Leipus, R. and Surgailis, D. (2003) Random coefficient autoregression, regime switching and long memory. *Adv. Appl. Probab.*, **35**, 1–18.

Leipus, R., Paulauskas, V. and Surgailis, D. (2003) Renewal regime switching and stable limit laws. Preprint.

Mikosch, T., Resnick, S., Rootzén, H. and Stegeman, A. (2002) Is network traffic approximated by stable Lévy motion or fractional Brownian motion? *Ann. Appl. Probab.*, **12**, 23–68.

Petrov, V.V. (1975) *Sums of Independent Random Variables.* New York: Springer-Verlag.

Pipiras, V., Taqqu, M.S. and Levy, J.B. (2003) Slow, fast and arbitrary growth conditions for the renewal–reward processes when both the renewals and the rewards are heavy-tailed. *Bernoulli*, **10**, 121–163.

Surgailis, D. (1982) Zones of attraction of self-similar multiple integrals. *Lithuanian Math. J.*, **22**, 327–340.

Surgailis, D. (2000) Long-range dependence and Appell rank. *Ann. Probab.*, **28**, 478–497.

Surgailis, D. (2002) Stable limits of empirical processes of moving averages with infinite variance. *Stochastic Process. Appl.*, **100**, 255–274.

Taqqu, M.S. (1975) Weak convergence to fractional Brownian motion and to the Rosenblatt process. *Z. Wahrscheinlichkeitstheorie Verw. Geb.*, **31**, 287–302.

Taqqu, M.S. (1979) Convergence of integrated processes of arbitrary Hermite rank. *Z. Wahrscheinlichkeitstheorie Verw. Geb.*, **50**, 53–83.

Taqqu, M.S. and Levy, J.B. (1986) Using renewal processes to generate long-range dependence and high variability. In E. Eberlein and M.S. Taqqu (eds), *Dependence in Probability and Statistics.* Boston: Birkhäuser.

Vaičiulis, M. (2003) Convergence of sums of Appell polynomials with infinite variance. *Lithuanian Math. J.*, **43**, 80–98.

von Bahr, B. and Esseen, C.-G. (1965) Inequalities for the rth absolute moment of the sum of random variables, $1 \leq r \leq 2$. *Ann. Math. Statist.*, **36**, 299–303.

Wu, W.B. (2003) Empirical processes of long-memory sequences. *Bernoulli*, **9**, 809–831.

Received September 2002 and revised July 2003