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A stochastic wave equation in dimension 3: smoothness of the law

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We prove the existence and regularity of the density of the real-valued solution to a three-dimensional stochastic wave equation. The noise is white in time and has a spatially homogeneous correlation whose spectral measure μ satisfies $\int_{\mathbb{R}^3} \mu(\mathrm{d}\xi)(1+|\xi|^2)^{-\eta} < \infty$, for some $\eta \in (0, \frac{1}{2})$. Our approach uses the mild formulation of the equation given by means of Dalang's extended version of Walsh's stochastic integration. We apply the tools of Malliavin calculus on the appropriate Gaussian space related to the noise. An extension of Dalang's stochastic integral to the Hilbert-valued setting is needed. Let S_3 be the fundamental solution to the three-dimensional wave equation. The assumption on the noise yields upper and lower bounds for the integral $\int_0^t \mathrm{d}s \int_{\mathbb{R}^3} \mu(\mathrm{d}\xi)|\mathcal{F}S_3(s)(\xi)|^2$ and upper bounds for $\int_0^t \mathrm{d}s \int_{\mathbb{R}^3} \mu(\mathrm{d}\xi)|\xi||\mathcal{F}S_3(s)(\xi)|^2$ in terms of powers of t. These estimates, together with a suitable mollifying procedure for S_3, are crucial in the analysis of the inverse of the Malliavin variance.

Keywords: Malliavin calculus; stochastic partial differential equations; wave equation

1. Introduction

In this paper we study the probability law of the real-valued solution to the stochastic wave equation

$$\left(\frac{\partial^2}{\partial t^2} - \Delta_3\right) u(t, x) = \sigma(u(t, x))\dot{F}(t, x) + b(u(t, x)), \qquad u(0, x) = \frac{\partial u}{\partial t}(0, x) = 0, \tag{1}$$

where $(t, x) \in (0, T] \times \mathbb{R}^3$, $T > 0$; Δ_3 denotes the Laplacian operator on $\mathbb{R}^3$ and $\dot{F}$ is a Gaussian noise white in time and correlated in space. Clearly, (1) is a particular case of a class of stochastic partial differential equations (SPDEs) of the form

$$Lu(t, x) = \sigma(u(t, x))\dot{F}(t, x) + b(u(t, x)), \qquad u(0, x) = \frac{\partial u}{\partial t}(0, x) = 0, \tag{2}$$

$(t, x) \in (0, T] \times \mathbb{R}^d$, $T > 0$, where L is a second-order partial differential operator and the fundamental solution of $Lu = 0$ is a non-negative distribution with rapid decrease Λ.

Assume that the coefficients σ and b are Lipschitz continuous real-valued functions and F is a mean-zero $L^2(\Omega, \mathcal{F}, P)$-valued Gaussian process indexed by the space of test functions $\mathcal{D}(\mathbb{R}^{d+1})$ with covariance functional $J(\varphi, \psi) = \int_{\mathbb{R}_+} \mathrm{d}s \int_{\mathbb{R}^d} \Gamma(\mathrm{d}x)(\varphi(s) * \tilde{\psi}(s))(x)$, where Γ is a non-negative, non-negative definite tempered measure and $\tilde{\psi}(s, x) = \psi(s, -x)$. Let $\mu = \mathcal{F}^{-1}\Gamma$, where $\mathcal{F}$ is the Fourier transform operator. Then

$$J(\varphi, \psi) = \int_{\mathbb{R}_+} \mathrm{d}s \int_{\mathbb{R}^d} \mu(\mathrm{d}\xi)\mathcal{F}\varphi(s)(\xi)\overline{\mathcal{F}\psi(s)(\xi)}.$$

In Dalang (1999) a suitable extension of Walsh's stochastic integral with respect to martingale measures is developed; with this tool a rigorous meaning is given to equation (2) in a mild form and a theorem on existence and uniqueness of solution is proved. More precisely, there exists a real-valued stochastic process $u = \{u(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$ which satisfies the equation

$$u(t, x) = \int_0^t \int_{\mathbb{R}^d} \Lambda(t - s, x - y)\sigma(u(s, y))M(\mathrm{d}s, \mathrm{d}y) + \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} b(u(t - s, x - y))\Lambda(s, \mathrm{d}y), \tag{3}$$

where M denotes the martingale measure extension of the process F (see Dalang and Frangos 1998).

Fix $(t, x) \in (0, T] \times \mathbb{R}^3$. Our purpose is to find sufficient conditions ensuring that the law of $u(t, x)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ and that the density is a $\mathcal{C}^\infty$ function. The existence of the density has been studied in the companion paper by Quer-Sardanyons and Sanz-Solé (2004).

Malliavin calculus provides a suitable tool for the analysis of these problems. The Gaussian family to be considered here is described as follows. Let $\mathcal{E}$ be the inner-product space consisting of functions $\varphi \in \mathcal{S}(\mathbb{R}^d)$, the Schwartz space of rapidly decreasing $\mathcal{C}^\infty$ test functions, endowed with the inner-product $\langle \varphi, \psi \rangle_{\mathcal{E}} := \int_{\mathbb{R}^d} \mu(\mathrm{d}\xi)\mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}$. Let $\mathcal{H}$ denote the completion of $(\mathcal{E}, \langle \cdot, \cdot \rangle_{\mathcal{E}})$ and set $\mathcal{H}_T = L^2([0, T]; \mathcal{H})$. Notice that $\mathcal{H}$ and $\mathcal{H}_T$ may contain distributions. The space $\mathcal{H}_T$ is a real Hilbert separable space. For $h \in \mathcal{H}_T$ we set $W(h) = \int_0^t \int_{\mathbb{R}^d} h(s, x)M(\mathrm{d}s, \mathrm{d}x)$, where the stochastic integral is interpreted in Dalang's sense. Then $\{W(h), h \in \mathcal{H}_T\}$ is a Gaussian process and we can apply the Malliavin calculus based on it (see, for instance, Nualart 1998).

In Theorem 1 of Quer-Sardanyons and Sanz-Solé (2004) we introduce an extension of Dalang's stochastic integral to integrators that are defined by stochastic integration of Hilbert-valued predictable processes with respect to martingale measures. Owing to this extension we have proved that the solution of (3) at any point (t, x) is once differentiable in the Malliavin sense and that the derivative belongs to any L^p and satisfies an SPDE.

We prove in Section 3 below that $u(t, x) \in \mathbb{D}^\infty$ and give the equation satisfied by $D^N u(t, x)$. The standard approach to this problem (see, for instance, Millet and Sanz-Solé 1999; Márquez-Carreras *et al*. 2001) cannot be used here. In fact, the difference of two positive distributions is not necessarily positive; but positivity is one of the requirements in the construction of Dalang's integral and, especially for obtaining L^p bounds, a useful tool for proving L^p convergences. We circumvent this difficulty as follows. We consider a sequence of regularized processes $u_n(t, x)$, $n \geqslant 1$, obtained by convolution of the fundamental solution Λ with an approximation of the identity. The L^p-limit of $u_n(t, x)$ as n tends to infinity is $u(t, x)$, as is proved in Proposition 1 of Quer-Sardanyons and Sanz-Solé (2003); in addition, $u_n(t, x) \in \mathbb{D}^\infty$. Then, since the iterated Malliavin derivative operator D^N is closed, it suffices to prove that the sequence $D^N u_n(t, x)$ converges in the topology of $L^p(\Omega; \mathcal{H}_T^{\otimes N})$, for any $N \geqslant 1$, $p \in [1, \infty)$. This can be achieved by first proving that the sequence is bounded in any L^p and then proving the convergence of order 2, which can be checked with techniques related to the isometry property of the stochastic integral.

The results of Section 4 concern the particular case of equation (1), the stochastic wave equation in spatial dimension 3. We prove that the inverse of the Malliavin variance of $u(t, x)$ belongs to any $L^p(\Omega)$ for all fixed $(t, x) \in (0, T] \times \mathbb{R}^3$. Then, by the results of Section 3, we conclude that the law of $u(t, x)$ has a smooth density.

The existence of moments of any order of the inverse of the Malliavin variance is assured by the integrability in a neighbourhood of zero of the function

$$\varepsilon \to \varepsilon^{-(1+p)} P\{\|Du(t, x)\|_{\mathcal{H}_T}^2 < \varepsilon\},$$

for any $p \in [0, \infty)$. Hence, the main issue is to obtain the *size* in ε of the factor $P\{\|Du(t, x)\|_{\mathcal{H}_T}^2 < \varepsilon\}$. The difficulties come from the fact that the fundamental solution of the wave equation is a Schwartz distribution. The natural idea is to smooth that distribution, as we did to study the differentiability. This time we introduce a regularization kernel which depends on ε in a suitable way so that the error in this approximation is a function of ε as well. This technique is complemented with upper and lower bounds of integrals involving the Fourier transform of the fundamental solution of the wave equation, which have also played a crucial role in the arguments of Quer-Sardanyons and Sanz-Solé (2004); these are presented in the Appendix.

All positive real constants are denoted by C, regardless of their values. In the following section we give some basic notation for Malliavin calculus used throughout the paper. We refer the reader to Nualart (1995) for a complete account of notions related to this topic.

2. Preliminaries

Consider the stochastic equation (3) as described in the Introduction. Assume that the following set of hypotheses is satisfied:

Hypothesis D. *Let Λ be the fundamental solution of $Lu = 0$. Then $\Lambda(t)$ is a non-negative distribution with rapid decrease such that*

$$\int_0^T \mathrm{d}t \int_{\mathbb{R}^d} \mu(\mathrm{d}\xi) |\mathcal{F}\Lambda(t)(\xi)|^2 < \infty \tag{4}$$

and

$$\lim_{h \downarrow 0} \int_0^T \mathrm{d}t \int_{\mathbb{R}^d} \mu(\mathrm{d}\xi) \sup_{t<r<t+h} |\mathcal{F}(\Lambda(r) - \Lambda(t))(\xi)|^2 + 0.$$

Moreover, Λ is a non-negative measure on $\mathbb{R}_+ \times \mathbb{R}^d$ of the form $\Lambda(t, \mathrm{d}y)\mathrm{d}t$ such that $\sup_{0 \leqslant t \leqslant T} \Lambda(t, \mathbb{R}^d) \leqslant C_T < \infty$.

Then Theorem 5 in Dalang (1999) establishes the existence of a unique progressively measurable process $\{u(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$ such that (3) holds; in addition, $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathrm{E}(|u(t,x)|^p) < \infty$, for any $p \in [1,\infty)$, and this has a spatial stationary covariance function. This process will be called the *solution* of (3).

We denote by D the Malliavin derivative operator defined in the framework of the Gaussian space described in the Introduction. Fix any positive integer N; then D^N denotes the Nth iteration of D. For any random variable X, the Nth derivative, if it exists, defines a random vector with values in $\mathcal{H}_T^{\otimes N}$. For any $p \in [1,\infty)$ we denote by $\mathbb{D}^{N,p}$ the Sobolev–Watanabe space of random variables X such that

$$\|X\|_{N,p}^p := \mathrm{E}(|X|^p) + \sum_{j=1}^{N} \mathrm{E}\left(\|D^j X\|_{\mathcal{H}_T^{\otimes j}}^p \right) < +\infty.$$

Let $\mathcal{A}$ be a separable real Hilbert space and $K = \{K(s,z), (s,z) \in [0,T] \times \mathbb{R}^d\}$ be an $\mathcal{A}$-valued predictable process. Set $K^j(s,z) = \langle K(s,z), e_j \rangle_{\mathcal{A}}$, where $\{e_j, j \geqslant 0\}$ is a complete orthonormal system of $\mathcal{A}$. Assume that:

1. $\sup_{(s,z)\in[0,T]\times\mathbb{R}^d} \mathrm{E}(\|K(s,z)\|_{\mathcal{A}}^2) < \infty$;
2. for all $j \geqslant 0$, $s \in [0,T]$, $x, y \in \mathbb{R}^d$,

$$\mathrm{E}(K^j(s,x)K^j(s,y)) = \mathrm{E}(K^j(s,0)K^j(s,y-x)).$$

For any $j \geqslant 0$, set

$$M_t^{K^j}(A) = \int_0^t \int_A K^j(s,z) M(\mathrm{d}s, \mathrm{d}z), \qquad t \in [0,T],\ A \in \mathcal{B}_b(\mathbb{R}^d).$$

The process $M_t^K(A) = \sum_{j\geqslant 0} M_t^{K^j}(A) e_j$ defines an $\mathcal{A}$-valued martingale measure.

Set $G_j^K(s,z) = \mathrm{E}(K^j(s,0)K^j(s,z))$. The measure

$$\Gamma_s^K(\mathrm{d}z) = \sum_{j\geqslant 0} G_j^K(s,z) \Gamma(\mathrm{d}z)$$

is non-negative and tempered. Let μ_s^K be the non-negative tempered measure such that $\mathcal{F}^{-1}\Gamma_s^K = \mu_s^K$.

The next result reproduces Theorem 1 in Quer-Sardanyons and Sanz-Solé (2004). It is an extension to the Hilbert setting of Theorems 2 and 5 in Dalang (1999).

Proposition 1. *Let $t \to S(t)$ be a deterministic function with values in the space of non-negative distributions with rapid decrease satisfying* (4). *Then the indefinite stochastic integral of S with respect to the martingale measure M^K, $(S \cdot M^K)_t$, $t \in [0,T]$, exists as an $\mathcal{A}$-valued process and satisfies*

$$\mathrm{E}(\|(S \cdot M^K)_t\|_{\mathcal{A}}^2) = \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mu_s^K(\mathrm{d}\xi) |\mathcal{F}S(s)(\xi)|^2.$$

Moreover, for any $p \in [2,\infty)$, $t \in [0,T]$,

$$\mathrm{E}(\|(S\cdot M^K)_t\|_{\mathcal{A}}^p)\leqslant C_t\int_0^t \mathrm{d}s \sup_{x\in\mathbb{R}^d}\mathrm{E}(\|K(s,x)\|_{\mathcal{A}}^p)\int_{\mathbb{R}^d}\mu(\mathrm{d}\xi)|\mathcal{F}S(s)(\xi)|^2, \tag{5}$$

with $C_t = (\int_0^t \mathrm{d}s \int_{\mathbb{R}^d}\mu(\mathrm{d}\xi)|\mathcal{F}S(s)(\xi)|^2)^{p/2-1}$.

We shall use the notation

$$\|S\|_{0,K}^2 = \int_0^T \mathrm{d}s\int_{\mathbb{R}^d}\mu_s^K(\mathrm{d}\xi)|\mathcal{F}S(s)(\xi)|^2.$$

In this paper we will apply this result to $\mathcal{A}:=\mathcal{H}_T^{\otimes j}$ and to $\mathcal{H}_T^{\otimes j}$-valued stochastic processes involving Malliavin derivatives up to order $j\geqslant 1$.

3. Malliavin differentiability of spatially homogeneous SPDEs

Suppose that the coefficients of equation (3) are $\mathcal{C}^1$ functions with bounded Lipschitz continuous derivatives. We have proved in Quer-Sardanyons and Sanz-Solé (2004) that, for any fixed $t\geqslant 0$ and $x\in\mathbb{R}^d$, $u(t,x)$ belongs to the space $\mathbb{D}^{1,p}$, for all $p\in[1,\infty)$. The purpose of this section is to extend this result to any differentiability order. That is to say, we wish to prove that $u(t,x)\in\mathbb{D}^\infty=\cap_{N\in\mathbb{N}}\cap_{p\in[1,\infty)}\mathbb{D}^{N,p}$. It is clear that a strengthening of the regularity of the coefficients is needed.

We shall use the notation

$$D^N_{((r_1,\varphi_1),\ldots,(r_N,\varphi_N))}X = \langle D^N_{(r_1,\ldots,r_N)}X,\ \varphi_1\otimes\ldots\otimes\varphi_N\rangle_{\mathcal{H}^{\otimes N}},$$

for $r_i\in[0,T]$, $\varphi_i\in\mathcal{H}$, $i=1,\ldots,N$. Thus, we have that

$$\|D^NX\|^2_{\mathcal{H}_T^{\otimes N}} = \int_{[0,T]^N}\mathrm{d}r_1\ldots\mathrm{d}r_N\sum_{j_1,\ldots,j_N}|D_{((r_1,e_{j_1}),\ldots,(r_N,e_{j_N}))}X|^2, \tag{6}$$

where $\{e_j\}_{j\geqslant 0}$ is a complete orthonormal system of $\mathcal{H}$.

Let $N\in\mathbb{N}$, fix a set $A_N=\{\alpha_i=(r_i,\varphi_i)\in\mathbb{R}_+\times\mathcal{H},\ i=1,\ldots,N\}$ and set $\bigvee_i r_i = \max(r_1,\ldots,r_N)$, $\alpha=(\alpha_1,\ldots,\alpha_N)$, $\hat{\alpha}_i=(\alpha_1,\ldots,\alpha_{i-1},\alpha_{i+1},\ldots,\alpha_N)$. Denote by $\mathcal{P}_m$ the set of partitions of A_N consisting of m disjoint subsets $p_1,\ldots,p_m$, $m=1,\ldots,N$, and by $|p_i|$ the cardinal of p_i. Let X be a random variable belonging to $\mathbb{D}^{N,2}$, $N\geqslant 1$, and g be a real $\mathcal{C}^N$-function with bounded derivatives up to order N. Leibniz's rule for Malliavin's derivatives yields

$$D^N_\alpha(g(X)) = \sum_{m=1}^N\sum_{\mathcal{P}_m} c_m g^{(m)}(X)\prod_{i=1}^m D^{|p_i|}_{p_i}X, \tag{7}$$

with positive coefficients c_m, $m=1,\ldots,N$, $c_1=1$. Let

$$\Delta^N_\alpha(g,X) := D^N_\alpha g(X) - g'(X)D^N_\alpha X.$$

Notice that $\Delta^N_\alpha(g,X)=0$ if $N=1$ and it only depends on the Malliavin derivatives up to the order $N-1$ if $N>1$.

We now state the main result of this section.

Theorem 1. *Assume Hypothesis D and that the coefficients σ and b are $\mathcal{C}^\infty$ functions with bounded derivatives of any order greater than or equal to one. Then, for every $(t, x) \in [0, T] \times \mathbb{R}^d$, the random variable $u(t, x)$ belongs to the space $\mathbb{D}^\infty$. Moreover, for any $p \geqslant 1$ and $N \geqslant 1$, there exists an $L^p(\Omega; \mathcal{H}_T^{\otimes N})$-valued random process $\{Z^N(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$ such that*

$$\begin{aligned} D^N u(t, x) = {} & Z^N(t, x) \\ & + \int_0^t \int_{\mathbb{R}^d} \Lambda(t - s, x - z)[\Delta^N(\sigma, u(s, z)) + D^N u(s, z)\sigma'(u(s, z))]M(\mathrm{d}s, \mathrm{d}z) \\ & + \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \Lambda(s, \mathrm{d}z)[\Delta^N(b, u(t - s, x - z)) \\ & + D^N u(t - s, x - z)b'(u(t - s, x - z))] \end{aligned} \tag{8}$$

and

$$\sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathrm{E}(\|D^N u(s, y)\|^p_{\mathcal{H}_T^{\otimes N}}) < +\infty.$$

We prove this theorem by applying the next lemma, which follows from the fact that D^N is a closed operator defined on $L^p(\Omega)$ with values in $L^p(\Omega; \mathcal{H}_T^{\otimes N})$.

Lemma 1. *Let $\{F_n\}_{n\geqslant1}$ be a sequence of random variables belonging to $\mathbb{D}^{N,p}$. Assume that:*

(a) *there exists a random variable F such that F_n converges to F in $L^p(\Omega)$ as n tends to ∞,*
(b) *the sequence $\{D^N F_n\}_{n\geqslant1}$ converges in $L^p(\Omega; \mathcal{H}_T^{\otimes N})$.*

Then F belongs to $\mathbb{D}^{N,p}$ and $D^N F = L^p(\Omega; \mathcal{H}_T^{\otimes N}) - \lim_{n\to\infty} D^N F_n$.

As in Quer-Sardanyons and Sanz-Solé (2004), we consider the sequence of processes $\{u_n(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$ solving the equation

$$\begin{aligned} u_n(t, x) = {} & \int_0^t \int_{\mathbb{R}^d} \Lambda_n(t - s, x - z)\sigma(u_n(s, z))M(\mathrm{d}s, \mathrm{d}z) \\ & + \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} b(u_n(t - s, x - z))\Lambda(s, \mathrm{d}z), \end{aligned}$$

where $\Lambda_n(t) = \psi_n * \Lambda(t)$, with $\psi_n(x) = n^d \psi(nx)$, $n \geqslant 1$, ψ being a non-negative function in $\mathcal{C}^\infty(\mathbb{R}^d)$ with support contained in the unit ball of $\mathbb{R}^d$ and such that $\int_{\mathbb{R}^d} \psi(x)\mathrm{d}x = 1$.

Since Λ_n is smooth, a standard proof (see, for instance, Millet and Sanz-Solé 1999; Márquez-Carreras *et al*. 2001) yields that $u_n(t, x) \in \mathbb{D}^\infty$, for all $n \geqslant 1$. Moreover, the derivative $D^N u_n(t, x)$ satisfies the equation

$$
\begin{aligned}
D_\alpha^N u_n(t,x) = &\sum_{i=1}^N \langle \Lambda_n(t-r_i, x-*) D_{\hat{\alpha}_i}^{N-1} \sigma(u_n(r_i,*)), \varphi_i\rangle_{\mathcal{H}} \\
&+ \int_{\bigvee_i r_i}^t \int_{\mathbb{R}^d} \Lambda_n(t-s, x-z)[\Delta_\alpha^N(\sigma, u_n(s,z)) \\
&+ D_\alpha^N u_n(s,z)\sigma'(u_n(s,z))] M(\mathrm{d}s, \mathrm{d}z) \\
&+ \int_{\bigvee_i r_i}^t \mathrm{d}s \int_{\mathbb{R}^d} \Lambda(s, \mathrm{d}z)[\Delta_\alpha^N(b, u_n(t-s, x-z)) \\
&+ D_\alpha^N u_n(t-s, x-z) b'(u_n(t-s, x-z))],
\end{aligned} \tag{9}
$$

where $\alpha = ((r_1, \varphi_1), \ldots, (r_N, \varphi_N))$, with $r_1, \ldots, r_N \geqslant 0$ and $\varphi_1, \ldots, \varphi_N \in \mathcal{H}$.

Lemma 2. *Assume the same hypothesis as in Theorem 1. Then, for all $p \in [1, \infty)$ and every $N \geqslant 1$,*

$$
\sup_{n \geqslant 1} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathrm{E}\Big(\|D^N u_n(t,x)\|^p_{\mathcal{H}_T^{\otimes N}} \Big) < +\infty. \tag{10}
$$

Proof. We will use an induction argument with respect to N with $p \geqslant 2$ fixed. For $N = 1$, the property (10) is proved in Quer-Sardanyons and Sanz-Solé (2003, Proposition 2). Assume that

$$
\sup_{n \geqslant 1} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathrm{E}\Big(\|D^k u_n(t,x)\|^p_{\mathcal{H}_T^{\otimes k}} \Big) < +\infty,
$$

for any $k = 1, \ldots, N-1$. Let $\alpha = ((r_1, e_{j_1}), \ldots, (r_N, e_{j_N}))$, $r = (r_1, \ldots, r_N)$, $\mathrm{d}r = \mathrm{d}r_1 \ldots \mathrm{d}r_N$. Then, by (6), we have that

$$
\begin{aligned}
\mathrm{E}\Big(\|D^N u_n(t,x)\|^p_{\mathcal{H}_T^{\otimes N}} \Big) &= \mathrm{E}\left(\int_{[0,T]^N} \mathrm{d}r \sum_{j_1,\ldots,j_N} |D_\alpha^N u_n(t,x)|^2 \right)^{p/2} \\
&\leqslant C \sum_{i=1}^5 N_i,
\end{aligned}
$$

where

$$N_1 = \mathrm{E}\left(\int_{[0,T]^N} \mathrm{d}r \sum_{j_1,\ldots,j_N} \left|\sum_{i=1}^{N} \langle \Lambda_n(t-r_i, x-*) \times D_{\hat{\alpha}_i}^{N-1}\sigma(u_n(r_i,*)), e_{j_i}\rangle_{\mathcal{H}}\right|^2\right)^{p/2},$$

$$N_2 = \mathrm{E}\left(\int_{[0,T]^N} \mathrm{d}r \sum_{j_1,\ldots,j_N} \left|\int_{\bigvee_i r_i}^{t}\int_{\mathbb{R}^d} \Lambda_n(t-s, x-z) \times \Delta_\alpha^N(\sigma, u_n(s,z)) M(\mathrm{d}s, \mathrm{d}z)\right|^2\right)^{p/2},$$

$$N_3 = \mathrm{E}\left(\int_{[0,T]^N} \mathrm{d}r \sum_{j_1,\ldots,j_N} \left|\int_{\bigvee_i r_i}^{t} \mathrm{d}s \int_{\mathbb{R}^d} \Lambda(s, \mathrm{d}z) \times \Delta_\alpha^N(b, u_n(t-s, x-z))\right|^2\right)^{p/2},$$

$$N_4 = \mathrm{E}\left(\int_{[0,T]^N} \mathrm{d}r \sum_{j_1,\ldots,j_N} \left|\int_{\bigvee_i r_i}^{t}\int_{\mathbb{R}^d} \Lambda_n(t-s, x-z) D_\alpha^N u_n(s,z) \times \sigma'(u_n(s,z)) M(\mathrm{d}s, \mathrm{d}z)\right|^2\right)^{p/2},$$

$$N_5 = \mathrm{E}\left(\int_{[0,T]^N} \mathrm{d}r \sum_{j_1,\ldots,j_N} \left|\int_{\bigvee_i r_i}^{t} \mathrm{d}s \int_{\mathbb{R}^d} \Lambda(s, \mathrm{d}z) D_\alpha^N u_n(t-s, x-z) \times b'(u_n(t-s, x-z))\right|^2\right)^{p/2}.$$

By Parseval's identity and the definition of the $\mathcal{H}$-norm, it follows that

$$\begin{aligned}
N_1 &\leqslant C \sum_{i=1}^{N} \mathrm{E}\left(\int_{[0,T]^N} \mathrm{d}r \sum_{j_1,\ldots,j_N} \left|\langle \Lambda_n(t-r_i, x-*) \times D_{\hat{\alpha}_i}^{N-1}\sigma(u_n(r_i,*)), e_{j_i}\rangle_{\mathcal{H}}\right|^2\right)^{p/2} \\
&= C \sum_{i=1}^{n} \mathrm{E}\left(\int_{[0,T]^N} \mathrm{d}r \sum_{\hat{j}_i} \|\Lambda_n(t-r_i, x-*) D_{\hat{\alpha}_i}^{N-1}\sigma(u_n(r_i,*))\|_{\mathcal{H}}^2\right)^{p/2} \\
&= C \sum_{i=1}^{n} \mathrm{E}\Bigg(\int_{[0,T]^N} \mathrm{d}r \int_{\mathbb{R}^d} \Gamma(\mathrm{d}z) \int_{\mathbb{R}^d} \mathrm{d}y \Lambda_n(t-r_i, x-y) \\
&\quad \times \Lambda_n(t-r_i, x-y+z)\left[\sum_{\hat{j}_i} D_{\hat{\alpha}_i}^{N-1}\sigma(u_n(r_i,y)) D_{\hat{\alpha}_i}^{N-1}\sigma(u_n(r_i, y-z))\right]\Bigg)^{p/2},
\end{aligned}$$

where $\hat{j}_i = j_1, \ldots, j_{i-1}, j_{i+1}, \ldots, j_N$. Then, by the Cauchy–Schwarz inequality and Hölder's inequality the preceding expression is bounded by

$$\sum_{i=1}^{n} \mathrm{E}\biggl(\int_0^T \mathrm{d}r_i \int_{\mathbb{R}^d} \Gamma(\mathrm{d}z) \int_{\mathbb{R}^d} \mathrm{d}y \Lambda_n(t-r_i, x-y)\Lambda_n(t-r_i, x-y+z)$$

$$\times \int_{[0,T]^{N-1}} \mathrm{d}\hat{r}_i \|D_{\hat{r}_i}^{N-1}\sigma(u_n(r_i, y))\|_{\mathcal{H}^{\otimes(N-1)}} \times \|D_{\hat{r}_i}^{N-1}\sigma(u_n(r_i, y-z))\|_{\mathcal{H}^{\otimes(N-1)}}\biggr)^{p/2}$$

$$\leqslant C \sum_{i=1}^{n} \int_0^T \mathrm{d}r_i \int_{\mathbb{R}^d} \Gamma(\mathrm{d}z) \int_{\mathbb{R}^d} \mathrm{d}y \Lambda_n(t-r_i, x-y)\Lambda_n(t-r_i, x-y+z)$$

$$\times \mathrm{E}\biggl(\int_{[0,T]^{N-1}} \mathrm{d}\hat{r}_i \|D_{\hat{r}_i}^{N-1}\sigma(u_n(r_i, y))\|_{\mathcal{H}^{\otimes(N-1)}} \times \|D_{\hat{r}_i}^{N-1}\sigma(u_n(r_i, y-z))\|_{\mathcal{H}^{\otimes(N-1)}}\biggr)^{p/2}$$

$$\leqslant C \sum_{i=1}^{n} \int_0^T \mathrm{d}r_i \int_{\mathbb{R}^d} \Gamma(\mathrm{d}z) \int_{\mathbb{R}^d} \mathrm{d}y \Lambda_n(t-r_i, x-y)\Lambda_n(t-r_i, x-y+z)$$

$$\times \sup_{v\in\mathbb{R}^d} \mathrm{E}\biggl(\int_{[0,T]^{N-1}} \mathrm{d}\hat{r}_i \|D_{\hat{r}_i}^{N-1}\sigma(u_n(r_i, v))\|^2_{\mathcal{H}^{\otimes(N-1)}}\biggr)^{p/2}$$

$$\leqslant C \sup_{(s,z)\in[0,T]\times\mathbb{R}^d} \mathrm{E}\Bigl(\|D^{N-1}\sigma(u_n(s, z))\|^p_{\mathcal{H}_T^{\otimes(N-1)}}\Bigr),$$

with $\mathrm{d}\hat{r}_i = \mathrm{d}r_1 \ldots \mathrm{d}r_{i-1}\mathrm{d}r_{i+1} \ldots \mathrm{d}r_N$. By (7), the assumptions on σ and the induction hypothesis, it follows that N_1 is uniformly bounded with respect to n, t and x.

In the remaining terms we can replace $\bigvee_i r_i$ by 0, because the Malliavin derivatives involved vanish for $t < \bigvee_i r_i$.

By Proposition 1 (see (5)),

$$N_2 = \mathrm{E}\biggl(\biggl|\int_0^t\int_{\mathbb{R}^d} \Lambda_n(t-s, x-z)\Delta^N(\sigma, u_n(s, z))M(\mathrm{d}s, \mathrm{d}z)\biggr|^p_{\mathcal{H}_T^{\otimes N}}\biggr)$$

$$\leqslant C\int_0^t \mathrm{d}s \sup_{y\in\mathbb{R}^d} \mathrm{E}\Bigl(\|\Delta^N(\sigma, u_n(s, y))\|^p_{\mathcal{H}_T^{\otimes N}}\Bigr) \int_{\mathbb{R}^d} \mu(\mathrm{d}\xi)|\mathcal{F}\Lambda(t-s)(\xi)|^2$$

$$\leqslant C\int_0^t \mathrm{d}s \sup_{(\tau,y)\in[0,s]\times\mathbb{R}^d} \mathrm{E}\Bigl(\|\Delta^N(\sigma, u_n(\tau, y))\|^p_{\mathcal{H}_T^{\otimes N}}\Bigr) J(t-s),$$

with $J(t) = \int_{\mathbb{R}^d} \mu(\mathrm{d}\xi)|\mathcal{F}\Lambda(t)(\xi)|^2$. According to the induction hypothesis, this last term is uniformly bounded with respect to n, t and x.

Using similar arguments – this time for deterministic integration of Hilbert-valued processes – Hölder's inequality and the assumptions on Λ, we obtain

$$N_3 \leqslant C\int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \Lambda(s, \mathrm{d}z) \mathrm{E}\|\Delta^N(b, u_n(t-s, x-z))\|^p_{\mathcal{H}_T^{\otimes N}}$$
$$\leqslant C \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathrm{E}\Big(\|\Delta^N(b, u_n(s, y))\|^p_{\mathcal{H}_T^{\otimes N}}\Big),$$

which again, by the induction hypothesis, is uniformly bounded in n, t and x.

For N_4 we proceed as for N_2; this yields

$$N_4 \leqslant C\int_0^t \mathrm{d}s \sup_{(\tau,y)\in[0,s]\times\mathbb{R}^d} \mathrm{E}\Big(\|D^N u_n(\tau, y))\|^p_{\mathcal{H}\underset{T}{\otimes}N}\Big) J(t-s).$$

Finally, as for N_3,

$$N_5 \leqslant C\int_0^t \mathrm{d}s \sup_{(\tau,y)\in[0,s]\times\mathbb{R}^d} \mathrm{E}\Big(\|D^N u_n(\tau, y))\|^p_{\mathcal{H}_T^{\otimes N}}\Big).$$

Summarizing the estimates obtained so far, we obtain

$$\sup_{(s,y)\in[0,t]\times\mathbb{R}^d} \mathrm{E}\Big(\|D^N u_n(s, y))\|^p_{\mathcal{H}_T^{\otimes N}}\Big)$$
$$\leqslant C\left[1 + \int_0^t \mathrm{d}s \sup_{(\tau,y)\in[0,s]\times\mathbb{R}^d} \mathrm{E}\Big(\|D^N u_n(\tau, y))\|^p_{\mathcal{H}_T^{\otimes N}}\Big)(J(t-s)+1)\right].$$

An application of a version of Gronwall's lemma (Dalang 1999, Lemma 15) concludes the proof. □

For $N \geqslant 1$, $n \geqslant 1$, $r = (r_1, \ldots, r_N)$, $\alpha = ((r_1, e_{j_1}), \ldots, (r_N, e_{j_N}))$ and $(t, x) \in [0, t] \times \mathbb{R}^d$, we define the $\mathcal{H}^{\otimes N}$-valued random variable $Z_r^{N,n}(t, x)$ as follows:

$$\langle Z_r^{N,n}(t, x), e_{j_1} \otimes \ldots \otimes e_{j_N}\rangle_{\mathcal{H}^{\otimes N}} = \sum_{i=1}^N \langle \Lambda_n(t - r_i, x - *) D^{N-1}_{\hat{\alpha}_i} \sigma(u_n(r_i, *)), e_{j_i}\rangle_{\mathcal{H}}.$$

Applying Lemma 2, it can easily be seen that $Z^{N,n}(t, x) \in L^p(\Omega; \mathcal{H}_T^{\otimes N})$ and

$$\sup_{n\geqslant 1} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathrm{E}\Big(\|Z^{N,n}(t, x)\|^p_{\mathcal{H}_T^{\otimes N}}\Big) < +\infty, \tag{11}$$

for every $p \in [1, \infty)$. Notice that $Z^{N,n}(t, x)$ coincides with the first term of the right-hand side of (9) for $\alpha = ((r_1, e_{j_1}), \ldots, (r_N, e_{j_N}))$.

On the other hand, for $N \geqslant 1$, we introduce the assumption that the sequence $\{D^j u_n(t, x), n \geqslant 1\}$ converges in $L^p(\Omega; \mathcal{H}_T^{\otimes j})$, $j = 1, \ldots, N-1$, with the convention that $L^p(\Omega; \mathcal{H}_T^{\otimes 0}) = L^p(\Omega)$ We denote this assumption by (H_{N-1}).

Proposition 1 in Quer-Sardanyons and Sanz-Solé (2004) yields the validity of (H_0). Moreover, for $N > 1$, (H_{N-1}) implies that $u(t, x) \in \mathbb{D}^{j,p}$ and the sequences $\{D^j u_n(t, x), n \geqslant 1\}$ converge in $L^p(\Omega; \mathcal{H}_T^{\otimes j})$ to $D^j u(t, x)$. In addition, by Lemma 2,

$$\sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathrm{E}(\|D^j u(s,\, y)\|^p_{\mathcal{H}_T^{\otimes j}}) < \infty, \tag{12}$$

$j = 1, \ldots, N-1$.

Lemma 3. *Fix $N \geqslant 1$. Assume the same hypothesis as in Theorem 1 and that (H_{N-1}) holds. Then the sequence $\{Z^{N,n}(t,\, x)\}_{n\geqslant 1}$ converges in $L^p(\Omega;\, \mathcal{H}_T^{\otimes N})$ to a random variable $Z^N(t,\, x)$.*

Proof. For $N = 1$ the result is proved in Quer-Sardanyons and Sanz-Solé (2004, Proposition 3). Assume $N > 1$. In view of (11), it suffices to show that $\{Z^{N,n}(t,\, x)\}_{n\geqslant 1}$ is a Cauchy sequence in $L^2(\Omega;\, \mathcal{H}_T^{\otimes N})$.

For $n,\, m \geqslant 1$, set

$$\begin{aligned} Z^{n,m} := \mathrm{E}\int_{[0,T]^N} \mathrm{d}r \sum_{j_1,\ldots,j_N} \Bigg| &\sum_{i=1}^N \langle \Lambda_n(t - r_i,\, x - *) D^{N-1}_{\hat{\alpha}_i} \sigma(u_n(r_i,\, *)),\, e_{j_i}\rangle_{\mathcal{H}} \\ &- \sum_{i=1}^N \langle \Lambda_m(t - r_i,\, x - *) D^{N-1}_{\hat{\alpha}_i} \sigma(u_m(r_i,\, *)),\, e_{j_i}\rangle_{\mathcal{H}} \Bigg|^2. \end{aligned}$$

Then

$$Z^{n,m} \leqslant C(Z_1^n + Z_2^{n,m} + Z_3^m),$$

where

$$Z_1^n = \sum_{i=1}^N \mathrm{E}\int_{[0,T]^N} \mathrm{d}r \sum_{j_1,\ldots,j_N} |\langle \Lambda_n(t - r_i,\, x - *) \times [D^{N-1}_{\hat{\alpha}_i} \sigma(u_n(r_i,\, *)) - D^{N-1}_{\hat{\alpha}_i} \sigma(u(r_i,\, *))],\, e_{j_i}\rangle_{\mathcal{H}}|^2,$$

$$Z_2^{n,m} = \sum_{i=1}^N \mathrm{E}\int_{[0,T]^N} \mathrm{d}r \sum_{j_1,\ldots,j_N} |\langle D^{N-1}_{\hat{\alpha}_i} \sigma(u(r_i,\, *)) \times [\Lambda_n(t - r_i,\, x - *) - \Lambda_m(t - r_i,\, x - *)],\, e_{j_i}\rangle_{\mathcal{H}}|^2,$$

$$Z_3^m = \sum_{i=1}^N \mathrm{E}\int_{[0,T]^N} \mathrm{d}r \sum_{j_1,\ldots,j_N} |\langle \Lambda_m(t - r_i,\, x - *) \times [D^{N-1}_{\hat{\alpha}_i} \sigma(u(r_i,\, *)) - D^{N-1}_{\hat{\alpha}_i} \sigma(u_m(r_i,\, *))],\, e_{j_i}\rangle_{\mathcal{H}}|^2.$$

Parseval's identity and the Cauchy–Schwarz inequality ensure that

$$
\begin{aligned}
Z_1^n &= \sum_{i=1}^{N} \mathrm{E} \int_{[0,T]^{N-1}} \mathrm{d}\hat{r}_i \sum_{\hat{j}_i} \|\Lambda_n(t-\cdot, x-*)[D_{\hat{\alpha}_i}^{N-1}\sigma(u_n(\cdot,*)) - D_{\hat{\alpha}_i}^{N-1}\sigma(u(\cdot,*))]\|_{\mathcal{H}_T}^2 \\
&\leqslant \sum_{i=1}^{N} \mathrm{E} \int_{[0,T]^{N-1}} \mathrm{d}\hat{r}_i \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \Gamma(\mathrm{d}z) \int_{\mathbb{R}^d} \mathrm{d}y \Lambda_n(t-s, x-y) \\
&\quad \times \Lambda_n(t-s, x-y+z) \| D_{\hat{r}_i}^{N-1}(\sigma(u_n(s,y)) - \sigma(u(s,y)))\|_{\mathcal{H}^{\otimes(N-1)}} \\
&\quad \times \| D_{\hat{r}_i}^{N-1}(\sigma(u_n(s,y-z)) - \sigma(u(s,y-z)))\|_{\mathcal{H}^{\otimes(N-1)}} \\
&\leqslant \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathrm{E}\Big(|D^{N-1}(\sigma(u_n(s,y)) - \sigma(u(s,y)))|_{\mathcal{H}_T^{\otimes(N-1)}}^2 \Big) \times \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mu(\mathrm{d}\xi) |\mathcal{F}\Lambda(t-s)(\xi)|^2 \\
&\leqslant C \sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathrm{E}\Big(\|D^{N-1}(\sigma(u_n(s,y)) - \sigma(u(s,y)))\|_{\mathcal{H}_T^{\otimes(N-1)}}^2 \Big).
\end{aligned}
$$

Equation (7), Lemma 3 and assumption (H_{N-1}) yield that the last term tends to zero as n goes to infinity. Analogously, Z_3^m tends to zero as m tends to infinity.

Using similar arguments, we obtain

$$
\begin{aligned}
Z_2^{n,m} &= \sum_{i=1}^{N} E \int_{[0,T]^{N-1}} \mathrm{d}\hat{r}_i \sum_{\hat{j}_i} \|D_{\hat{\alpha}_i}^{N-1}\sigma(u(\cdot,*))[\Lambda_n(t-\cdot, x-*) - \Lambda_m(t-\cdot, x-*)]\|_{\mathcal{H}_T}^2 \\
&= \sum_{i=1}^{N} \mathrm{E} \int_{[0,T]^{N-1}} \mathrm{d}\hat{r}_i \sum_{\hat{j}_i} \int_0^T \mathrm{d}s \int_{\mathbb{R}^d} \Gamma(\mathrm{d}z) \int_{\mathbb{R}^d} \mathrm{d}y D_{\hat{\alpha}_i}^{N-1}\sigma(u(s,y)) \\
&\quad \times D_{\hat{\alpha}_i}^{N-1}\sigma(u(s,y-z))[\Lambda_n(t-s,x-y) - \Lambda_m(t-s,x-y)] \\
&\quad \times [\Lambda_n(t-s,x-y+z) - \Lambda_m(t-s,x-y+z)] \\
&= \sum_{i=1}^{N} \mathrm{E} \int_{[0,T]^{N-1}} \mathrm{d}\hat{r}_i \sum_{\hat{j}_i} \int_0^T \mathrm{d}s \int_{\mathbb{R}^d} \mu_s^{D_{\hat{\alpha}_i}^{N-1}\sigma(u)}(\mathrm{d}\xi) |\mathcal{F}(\Lambda_n(t-s) - \Lambda_m(t-s))(\xi)|^2.
\end{aligned}
$$

This term tends to zero as m and n go to infinity. Indeed, arguing as in the proof of Theorem 2 from Dalang (1999), we have that

$$
\|\Lambda(t-\cdot)\|_{0,D_{\hat{\alpha}_i}^{N-1}\sigma(u)}^2 \leqslant \liminf_{k\to\infty} \|\Lambda_k(t-\cdot)\|_{0,D_{\hat{\alpha}_i}^{N-1}\sigma(u)}^2.
$$

Then, by Fatou's lemma,

$$
\begin{aligned}
&\mathrm{E}\int_{[0,T]^{N-1}} \mathrm{d}\hat{r}_i \sum_{j_{\hat{i}}} \int_0^T \mathrm{d}s \int_{\mathbb{R}^d} \mu_s^{D_{\hat{\alpha}_i}^{N-1}\sigma(u)}(\mathrm{d}\xi)|\mathcal{F}\Lambda(t-s)(\xi)|^2 \\
&\qquad = \int_{[0,T]^{N-1}} \mathrm{d}\hat{r}_i \sum_{j_{\hat{i}}} \|\Lambda(t-\cdot)\|^2_{0,D_{\hat{\alpha}_i}^{N-1}\sigma(u)} \\
&\qquad \leqslant \liminf_{k\to\infty} \int_{[0,T]^{N-1}} \mathrm{d}\hat{r}_i \sum_{\hat{j}_i} \|\Lambda_k(t-\cdot)\|^2_{0,D_{\hat{\alpha}_i}^{N-1}\sigma(u)}.
\end{aligned}
$$

This last term is bounded by a finite constant not depending on k, as can easily be seen using (12). Then we conclude by bounded convergence. □

Proof of Theorem 1. Fix $(t, x) \in (0, T] \times \mathbb{R}^d$, $p \in [2, \infty)$. We apply Lemma 1 to $F_n := u_n(t, x)$ and $F := u(t, x)$. We know that assumption (a) of the lemma is satisfied.

Let us check that the sequence $\{D^N u_n(t, x)\}_{n\geqslant 1}$ converges in the space $L^p(\Omega; \mathcal{H}_T^{\otimes N})$, for every $N \geqslant 1$ and $p \geqslant 2$, which implies that the random variable $D^N u(t, x)$ exists, belongs to $L^p(\Omega; \mathcal{H}_T^{\otimes N})$ and, by Lemma 2, satisfies

$$
\sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathrm{E}(\|D^N u(s, y)\|^p_{\mathcal{H}_T^{\otimes N}}) < +\infty.
$$

Owing to Lemma 2, it suffices to check the assertion for $p = 2$. We will use an induction argument on N. For $N = 1$ the proof is given in Theorem 2 of Quer-Sardanyons and Sanz-Solé (2004).

Assume the induction hypothesis (H_{N-1}). Let $\mathcal{B}_{p,N}$ be the class of progressively measurable $\mathcal{H}_T^{\otimes N}$-valued processes $\{\Psi(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$ with spatially homogeneous covariance function and satisfying

$$
\sup_{(s,y)\in[0,T]\times\mathbb{R}^d} \mathrm{E}(\|\Psi(s, y)\|_{\mathcal{H}_T^{\otimes N}}) < +\infty.
$$

We consider the stochastic integral equation in $\mathcal{B}_{p,N}$,

$$
\begin{aligned}
U(t, x) = Z^N(t, x) &+ \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-z)[\Delta(\sigma, u(s, z)) + U(s, z)\sigma'(u(s, z))]M(\mathrm{d}s, \mathrm{d}z) \\
&+ \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \Lambda(s, \mathrm{d}z)[\Delta(b, u(t-s, x-z)) + U(t-s, x-z)b'(u(t-s, x-z))],
\end{aligned}
$$

with $Z^N(t, x)$ given in Lemma 3. There exists a unique solution to this equation. Moreover, following arguments similar to those in the proof of Theorem 2 in Quer-Sardanyons and Sanz-Solé (2004), owing to Lemma 3 and (H_{N-1}) it is easy to prove that

$$
U(t, x) = L^2(\Omega; \mathcal{H}_T^{\otimes N}) - \lim_{n\to\infty} D^N u_n(t, x),
$$

the limit being uniform in (t, x). Then by uniqueness of the solution $U \equiv D^N u$, and the process $D^N u(t, x)$ satisfies equation (8). □

4. Study of the inverse of the Malliavin matrix

In this section we consider the stochastic wave equation (1). Let S_3 be the fundamental solution of $Lu = 0$ where $L = \partial^2/\partial t^2 - \Delta_3$. In this case condition (4) is equivalent to

$$\int_{\mathbb{R}^3} \frac{\mu(\mathrm{d}\xi)}{1+|\xi|^2} < \infty, \tag{13}$$

and this implies Hypothesis D (for details, see Dalang 1999).

Let $\{u(t, x), (t, x) \in [0, T] \times \mathbb{R}^3\}$ be the real-valued process solving (1). The purpose of this section is to study the L^p-integrability of the inverse of the Malliavin variance of $u(t, x)$ for any fixed $(t, x) \in (0, T] \times \mathbb{R}^3$. More precisely, we prove the following result.

Theorem 2. *Assume that the coefficients σ and b are $\mathcal{C}^1$ functions with bounded Lipschitz continuous derivatives and, in addition, that:*

(a) *there exists $\sigma_0 > 0$ such that* $\inf\{|\sigma(z)|, z \in \mathbb{R}\} \geqslant \sigma_0$*;*
(b) *there exists $\eta \in (0, \frac{1}{2})$ such that*

$$\sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} \Gamma(\mathrm{d}x) \mathcal{F}^{-1}\left(\frac{1}{(1+|\xi|^2)^\eta}\right)(x-y) < \infty.$$

Then, for any $p > 0$,

$$\mathrm{E}\Big(\|Du(t, x)\|_{\mathcal{H}_T}^{-p}\Big) < \infty.$$

This result, together with Theorem 1 applied to equation (1), yields the main result of the paper, as follows.

Theorem 3. *Assume that the coefficients σ and b are $\mathcal{C}^\infty$ functions with bounded derivatives of any order greater than or equal to one, and that hypotheses* (a) *and* (b) *of Theorem* 2 *are satisfied. Then the random variable $u(t, x)$, $(t, x) \in (0, T] \times \mathbb{R}^3$, has a density which is a $\mathcal{C}^\infty$ function.*

We notice that assumption (b) in Theorem 2 implies (13) (Lévêque 2001, Proposition 4.4.1).

Recall that the Malliavin derivative $Du(t, x)$ of the solution to (1) satisfies the equation

$$\begin{aligned} Du(t, x) = Z(t, x) &+ \int_0^t \int_{\mathbb{R}^3} S_3(t-s, x-z)\sigma'(u(s, z))Du(s, z)M(\mathrm{d}s, \mathrm{d}z) \\ &+ \int_0^t \mathrm{d}s \int_{\mathbb{R}^3} S_3(t-s, \mathrm{d}z) b'(u(s, x-z))Du(s, x-z), \end{aligned} \tag{14}$$

where $\{Z(t, x), (t, x) \in [0, T] \times \mathbb{R}^3\}$ is the $\mathcal{H}_T$-valued random process given by

$$Z(t, x) = L^p(\Omega; \mathcal{H}_T) - \lim_{n\to\infty} Z^n(t, x),$$

$p \geqslant 1$, where $Z^n(t, x) := S_{3,n}(t - \cdot, x - *)\sigma(u(\cdot, *))$ with $S_{3,n} = S_3 * \psi_n$; see either Theorem 1 or Quer-Sardanyons and Sanz-Solé (2004, Theorem 2).

Lemma 4. *Assume that σ is Lipschitz continuous and that condition* (13) *is satisfied. Then, for any $(t, x) \in (0, T] \times \mathbb{R}^3$, $v \in (0, t]$ and $q \geqslant 1$,*

$$\mathrm{E}\Big(\| Z_{t-\cdot,*}(t, x)\|^{2q}_{\mathcal{H}_v}\Big) \leqslant C\bigg(\int_0^v \mathrm{d}s \int_{\mathbb{R}^3} \mu(\mathrm{d}\xi)|\mathcal{F}S_3(s)(\xi)|^2\bigg)^q.$$

Proof. Hölder's inequality with respect to the non-negative finite measure $S_{3,n}(s, x-y)S_{3,n}(s, x-y+z)\mathrm{d}s\Gamma(\mathrm{d}z)\mathrm{d}y$ yields

$$\begin{aligned}
\mathrm{E}(\| Z_{t-\cdot,*}(t, x)\|^{2q}_{\mathcal{H}_v}) &= \lim_{n\to\infty} \mathrm{E}(\| Z^n_{t-\cdot,*}(t, x)\|^{2q}_{\mathcal{H}_v}) \\
&= \lim_{n\to\infty} \mathrm{E}(\|S_{3,n}(\cdot, x - *)\sigma(u(t-\cdot, *))\|^{2q}_{\mathcal{H}_v}) \\
&= \lim_{n\to\infty} \mathrm{E}\bigg(\bigg|\int_0^v \mathrm{d}s \int_{\mathbb{R}^3} \Gamma(\mathrm{d}z) \int_{\mathbb{R}^3} \mathrm{d}y S_{3,n}(s, x-y)\sigma(u(t-s, y)) \\
&\qquad \times S_{3,n}(s, x-y+z)\sigma(u(t-s, y-z))\bigg|^q\bigg) \\
&\leqslant \lim_{n\to\infty} \bigg(\int_0^v \mathrm{d}s \int_{\mathbb{R}^3} \Gamma(\mathrm{d}z) \int_{\mathbb{R}^3} \mathrm{d}y S_{3,n}(s, x-y) S_{3,n}(s, x-y+z)\bigg)^{q-1} \\
&\qquad \times \int_0^v \mathrm{d}s \int_{\mathbb{R}^3} \Gamma(\mathrm{d}z) \int_{\mathbb{R}^3} \mathrm{d}y S_{3,n}(s, x-y) S_{3,n}(s, x-y+z) \\
&\qquad \times \mathrm{E}(|\sigma(u(t-s, y))\sigma(u(t-s, y-z)|^q) \\
&\leqslant C\bigg(1 + \sup_{(s,z)\in[0,T]\times\mathbb{R}^3} \mathrm{E}(|u(s, z)|^{2q})\bigg) \\
&\qquad \times \lim_{n\to\infty} \bigg(\int_0^v \mathrm{d}s \int_{\mathbb{R}^3} \Gamma(\mathrm{d}z)(S_{3,n}(s) * \tilde{S}_{3,n}(s))(z)\bigg)^q \\
&\leqslant C(\int_0^v \mathrm{d}s \int_{\mathbb{R}^3} \mu(\mathrm{d}\xi)|\mathcal{F}S_3(s)(\xi)|^2)^q,
\end{aligned}$$

where in the last inequality we have used the L^q uniform boundedness of $u(t, x)$. □

Owing to Lemma 4 and Proposition 1, we obtain the following technical result.

Lemma 5. *Under the same hypothesis as in Lemma 4, we have that*

$$\sup_{t-v\leqslant s\leqslant t}\ \sup_{y\in\mathbb{R}^3} \mathrm{E}(\|D_{t-\cdot,*}u(s,\ y)\|_{\mathcal{H}_v}^{2q}) \leqslant C\left(\int_0^v \mathrm{d}s\int_{\mathbb{R}^3}\mu(\mathrm{d}\xi)|\mathcal{F}S_3(s)(\xi)|^2\right)^q,$$

for all $t\in(0,\ T]$ *and* $q\geqslant 1$.

We remark that both of the preceding lemmas also hold in the more general setting of Section 3.

Proof of Theorem 2. Fix $p>0$; it suffices to check that, for some $\epsilon_0>0$,

$$\int_0^{\epsilon_0}\epsilon^{-(1+p)}P\{\|Du(t,\ x)\|_{\mathcal{H}_T}^2<\epsilon\}\mathrm{d}\epsilon<\infty.$$

Let $\epsilon_1,\ \delta>0$ be such that, for any $\epsilon\in(0,\ \epsilon_1]$, $t-\epsilon^\delta>0$. Owing to (14), we consider the decomposition

$$P\{\|Du(t,\ x)\|_{\mathcal{H}_T}^2<\epsilon\}\leqslant P^1(\epsilon,\ \delta)+P^{2,1}(\epsilon,\ \delta,\ \nu)+P^{2,2}(\epsilon,\ \delta,\ \nu),$$

where

$$P^1(\epsilon,\ \delta)=P\left\{\left|\int_{t-\epsilon^\delta}^t \mathrm{d}rU(t,\ r,\ x)\right|\geqslant\epsilon\right\},$$

$$P^{2,1}(\epsilon,\ \delta,\ \nu)=P\{\|S_{\epsilon^{-\nu}}(\cdot,\ x-*)\sigma(u(t-\cdot,\ *))\|_{\mathcal{H}_{\epsilon^\delta}}^2<6\epsilon\},$$

$$P^{2,2}(\epsilon,\ \delta,\ \nu)=P\{\|Z_{t-\cdot,*}(t,\ x)-S_{\epsilon^{-\nu}}(\cdot,\ x-*)\sigma(u(t-\cdot,\ *))\|_{\mathcal{H}_{\epsilon^\delta}}^2\geqslant\epsilon\},$$

with $U(t,\ r,\ x)=\|D_{r,*}u(t,\ x)\|_{\mathcal{H}}^2-\|Z_{r,*}(t,\ x)\|_{\mathcal{H}}^2$ and $S_{\epsilon^{-\nu}}=\psi_{\epsilon^{-\nu}}*S_3$, $\psi_{\epsilon^{-\nu}}(x)=\epsilon^{-3\nu}\psi(\epsilon^{-\nu}x)$, $\nu>0$ and ψ a non-negative function in $\mathcal{C}^\infty(\mathbb{R}^3)$ with support contained in the unit ball of $\mathbb{R}^3$ and such that $\int_{\mathbb{R}^3}\psi(x)\mathrm{d}x=1$.

Let us first consider the term $P^1(\epsilon,\ \delta)$. By Chebyshev's inequality, for every $q\geqslant 1$ we have that

$$P^1(\epsilon,\ \delta)\leqslant\epsilon^{-q}\mathrm{E}\left(\left|\int_{t-\epsilon^\delta}^t \mathrm{d}rU(t,\ r,\ x)\right|^q\right)\leqslant C\epsilon^{-q}\sum_{k=1}^5 T_k, \tag{15}$$

with

$$T_1 = \mathrm{E}\left(\left|\int_{t-\epsilon^\delta}^{t} \mathrm{d}r \left\langle Z_{r,*}(t, x), \int_{t-\epsilon^\delta}^{t}\int_{\mathbb{R}^3} S_3(t-s, x-z) D_{r,*}u(s, z) \times \sigma'(u(s, z)) M(\mathrm{d}s, \mathrm{d}z)\right\rangle_{\mathcal{H}}\right|^q\right),$$

$$T_2 = \mathrm{E}\left(\left|\int_{t-\epsilon^\delta}^{t} \mathrm{d}r \left\langle Z_{r,*}(t, x), \int_{t-\epsilon^\delta}^{t} \mathrm{d}s \int_{\mathbb{R}^3} S_3(t-s, \mathrm{d}z) D_{r,*}u(s, x-z) \times b'(u(s, x-z))\right\rangle_{\mathcal{H}}\right|^q\right),$$

$$T_3 = \mathrm{E}\left(\left|\int_{t-\epsilon^\delta}^{t} \mathrm{d}r \left\|\int_{t-\epsilon^\delta}^{t}\int_{\mathbb{R}^3} S_3(t-s, x-z) D_{r,*}u(s, z) \times \sigma'(u(s, z)) M(\mathrm{d}s, \mathrm{d}z)\right\|_{\mathcal{H}}^2\right|^q\right),$$

$$T_4 = \mathrm{E}\left(\left|\int_{t-\epsilon^\delta}^{t} \mathrm{d}r \left\langle \int_{t-\epsilon^\delta}^{t}\int_{\mathbb{R}^3} S_3(t-s, x-z) D_{r,*}u(s, z)\sigma'(u(s, z)) M(\mathrm{d}s, \mathrm{d}z),\right.\right.\right.$$
$$\left.\left.\left.\times \int_{t-\epsilon^\delta}^{t} \mathrm{d}s \int_{\mathbb{R}^3} S_3(t-s, \mathrm{d}z) D_{r,*}u(s, x-z) b'(u(s, x-z))\right\rangle_{\mathcal{H}}\right|^q\right),$$

$$T_5 = \mathrm{E}\left(\left|\int_{t-\epsilon^\delta}^{t} \mathrm{d}r \left\|\int_{t-\epsilon^\delta}^{t} \mathrm{d}s \int_{\mathbb{R}^3} S_3(t-s, \mathrm{d}z) D_{r,*}u(s, x-z) \times b'(u(s, x-z))\right\|_{\mathcal{H}}^2\right|^q\right).$$

Schwarz's inequality yields

$$T_1 \leqslant T_{11}^{1/2} T_{12}^{1/2},$$

with

$$T_{11} = \mathrm{E}\left(\left|\int_{t-\epsilon^\delta}^{t} \mathrm{d}r \| Z_{r,*}(t, x)\|_{\mathcal{H}}^2\right|^q\right),$$

$$T_{12} = \mathrm{E}\left(\left|\int_{t-\epsilon^\delta}^{t} \mathrm{d}r \left\|\int_{t-\epsilon^\delta}^{t}\int_{\mathbb{R}^3} S_3(t-s, x-z) D_{r,*}u(s, z) \times \sigma'(u(s, z)) M(\mathrm{d}s, \mathrm{d}z)\right\|_{\mathcal{H}}^2\right|^q\right).$$

By Lemma 4 and (27),

$$T_{11} = \mathrm{E}\left(\| Z_{t-\cdot,*}(t, x)\|_{\mathcal{H}_{\epsilon^\delta}}^{2q}\right) \leqslant C\epsilon^{q\delta(3-2\eta)}. \tag{16}$$

We have that

$$T_{12} = \mathrm{E}\left(\left\|\int_{t-\epsilon^\delta}^{t}\int_{\mathbb{R}^3} S_3(t-s, x-z) D_{t-\cdot,*}u(s, z)\sigma'(u(s, z)) M(\mathrm{d}s, \mathrm{d}z)\right\|_{\mathcal{H}_{\epsilon^\delta}}^{2q}\right).$$

Here we apply Proposition 1 to $\mathcal{A} := \mathcal{H}_{\epsilon^\delta}$, $K(s, z) := D_{t-\cdot,*}u(s, z)\sigma'(u(s, z))$ and $S := S_3$. Thus, Lemma 5 and (27) ensure that

$$T_{12} \leqslant C\left(\int_0^{\epsilon^\delta} \mathrm{d}s \int_{\mathbb{R}^3} \mu(\mathrm{d}\xi) |\mathcal{F}S_3(s)(\xi)|^2\right)^{2q} \leqslant \epsilon^{2q\delta(3-2\eta)}.$$

Hence,

$$T_1 \leqslant C\epsilon^{3q\delta(3-2\eta)/2}. \tag{17}$$

We now consider the term

$$T_{22} := \mathrm{E}\left(\left\|\int_{t-\epsilon^\delta}^t \mathrm{d}s\int_{\mathbb{R}^3} S_3(t-s,\,\mathrm{d}z)D_{t-\cdot,*}u(s,\,x-z)b'(u(s,\,x-z))\right\|_{\mathcal{H}_{\epsilon^\delta}}^{2q}\right).$$

Hölder's inequality with respect to the finite measure $S_3(t-s,\,\mathrm{d}z)\mathrm{d}s$ on $[t-\epsilon^\delta,\,t]\times\mathbb{R}^3$ yields

$$T_{22} \leqslant \left(\int_{t-\epsilon^\delta}^t \mathrm{d}s\int_{\mathbb{R}^3} S_3(t-s,\,\mathrm{d}z)\right)^{2q-1}$$
$$\times\,\mathrm{E}\left(\int_{t-\epsilon^\delta}^t \mathrm{d}s\int_{\mathbb{R}^3} S_3(t-s,\,\mathrm{d}z)\|D_{t-\cdot,*}u(s,\,x-z)b'(u(s,\,x-z))\|_{\mathcal{H}_{\epsilon^\delta}}^{2q}\right).$$

Notice that

$$\int_{t-\epsilon^\delta}^t \mathrm{d}s\int_{\mathbb{R}^3} S_3(t-s,\,\mathrm{d}z) = \int_0^{\epsilon^\delta}\mathrm{d}s\int_{\mathbb{R}^3} S_3(s,\,\mathrm{d}z) \leqslant C\epsilon^{2\delta},$$

because $S_3(t) = \sigma_t/4\pi t$, where σ_t denotes the uniform measure on the three-dimensional sphere of radius t. Then, since b' is bounded, Lemma 5 and (27) imply

$$T_{22} \leqslant C\epsilon^{4q\delta+q\delta(3-2\eta)} = C\epsilon^{q\delta(7-2\eta)}. \tag{18}$$

Schwarz's inequality and the estimates (16), (17), (18) yield

$$T_2 \leqslant T_{11}^{1/2}T_{22}^{1/2} \leqslant C\epsilon^{q\delta(5-2\eta)},$$
$$T_3 = T_{12} \leqslant C\epsilon^{2q\delta(3-2\eta)},$$
$$T_4 \leqslant T_{12}^{1/2}T_{22}^{1/2} \leqslant C\epsilon^{q\delta(13/2-3\eta)},$$
$$T_5 = T_{22} \leqslant C\epsilon^{q\delta(7-2\eta)}. \tag{19}$$

Therefore, (15), (17) and (19) imply

$$P^1(\epsilon,\,\delta) \leqslant C\epsilon^{q(-1+3\delta(3-2\eta)/2)}.$$

Consequently, $\int_0^{\epsilon_0} P^1(\epsilon,\,\delta)\epsilon^{-(1+p)}\mathrm{d}\epsilon < \infty$ if

$$\frac{1}{\delta} < \frac{\frac{3}{2}q(3-2\eta)}{p+q}. \tag{20}$$

We now study the term $P^{2,1}(\epsilon,\,\delta,\,\nu)$. Our purpose is to choose some positive δ and ν such that, for ϵ sufficiently small, $\{\|S_{\epsilon^{-\nu}}(\cdot,\,x-*)\sigma(u(t-\cdot,\,*))\|_{\mathcal{H}_{\epsilon^\delta}}^2 < 6\epsilon\}$ is the empty set and therefore $P^{2,1}(\epsilon,\,\delta,\,\nu) = 0$. Assumption (a) in Theorem 2 yields

$$\|S_{\epsilon^{-\nu}}(r, x-*)\sigma(u(t-r,*))\|^2_{\mathcal{H}} \geqslant \sigma_0^2\int_{\mathbb{R}^3}\mu(\mathrm{d}\xi)|\mathcal{F}S_{\epsilon^{-\nu}}(r)(\xi)|^2$$

$$\geqslant \sigma_0^2\left(\frac{1}{2}\int_{\mathbb{R}^3}\mu(\mathrm{d}\xi)|\mathcal{F}S_3(r)(\xi)|^2 - \int_{\mathbb{R}^3}\mu(\mathrm{d}\xi)|\mathcal{F}(S_{\epsilon^{-\nu}}-S_3)(r)(\xi)|^2\right).$$

We have that

$$|\mathcal{F}(S_{\epsilon^{-\nu}}-S_3)(r)(\xi)|^2 = |\mathcal{F}\psi_{\epsilon^{-\nu}}(\xi)-1|^2|\mathcal{F}S_3(r)(\xi)|^2$$

$$\leqslant 4\pi|\mathcal{F}S_3(r)(\xi)|^2|\xi|\epsilon^{\nu}.$$

Therefore, the lower bounds (26) and (28) yield

$$\|S_{\epsilon^{-\nu}}(\cdot, x-*)\sigma(u(t-\cdot,*))\|^2_{\mathcal{H}_{\epsilon^\delta}}$$

$$\geqslant \sigma_0^2\left(\frac{1}{2}\int_0^{\epsilon^\delta}\mathrm{d}r\int_{\mathbb{R}^3}\mu(\mathrm{d}\xi)|\mathcal{F}S_3(r)(\xi)|^2 - 4\pi\epsilon^\nu\int_0^{\epsilon^\delta}\mathrm{d}r\int_{\mathbb{R}^3}\mu(\mathrm{d}\xi)|\xi||\mathcal{F}S_3(r)(\xi)|^2\right)$$

$$\geqslant \sigma_0^2\left(\frac{1}{2}C_1\epsilon^{3\delta} - C_2\epsilon^{\nu+\delta(2-2\eta)}\right),$$

for some positive constants C_1, C_2. Let ν, $\delta > 0$ be such that

$$\frac{1+2\eta}{\nu} < \frac{1}{\delta}; \tag{21}$$

then

$$\frac{1}{2}C_1\epsilon^{3\delta} - C_2\epsilon^{\nu+\delta(2-2\eta)} \geqslant \tfrac{1}{4}C_1\epsilon^{3\delta}, \qquad \text{for all } \epsilon \leqslant \epsilon_2 := \left(\frac{C_1}{4C_2}\right)^{1/(\nu-\delta(1+2\eta))}.$$

Thus, for any $\epsilon \leqslant \epsilon_2$,

$$\|S_{\epsilon^{-\nu}}(\cdot, x-*)\sigma(u(t-\cdot,*))\|^2_{\mathcal{H}_{\epsilon^\delta}} \geqslant \sigma_0^2\tfrac{1}{4}C_1\epsilon^{3\delta}.$$

Moreover, the condition

$$3\delta < 1 \tag{22}$$

implies

$$6\epsilon < \sigma_0^2\tfrac{C_1}{4}\epsilon^{3\delta}, \qquad \text{for } \epsilon \leqslant \epsilon_3 := \left(\frac{C_1\sigma_0^2}{24}\right)^{1/(1-3\delta)}.$$

Hence, if ν, $\delta > 0$ satisfy (21) and (22) then $P^{2,1}(\epsilon, \delta, \nu) = 0$, for any $\epsilon \leqslant \epsilon_2 \wedge \epsilon_3$.

Consider now the term $P^{2,2}(\epsilon, \delta, \nu)$. By Chebyshev's inequality and (29), we have that

$$
\begin{aligned}
P^{2,2}(\epsilon, \delta, \nu) &\leqslant \epsilon^{-1} \mathrm{E}(\| Z_{t-\cdot,*}(t, x) - S_{\epsilon^{-\nu}}(\cdot, x - *)\sigma(u(t - \cdot, *))\|^2_{\mathcal{H}_{\epsilon^\delta}}) \\
&= \epsilon^{-1} \int_0^{\epsilon^\delta} \mathrm{d}s \int_{\mathbb{R}^3} \mu_s^{\overline{\sigma}}(\mathrm{d}\xi) |\mathcal{F}(S_3(s) - S_{\epsilon^{-\nu}}(s))(\xi)|^2 \\
&\leqslant 4\pi\epsilon^{-1+\nu} \int_0^{\epsilon^\delta} \mathrm{d}s \int_{\mathbb{R}^3} \mu_s^{\overline{\sigma}}(\mathrm{d}\xi) |\xi| |\mathcal{F} S_3(s)(\xi)|^2 \\
&\leqslant C\epsilon^{-1+\nu+\delta(2-2\eta)},
\end{aligned}
$$

for some positive constant C, where $\overline{\sigma}$ denotes the process $\{\sigma(u(t - r, x)), (r, x) \in [0, t] \times \mathbb{R}^3\}$.

Thus, $\int_0^{\epsilon_0} \epsilon^{-(1+p)} P^{2,2}(\epsilon, \delta, \nu) \mathrm{d}\epsilon < \infty$ if and only if

$$
-1 - p + \nu + \delta(2 - 2\eta) > 0. \tag{23}
$$

We finish the proof by analysing the compatibility of the conditions (20)–(23). We recall that $\eta \in (0, \frac{1}{2})$ and $p \in [0, \infty)$ are fixed. Choose $\nu > 0$ such that

$$
\frac{1 + 2\eta}{3} < \nu. \tag{24}
$$

Then (20)–(23) are equivalent to (23) and

$$
3 < \frac{1}{\delta} < \frac{\frac{3}{2} q(3 - 2\eta)}{p + q}. \tag{25}
$$

Let $q_0 \geqslant 1$ be such that $3 < \frac{3}{2} q_0(3 - 2\eta)/(p + q_0)$, or equivalently $2p/(1 - 2\eta) < q_0$. Then let $\delta_0 > 0$ satisfy (25) with $q = q_0$. For this δ_0, choose $\nu_0 > 0$ sufficiently large such that (23) and (24) hold. The proof of the theorem is complete. □

Appendix

In this appendix we present some of the technical results that have been used in the proofs of Section 4. These provide bounds for integrals involving the Fourier transform of the fundamental solution of the wave equation in any spatial dimension d, denoted here by S_d. The proofs of these results are given in Quer-Sardanyons and Sanz-Solé (2004, Appendix).

We recall that, for every $d \geqslant 1$,

$$
\mathcal{F} S_d(t)(\xi) = \frac{\sin(2\pi t|\xi|)}{2\pi|\xi|}.
$$

For any $\eta \in (0, 1]$, we introduce the assumption

$$
\int_{\mathbb{R}^d} \frac{\mu(\mathrm{d}\xi)}{(1 + |\xi|^2)^\eta} < \infty
$$

which we denote by (H_η). We observe that (H_η) is weaker than assumption (b) of Theorem 2 (Lévêque 2001, Proposition 4.4.1).

Assume that (H_η) holds for $\eta = 1$. Then there exist two positive constants C_i, $i = 1, 2$, such that, for any $t \in (0, 1)$,

$$C_1 t^3 \leqslant \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mu(\mathrm{d}\xi) |\mathcal{F} S_d(s)(\xi)|^2 \leqslant C_2 t. \tag{26}$$

Suppose that (H_η) holds for some $\eta \in (0, 1)$. Then, there exists a positive constant C_3, such that for any $t \in [0, T]$,

$$\int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mu(\mathrm{d}\xi) |\mathcal{F} S_d(s)(\xi)|^2 \leqslant C_3 t^{3-2\eta}. \tag{27}$$

Assume that (H_η) holds for some $\eta \in (0, \frac{1}{2})$. Then there exists a positive constant C_4 such that for any $t \in [0, T]$,

$$\int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mu(\mathrm{d}\xi) |\xi| |\mathcal{F} S_d(s)(\xi)|^2 \leqslant C_4 t^{2-2\eta}. \tag{28}$$

Let $\{Z(t, x), (t, x) \in [0, T] \times \mathbb{R}^d\}$ be a predictable L^2-valued process with stationary covariance function and such that $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathrm{E}(|Z(t, x)|^2) < \infty$. Assume that hypothesis (b) of Theorem 2 holds. Then, there exists a positive constant C_5 such that

$$\int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \mu_s^Z(\mathrm{d}\xi) |\xi| |\mathcal{F} S_d(s)(\xi)|^2 \leqslant C_4 t^{2-2\eta}. \tag{29}$$

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