Bernoulli **10**(1), 2004, 121–163

Slow, fast and arbitrary growth conditions for renewal–reward processes when both the renewals and the rewards are heavy-tailed

VLADAS PIPIRAS[1], MURAD S. TAQQU[2] and JOSHUA B. LEVY[3]

[1]*Department of Statistics, University of North Carolina, New West, CB #3260, Chapel Hill NC 27599, USA. E-mail: pipiras@email.unc.edu*
[2]*Department of Mathematics, Boston University, 111 Cummington St., Boston MA 02215, USA. E-mail: murad@math.bu.edu*
[3]*Department of Mathematics, University of Minnesota Morris, 600 East 4th St., Morris MN 56267, USA. E-mail: levyj@mrs.umn.edu*

Consider M independent and identically distributed renewal–reward processes with heavy-tailed renewals and rewards that have either finite variance or heavy tails. Let $W^*(Ty, M)$, $y \in [0, 1]$, denote the total reward process computed as the sum of all rewards in M renewal–reward processes over the time interval $[0, T]$. If $T \to \infty$ and then $M \to \infty$, Taqqu and Levy have shown that the properly normalized total reward process $W^*(T\cdot, M)$ converges to the stable Lévy motion, but, if $M \to \infty$ followed by $T \to \infty$, the limit depends on whether the tails of the rewards are lighter or heavier than those of renewals. If they are lighter, then the limit is a self-similar process with stationary and dependent increments. If the rewards have finite variance, this self-similar process is fractional Brownian motion, and if they are heavy-tailed rewards, it is a stable non-Gaussian process with infinite variance. We consider asymmetric rewards and investigate what happens when M and T go to infinity jointly, that is, when M is a function of T and $M = M(T) \to \infty$ as $T \to \infty$. We provide conditions on the growth of M for the total reward process $W^*(T\cdot, M(T))$ to converge to any of the limits stated above, as $T \to \infty$. We also show that when the tails of the rewards are heavier than the tails of the renewals, the limit is stable Lévy motion as $M = M(T) \to \infty$, irrespective of the function $M(T)$.

Keywords: fractional Brownian motion; heavy tails; renewal–reward processes; self-similar processes; stable processes

1. Introduction

A renewal–reward process can be described by two sequences of random variables. The sequence of renewals $\{S_n\}_{n\geqslant 0}$ marks the consecutive renewal times and defines corresponding inter-renewal intervals. The sequence of rewards $\{W_n\}_{n\geqslant 1}$ attaches a (random) number to each inter-renewal interval. We focus here on renewal–reward

1350–7265

processes with heavy-tailed inter-renewal intervals and with either finite variance or heavy-tailed rewards.

Consider a stochastic process, denoted by $W^* = W^*(Ty, M)$, $y \in [0, 1]$, which is the aggregate reward process of M independent and identically distributed (i.i.d.) renewal–reward processes over a time interval $[0, T]$. By a central limit theorem type argument, one expects the properly normalized processes $W^*(T\cdot, M)$ to have a limit as M and T grow to infinity. Suppose that M tends to infinity first and then T tends to infinity (we write $T \to \infty$ (second), $M \to \infty$ (first)). It is well known that if the rewards have finite variance, then the limit of properly normalized processes $W^*(T\cdot, M)$ is fractional Brownian motion (see Taqqu and Levy 1986). The limit process is stable in the case of heavy-tailed, that is, infinite-variance rewards. This stable process can have independent or dependent increments. It has independent increments (and hence is the stable Lévy motion) if the tails of the rewards are heavier than the tails of the renewals, but it has dependent increments if the tails of the rewards are lighter than the tails of the renewals (Levy and Taqqu 1987; 2000). On the other hand, if the limit in M and T is taken in the reverse order, that is, $M \to \infty$ (second), $T \to \infty$ (first), then the limit is always stable Lévy motion.

In this paper we study what happens as M and T tend to infinity simultaneously, that is, we assume that M is a function of T such that $M(T) \to \infty$ as $T \to \infty$. This perspective has relevance in the context of the modelling of computer networks where M represents the number of computer workstations sending packets to the network and where the renewals represent changes of regime (see Leland *et al*. 1994; Willinger *et al*. 1997). When the limit is the stable Lévy motion irrespective of the order in which the limits in M and T are taken, one expects to obtain that limit irrespective of the nature of the function $M(T) \to \infty$. The proof of this fact turns out to be quite delicate. When the limit of $W^*(T\cdot, M)$ depends on the order of the limits in M and T, we expect to obtain one or the other limit depending on the rate at which $M(T)$ goes to infinity as $T \to \infty$. We will indicate below what these rates are. As we will show, there are two regimes governing the growth of $M(T)$ as $T \to \infty$, one regime yielding one limit, the second regime yielding the other limit.

This significantly extends the work of Mikosch *et al*. (2002) who considered the on–off version of the model, where the rewards are bounded and alternate between 1 and 0.

We begin by introducing our assumptions and notation in Section 1.1, and by providing an overview of related work in Section 1.2. In Section 2 we state our results. These results are proved in Sections 3 and 4.

1.1. Assumptions and other preliminaries

We begin with some assumptions on renewal times. Let $\{U_i\}_{i\geqslant 1}$ be a sequence of i.i.d. random variables with range the positive integers, having a common distribution U such that either

$$P(U \geqslant u) = u^{-\alpha} L_U(u), \qquad u = 1, 2, \ldots, \alpha \in (1, 2), \tag{U1}$$

or

$$P(U = u) = \alpha u^{-\alpha-1} l_U(u), \qquad u = 1, 2, \ldots, \alpha \in (1, 2), \tag{U2}$$

where L_U, l_U are slowly varying functions at infinity. Let $\mu = \mathrm{E}U$. The random variable U_0 will be the first arrival time, independent of the sequence $\{U_i\}_{i\geqslant 1}$ and having the distribution

$$P(U_0 = u) = \frac{1}{\mu} P(U \geqslant u + 1), \qquad u = 0, 1, 2, \ldots. \tag{1.1}$$

The random variables U_i will be called *inter-renewal times*. The sequence of *renewal times* $\{S_n\}_{n\geqslant 0}$ is then defined by $S_n = \sum_{k=0}^{n} U_k$. The term *renewal* will be used generically to refer to the inter-renewal times U_i or the renewal times S_n. The special choice of U_0 allows the counting process $\sum_n 1_{\{S_n \leqslant t\}}$, $t \geqslant 0$, to have stationary increments. It is well known that condition (U2) implies (U1) with

$$\lim_{u\to\infty} \frac{L_U(u)}{l_U(u)} = 1, \tag{1.2}$$

while, for (U1) to imply (U2), the function L_U has to satisfy additional assumptions (see, for example, Bingham *et al*. 1987).

Turning now to our assumptions on the rewards, let $\{W_n\}_{n\geqslant 0}$ be a sequence of i.i.d. random variables, referred to as *rewards*, independent of the inter-renewal times sequence $\{U_n\}_{n\geqslant 0}$ and having a common distribution W such that either

$$\sigma^2 = \mathrm{E}W^2 < \infty \tag{FVR}$$

or

$$P(W \leqslant -w) \sim c^- w^{-\beta} L_W(w), \quad P(W \geqslant w) \sim c^+ w^{-\beta} L_W(w), \qquad \text{as } w \to \infty, \tag{IVRL}$$

where c^-, $c^+ \geqslant 0$, $c^+ + c^- > 0$ and L_W is a slowly varying function at infinity, and either

$$\alpha < \beta < 2 \tag{IVR}$$

(the tail of the reward is lighter than the tail of the renewal) or

$$0 < \beta < \alpha \tag{IVRH}$$

(the tail of the reward is heavier than the tail of the renewal). If, for example, $c^- = 0$ but $c^+ > 0$, then the first condition in (IVR) should be interpreted as $P(W \leqslant -w)/(w^{-\beta} L_W(w)) \to 0$ as $w \to \infty$. When $\beta = 1$, we suppose that rewards are symmetric and, for centring purposes, we also assume that

$$\mathrm{E}W = 0$$

either when $1 < \beta < 2$ or when $\mathrm{E}W^2 < \infty$. Observe also that condition (IVR) implies that W has infinite variance $\mathrm{E}W^2 = \infty$.

Let us explain some of our assumptions. When $\beta = 1$, we suppose that rewards are symmetric because our proofs rely on the characteristic function representation of W and this representation is quite involved in general when $\beta = 1$ (see Aaronson and Denker 1998). We exclude the case $\beta = \alpha$ from the assumptions on rewards for the following reason. The aggregated reward process which we will consider, involves a sum of rewards W_i over renewal intervals of length U_i, that is, products $W_i U_i$. Since we wish to apply the central limit theorem for these sums, we need to know the tail behaviour of the product random variable WU. When W and U satisfy the assumptions above and $\alpha \neq \beta$ or $\mathrm{E}W^2 < \infty$, we will apply the following well-known result attributed to Breiman (1965).

Lemma 1.1. *Let X and Y be two independent random variables such that*

$$P(X \leqslant -x) \sim c_1 x^{-\gamma} L(x), \qquad P(X \geqslant x) \sim c_2 x^{-\gamma} L(x),$$

as $x \to \infty$, where $\gamma > 0$, $c_1, c_2 \geqslant 0$, $c_1 + c_2 > 0$ and L is a slowly varying function at infinity, and $\mathrm{E}|Y|^\delta < \infty$ for some $\delta > \gamma$. Then, as $z \to \infty$,

$$P(XY \leqslant -z) \sim (c_1 \mathrm{E}Y_+^\gamma + c_2 \mathrm{E}Y_-^\gamma) z^{-\gamma} L(z), \qquad P(XY \geqslant z) \sim (c_1 \mathrm{E}Y_-^\gamma + c_2 \mathrm{E}Y_+^\gamma) z^{-\gamma} L(z),$$

where $Y_+ = Y 1_{\{Y>0\}}$ and $Y_- = (-Y) 1_{\{Y<0\}}$.

When W and U satisfy the assumptions above and $\alpha = \beta$, the variable WU still has a regularly varying tail with exponent α but there is no such explicit and simple formula as in Lemma 1.1 to describe the tail behaviour of WU. See Cline (1986) for additional information.

To help the reader, we use special labels to distinguish between the various assumptions on the rewards (finite variance versus infinite variance) and on the heaviness of the tails. The labels (FVR) and (IVR) stand for 'finite-variance rewards' and 'infinite-variance rewards', respectively, and the labels (IVRL) and (IVRH) indicate that, in addition, the tails of rewards are lighter or heavier than those of the inter-renewal times, respectively. Because the exponents appear with a negative sign in (IVR) and (U1), (U2), the tails of the rewards are lighter if their index β is greater than the index α of the inter-renewal times. The labels (U1) and (U2) refer to assumptions on the inter-renewal times U.

The renewal–reward process associated with the sequence of renewal times $\{S_n\}_{n \geqslant 0}$ and the sequence of rewards $\{W_n\}_{n \geqslant 0}$ is then defined as

$$W(t) = W_0 1_{(0,S_0]}(t) + \sum_{n=1}^{\infty} W_n 1_{(S_{n-1},S_n]}(t), \qquad t = 0, 1, \ldots. \tag{1.3}$$

The cumulative reward process $W^*(T)$, $T = 1, 2, \ldots,$ is defined as

$$W^*(T) = \sum_{t=1}^{T} W(t). \tag{1.4}$$

If $L : (0, \infty) \mapsto (0, \infty)$ is a slowly varying function at infinity and $\gamma > 0$, we also denote by L_γ^* a slowly varying function such that, for all $x > 0$,

$$L_\gamma^*(u)^{-\gamma} L(u^{1/\gamma} L_\gamma^*(u) x) \to 1, \tag{1.5}$$

as $u \to \infty$. We will write $L_\gamma^* = L_U^*$ when $\gamma = \alpha$ and $L = L_U$, and $L_\gamma^* = L_W^*$ when $\gamma = \beta$ and $L = L_W$ in (1.5). It is well known that the functions L_U^* and L_W^* appear in the normalization term for the partial sums $\sum_{k=1}^n U_k$ and $\sum_{k=1}^n W_k$ to converge to a stable random variable as $n \to \infty$.

Consider now a sequence of renewal–reward processes $\{W_m(t), t = 0, 1, \ldots\}$, $m =$

1, 2, ..., which are i.i.d. copies of $W(t)$, and a sequence of their cumulative reward processes $\{W_m^*(T),\ T = 1, 2, \ldots\}$, $m = 1, 2, \ldots,$ which are i.i.d. copies of $W^*(T)$. Let

$$W^*(Ty, M) := \sum_{m=1}^{M} W_m^*(Ty) := \sum_{m=1}^{M} W_m^*([Ty]) = \sum_{m=1}^{M}\sum_{t=1}^{[Ty]} W_m(t)$$
$$:= \sum_{m=1}^{M}\sum_{t=1}^{[Ty]}\left\{\sum_{n=0}^{\infty} W_n^m 1_{(S_{n-1}^m, S_n^m]}(t)\right\} \tag{1.6}$$

be the total reward process ($[\cdot]$ denotes the integer-part function). Here, $T = 1, 2, \ldots, 0 \leqslant y \leqslant 1$, $M = 1, 2, \ldots$ (and $S_{-1}^m = 0$) and W_n^m denotes the reward of index n in the mth copy.

Remark. The total reward process is often defined in the probability literature as an integral of the reward processes which are themselves defined in continuous time. There is no essential difference between working in continuous time and discrete time. We work in discrete time because our framework nicely illustrates how continuous-time processes arise as limits of discrete-time ones and because the following results on which we rely were stated in discrete time.

1.2. Overview of related work

The following known results describe asymptotics of the total reward process $W^*(T\cdot, M)$, as M and T grow to infinity. They can be briefly summarized as follows. Suppose $T \to \infty$ (second), $M \to \infty$ (first). If the rewards have a lighter tail than the inter-renewal times, one obtains in the limit fractional Brownian motion if $\mathrm{E}W^2 < \infty$ and a dependent stable process if $\mathrm{E}W^2 = \infty$ (these processes are defined below). If the rewards have a heavier tail than the inter-renewal times, one obtains the stable Lévy motion in the limit. If $M \to \infty$ (second), $T \to \infty$ (first), then the limit is always the stable Lévy motion. Here is a precise statement of these results.

The first theorem considers $T \to \infty$ (second) and deals with finite-variance rewards ($\sigma^2 = \mathrm{E}W^2 < \infty$).

Theorem 1.1. (Taqqu and Levy 1986, Theorem 6, (ii)). *Under assumptions* (U1) *on the renewals and* (FVR) *on the rewards,*

$$\mathcal{L} - \lim_{T\to\infty}\lim_{M\to\infty} \frac{W^*(Ty, M)}{T^{(3-\alpha)/2}(L_U(T))^{1/2}M^{1/2}} \stackrel{d}{=} \sigma_0 B_H(y), \tag{1.7}$$

where $y \in [0, 1]$, B_H *is a standard fractional Brownian motion with parameter* $H = (3-\alpha)/2$ *and* $\sigma_0^2 = 2\sigma^2(\mu(\alpha-1)(2-\alpha)(3-\alpha))^{-1}$.

Here, $\mathcal{L}-$ and $\stackrel{d}{=}$ refer, respectively, to convergence and equality of the finite-dimensional distributions. Recall that a stochastic process $\{B_H(t)\}_{t\in\mathbb{R}}$ with $H \in (0, 1)$ is called a *fractional Brownian motion* if it is a Gaussian zero-mean process with covariance function

$$\mathrm{E}B_H(u)B_H(v) = \frac{\mathrm{E}B_H^2(1)}{2}\{|u|^{2H} + |v|^{2H} - |u-v|^{2H}\}, \qquad u, v \in \mathbb{R}. \tag{1.8}$$

It is called *standard* if $\mathrm{E}B_H^2(1) = 1$. The fractional Brownian motion B_H has stationary dependent (unless $H = \frac{1}{2}$) increments and is self-similar with exponent H, that is, the processes $B_H(at)$ and $a^H B_H(t)$ have the same finite-dimensional distributions for any $a > 0$.

The next two theorems characterize the limit when the rewards in Theorem 1.1 have infinite variance instead.

Theorem 1.2. (Levy and Taqqu 2000, Theorem 2.1; Pipiras and Taqqu 2000, Proposition 2.1). *Under assumptions* (U2) *on the renewals and* (IVRL) *on the rewards with symmetric rewards,*

$$\mathcal{L} - \lim_{T\to\infty} \lim_{M\to\infty} \frac{W^*(Ty, M)}{T^{(\beta-\alpha+1)/\beta}(l_U(T))^{1/\beta} M^{1/\beta} L_W^*(M)} \stackrel{d}{=} Z_\beta(y), \tag{1.9}$$

where $y \in [0, 1]$ *and* Z_β *is a symmetric* β*-stable process described below.*

The limit process Z_β in Theorem 1.2 is a symmetric β-stable process characterized by

$$\mathrm{E}\exp\left\{ \mathrm{i}\sum_{j=1}^{d} \theta_j Z_\beta(y_j) \right\} = \exp\{-\sigma^\beta(\boldsymbol{\theta}, \boldsymbol{y})\},$$

where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_d) \in \mathbb{R}^d$, $\boldsymbol{y} = (y_1, \ldots, y_d) \in [0, 1]^d$, $d \in \mathbb{N}$, and

$$\sigma^\beta(\boldsymbol{\theta}, \boldsymbol{y}) = (\mu C_\beta)^{-1} 2c \int_{\mathbb{R}}\int_{\mathbb{R}} \left| \sum_{j=1}^{d} \theta_j((y_j \wedge v - u)_+ - (0 \wedge v - u)_+) \right|^\beta \alpha(v-u)_+^{-\alpha-1} \mathrm{d}u\,\mathrm{d}v \tag{1.10}$$

with

$$C_\beta^{-1} = \frac{\Gamma(2-\beta)\cos(\beta\pi/2)}{1-\beta}$$

and $c = c^+ = c^-$. (For more information on stable processes, see Samorodnitsky and Taqqu 1994). As shown in Levy and Taqqu (2000), the process Z_β has stationary *dependent* increments and is self-similar with exponent

$$H = \frac{\beta - \alpha + 1}{\beta}.$$

Note that, if one sets $\beta = 2$ (the case of finite variance), one recovers the self-similarity exponent H given in Theorem 1.1. In fact, supposing that all slowly varying functions asymptotically equal 1, by setting $\beta = 2$ in the normalization $T^{(\beta-\alpha+1)/\beta} M^{1/\beta}$ of Theorem

1.2, one recovers the normalization $T^{(3-\alpha)/2}M^{1/2}$ used in Theorem 1.1. One may thus view Theorem 1.1 as the boundary case $\beta = 2$ of Theorem 1.2.

Theorem 1.3. (Levy and Taqqu 2000, Theorem 2.1). *Under assumptions* (U2) *on the renewals and* (IVRH) *on the rewards with symmetric rewards,*

$$\mathcal{L} - \lim_{T\to\infty}\lim_{M\to\infty} \frac{W^*(Ty, M)}{T^{1/\beta}M^{1/\beta}L_W^*(M)} \stackrel{d}{=} \Lambda_\beta(y), \tag{1.11}$$

where $y \in [0, 1]$ *and* Λ_β *is a* β*-stable Lévy motion satisfying*

$$P(\Lambda_\beta(1) \leqslant -x) \sim c\mu^{-1}\mathrm{E}U^\beta x^{-\beta}, \quad P(\Lambda_\beta(1) \geqslant x) \sim c\mu^{-1}\mathrm{E}U^\beta x^{-\beta}, \qquad \text{as } x \to \infty, \tag{1.12}$$

with $c = c^+ = c^-$.

A β-stable Lévy motion with $\beta \in (0, 2)$ is a β-stable stochastic process with independent and stationary increments. It is self-similar with exponent $1/\beta$. While Theorems 1.2 and 1.3 concern symmetric rewards only, in this work we will consider asymmetric rewards as well.

Finally, the fourth result characterizes the asymptotics of $W^*(T\cdot, M)$ when the limit in Theorems 1.1, 1.2 and 1.3 is reversed, that is, $M \to \infty$ (second), $T \to \infty$ (first). In this case, the rewards have either finite variance or heavy tails.

Theorem 1.4. (Taqqu and Levy 1986, Theorem 6, (i); and Levy and Taqqu 1987, Theorem 1). *Under assumptions* (U1) *on the renewals and either* (FVR) *or* (IVR) *on the rewards, one has*

$$\mathcal{L} - \lim_{M\to\infty}\lim_{T\to\infty} \frac{W^*(Ty, M)}{M^{1/\alpha}T^{1/\alpha}L_U^*(T)} \stackrel{d}{=} \Lambda_\alpha(y), \tag{1.13}$$

if $\mathrm{E}W^2 < \infty$ *(assumption* (FVR)*) or* $1 < \alpha < \beta < 2$ *(assumption* (IVRL)*), where* Λ_α *is an* α*-stable Lévy motion satisfying*

$$P(\Lambda_\alpha(1) \leqslant -x) \sim \mu^{-1}\mathrm{E}W_-^\alpha x^{-\alpha}, \quad P(\Lambda_\alpha(1) \geqslant x) \sim \mu^{-1}\mathrm{E}W_+^\alpha x^{-\alpha}, \qquad \text{as } x \to \infty; \tag{1.14}$$

and also

$$\mathcal{L} - \lim_{M\to\infty}\lim_{T\to\infty} \frac{W^*(Ty, M)}{M^{1/\alpha}T^{1/\alpha}L_U^*(T)} \stackrel{d}{=} \Lambda_\beta(y), \tag{1.15}$$

if $0 < \beta < \alpha < 2$ *(assumption* (IVRH)*), where* Λ_β *is a* β*-stable Lévy motion satisfying*

$$P(\Lambda_\beta(1) \leqslant -x) \sim c^-\mu^{-1}\mathrm{E}U^\beta x^{-\beta}, \quad P(\Lambda_\beta(1) \geqslant x) \sim c^+\mu^{-1}\mathrm{E}U^\beta x^{-\beta}, \qquad \text{as } x \to \infty. \tag{1.16}$$

Remark. When $0 < \beta < \alpha$ (assumption (IVRH)) and the rewards are symmetric, the limits in Theorems 1.3 and 1.4 have the same finite-dimensional distributions. That is, one obtains in the limit the stable Lévy motion with index β whether $T \to \infty$, $M \to \infty$ or $M \to \infty$,

$T \to \infty$. This β-stable Lévy motion is described by its tail behaviour in (1.12) and (1.16). Alternatively, it can be characterized by its characteristic function

$$\mathrm{E}\exp\left\{\mathrm{i}\sum_{j=1}^{d}\theta_j\Lambda_\beta(y_j)\right\} = \exp\{-\sigma^\beta(\boldsymbol{\theta}, \boldsymbol{y})(1 - \mathrm{i}\zeta(\boldsymbol{\theta}, \boldsymbol{y})\tan\beta\pi/2)\},$$

where

$$\sigma^\beta(\boldsymbol{\theta}, \boldsymbol{y}) = (\mu C_\beta)^{-1}(c^+ + c^-)\mathrm{E}U^\beta\sum_{j=1}^{d}|\phi_j|^\beta(y_j - y_{j-1}), \tag{1.17}$$

$$\zeta(\boldsymbol{\theta}, \boldsymbol{y}) = \frac{(c^+ - c^-)}{(c^+ + c^-)}\frac{\sum_{j=1}^{d}\phi_j^{\langle\beta\rangle}(y_j - y_{j-1})}{\sum_{j=1}^{d}|\phi_j|^\beta(y_j - y_{j-1})}, \tag{1.18}$$

with $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_d) \in \mathbb{R}^d$, $\boldsymbol{y} = (y_1, \ldots, y_d) \in (0, 1]^d$, $0 < y_1 < \ldots < y_d \leqslant 1$, $d \geqslant 1$,

$$\phi_j = \theta_j + \theta_{j+1} + \ldots + \theta_d$$

and

$$a^{\langle\beta\rangle} = \mathrm{sign}(a)|a|^\beta, \qquad a \in \mathbb{R}.$$

The α-stable Lévy motion Λ_α which appears in (1.14) can be described through its characteristic function in a similar way.

Suppose now that M is a function of T and that $M = M(T) \to \infty$ as $T \to \infty$. We want to know when the total reward process $W^*(T\cdot, M(T))$ converges to any of the above limits as $T \to \infty$. As previously mentioned, we have to distinguish between two cases. Since the limit process in Theorems 1.3 and 1.4 is the same under the assumption (IVRH), we may expect $W^*(T\cdot, M(T))$ to always converge to this limit as $T \to \infty$. We will show below that this is indeed the case. Under the assumptions (FVR) and (IVRL), however, the limit processes in Theorems 1.1 or 1.2 and in Theorem 1.4 are different. In this case, we will find conditions on $M(T)$ for the normalized total reward process $W^*(T\cdot, M(T))$ to converge to any of the three limits obtained in the above theorems.

This result extends that of Mikosch *et al*. (2002) who considered the so-called on–off version of the model. In the on–off model, rewards are constant (say, 1) but renewals, which have heavy tails as in our model, alternate between busy (or 'on') periods, when there is a reward, and idle (or 'off') periods, when there is no reward. One is then interested in the fluctuations of the total reward process $W^*(T\cdot, M)$ around the mean which, contrary to the case considered here, is no longer zero. One can show (see, for example, Taqqu *et al*. 1997) that the asymptotics of $W^*(T\cdot, M)$ are similar in this case to those described in Theorems 1.1 and 1.4, namely, if $T \to \infty$ (second), $M \to \infty$ (first), the properly normalized and centred $W^*(T\cdot, M)$ converges to fractional Brownian motion, whereas, if $M \to \infty$ (second), $T \to \infty$ (first), the limit is stable Lévy motion. Mikosch *et al*. (2002) assumed that $M = M(T) \to \infty$ as $T \to \infty$, and found conditions on the growth of $M(T)$ which distinguish between fractional Brownian motion and stable Lévy motion in the limit as $T \to \infty$. We should also mention here a recent paper by Gaigalas and Kaj (2003). These authors consider a growth regime of $M = M(T)$ which is intermediate to those of Mikosch *et al.* (2002), and find a new process in the limit as $T \to \infty$ which is neither fractional Brownian motion nor stable Lévy motion.

A number of models with random heavy-tailed rewards have already appeared in the telecommunications literature. See, for example, Maulik *et al.* (2002) and Guerin *et al.* (2003) where a reward, called a random transmission time, is defined as a ratio of the size of a transferred file and the transfer time. Although the model considered by these authors is the so-called infinite-source Poisson, all these models – the infinite-source Poisson model, the renewal–reward model and the so-called on–off model – will have the same asymptotics and the results of our paper show what can happen.

2. Main results

We first consider the convergence of the total reward process under the assumptions (FVR) and (IVRL). To get an idea about conditions on $M(T)$ needed for the convergence of $W^*(T\cdot, M(T))$, suppose for simplicity that all slowly varying functions asymptotically equal 1. In view of Theorem 1.4, when T is large compared to M, we still expect $W^*(T\cdot, M(T))/M^{1/\alpha}T^{1/\alpha}$ to converge to stable Lévy motion. The question is how fast the parameter $M(T)$ should grow with T. Observe that, for finite T, the process $W^*(T\cdot, M)$ involves only inter-renewal times that cannot be greater than T, and, in addition, $\mathrm{E}|W|^\alpha < \infty$ for $\alpha < \beta$. Therefore the process $W^*(T, M)$ has always finite αth moment and the convergence (1.9) suggests that

$$\big(\mathrm{E}|W^*(T, M)|^\alpha\big)^{1/\alpha} \sim C\, T^{(\beta-\alpha+1)/\beta} M^{1/\beta}$$

for large T and M. Therefore, if we set

$$M = M(T)$$

and expect $W^*(T\cdot, M(T))/M^{1/\alpha}T^{1/\alpha}$ to converge, as $T \to \infty$, to α-stable Lévy motion which has an infinite αth moment, then the moment of order α of $W^*(T\cdot, M)/M^{1/\alpha}T^{1/\alpha}$ should diverge. In other words,

$$\frac{(\mathrm{E}|W^*(T, M)|^\alpha)^{1/\alpha}}{M^{1/\alpha}T^{1/\alpha}} \sim C\, \frac{T^{(\beta-\alpha+1)/\beta} M^{1/\beta}}{M^{1/\alpha}T^{1/\alpha}} \gg 1$$

or $T^{(\beta-\alpha+1)/\beta} M^{1/\beta} \gg T^{1/\alpha}M^{1/\alpha}$. It is easy to see that this last condition reduces to

$$T^{\alpha-1} \gg M$$

which, interestingly, does not depend on β. When $M \gg T^{\alpha-1}$, we expect the normalization of $W^*(T\cdot, M(T))$ to be $T^{(\beta-\alpha+1)/\beta} M(T)^{1/\beta}$ and the limiting process to be Z_β (or fractional Brownian motion in the case $\beta = 2$). While this heuristic argument is suggestive, it provides neither the limits nor the exact conditions under which convergence holds. To formulate the results, we introduce the following two regimes: a fast-growth one,

$$\lim_{T\to\infty} \frac{M}{T^{\alpha-1}} (L_U^*(MT))^\alpha = \infty; \tag{2.1}$$

and a slow-growth one,

$$\lim_{T\to\infty} \frac{M}{T^{\alpha-1}} (L_U^*(MT))^\alpha = 0. \tag{2.2}$$

The following theorems are our main results under the assumptions (FVR) and (IVRL).

Theorem 2.1. *Under the fast-growth condition* (2.1) *and assumptions* (U1) *on the renewals and* (FVR) *on the rewards,*

$$\mathcal{L} - \lim_{T\to\infty} \frac{W^*(Ty, M)}{T^{(3-\alpha)/2} M^{1/2} (L_U(T))^{1/2}} \stackrel{d}{=} \sigma_0 B_H(y), \tag{2.3}$$

where $y \in [0, 1]$ *and* B_H *is a standard fractional Brownian motion as in Theorem* 1.1.

In the following result, it is assumed that the rewards are heavy-tailed and possibly asymmetric.

Theorem 2.2. *Under the fast-growth condition* (2.1) *above and assumptions* (U2) *on the renewals and* (IVRL) *on the rewards,*

$$\mathcal{L} - \lim_{T\to\infty} \frac{W^*(Ty, M)}{T^{(\beta-\alpha+1)/\beta} M^{1/\beta} (l_U(T))^{1/\beta} L_W^*(T^{-\alpha+1} M l_U(T))} \stackrel{d}{=} Z_\beta(y), \tag{2.4}$$

where $y \in [0, 1]$ *and* Z_β *is the* β*-stable process characterized by*

$$\mathrm{E}\exp\left\{ i \sum_{j=1}^d \theta_j Z_\beta(y_j) \right\} = \exp\left\{ -\sigma^\beta(\boldsymbol{\theta}, \boldsymbol{y}) \left(1 - \mathrm{i}\zeta(\boldsymbol{\theta}, \boldsymbol{y}) \tan \frac{\beta\pi}{2} \right) \right\}, \tag{2.5}$$

where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_d) \in \mathbb{R}^d$, $\boldsymbol{y} = (y_1, \ldots, y_d) \in [0, 1]^d$, $d \in \mathbb{N}$,

$$\sigma^\beta(\boldsymbol{\theta}, \boldsymbol{y}) = (\mu C_\beta)^{-1}(c^+ + c^-) \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \sum_{j=1}^d \theta_j ((y_j \wedge v - u)_+ - (0 \wedge v - u)_+) \right|^\beta \alpha (v-u)_+^{-\alpha-1} \mathrm{d}u\, \mathrm{d}v \tag{2.6}$$

and skewness term (with the notation $a^{\langle\beta\rangle} = |a|^\beta \operatorname{sign}(a)$*)*

$$\zeta(\boldsymbol{\theta}, \boldsymbol{y}) = \frac{(c^+ - c^-)}{(c^+ + c^-)} \frac{\int_{\mathbb{R}} \int_{\mathbb{R}} \left(\sum_{j=1}^d \theta_j ((y_j \wedge v - u)_+ - (0 \wedge v - u)_+) \right)^{\langle\beta\rangle} \alpha (v-u)_+^{-\alpha-1} \mathrm{d}u\, \mathrm{d}v}{\int_{\mathbb{R}} \int_{\mathbb{R}} \left| \sum_{j=1}^d \theta_j ((y_j \wedge v - u)_+ - (0 \wedge v - u)_+) \right|^\beta \alpha (v-u)_+^{-\alpha-1} \mathrm{d}u\, \mathrm{d}v}. \tag{2.7}$$

Theorem 2.3. *Under the slow-growth condition* (2.2) *and assumptions* (U1) *on the renewals and either* (FVR) *or* (IVRL) *on the rewards,*

$$\mathcal{L} - \lim_{T\to\infty} \frac{W^*(Ty, M)}{T^{1/\alpha}M^{1/\alpha}L_U^*(TM)} \stackrel{d}{=} \Lambda_\alpha(y), \tag{2.8}$$

where $y \in [0, 1]$ *and* Λ_α *is a* α*-stable Lévy motion satisfying* (1.14).

Theorems 2.1, 2.2 and 2.3 are proved in Section 3. Observe also that the normalizations in (2.4) and (2.8) have slightly changed from those in (1.9) and (1.13). We will now provide a number of equivalent ways to express the slow- and fast-growth conditions stated above. Let

$$\overline{F}_U(s) = P(U \geqslant s) = s^{-\alpha}L_U(s), \qquad s > 0,$$

and let

$$b(t) = (1/\overline{F}_U)^{\leftarrow}(t)$$

be the generalized inverse of $1/\overline{F}_U$ (the generalized inverse $f^{\leftarrow}$ of a function f is given by $f^{\leftarrow}(t) = \inf\{s > 0 : f(s) > t\}$). Then one has

$$(2.1) \Leftrightarrow \lim_T \frac{M^{1/\alpha}T^{1/\alpha}L_U^*(MT)}{T} = \infty \Leftrightarrow \lim_T \frac{b(MT)}{T} = \infty \Leftrightarrow \lim_T MT^{1-\alpha}L_U(T) = \infty, \tag{2.9}$$

$$(2.2) \Leftrightarrow \lim_T \frac{M^{1/\alpha}T^{1/\alpha}L_U^*(MT)}{T} = 0 \Leftrightarrow \lim_T \frac{b(MT)}{T} = 0 \Leftrightarrow \lim_T MT^{1-\alpha}L_U(T) = 0. \tag{2.10}$$

The first equivalence relations in (2.9) and (2.10) follow by taking the power $1/\alpha$. The second conditions follow from the fact that

$$n^{1/\alpha}L_U^*(n) \sim b(n), \tag{2.11}$$

as $n \to \infty$. Indeed, by Theorem 1.5.12 in Bingham *et al.* (1987),

$$\overline{F}_U(n^{1/\alpha}L_U^*(n)) = (n^{1/\alpha}L_U^*(n))^{-\alpha}L_U(n^{1/\alpha}L_U^*(n)) = n^{-1}L_U^*(n)^{-\alpha}L_U(n^{1/\alpha}L_U^*(n))$$

$$\sim n^{-1} \sim \left(\frac{1}{\overline{F}_U}\left(\frac{1}{\overline{F}_U}^{\leftarrow}(n)\right)\right)^{-1} = \overline{F}_U(b(n)).$$

Now (2.11) follows by taking $\overline{F}_U^{\leftarrow}$ of both sides and again using Theorem 1.5.12 in Bingham *et al.* (1987). As for the third equivalence conditions in (2.9) and (2.10), they are proved in Mikosch *et al.* (2002, Lemma 1).

Finally, we consider the (IVRH) assumption, namely $0 < \beta < \alpha$, and state a result which requires no assumptions on the growth of the function $M = M(T)$.

Theorem 2.4. *Suppose that* $M = M(T) \to \infty$ *as* $T \to \infty$. *Then, under the assumptions* (U2) *on the renewals and* (IVRH) *on the rewards,*

$$\mathcal{L} - \lim_{T\to\infty} \frac{W^*(Ty, M)}{T^{1/\beta} M^{1/\beta} L_W^*(TM)} \stackrel{d}{=} \Lambda_\beta(y), \tag{2.12}$$

where $y \in [0, 1]$ *and* Λ_β *is a* β*-stable Lévy motion satisfying* (1.16).

This theorem is proved in Section 4 (the cases $0 < \beta < 1 < \alpha$ and $1 \leqslant \beta < \alpha$ are treated separately).

Remark. In some cases one may need a stronger convergence result than convergence in the sense of the finite-dimensional distributions. We show in Appendix B that (2.3) and (2.4) can be extended to weak convergence in the space $D[0, 1]$ equipped with the usual Skorokhod J_1 topology. (Recall that $D[0, 1]$ is the space of right-continuous functions on [0, 1] which have left limits.) We can make this extension quite easily because the limit processes B_H and Z_β are smooth enough, and their path regularity is easy to establish. For example, since $H = (3 - \alpha)/2 > 1/2$, fractional Brownian motion B_H is long-range dependent with smoother paths as H increases and the well-known Kolmogorov criterion applies to $\mathrm{E}|B_H(t) - B_H(s)|^2$ (there is no need to consider a power higher than 2). It is more difficult to extend (2.8) and (2.12) where the limit is Lévy motion. This could perhaps be done, as indicated in Mikosch *et al*. (2002), in the space $D[0, 1]$ equipped with the M_1 topology by following the arguments of Resnick and van der Berg (2000).

3. Proofs under the fast and slow growth conditions

In this section we prove Theorems 2.1, 2.2 and 2.3. The proof of Theorem 2.1 (fast-growth, (FVR), convergence to fractional Brownian motion) is the simplest of the three and uses ideas of Mikosch *et al*. (2002). That of Theorem 2.3 (slow-growth, (FVR) or (IVRL), convergence to stable Lévy motion) also uses ideas and results of Mikosch *et al*. (2002). Finally, the proof of Theorem 2.2 (fast-growth, (IVRL), convergence to the stable process with dependent increments) deals with a totally novel situation.

3.1. Proof of Theorem 2.1

Recall from (1.6) that the total reward process associated with the rewards W_n^m and renewals S_n^m is

$$W^*(Ty, M) = \sum_{m=1}^{M} W_m^*(Ty) = \sum_{m=1}^{M} \sum_{t=1}^{[Ty]} \left(W_0^m 1_{(0, S_0^m]}(t) + \sum_{n=1}^{\infty} W_n^m 1_{(S_{n-1}^m, S_n^m]}(t) \right).$$

Denote the normalization used in Theorem 2.1 by

$$N(T) = T^{(3-\alpha)/2} M^{1/2} (L_U(T))^{1/2}$$

and let $0 < y_1 < \ldots < y_d \leqslant 1$. We have to show that, as $T \to \infty$,

$$\big(N(T)^{-1}W^*(Ty_1, M), \ldots, N(T)^{-1}W^*(Ty_d, M)\big) \xrightarrow{d} \sigma_0(B_H(y_1), \ldots, B_H(y_d)), \tag{3.1}$$

where $\xrightarrow{d}$ denotes convergence in distribution. We will show the convergence (3.1) only when W has a continuous distribution function, that is, W is atomless. A general case can be proved similarly by using the usual 'perturbation' trick: take a sequence X_n^m, $m \geqslant 1$, $n \geqslant 0$, of i.i.d. $\mathcal{N}(0, 1)$ random variables, independent of W_n^m and S_n^m, $m \geqslant 1$, $n \geqslant 0$, consider the total reward process with new rewards $W_{n,\epsilon}^m = W_n^m + \epsilon X_n^m$ which now have a continuous distribution function, apply the already established result and then show that the variance of the total reward from variables ϵX_n^m is negligible as $\epsilon \to 0$. Now, since $\mathrm{E}W = 0$ and W is atomless, one can choose $a_k \to -\infty$ and $b_k \to +\infty$ such that $\mathrm{E}W1_{\{a_k<W<b_k\}} = 0$. Observe that the random variables $W_n^{m,k} = W_n^m 1_{\{a_k<W_n^m<b_k\}}$ are bounded. Define

$$W_{m,k}^*(Ty) := \sum_{t=1}^{[Ty]}\left(W_0^{m,k}1_{[0,S_0^m)}(t) + \sum_{n=1}^{\infty} W_n^{m,k}1_{[S_{n-1}^m, S_n^m)}(t)\right)$$

and $W_k^*(Ty, M) := \sum_{m=1}^M W_{m,k}^*(Ty)$. The proof of the convergence is in three parts.

(a) Convergence for $d = 1$ at $y = y_1$. By Theorem 4.2 in Billingsley (1968), it is enough to show the following three steps:

(1) $(N(T))^{-1}W_k^*(Ty, M) \to_d \sigma_{0,k}B_H(y)$, where $\sigma_{0,k}^2 = 2\mathrm{E}W^2 1_{\{a_k<W<b_k\}}(\mu(\alpha-1)(2-\alpha)(3-\alpha))^{-1}$.
(2) $\sigma_{0,k}B_H(y) \to_d \sigma_0 B_H(y)$, as $k \to \infty$
(3) $\limsup_{k\to\infty}\limsup_{T\to\infty} P(|W^*(Ty, M) - W_k^*(Ty, M)| \geqslant N(T)\epsilon) = 0$, for all $\epsilon > 0$.

For step (1), we adapt Mikosch *et al*. (2002, Lemma 13). We need to show that

(i) $MP(|W_{1,k}^*(Ty)| \geqslant N(T)\epsilon) \to 0$, for all $\epsilon > 0$;
(ii) $M(N(T))^{-2}\operatorname{var}(W_{1,k}^*(Ty)1_{\{|W_{1,k}^*(Ty)|\leqslant N(T)\tau\}}) \to \sigma_{0,k}^2 y^{3-\alpha}$, for some $\tau > 0$; and
(iii) $M(N(T))^{-1}\mathrm{E}(W_{1,k}^*(Ty)1_{\{|W_{1,k}^*(Ty)|\leqslant N(T)\tau\}}) \to 0$, for some $\tau > 0$.

To show (i), use the fact that $|W_{1,k}^*(Ty)| \leqslant \max\{|a_k|, |b_k|\}[Ty]$ and that, by the fast-growth condition, $T^{-1}N(T) = (MT^{1-\alpha}L_U(T))^{1/2} \to \infty$. Then $P(|W_{1,k}^*(Ty)| \geqslant N(T)\epsilon) = 0$, for large enough T. To verify (ii), observe that, for large enough T, as in Taqqu and Levy (1986, p. 87),

$$\begin{aligned}
&M(N(T))^{-2}\operatorname{var}(W_{1,k}^*(Ty)1_{\{|W_{1,k}^*(Ty)|\leqslant N(T)\tau\}})\\
&\quad= M(N(T))^{-2}\operatorname{var}(W_{1,k}^*(Ty))\\
&\quad\sim M(N(T))^{-2}2\mathrm{E}W^2 1_{\{a_k<W<b_k\}}(\mu(\alpha-1)(2-\alpha)(3-\alpha))^{-1}[Ty]^{3-\alpha}L_U^*(Ty)\\
&\quad= \frac{M}{T^{3-\alpha}ML_U^*(T)}2\mathrm{E}W^2 1_{\{a_k<W<b_k\}}(\mu(\alpha-1)(2-\alpha)(3-\alpha))^{-1}[Ty]^{3-\alpha}L_U^*(Ty)\\
&\quad\sim 2\mathrm{E}W^2 1_{\{a_k<W<b_k\}}(\mu(\alpha-1)(2-\alpha)(3-\alpha))^{-1}y^{3-\alpha} = \sigma_{0,k}^2 y^{3-\alpha}.
\end{aligned}$$

As for condition (iii), we have, for large enough T,

$$M(N(T))^{-1}\mathrm{E}(W^*_{1,k}(Ty)1_{\{|W^*_{1,k}(Ty)|\leqslant N(T)\tau\}}) = M(N(T))^{-1}\mathrm{E}W^*_{1,k}(Ty) = 0,$$

since $\mathrm{E}W1_{\{a_k<W<b_k\}} = 0$. Step (2) follows since $\sigma^2_{0,k} \to \sigma^2_0$, as $k \to \infty$. For step (3), we have

$$\begin{aligned} P\big(|W^*(Ty, M) - W^*_k(Ty, M)| \geqslant N(T)\epsilon\big) &\\ &\leqslant (N(T)\epsilon)^{-2}\mathrm{E}|W^*(Ty, M) - W^*_k(Ty, M)|^2 \\ &= \epsilon^{-2}M(N(T))^{-2}\mathrm{E}\left|\sum_{t=1}^{[Ty]}\sum_{n=0}^{\infty} W_n 1_{\{W_n>b_k \text{ or } W_n<a_k\}}1_{[S_{n-1},S_n)}(t)\right|^2. \end{aligned}$$

Then, as in the proof of (ii) in step (1), we obtain

$$\begin{aligned} \limsup_{T\to\infty} P(|W^*(Ty, M) - W^*_k(Ty, M)| \geqslant N(T)\epsilon) &\\ &\leqslant \epsilon^{-2}2\mathrm{E}W^2 1_{\{W>b_k \text{ or } W<a_k\}}(\mu(\alpha-1)(2-\alpha)(3-\alpha))^{-1}y^{3-\alpha}. \end{aligned}$$

The conclusion follows since $\mathrm{E}W^2 1_{\{W>b_k \text{ or } W<a_k\}} \to 0$ as $k \to \infty$.

(b) Convergence for $d = 2$ at $0 < y_1 < y_2 \leqslant 1$. One needs to verify that, as $T \to \infty$,

$$\frac{1}{N(T)}\sum_{m=1}^{M}\big(\theta_1 W^*_m(Ty_1) + \theta_2 W^*_m(Ty_2)\big) \xrightarrow{d} \sigma_0(\theta_1 B_H(y_1) + \theta_2 B_H(y_2)),$$

for all $\theta_1, \theta_2 \in \mathbb{R}$. Without loss of generality we may assume that the rewards are bounded (otherwise carry out the three steps in (a)). We then have to show conditions (i)–(iii) of step (1) in (a), where $W^*_{1,k}(Ty)$ is replaced by $\theta_1 W^*_1(Ty_1) + \theta_2 W^*_1(Ty_2)$. Conditions (i) and (iii) are obvious for the same reasons as in (a). To verify (ii), note that, for large enough T,

$$\begin{aligned} M(N(T))^{-2}\,\mathrm{var}((\theta_1 W^*_1(Ty_1) + \theta_2 W^*_1(Ty_2))1_{\{|\theta_1 W^*_1(Ty_1)+\theta_2 W^*_1(Ty_2)|\leqslant N(T)\tau\}}) &\\ &= M(N(T))^{-2}\mathrm{E}(\theta_1 W^*_1(Ty_1) + \theta_2 W^*_1(Ty_2))^2 \\ &= M(N(T))^{-2}\big(\theta_1^2\mathrm{E}W^*_1(Ty_1)^2 + 2\theta_1\theta_2\mathrm{E}W^*_1(Ty_1)W^*_1(Ty_2) + \theta_2^2\mathrm{E}W^*_1(Ty_2)^2\big). \end{aligned}$$

Since, by using stationarity,

$$\begin{aligned} 2\mathrm{E}W^*_1(Ty_1)W^*_1(Ty_2) &= \mathrm{E}W^*_1(Ty_1)^2 + \mathrm{E}W^*_1(Ty_2)^2 - E(W^*_1(Ty_2) - W^*_1(Ty_1))^2 \\ &= \mathrm{E}W^*_1(Ty_1)^2 + \mathrm{E}W^*_1(Ty_2)^2 - E(W^*_1(Ty_2 - Ty_1))^2, \end{aligned}$$

it follows as in (a) that

$$\begin{aligned} M(N(T))^{-2}\,\mathrm{var}((\theta_1 W^*_1(Ty_1) + \theta_2 W^*_1(Ty_2))1_{\{|\theta_1 W^*_1(Ty_1)+\theta_2 W^*_1(Ty_2)|\leqslant N(T)\tau\}}) &\\ &\to \sigma_0^2(\theta_1^2 y_1^{3-\alpha} + 2\theta_1\theta_2\tfrac{1}{2}(y_1^{3-\alpha} + y_2^{3-\alpha} - (y_2-y_1)^{3-\alpha}) + \theta_2^2 y_2^{3-\alpha}). \end{aligned}$$

Finally, observe that, by (1.8), the last expression is equal to $\sigma_0^2\mathrm{E}(\theta_1 B_{(3-\alpha)/2}(y_1) + \theta_2 B_{(3-\alpha)/2}(y_2))^2$.

(c) The convergence (3.1) for any d at $0 < y_1 < \ldots < y_d \leqslant 1$ can be established as in (b).

3.2. Proof of Theorem 2.2

We will first consider the case of symmetric rewards (Section 3.2.1) and we will assume without loss of generality that $c^+ = c^- = \frac{1}{2}$. In this case, the skewness parameters $\zeta(\boldsymbol{\theta}, \boldsymbol{y}) \equiv 0$ and the limit process Z_β is characterized by the scale parameters $\sigma(\boldsymbol{\theta}, \boldsymbol{y})$ alone, which can also be expressed as in (1.10). The case of asymmetric rewards is dealt with in Section 3.2.2.

3.2.1. Symmetric rewards

Observe first that

$$W^*(T) = \sum_{k=0}^{\infty}(T \wedge S_k - S_{k-1})_+ W_k. \tag{3.2}$$

In order to express the finite-dimensional characteristic function, introduce $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_d) \in \mathbb{R}^d$, $\boldsymbol{y} = (y_1, \ldots, y_d) \in (0, 1]^d$ with $0 < y_1 < \ldots < y_d \leqslant 1$, $d \geqslant 1$, and $T_j = [Ty_j]$, $j = 1, \ldots, d$. As in Levy and Taqqu (2000), let

$$\sigma^\beta := \sigma^\beta(\boldsymbol{\theta}, \boldsymbol{y}) = C_\beta^{-1}(I(\boldsymbol{\theta}, \boldsymbol{y}) + J(\boldsymbol{\theta}, \boldsymbol{y})) =: C_\beta^{-1}(I + J), \tag{3.3}$$

where $\sigma^\beta = \sigma^\beta(\boldsymbol{\theta}, \boldsymbol{y})$ and C_β are defined after Theorem 1.2, and

$$I = \mu^{-1}\int_0^\infty \left|\sum_{j=1}^d \theta_j(y_j \wedge u)\right|^\beta u^{-\alpha}\,\mathrm{d}u, \tag{3.4}$$

$$J = \mu^{-1}\int_0^\infty\int_0^\infty \left|\sum_{j=1}^d \theta_j(y_j \wedge v - u)_+\right|^\beta \alpha(v-u)_+^{-\alpha-1}\mathrm{d}u\,\mathrm{d}v. \tag{3.5}$$

In the proof below we will use the following facts from Levy and Taqqu (2000):

$$\mathrm{E}\left|\sum_{j=1}^d \theta_j(T_j \wedge S_0)\right|^\beta \frac{1}{T^{\beta-\alpha+1}l_U(T)} = \mu^{-1}\sum_{x=0}^\infty\left|\sum_{j=1}^d \theta_j(T_j \wedge x)\right|^\beta \frac{P(U > x)}{T^{\beta-\alpha+1}l_U(T)} \to I, \tag{3.6}$$

as $T \to \infty$ (see Proposition 5.1 in Levy and Taqqu 2000), and

$$\mathrm{E}\sum_{k=1}^\infty\left|\sum_{j=1}^d \theta_j(T_j \wedge S_k - S_{k-1})_+\right|^\beta \frac{1}{T^{\beta-\alpha+1}l_U(T)}$$

$$= \mu^{-1}\sum_{y=0}^\infty\sum_{x=0}^y\left|\sum_{j=1}^d \theta_j(T_j \wedge y - x)_+\right|^\beta \frac{P(U = y-x)}{T^{\beta-\alpha+1}l_U(T)} \to J, \tag{3.7}$$

as $T \to \infty$ (see Propositions 5.2–5.4 in Levy and Taqqu 2000; the equality in (3.7) is shown on p. 32 of that paper). Let also

$$N(T) = T^{(\beta-\alpha+1)/\beta} M^{1/\beta} (l_U(T))^{1/\beta} L_W^*(T^{-\alpha+1} M l_U(T)) = T(Q(T))^{1/\beta} L_W^*(Q(T)), \tag{3.8}$$

where

$$Q(T) = T^{-\alpha+1} M l_U(T) \to \infty, \tag{3.9}$$

by the fast-growth condition (see (2.9) and (1.2)).

To prove Theorem 2, it is enough to show the following result.

Lemma 3.1. *As* $T \to \infty$,

$$D := \left| \mathrm{E}\exp\left\{ \mathrm{i} \sum_{j=1}^{d} \theta_j W^*(Ty_j, M)/N(T) \right\} - \mathrm{E}\exp\left\{ \mathrm{i} \sum_{j=1}^{d} \theta_j Z_\beta(y_j) \right\} \right| \to 0. \tag{3.10}$$

Proof. Using (3.2), independence of the M renewal–reward processes and also the fact that they are identically distributed, we have

$$D = \left| \prod_{m=1}^{M} \mathrm{E}\exp\left\{ \mathrm{i} \sum_{k=0}^{\infty} \sum_{j=1}^{d} \theta_j (T_j \wedge S_k - S_{k-1})_+ W_k / N(T) \right\} - \mathrm{E}\exp\left\{ \mathrm{i} \sum_{j=1}^{d} \theta_j Z_\beta(y_j) \right\} \right|.$$

Using independence of the sequences $\{W_k\}_{k \geqslant 0}$ and $\{S_k\}_{k \geqslant 0}$ and also that of the W_k, we obtain

$$D = \left| \prod_{m=1}^{M} \mathrm{E}\left(\mathrm{E}\exp\left\{ \mathrm{i} \sum_{k=0}^{\infty} \sum_{j=1}^{d} \theta_j (T_j \wedge s_k - s_{k-1})_+ W_k / N(T) \right\} \Bigg|_{(s_k)=(S_k)} \right) - \exp\{-\sigma^\beta\} \right|$$

$$= \left| \prod_{m=1}^{M} \mathrm{E}\left(\left(\prod_{k=0}^{\infty} \mathrm{E}\exp\left\{ \mathrm{i} \sum_{j=1}^{d} \theta_j (T_j \wedge s_k - s_{k-1})_+ W_k / N(T) \right\} \right) \Bigg|_{(s_k)=(S_k)} \right) - \exp\{-\sigma^\beta\} \right|.$$

By Theorem 2.6.5 in Ibragimov and Linnik (1971) (see also Aaronson and Denker 1998, Theorem 1), for the random variable W in the domain of attraction of a symmetric β-stable random variable,

$$\mathrm{E}\exp\{\mathrm{i}uW\} = \exp\{-C_\beta^{-1} |u|^\beta L_W(|u|^{-1}) h(u)\}, \qquad u \in \mathbb{R}, \tag{3.11}$$

where $\lim_{u \to 0} h(u) = 1$. Then, by applying (3.11), we express the identity above as

$$D = \left| \prod_{m=1}^{M} \mathrm{E}\exp\left\{ -C_\beta^{-1} \sum_{k=0}^{\infty} |\vartheta_k|^\beta L_W(|\vartheta_k|^{-1}) h(\vartheta_k) \right\} - \exp\{-\sigma^\beta\} \right|, \tag{3.12}$$

where

$$\vartheta_k = \frac{1}{N(T)} \sum_{j=1}^{d} \theta_j (T_j \wedge S_k - S_{k-1})_+. \tag{3.13}$$

Now, using the inequality $|\prod_{m=1}^{M} a_m - \prod_{m=1}^{M} b_m| \leqslant \sum_{m=1}^{M} |a_m - b_m|$, valid for $|a_m|, |b_m| \leqslant 1$ (it is enough to prove the inequality when $M = 2$, and this is done by a simple application of the triangle inequality to $|(a_1 a_2 - a_2 b_1) + (a_2 b_1 - b_1 b_2)|$), we have

$$\begin{aligned} D &\leqslant M \left| \mathrm{E} \exp\left\{ -C_\beta^{-1} \sum_{k=0}^{\infty} |\vartheta_k|^\beta L_W(|\vartheta_k|^{-1}) h(\vartheta_k) \right\} - \exp\left\{ -\frac{\sigma^\beta}{M} \right\} \right| \\ &\leqslant M \left| \mathrm{E} \exp\left\{ \frac{\sigma^\beta}{M} - C_\beta^{-1} \sum_{k=0}^{\infty} |\vartheta_k|^\beta L_W(|\vartheta_k|^{-1}) h(\vartheta_k) \right\} - 1 \right|. \end{aligned} \tag{3.14}$$

Since, by Taylor's formula, $|\mathrm{e}^x - 1 - x| \leqslant \mathrm{e}^{x_0} x^2/2 \leqslant \mathrm{e}^{|x|} x^2/2$ for some $|x_0| \leqslant |x|$, we have $|\mathrm{E}\mathrm{e}^X - 1| \leqslant |\mathrm{E}X| + \mathrm{E}\mathrm{e}^{|X|} X^2/2$ and hence

$$\begin{aligned} D \leqslant{} & \left| M \mathrm{E} C_\beta^{-1} \sum_{k=0}^{\infty} |\vartheta_k|^\beta L_W(|\vartheta_k|^{-1}) h(\vartheta_k) - \sigma^\beta \right| \\ & + \frac{M}{2} \mathrm{E} \exp\left\{ \frac{\sigma^\beta}{M} + C_\beta^{-1} \sum_{k=0}^{\infty} |\vartheta_k|^\beta L_W(|\vartheta_k|^{-1}) |h(\vartheta_k)| \right\} \\ & \times \left(C_\beta^{-1} \sum_{k=0}^{\infty} |\vartheta_k|^\beta L_W(|\vartheta_k|^{-1}) h(\vartheta_k) - \frac{\sigma^\beta}{M} \right)^2. \end{aligned} \tag{3.15}$$

Focus now on the second term in the bound above. Observe first that, by the fast-growth condition (2.9),

$$|\vartheta_k| = \frac{1}{N(T)} \left| \sum_{j=1}^{d} \theta_j (T_j \wedge S_k - S_{k-1})_+ \right| \leqslant \frac{C\,T}{N(T)} = \frac{C}{(Q(T) L_W^*(Q(T))^\beta)^{1/\beta}} \to 0$$

(use (3.8), (3.9) and the fact that L_W^* is a slowly varying function). Consequently,

$$h(\vartheta_k) \to 1, \tag{3.16}$$

as $T \to \infty$, uniformly in k. Using Lemma 3.2 below, we have

$$F := \sum_{k=0}^{\infty} |\vartheta_k|^\beta L_W(|\vartheta_k|^{-1}) \leqslant \frac{C}{Q(T)} \to 0, \tag{3.17}$$

as $T \to \infty$. Using (3.16) and (3.17), we can now bound the exponential in (3.15) by a constant to obtain

$$D \leqslant \left| M\mathrm{E} C_\beta^{-1} \sum_{k=0}^{\infty} |\vartheta_k|^\beta L_W(|\vartheta_k|^{-1}) h(\vartheta_k) - \sigma^\beta \right|$$

$$+ CM\mathrm{E}\left(C_\beta^{-1} \sum_{k=0}^{\infty} |\vartheta_k|^\beta L_W(|\vartheta_k|^{-1}) h(\vartheta_k) - \frac{\sigma^\beta}{M} \right)^2.$$

By the triangle inequality, the relation $(a+b)^2 \leqslant 2(a^2+b^2)$ and (3.17), we bound D further as

$$D \leqslant \left| M\mathrm{E} C_\beta^{-1} \sum_{k=0}^{\infty} |\vartheta_k|^\beta L_W(|\vartheta_k|^{-1}) - \sigma^\beta \right| + \sup_{k\geqslant 0} |h(\vartheta_k) - 1| M\mathrm{E} C_\beta^{-1} \sum_{k=0}^{\infty} |\vartheta_k|^\beta L_W(|\vartheta_k|^{-1})$$

$$+ \frac{C}{Q(T)} M\mathrm{E} C_\beta^{-1} \sum_{k=0}^{\infty} |\vartheta_k|^\beta L_W(|\vartheta_k|^{-1}) + M\frac{C}{M^2}.$$

Since by (3.9), $Q(T) \to \infty$ and by (3.16), $\sup_k |h(\vartheta_k) - 1| \to 0$, as $T \to \infty$, to prove $D \to 0$, it is enough to show that $\mathrm{M}EC_\beta^{-1}\sum_{k=0}^{\infty} |\vartheta_k|^\beta L_W(|\vartheta_k|^{-1}) \to \sigma^\beta$. By separating the terms $k = 0$ and $k \geqslant 1$ and using (3.3), it is enough to show that, as $T \to \infty$,

$$R_1 := \left| M\mathrm{E}|\vartheta_0|^\beta L_W(|\vartheta_0|^{-1}) - I \right| \to 0 \tag{3.18}$$

and

$$R_2 := \left| M\mathrm{E} \sum_{k=1}^{\infty} |\vartheta_k|^\beta L_W(|\vartheta_k|^{-1}) - J \right| \to 0. \tag{3.19}$$

This is established in Lemmas 3.3 and 3.4 below. □

Remark. If $l_U(u) \sim 1$ and $L_W(u) \sim 1$, as $u \to \infty$, then the conditions (3.18) and (3.19) to prove become

$$\mathrm{E} C_\beta^{-1} \sum_{k=0}^{\infty} \left| \sum_{j=1}^{d} \theta_j (T_j \wedge S_k - S_{k-1})_+ \right|^\beta \frac{1}{T^{\beta-\alpha+1}} \to \sigma^\beta,$$

as $T \to \infty$, which follows immediately from (3.6) and (3.7). We thus need to show that the slowly varying functions have the correct expression in (2.4).

We next establish three lemmas used in the proof above.

Lemma 3.2. *$F \leqslant C(Q(T))^{-1}$, where $Q(T)$ and F are defined in* (3.9) *and* (3.17)*, respectively.*

Proof. Set

$$\eta_k = Q(T)^{1/\beta} L_W^*(Q(T)) \vartheta_k = \left| \sum_{j=1}^{d} ((T_j/T) \wedge (S_k/T) - (S_{k-1}/T))_+ \right|,$$

where ϑ_k is defined in (3.13). Then

$$F = \frac{1}{Q(T)}\sum_{k=0}^{\infty}|\eta_k|^{\beta}L_W^*(Q(T))^{-\beta}L_W\left(|\eta_k|^{-1}Q(T)^{1/\beta}L_W^*(Q(T))\right) = \frac{1}{Q(T)}\sum_{k=0}^{\infty}|\eta_k|^{\beta}F^1(T)F_k^2(T),$$

where

$$F^1(T) = L_W^*(Q(T))^{-\beta}\left\{L_W(Q(T)^{1/\beta}L_W^*(Q(T)))\right\}$$

and

$$F_k^2(T) = \left\{L_W(Q(T)^{1/\beta}L_W^*(Q(T)))\right\}^{-1}L_W\left(|\eta_k|^{-1}Q(T)^{1/\beta}L_W^*(Q(T))\right).$$

By (1.5), $F^1(T) \to 1$ as $T \to \infty$. Moreover, by fixing $\delta > 0$ and using Potter's bounds (Bingham *et al.* 1987, Theorem 1.5.6), we obtain, for large enough T, $F_k^2(T) \leqslant 2\max\{|\eta_k|^{-\delta}, |\eta_k|^{\delta}\}$, since by (3.9), $Q(T)^{1/\beta}L_W^*(Q(T)) \to \infty$ and $|\eta_k|^{-1}$ is bounded from below. Therefore, for large enough T,

$$F \leqslant C(Q(T))^{-1}\sum_{k=0}^{\infty}|\eta_k|^{\beta-\delta}$$

(the term with $+\delta$ is bounded by that with $-\delta$ because $|\eta_k|$ is bounded from above). Now choosing δ such that $\beta - \delta > 1$, observe that

$$\sum_{k=0}^{\infty}|\eta_k|^{\beta-\delta} \leqslant C\sum_{k=0}^{\infty}\left(1\wedge\frac{S_k}{T}-\frac{S_{k-1}}{T}\right)_+^{\beta-\delta} \leqslant C\sum_{k=0}^{\infty}\left(1\wedge\frac{S_k}{T}-\frac{S_{k-1}}{T}\right)_+,$$

because $\beta - \delta > 1$ and $0 \leqslant (1\wedge(S_k/T)-(S_{k-1}/T))_+ \leqslant 1$. Since $S_{k-1} \leqslant S_k$ for all $k \geqslant 1$ and $S_k \to \infty$ almost surely, the last sum equals 1, and therefore the proof is complete. □

Lemma 3.3. *The convergence* (3.18) *holds as* $T \to \infty$.

Proof. Since by (3.13), $\vartheta_0 = \sum_{j=1}^{d}\theta_j(T_j\wedge S_0)/N(T)$ and, by (1.1), $P(S_0 = x) = P(U_0 = x) = \mu^{-1}P(U > x)$, $x = 0, 1, \ldots,$ we obtain

$$M\mathrm{E}|\vartheta_0|^{\beta}L_W(|\vartheta_0|^{-1}) = M\sum_{x=0}^{\infty}\left|\sum_{j=1}^{d}\theta_j(T_j\wedge x)\right|^{\beta}N(T)^{-\beta}L_W\left(\frac{N(T)}{\left|\sum_{j=1}^{d}\theta_j(T_j\wedge x)\right|}\right)\mu^{-1}P(U > x).$$

To simplify notation, we set

$$A_x = \sum_{j=1}^{d} \theta_j((T_j/T) \wedge (x/T)), \qquad x = 0, 1, \ldots,$$

$$G^1(T) = L_W^*(Q(T))^{-\beta}\Big\{L_W(Q(T)^{1/\beta}L_W^*(Q(T)))\Big\}, \tag{3.20}$$

$$G_x^2(T) = \Big\{L_W(Q(T)^{1/\beta}L_W^*(Q(T)))\Big\}^{-1} L_W\Big(|A_x|^{-1}Q(T)^{1/\beta}L_W^*(Q(T))\Big). \tag{3.21}$$

Observe that

$$\left|\sum_{j=1}^{d}\theta_j(T_j \wedge x)\right|^{\beta} N(T)^{-\beta} = |A_x|^{\beta} Q(T)^{-1}\big\{L_W^*(Q(T))\big\}^{-\beta}.$$

Then, by the triangle inequality, we obtain, for $\epsilon > 0$,

$$R_1 = \left|\frac{M\mu^{-1}}{Q(T)}\sum_{x=0}^{\infty}|A_x|^{\beta}G^1(T)G_x^2(T)P(U>x) - I\right|$$

$$\leqslant \frac{M\mu^{-1}}{Q(T)}\sum_{x:|A_x|\leqslant\epsilon}|A_x|^{\beta}G^1(T)G_x^2(T)P(U>x) + \frac{M\mu^{-1}}{Q(T)}\sum_{x:|A_x|>\epsilon}|A_x|^{\beta}|G^1(T)G_x^2(T)-1|P(U>x)$$

$$+\frac{M\mu^{-1}}{Q(T)}\sum_{x:|A_x|\leqslant\epsilon}|A_x|^{\beta}P(U>x) + \left|\frac{M\mu^{-1}}{Q(T)}\sum_{x=0}^{\infty}|A_x|^{\beta}P(U>x) - I\right|. \tag{3.22}$$

Denote the four terms in the above bound by $R_{1,1}$, $R_{1,2}$, $R_{1,3}$ and $R_{1,4}$. We will show that, for small enough $\epsilon > 0$ and large enough T, $R_{1,1}$ and $R_{1,3}$ are arbitrary small, and that, for fixed $\epsilon > 0$, $R_{1,2}$ and $R_{1,4}$ converge to 0. This will prove the convergence of R_1.

Observe that $G^1(T) \to 1$ as $T \to \infty$, since $Q(T) \to \infty$ and (1.5) holds. Using Potter's bounds (see Bingham *et al*. 1987, Theorem 1.5.6, (i)), we obtain, for sufficiently large T and fixed $\delta > 0$, that $G_x^2(T) \leqslant C\max\{|A_x|^{-\delta}, |A_x|^{\delta}\}$ (this result applies since $Q(T) \to \infty$ as $T \to \infty$, and $|A_x| \leqslant c < \infty$, so that $|A_x|^{-1} \geqslant c^{-1} > 0$). Then, for such T and δ,

$$R_{1,1} \leqslant C\frac{M}{Q(T)}\sum_{x:|A_x|\leqslant\epsilon}|A_x|^{\beta}\max\big\{|A_x|^{-\delta}, |A_x|^{\delta}\big\}\mathrm{P(U>x)}$$

$$\leqslant C\sum_{x:|A_x|\leqslant\epsilon}\big(|A_x|^{\beta-\delta} + |A_x|^{\beta+\delta}\big)\frac{P(U>x)}{T^{-\alpha+1}l_U(T)}. \tag{3.23}$$

Suppose now that $\delta > 0$ is such that $\alpha < \beta - \delta < \beta + \delta < 2$. Let us show that (3.6) implies

$$S_{T,\epsilon} := \sum_{x:|A_x|\leqslant\epsilon}|A_x|^{\beta\mp\delta}\frac{P(U>x)}{T^{-\alpha+1}l_U(T)} \to \int_{u:|\sum_{j=1}^{d}\theta_j(y_j\wedge u)|\leqslant\epsilon}\left|\sum_{j=1}^{d}\theta_j(y_j\wedge u)\right|^{\beta\mp\delta}u_+^{-\alpha}\,\mathrm{d}u, \tag{3.24}$$

as $T \to \infty$. Observe first that $S_{T,\epsilon} = \int_0^{\infty} f_{T,\epsilon}(u)\mathrm{d}u$, where

$$f_{T,\epsilon}(u) = |A_{[Tu]}|^{\beta\mp\delta} \frac{P(U > [Tu])}{T^{-\alpha} l_U(T)} 1_{\{u: |A_{[Tu]}| \leqslant \epsilon\}}(u).$$

By (1.2), we obtain

$$\frac{P(U > [Tu])}{T^{-\alpha} l_U(T)} = \left(\frac{[Tu]}{T}\right)^{-\alpha} \frac{L_U(T[Tu]/T)}{L_U(T)} \frac{L_U(T)}{l_U(T)} \to u^{-\alpha},$$

as $T \to \infty$, for $u > 0$ (using Theorem 1.2.1 in Bingham *et al*. 1987). Since also $A_{[Tu]} \to \sum_{j=1}^{d} \theta_j(y_j \wedge u)$, we see that

$$f_{T,\epsilon}(u) \to \left| \sum_{j=1}^{d} \theta_j(y_j \wedge u) \right|^{\beta\mp\delta} u_+^{-\alpha} 1_{\{u: |\sum_{j=1}^{d} \theta_j(y_j \wedge u)| \leqslant \epsilon\}}(u). \tag{3.25}$$

Observe now that

$$0 \leqslant f_{T,\epsilon}(u) \leqslant f_T(u) := |A_{[Tu]}|^{\beta\mp\delta} \frac{P(U > [Tu])}{T^{-\alpha} l_U(T)} \tag{3.26}$$

and that, by (3.6) and (3.4) (where β is replaced by $\beta \mp \delta$),

$$\int_0^\infty f_T(u) \mathrm{d}u = \sum_{x=0}^{\infty} |A_x|^{\beta\mp\delta} \frac{P(U > x)}{T^{-\alpha+1} l_U(T)} \to \int_0^\infty \left| \sum_{j=1}^{d} \theta_j(y_j \wedge u) \right|^{\beta\mp\delta} u^{-\alpha} \,\mathrm{d}u. \tag{3.27}$$

In view of (3.25), (3.26) and (3.27), the convergence (3.24) is a consequence of the following result: if f, g, f_n, g_n, $n \geqslant 1$, are measurable functions on (E, μ) such that $0 \leqslant f_n \leqslant g_n$, $f_n \to f$, $g_n \to g$ (in the almost everywhere sense) and $\int g_n \,\mathrm{d}\mu \to \int g \,\mathrm{d}\mu < \infty$, then $\int f_n \,\mathrm{d}\mu \to \int f \,\mathrm{d}\mu$ (see Proposition 18 in Royden 1988, p. 270). By choosing $\epsilon > 0$ small enough, the limit in (3.24) can be made arbitrarily small, and hence (3.23) and (3.24) imply that $R_{1,1}$ is arbitrarily small for large T.

For $R_{1,2}$, since $0 < C \leqslant |A_x|^{-1} \leqslant \epsilon^{-1}$ is bounded, its presence in the argument of L_W can be ignored, and in view of (1.5) and (3.6), we obtain $R_{1,2} \to 0$ as $T \to \infty$. For $R_{1,3}$, one can show as in the case of $R_{1,1}$ that, by choosing small enough $\epsilon > 0$, $R_{1,3}$ is arbitrarily small for large T. For $R_{1,4}$ we have, by (3.6),

$$R_{1,4} = \left| \sum_{x=0}^{\infty} |A_x|^{\beta} \frac{\mu^{-1} P(U > x)}{T^{-\alpha+1} l_U(T)} - I \right| \to 0.$$

□

Lemma 3.4. *The convergence* (3.19) *holds as* $T \to \infty$.

Proof. Note first that, by stationarity,

$$\sum_{k=1}^{\infty} P(S_{k-1} = x) = \sum_{k=1}^{\infty} P(S_{k-1} = 0) = P(S_0 = 0) = \mu^{-1}.$$

Since $S_k = S_{k-1} + U_k$, we obtain

$$\mathrm{E}\sum_{k=1}^{\infty} |\vartheta_k|^\beta L_W(|\vartheta_k|^{-1}) = \sum_{k=1}^{\infty}\sum_{x=0}^{\infty}\sum_{z=0}^{\infty}\left|\sum_{j=1}^{d}\theta_j(T_j \wedge (x+z) - x)_+\right|^\beta$$

$$\cdot N(T)^{-\beta} L_W\left(\frac{N(T)}{\left|\sum_{j=1}^{d}\theta_j(T_j \wedge (x+z) - x)_+\right|}\right) P(S_{k-1} = x)P(U = z)$$

$$= \mu^{-1}\sum_{y=0}^{\infty}\sum_{x=0}^{y}\left|\sum_{j=1}^{d}\theta_j(T_j \wedge y - x)_+\right|^\beta N(T)^{-\beta} L_W\left(\frac{N(T)}{\left|\sum_{j=1}^{d}\theta_j(T_j \wedge y - x)_+\right|}\right) P(U = y - x).$$

We will proceed as in the case of R_1 considered in Lemma 3.3. For ease of notation, we set $B_{x,y} = \sum_{j=1}^{d}\theta_j((T_j/T) \wedge (y/T) - (x/T))_+$, and also define $H^1(T)$ and $H^2_{x,y}(T)$ as in (3.20) and (3.21), by changing A_x in (3.21) to $B_{x,y}$. Then, for some $\epsilon > 0$,

$$R_2 = \left|\frac{M\mu^{-1}}{Q(T)}\sum_{y=0}^{\infty}\sum_{x=0}^{y}|B_{x,y}|^\beta H^1(T)H^2_{x,y}(T)P(U = y - x) - J\right| \leqslant R_{2,1} + R_{2,2} + R_{2,3} + R_{2,4},$$

where the first term $R_{2,1}$ is

$$\frac{M\mu^{-1}}{Q(T)}\sum_{(x,y):|B_{x,y}|\leqslant\epsilon}|B_{x,y}|^\beta H^1(T)H^2_{x,y}(T)P(U = y - x)$$

and the other terms are defined in a corresponding fashion as in (3.22). For $R_{2,1}$ we have, as in the case of $R_{1,1}$ in Lemma 3.3, that

$$R_{2,1} \leqslant C\sum_{(x,y):|B_{x,y}|\leqslant\epsilon}\left(|B_{x,y}|^{\beta-\delta} + |B_{x,y}|^{\beta+\delta}\right)\frac{P(U = y - x)}{T^{-\alpha+1}l_U(T)}$$

$$\to C\int_{(u,v):|\sum_{j=1}^{d}\theta_j(y_j \wedge v - u)_+|\leqslant\epsilon}\left(\left|\sum_{j=1}^{d}\theta_j(y_j \wedge v - u)_+\right|^{\beta-\delta} + \left|\sum_{j=1}^{d}\theta_j(y_j \wedge v - u)_+\right|^{\beta+\delta}\right)$$

$$\times\, \alpha(v - u)_+^{-\alpha-1}\,\mathrm{d}u\,\mathrm{d}v,$$

where $\delta > 0$ is such that $\alpha < \beta - \delta < \beta + \delta < 2$. (To show the convergence, use (3.7) instead of (3.6).) Then, by choosing small enough $\epsilon > 0$, we obtain that $R_{2,1}$ is arbitrarily small. For the terms $R_{2,2}$, $R_{2,3}$ and $R_{2,4}$, one argues in the same way as for $R_{1,2}$, $R_{1,3}$ and $R_{1,4}$ in Lemma 3.3. □

3.2.2. Asymmetric rewards

As in the case of symmetric rewards, by using the expression for a characteristic function of a random variable W in the domain of attraction of a β-stable random variable,

$$\mathrm{E}\exp\{\mathrm{i}uW\} = \exp\left\{-C_\beta^{-1}(c^+ + c^-)|u|^\beta L_W\left(|u|^{-1}\right)\left(1 - \mathrm{i}\frac{c^+ - c^-}{c^+ + c^-}\,\mathrm{sign}(u)\tan\frac{\beta\pi}{2}\right)h(u)\right\},$$

$u \in \mathbb{R}$, where $\lim_{u\to 0} h(u) = 1$ (see Theorem 2.6.5 in Ibragimov and Linnik 1971; or Theorem 1 in Aaronson and Denker 1998), we can write the difference between the characteristic functions of $\sum_{j=1}^d \theta_j W^*(Ty_j, M)/N(T)$ and $\sum_{j=1}^d \theta_j Z_\beta(y_j)$ as (see (3.12) and (3.14))

$$D = \Bigg| \prod_{m=1}^{M} \mathrm{E}\exp\left\{-C_\beta^{-1}(c^+ + c^-)\sum_{k=0}^{\infty}|\vartheta_k|^\beta L_W(|\vartheta_k|^{-1})\left(1 - \mathrm{i}\frac{c^+ - c^-}{c^+ + c^-}\,\mathrm{sign}(\vartheta_k)\tan\frac{\beta\pi}{2}\right)h(\vartheta_k)\right\}$$
$$- \exp\left\{-\sigma^\beta\left(1 - \mathrm{i}\zeta\tan\frac{\beta\pi}{2}\right)\right\}\Bigg| \leqslant M\left|\mathrm{E}\mathrm{e}^Z - 1\right|,$$

where $\sigma^\beta = \sigma^\beta(\boldsymbol{\theta}, \boldsymbol{y})$, $\zeta = \zeta(\boldsymbol{\theta}, \boldsymbol{y})$ and

$$Z = \frac{\sigma^\beta}{M}\left(1 - \mathrm{i}\zeta\tan\frac{\beta\pi}{2}\right) - C_\beta^{-1}(c^+ + c^-)\mathrm{E}\sum_{k=0}^{\infty}|\vartheta_k|^\beta L_W(|\vartheta_k|^{-1})$$
$$\times\left(1 - \mathrm{i}\frac{c^+ - c^-}{c^+ + c^-}\,\mathrm{sign}(\vartheta_k)\tan\frac{\beta\pi}{2}\right)h(\vartheta_k).$$

We now use the inequality $|\mathrm{E}\mathrm{e}^Z - 1| \leqslant |\mathrm{E}Z| + \mathrm{E}\mathrm{e}^{|Z|}|Z|^2/2$, where Z is a complex random variable, and proceed as in the case of symmetric rewards. Since the term $M\mathrm{E}\mathrm{e}^{|Z|}|Z|^2/2$ tends to zero as $T \to \infty$, as in the case of symmetric rewards, we are left with $M|\mathrm{E}Z|$. Thus we have $D \to 0$ as long as $M\mathrm{E}Z \to 0$ or

$$C_\beta^{-1}(c^+ + c^-)M\mathrm{E}\sum_{k=0}^{\infty}|\vartheta_k|^\beta L_W(|\vartheta_k|^{-1}) \to \sigma^\beta \tag{3.28}$$

and

$$C_\beta^{-1}(c^+ - c^-)M\mathrm{E}\sum_{k=0}^{\infty}(\vartheta_k)^{\langle\beta\rangle} L_W(|\vartheta_k|^{-1}) \to \sigma^\beta\zeta. \tag{3.29}$$

The convergence (3.28) has been established in the case of symmetric rewards. The convergence (3.29) can be proved in a similar way by writing

$$M\mathrm{E}\sum_{k=0}^{\infty}(\vartheta_k)^{\langle\beta\rangle}L_W(|\vartheta_k|^{-1})$$

$$= \frac{1}{T^{\beta-\alpha+1}l_U(T)}\mathrm{E}\sum_{k=0}^{\infty}\left(\sum_{j=1}^{d}\theta_j(T_j\wedge S_k - S_{k-1})_+\right)^{\langle\beta\rangle}$$

$$\cdot\,(L_W^*(T^{-\alpha+1}Ml_U(T)))^{-\beta}L_W\left(\frac{(T^{-\alpha+1}Ml_U(T))^{1/\beta}L_W^*(T^{-\alpha+1}Ml_U(T))}{\left|T^{-1}\sum_{j=1}^{d}\theta_j(T_j\wedge S_k - S_{k-1})_+\right|}\right),$$

by arguing that the last two multiplicative terms in the above sum can be disregarded and by showing that

$$\frac{1}{T^{\beta-\alpha+1}l_U(T)}\mathrm{E}\sum_{k=0}^{\infty}\left(\sum_{j=1}^{d}\theta_j(T_j\wedge S_k - S_{k-1})_+\right)^{\langle\beta\rangle}$$

$$\to \mu^{-1}\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\sum_{j=1}^{d}\theta_j((y_j\wedge v-u)_+ - (0\wedge v-u)_+)\right)^{\langle\beta\rangle}\alpha(v-u)_+^{-\alpha-1}\,\mathrm{d}u\,\mathrm{d}v.$$

3.3. Proof of Theorem 2.3

We will adapt the proof of Theorem 1 in Mikosch *et al*. (2002) to our context. Consider, first, a single time $y=1$ and let $\xi_T^m = \sum_{n=0}^{\infty}1_{[0,T]}(S_n^m)$ be the total number of renewals in $[0, T]$ in the mth sample. Since S_0^m is the first renewal, $S_{\xi_T^m-1}^m$ is the last renewal before or at time T and $S_{\xi_T^m}^m$ is the first renewal after T. To show the convergence, we will use the following decomposition of (1.6):

$$W^*(T, M) = \sum_{m=1}^{M}W_0^m\min(T, S_0^m) + \sum_{m=1}^{M}\sum_{k=1}^{\xi_T^m}W_k^mU_k^m - \sum_{m=1}^{M}\Big(S_{\xi_T^m-1}^m + U_{\xi_T^m}^m - T\Big)1_{\{\xi_T^m\geqslant 1\}}W_{\xi_T^m}^m$$

$$=: A_1(T) + A_2(T) - A_3(T). \tag{3.30}$$

The term $S_{\xi_T^m-1}^m + U_{\xi_T^m}^m - T$ is the time between T and the first renewal after T. Let $N(T) = T^{1/\alpha}M^{1/\alpha}L_U^*(TM)(\sim b(MT))$. We will show that

1. $(N(T))^{-1}A_1(T)\to 0$ in probability,
2. $(N(T))^{-1}A_2(T)\to_d\Lambda_\alpha(1)$, and
3. $(N(T))^{-1}A_3(T)\to 0$ in probability,

so that, by Theorem 4.1 in Billingsley (1968), $(N(T))^{-1}W^*(T, M) \to_d \Lambda_\alpha(1)$ as $T \to \infty$.

1. This step follows from

$$\frac{\mathrm{E}|A_1(T)|}{N(T)} \leqslant \frac{M\mathrm{E}|W|}{N(T)} \mathrm{E}\min(T, S_0) = \frac{M\mathrm{E}|W|}{N(T)}\left(\sum_{u=0}^{T} uP(S_0 = u) + TP(S_0 > T)\right)$$

$$= \frac{M\mathrm{E}|W|}{\mu N(T)}\left(\sum_{u=0}^{T} u(1+u)^{-\alpha}L_U(1+u) + T\sum_{u=T+1}^{\infty}(1+u)^{-\alpha}L_U(1+u)\right)$$

$$\leqslant C\,\frac{MT^{2-\alpha}L_U(T)}{N(T)}, \tag{3.31}$$

where we have used Karamata's theorem (see Bingham *et al.* 1987), and

$$\frac{MT^{2-\alpha}L_U(T)}{M^{1/\alpha}T^{1/\alpha}L_U^*(MT)}$$

$$= M^{(\alpha-1)/\alpha}T^{((2-\alpha)\alpha-1)/\alpha}\frac{L_U(T)}{L_U^*(MT)} = M^{(\alpha-1)/\alpha}T^{-(\alpha-1)^2/\alpha}\frac{L_U(T)}{L_U^*(MT)}$$

$$= (M^{1/\alpha}T^{1/\alpha-1}L_U^*(MT))^{\alpha-1}L_U^*(MT)^{-\alpha}L_U(M^{1/\alpha}T^{1/\alpha}L_U^*(MT))\frac{L_U(T)}{L_U(M^{1/\alpha}T^{1/\alpha}L_U^*(MT))}.$$

Using Potter's bounds, the slow-growth condition and (1.5), we obtain

$$\frac{MT^{2-\alpha}L_U(T)}{M^{1/\alpha}T^{1/\alpha}L_U^*(T)} \leqslant (M^{1/\alpha}T^{1/\alpha-1}L_U^*(MT))^{\alpha-1}L_U^*(MT)^{-\alpha}L_U(M^{1/\alpha}T^{1/\alpha}L_U^*(MT))$$

$$\cdot 2\max\{(M^{1/\alpha}T^{1/\alpha-1}L_U^*(MT))^{\delta}, (M^{1/\alpha}T^{1/\alpha-1}L_U^*(MT))^{-\delta}\} \to 0,$$

where δ is such that $\alpha - 1 - \delta > 0$. (To prove this step, one may also use Lemma 3 in Mikosch *et al.* 2002.)

2. We start by introducing the notation

$$A_2(T) = \sum_{m=1}^{M}\sum_{k=1}^{\xi_T^m} W_k^m U_k^m =: \sum_{m=1}^{M}\sum_{k=1}^{\xi_T^m} Y_k^m =: \sum_{m=1}^{M} S_m(T).$$

We want to show that $b(MT)^{-1}A_2(T) \to_d \Lambda_\alpha(1)$. The necessary and sufficient conditions for this convergence are (see Petrov 1975, Theorem 8, p. 81):

(i) $MP(S_1(T) > xb(MT)) \to \mu^{-1}\mathrm{E}W_+^\alpha x^{-\alpha}$, for all $x > 0$, as $T \to \infty$;
(ii) $MP(S_1(T) < -xb(MT)) \to \mu^{-1}\mathrm{E}W_-^\alpha x^{-\alpha}$, for all $x > 0$, as $T \to \infty$;
(iii) $\lim_{\epsilon\downarrow 0}\limsup_{T\to\infty} M(b(MT))^{-2}\mathrm{var}(S_1(T)1_{\{|S_1(T)|<\epsilon b(MT)\}}) = 0$.

The proof of (i) is similar to that of Lemma 10 in Mikosch *et al.* (2002). The idea is to replace ξ_T^m by its mean, which is, by stationarity, $\mathrm{E}\xi_T = (T+1)/\mu =: \mu_T$. Consider

$S(\xi_T) = \sum_{k=1}^{\xi_T} Y_k$ and $S([\mu_T]) = \sum_{k=1}^{[\mu_T]} Y_k$, where $(\xi_T, (Y_k)_{k\geqslant 1}) =_d (\xi_T^1, (Y_k^1)_{k\geqslant 1})$. Then, for some $\epsilon_T > 0$,

$$MP(S(\xi_T) > xb(MT)) = MP(S(\xi_T) > xb(MT), |\xi_T - \mu_T| \leqslant \epsilon_T\mu_T)$$
$$+ MP(S(\xi_T) > xb(MT), |\xi_T - \mu_T| > \epsilon_T\mu_T).$$

Following Mikosch *et al*. (2002), by virtue of the slow-growth condition, we can take $\epsilon_T \to 0$ such that

$$b(MT) = o(\epsilon_T T), \qquad \frac{1}{\log T} = o(\epsilon_T), \tag{3.32}$$

as $T \to \infty$. By Lemma 4 in Mikosch *et al*. (2002), for ϵ_T satisfying (3.32), we have $MP(|\xi_T - \mu_T| > \epsilon_T\mu_T) \to 0$ as $T \to \infty$. Therefore, it is enough to show that

$$MP(S(\xi_T) > xb(MT), |\xi_T - \mu_T| \leqslant \epsilon_T\mu_T) \to \mu^{-1}\mathrm{E}W_+^\alpha x^{-\alpha}. \tag{3.33}$$

This convergence will follow by finding proper bounds. For the upper bound, observe that, for $\delta \in (0, 1)$,

$$MP(S(\xi_T) > xb(MT), |\xi_T - \mu_T| \leqslant \epsilon_T\mu_T)$$
$$\leqslant MP(S(\xi_T) - S([\mu_T]) > \delta xb(MT), |\xi_T - \mu_T| \leqslant \epsilon_T\mu_T) + MP(S([\mu_T]) > (1-\delta)xb(MT)).$$

By Lemma 8 in Mikosch *et al*. (2002), $MP(S(\xi_T) - S([\mu_T]) > \delta xb(MT), |\xi_T - \mu_T| \leqslant \epsilon_T\mu_T) \to 0$ as $T \to \infty$. As for the second term,

$$MP(S([\mu_T]) > (1-\delta)xb(MT)) = MP\left(\sum_{k=1}^{[\mu_T]} Y_k > (1-\delta)xb(MT)\right),$$

observe that both μ_T and $b(MT)$ tend to infinity as $T \to \infty$, and that, by Lemma 1.1, $P(Y_k \geqslant x) \sim \mathrm{E}W_+^\alpha L_U(x)x^{-\alpha}$ as $x \to \infty$. Applying Corollary A.1 in Appendix A, we obtain

$$MP\left(\sum_{k=1}^{[\mu_T]} Y_k > (1-\delta)xb(MT)\right) \sim M[\mu_T]\mathrm{E}W_+^\alpha \overline{F}_U(b(MT))(1-\delta)^{-\alpha}x^{-\alpha}. \tag{3.34}$$

(To apply the corollary, suppose first that $\mathrm{E}W_+^\alpha \neq 0$. Then set $a_T = b(\mu_T\mathrm{E}W_+^\alpha)$, so that $[\mu_T]\overline{F}_Y(a_T) \sim [\mu_T]\mathrm{E}W_+^\alpha\overline{F}_U(b(\mu_T\mathrm{E}W_+^\alpha)) \sim 1$, and observe that $h(T) = (1-\delta)xb(MT)/b(\mu_T\mathrm{E}W_+^\alpha) \to \infty$. This last condition follows from the fact that $b(MT)/b(T) \to \infty$, which is a consequence of (2.11). The case $\mathrm{E}W_+^\alpha = 0$ can be considered in a similar way.) Finally, since

$$MT\overline{F}_U(b(MT)) = MT\left(\frac{1}{\overline{F}_U}\left(\frac{1}{\overline{F}_U}^{\leftarrow}(MT)\right)\right)^{-1} \sim MT(MT)^{-1} = 1,$$

one obtains

$$MP(S([\mu_T]) > (1-\delta)xb(MT)) \sim \mu^{-1}\mathrm{E}W_+^\alpha(1-\delta)^{-\alpha}x^{-\alpha}, \qquad \text{as } T \to \infty.$$

To obtain the lower bound of (3.33), observe that

$$MP(S(\xi_T) > xb(MT),\ |\xi_T - \mu_T| \leqslant \epsilon_T\mu_T)$$

$$\geqslant MP(S(\xi_T) - S([\mu_T]) > -\delta xb(MT),\ S([\mu_T]) > (1+\delta)xb(MT),\ |\xi_T - \mu_T| \leqslant \epsilon_T\mu_T)$$

$$\geqslant MP(S([\mu_T]) > (1+\delta)xb(MT)) - MP(S(\xi_T) - S([\mu_T]) \leqslant -\delta xb(MT),\ |\xi_T - \mu_T| \leqslant \epsilon_T\mu_T)$$

$$- MP(|\xi_T - \mu_T| > \epsilon_T\mu_T).$$

Again, Lemmas 4 and 8 in Mikosch *et al*. (2002) imply that $MP(|\xi_T - \mu_T| > \epsilon_T\mu_T) \to 0$, $MP(S(\xi_T) - S([\mu_T]) < -\delta xb(MT),\ |\xi_T - \mu_T| \leqslant \epsilon_T\mu_T) \to 0$. Moreover, as for the upper bound, we have that $MP(S([\mu_T]) > (1+\delta)xb(MT)) \sim \mu^{-1}\mathrm{E}W_+^{\alpha}(1+\delta)^{-\alpha}x^{-\alpha}$ as $T \to \infty$. Finally, by letting $\delta \to 0$ in the upper and the lower bounds, we obtain (i).

The proof of (ii) is similar to that of (i). To prove (iii), we will proceed as in the proof of Lemma 10 in Mikosch *et al*. (2002). Observe that

$$\operatorname{var}(S(\xi_T)1_{\{|S(\xi_T)|<\epsilon b(MT)\}}) \leqslant \mathrm{E}S(\xi_T)^2 1_{\{|S(\xi_T)|^2<(\epsilon b(MT))^2\}} = \int_0^{\epsilon^2 b(MT)^2} P(|S(\xi_T)|^2 > x)\mathrm{d}x$$

$$= \int_0^{\epsilon^2 b(MT)^2} P(S(\xi_T) > \sqrt{x})\mathrm{d}x + \int_0^{\epsilon^2 b(MT)^2} P(S(\xi_T) < -\sqrt{x})\mathrm{d}x.$$

We will deal with the first term only, since the arguments for the second are analogous. We have

$$\frac{M}{(b(MT))^2}\int_0^{\epsilon^2 b(MT)^2} P(S(\xi_T) > \sqrt{x})\mathrm{d}x$$

$$= \frac{M}{(b(MT))^2}\int_0^{\epsilon^2 b(MT)^2/M} P(S(\xi_T) > \sqrt{x})\mathrm{d}x + \frac{M}{(b(MT))^2}\int_{\epsilon^2 b(MT)^2/M}^{\epsilon^2 b(MT)^2} P(S(\xi_T) > \sqrt{x})\mathrm{d}x.$$

Since the first term is bounded by ϵ^2, it is enough to consider the second term, for which

$$\frac{M}{(b(MT))^2}\int_{\epsilon^2 b(MT)^2/M}^{\epsilon^2 b(MT)^2} P(S(\xi_T) > \sqrt{x})\mathrm{d}x \leqslant \frac{M}{(b(MT))^2}\int_{\epsilon^2 b(MT)^2/M}^{\epsilon^2 b(MT)^2} P(|\xi_T - \mu_T| > \epsilon_T\mu_T)\mathrm{d}x$$

$$+ \frac{M}{(b(MT))^2}\int_{\epsilon^2 b(MT)^2/M}^{\epsilon^2 b(MT)^2} P(S(\xi_T) > \sqrt{x},\ |\xi_T - \mu_T| \leqslant \epsilon_T\mu_T)\mathrm{d}x.$$

The first term equals $\epsilon^2 M(1 - M^{-1})P(|\xi_T - \mu_T| > \epsilon_T\mu_T) \to 0$ as $T \to \infty$, by Mikosch *et al*. (2002). Therefore, we need to deal with the second term only. We have

$$\frac{M}{(b(MT))^2}\int_{\epsilon^2 b(MT)^2/M}^{\epsilon^2 b(MT)^2} P(S(\xi_T) > \sqrt{x}, |\xi_T - \mu_T| \leqslant \epsilon_T\mu_T)\mathrm{d}x$$

$$\leqslant \frac{M}{(b(MT))^2}\int_{\epsilon^2 b(MT)^2/M}^{\epsilon^2 b(MT)^2} P(S(\xi_T) - S([\mu_T]) > \sqrt{x}/2, |\xi_T - \mu_T| \leqslant \epsilon_T\mu_T)\mathrm{d}x$$

$$+ \frac{M}{(b(MT))^2}\int_{\epsilon^2 b(MT)^2/M}^{\epsilon^2 b(MT)^2} P(S([\mu_T]) > \sqrt{x}/2)\mathrm{d}x.$$

Now use

$$P\big(S(\xi_T) - S([\mu_T]) > \sqrt{x}/2, |\xi_T - \mu_T| \leqslant \epsilon_T\mu_T\big) \leqslant P\left(\max_{|j-\mu_T|\leqslant\epsilon_T\mu_T} |S(j) - S([\mu_T])| > \sqrt{x}/2\right)$$

$$= P\left(\max_{1\leqslant k\leqslant\epsilon_T\mu_T} |S(k)| > \sqrt{x}/2\right) \leqslant CP\big(|S([\epsilon_T\mu_T])| > \sqrt{x}/2\big)$$

(see Petrov 1995, Theorem 2.2 or 2.3, in the asymmetric case) and conclude the proof of (ii) as at the end of the proof of Lemma 10 in Mikosch *et al*. (2002).

3. By decomposing $A_3(T)$ into

$$A_3(T) = \sum_{m=1}^{M}\left(S^m_{\xi^m_T-1} + U^m_{\xi^m_T} - T\right)1_{\{\xi^m_T\geqslant 1\}} W^m_{\xi^m_T} 1_{\{|U^m_{\xi^m_T} W^m_{\xi^m_T}|>b(MT)\}}$$

$$+ \sum_{m=1}^{M}\left(S^m_{\xi^m_T-1} + U^m_{\xi^m_T} - T\right)1_{\{\xi^m_T\geqslant 1\}} W^m_{\xi^m_T} 1_{\{|U^m_{\xi^m_T} W^m_{\xi^m_T}|\leqslant b(MT)\}} =: A_{3,1}(T) + A_{3,2}(T), \quad (3.35)$$

it is enough to show that $(b(MT))^{-1}\mathrm{E}|A_{3,1}(T)| \to 0$ and $(b(MT))^{-2}\mathrm{E}|A_{3,2}(T)|^2 \to 0$, as $T \to \infty$. For $A_{3,1}(T)$ observe that, since $S^m_{\xi^m_T-1} + U^m_{\xi^m_T} - T$ measures the time between the terminal time T and the renewal time which immediately follows it, one has

$$(b(MT))^{-1}\mathrm{E}|A_{3,1}(T)| \leqslant M(b(MT))^{-1}\mathrm{E}|U_{\xi_T} W_{\xi_T}| 1_{\{|U_{\xi_T} W_{\xi_T}|>b(MT)\}} 1_{\{\xi_T\geqslant 1\}}. \quad (3.36)$$

Since, by Lemma 1.1, the random variables $|U_k W_k|$, $k \geqslant 1$, are independent and have heavy tails, one obtains, as in Lemma 5 in Mikosch *et al*. (2002), that

$$M(b(MT))^{-1}\mathrm{E}|U_{\xi_T} W_{\xi_T}| 1_{\{|U_{\xi_T} W_{\xi_T}|>b(MT)\}} 1_{\{\xi_T\geqslant 1\}} \to 0,$$

and hence $(b(MT))^{-1}\mathrm{E}|A_{3,1}(T)| \to 0$, as $T \to \infty$. The term $1_{\{\xi_T\geqslant 1\}}$ must be included in the preceding relation because if $\xi_T = 0$, the 'first' renewal time has infinite mean (see (1.1)). As for $A_{3,2}(T)$, we have

$$
\begin{aligned}
&(b(MT))^{-2}\mathrm{E}|A_{3,2}(T)|^2 \\
&\quad \leqslant M(b(MT))^{-2}\mathrm{E}|U_{\xi_T}W_{\xi_T}|^2 1_{\{|U_{\xi_T}W_{\xi_T}|\leqslant b(MT)\}}1_{\{\xi_T\geqslant 1\}} \\
&\quad = M(b(MT))^{-2}\int_0^{b(MT)^2} P(|U_{\xi_T}W_{\xi_T}| > \sqrt{x},\, \xi_T \geqslant 1)\mathrm{d}x \\
&\quad \leqslant MP(|\xi_T - \mu_T| > \mu_T\epsilon_T) + M(b(MT))^{-2}\int_0^{b(MT)^2} P(|U_{\xi_T}W_{\xi_T}| > \sqrt{x},\, |\xi_T - \mu_T| \leqslant \mu_T\epsilon_T)\mathrm{d}x.
\end{aligned}
$$

The first term in the bound tends to 0 by Lemma 4 in Mikosch *et al*. (2002). For the second term, bound the probability by

$$
P\left(\max_{|k-\mu_T|\leqslant\mu_T\epsilon_T} |U_kW_k| > \sqrt{x}\right) \leqslant \sum_{|k-\mu_T|\leqslant\mu_T\epsilon_T} P(|U_kW_k| > \sqrt{x}) = [2\mu_T\epsilon_T]P(|UW| > \sqrt{x}),
$$

so that, by Karamata's theorem and Lemma 1.1,

$$
\begin{aligned}
M(b(MT))^{-2}&\int_0^{b(MT)^2} P(|U_{\xi_T}W_{\xi_T}| > \sqrt{x},\, |\xi_T - \mu_T| \leqslant \mu_T\epsilon_T)\mathrm{d}x \\
&\leqslant C\,[2\mu_T\epsilon_T]\,M(b(MT))^{-2}b(MT)^2P(|UW| > b(MT)) \\
&\sim C\,\epsilon_T MT\overline{F}_U(b(MT)) \sim C\,\epsilon_T \to 0.
\end{aligned}
$$

The convergence of the finite-dimensional distributions in Theorem 2 can be shown as in Lemmas 11 and 12 in Mikosch *et al*. (2002). More specifically, consider for example the case of convergence of two-dimensional distributions. It is then enough to show that $b_1(N(T))^{-1}A_2(Tt_1) + b_2(N(T))^{-1}(A_2(Tt_2) - A_2(Tt_1)) \to_d b_1\Lambda_\alpha(t_1) + b_2(\Lambda_\alpha(t_2) - \Lambda_\alpha(t_1))$ as $T \to \infty$, for $b_1, b_2 \in \mathbb{R}$, $t_2 > t_1 \geqslant 0$, where $A_2(\cdot)$ is defined in (3.30). Expressing the latter sequence as a normalized partial sum of i.i.d. random variables

$$
S_m(T,\, t_1,\, t_2) = b_1S_m(Tt_1) + b_2(S_m(Tt_2) - S_m(Tt_1)),
$$

we need to prove the three conditions (i), (ii) and (iii) analogous to those in step 2.3 above. For example, in condition (i), we need to show that the sequence $MP(S_1(T,\, t_1,\, t_2) > xb(MT))$ converges to

$$
\mu^{-1}(\mathrm{E}W_+^\alpha 1_{\{b_1>0\}} + \mathrm{E}W_-^\alpha 1_{\{b_1<0\}})|b_1|^\alpha x^{-\alpha}t_1 + \mu^{-1}(\mathrm{E}W_+^\alpha 1_{\{b_2>0\}} + \mathrm{E}W_-^\alpha 1_{\{b_2<0\}})|b_2|^\alpha x^{-\alpha}(t_2 - t_1), \tag{3.37}
$$

for all $x > 0$, as $T \to \infty$. By introducing the set $\Theta = \{|\xi_{Tt_j} - \mu_{Tt_j}| \leqslant \epsilon_T\mu_{Tt_j},\, j = 1, 2\}$, where $\epsilon_T \to 0$ satisfies (3.32), and using Lemma 4 in Mikosch *et al*. (2002), we have $MP(S_1(T,\, t_1,\, t_2) > xb(MT)) \sim MP(S_1(T,\, t_1,\, t_2) > xb(MT),\, \Theta)$, as $T \to \infty$. Following the idea and the notation of step 2 above, we can then show that

$$
MP(S_1(T,\, t_1,\, t_2) > xb(MT),\, \Theta) \sim MP(b_1S([\mu_{Tt_1}]) + b_2(S([\mu_{Tt_2}]) - S([\mu_{Tt_1}])) > xb(MT)),
$$

as $T \to \infty$. Arguing as in Lemma 12 of Mikosch *et al*. (2002), the last expression is asymptotically equal to $MP(b_1S([\mu_{Tt_1}]) > xb(MT)) + MP(b_2(S([\mu_{Tt_2}]) - S([\mu_{Tt_1}])) > xb(MT)) = MP(b_1S([\mu_{Tt_1}]) > xb(MT)) + MP(b_2S([\mu_{Tt_2}] - [\mu_{Tt_1}]) > xb(MT))$. The limit (3.37) follows by arguing as in step 2. Condition (ii) concerning the left tail can be proved in a similar way, and condition (iii) follows as in step 2.

4. Proof under arbitrary growth condition

The proof of Theorem 2.4 is given separately for the cases $0 < \beta < 1 < \alpha$ and $1 \leqslant \beta < \alpha$. The case $0 < \beta < 1 < \alpha$ is proved by using the ideas of Section 3.2. In the case $1 \leqslant \beta < \alpha$, by using in addition the arguments of Section 3.3, we prove the convergence (2.12) under two complementary regimes.

4.1. The case $0 < \beta < 1 < \alpha$

The proof in this case is structured like that of Theorem 2.2. Consider, first, symmetric rewards with $c^+ = c^- = \frac{1}{2}$. Let

$$N(T) = T^{1/\beta} M^{1/\beta} L_W^*(MT) \tag{4.1}$$

be the normalization used in Theorem 2.4. Then we need to show (3.10), where Z_β is replaced by Λ_β, that is, the β-stable Lévy motion satisfying (1.16) with $c^+ = c^- = \frac{1}{2}$. Such Lévy motion is characterized by $\mathrm{E}\exp\{\mathrm{i}\sum_{j=1}^d \theta_j \Lambda_\beta(y_j)\} = \exp\{-\sigma^\beta(\boldsymbol{\theta}, \boldsymbol{y})\}$, where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_d) \in \mathbb{R}^d$, $\boldsymbol{y} = (y_1, \ldots, y_d) \in (0, 1]^d$ with $0 < y_1 < \ldots < y_d \leqslant 1$, $d \geqslant 1$, and $\sigma^\beta(\boldsymbol{\theta}, \boldsymbol{y}) = \sigma^\beta$ is given by (1.17). In the notation of Section 3.2 (see, in particular, (3.13)), observe first that, as $T \to \infty$,

$$|\vartheta_k| \leqslant \frac{CT}{T^{1/\beta} M^{1/\beta} L_W^*(MT)} = \frac{C}{T^{1/\beta - 1} M^{1/\beta} L_W^*(MT)} \to 0, \tag{4.2}$$

since $1/\beta - 1 > 0$ and hence $\sup_k |h(\vartheta_k) - 1| \to 0$ (see (3.11)). We will show next that $F \to 0$ as $T \to \infty$, where F is defined in (3.17). Then, as in Section 3.2.1, we will only need to prove that, as $T \to \infty$,

$$M \mathrm{E} C_\beta^{-1} \sum_{k=0}^{\infty} |\vartheta_k|^\beta L_W(|\vartheta_k|^{-1}) \to \sigma^\beta, \tag{4.3}$$

where σ^β is now defined by (1.17). Using (3.13), we can express F in (3.17) as

$$F = \frac{1}{TM} \sum_{k=0}^{\infty} \left| \sum_{j=1}^{d} \theta_j (T_j \wedge S_k - S_{k-1})_+ \right|^\beta F^1(T) F_k^2(T),$$

where

$$F^1(T) = L_W^*(TM)^{-\beta} L_W(T^{1/\beta} M^{1/\beta} L_W^*(TM)),$$

$$F_k^2(T) = \{L_W(T^{1/\beta} M^{1/\beta} L_W^*(TM))\}^{-1} L_W\left(\frac{T^{1/\beta} M^{1/\beta} L_W^*(TM)}{\left|\sum_{j=1}^d \theta_j (T_j \wedge S_k - S_{k-1})_+\right|}\right).$$

By (1.5), $F^1(T) \to 1$ as $T \to \infty$, and by Potter's bounds, for $\delta > 0$, $F_k^2(T) \leqslant 2|\sum_{j=1}^d \theta_j (T_j \wedge S_k - S_{k-1})_+|^{\pm\delta}$ for large enough T, where $a^{\pm\delta} = \max\{a^\delta, a^{-\delta}\}$ for $a > 0$. Then, for large enough T,

$$F \leqslant \frac{C}{MT} \sum_{k=0}^{\infty} \left| \sum_{j=1}^d \theta_j (T_j \wedge S_k - S_{k-1})_+ \right|^{\beta\pm\delta} \leqslant \frac{C}{MT} \sum_{k=0}^{\infty} (T \wedge S_k - S_{k-1})_+^{\beta\pm\delta}. \tag{4.4}$$

Fix $\delta > 0$ such that $\beta \pm \delta \in (0, 1)$. Then, since $(T \wedge S_k - S_{k-1})_+$ is a positive integer, we have $(T \wedge S_k - S_{k-1})_+^{\beta\pm\delta} \leqslant (T \wedge S_k - S_{k-1})_+$ and hence

$$F \leqslant \frac{C}{MT} \sum_{k=0}^{\infty} (T \wedge S_k - S_{k-1})_+ = \frac{C}{MT}\, T = \frac{C}{M}.$$

Since $M = M(T) \to \infty$ as $T \to \infty$, we obtain the convergence $F \to 0$.

To show the convergence (4.3), we study the sum over $k = 0$ and $k \geqslant 1$ separately. We first show that as $T \to \infty$, $M\mathrm{E}|\vartheta_0|^\beta L_W(|\vartheta_0|^{-1}) \to 0$, which is the term with $k = 0$. Arguing as above, we can bound it by

$$M \frac{C}{MT} \mathrm{E} \left| \sum_{j=1}^d \theta_j (T_j \wedge S_0) \right|^{\beta\pm\delta} \leqslant CT^{\beta+\delta-1},$$

for large enough T. The last bound then tends to 0 if we take $\delta > 0$ such that $\beta + \delta < 1$. We now turn to the sum over $k \geqslant 1$. As in Section 3.2 (see (3.6) and (3.7)), this sum equals

$$M\mu^{-1} \sum_{y=0}^{\infty} \sum_{x=0}^{\infty} \left| \sum_{j=1}^d \theta_j (T_j \wedge y - x)_+ \right|^\beta N(T)^{-\beta} L_W\left(\frac{N(T)}{\left|\sum_{j=1}^d \theta_j (T_j \wedge y - x)_+\right|}\right) P(U = y - x). \tag{4.5}$$

The idea now, following Levy and Taqqu (2000), is to introduce four sets of indices and show that the sum over only one of them contributes to the limit as $T \to \infty$. Let

$$\mathcal{A}_1 = \{(i_1, i_1) : i_1 = i_2 = i, 1 \leqslant i \leqslant d + 1\},$$

$$\mathcal{A}_2 = \{(i_1, i_2) : i_1 < i_2 - 1 = i, 1 \leqslant i \leqslant d\},$$

$$\mathcal{A}_3 = \{(i_1, i_2) : i_1 = i_2 - 1 = i, 1 \leqslant i \leqslant d\}$$

and split the sum (4.5) into four terms summing over

$$\sum_{\mathcal{A}_l} \sum_{y=T_{i_2-1}+1}^{T_{i_2}} \sum_{x=T_{i_1-1}+1}^{T_{i_1}}$$

with $l = 1, 2, 3$, and $\sum_{y=0}^{\infty} 1_{\{x=0\}}$, denoted by J_l, $l = 1, 2, 3, 4$ (we let $T_0 = 0$ and $T_{d+1} = \infty$). In view of the identity (3.7), the indices x and y correspond to variables S_{k-1} and S_k, respectively. Hence, the terms J_1, J_2 and J_3 concern the cases when S_{k-1} and S_k belong to the same interval $(T_{i-1}, T_i]$, two non-adjacent intervals $(T_{i-1}, T_i]$ and $(T_{j-1}, T_j]$, and two consecutive intervals $(T_{i-1}, T_i]$ and $(T_i, T_{i+1}]$, respectively. Since, as T_i increases with T, S_{k-1} and S_k are more likely to fall in the same interval $(T_{i-1}, T_i]$, we expect that only the term J_1 contributes to the limit.

Consider, first, J_l with $l = 2, 3$. As in the case $k = 0$, for $\beta \pm \delta \in (0, 1)$ and large enough T,

$$J_l \leqslant M \frac{C}{MT} \sum_{\mathcal{A}_l} \sum_{y=T_{i_2-1}+1}^{T_{i_2}} \sum_{x=T_{i_1-1}+1}^{T_{i_1}} \left| \sum_{j=1}^{d} \theta_j (T_j \wedge y - x)_+ \right|^{\beta \pm \delta} P(U = y - x).$$

The convergence of the bound to 0 follows from Propositions 5.3 and 5.4 in Levy and Taqqu (2000). For J_4, we similarly have that

$$J_4 \leqslant \frac{C}{T} \sum_{y=0}^{\infty} \left| \sum_{j=1}^{d} \theta_j (T_j \wedge y) \right|^{\beta \pm \delta} P(U = y) \leqslant C T^{\beta + \delta - 1} \to 0,$$

as $T \to \infty$, as long as $\beta + \delta < 1$.

To conclude the proof, we still need to show that the difference between J_1 and $C_\beta \sigma^\beta$ tends to 0. By the definition of $\mathcal{A}_1$, J_1 equals

$$M\mu^{-1} \sum_{i=1}^{d} \sum_{y=T_{i-1}+1}^{T_i} \sum_{x=T_{i-1}+1}^{T_i} \left| \sum_{j=1}^{d} \theta_j (T_j \wedge y - x)_+ \right|^{\beta} N(T)^{-\beta}$$
$$\times L_W \left(\frac{N(T)}{\left| \sum_{j=1}^{d} \theta_j (T_j \wedge y - x)_+ \right|} \right) P(U = y - x).$$

Observe now that, for $T_{i-1} + 1 \leqslant x, y \leqslant T_i$, $i = 1, \ldots, d$,

$$\sum_{j=1}^{d} \theta_j (T_j \wedge y - x)_+ = \sum_{j=i}^{d} \theta_j (T_j \wedge y - x)_+ = \sum_{j=i}^{d} \theta_j (y - x)_+ = \phi_i (y - x)_+,$$

and hence J_1 becomes

$$M\mu^{-1}\sum_{i=1}^{d}\sum_{y=T_{i-1}+1}^{T_i}\sum_{x=T_{i-1}+1}^{T_i}|\phi_i|^\beta(y-x)_+^\beta N(T)^{-\beta}L_W\left(\frac{N(T)}{|\phi_i|(y-x)_+}\right)P(U=y-x),$$

or, by making a simple change of variables,

$$M\mu^{-1}\sum_{i=1}^{d}\sum_{x=1}^{T_i-T_{i-1}-1}\sum_{u=1}^{x}|\phi_i|^\beta u^\beta N(T)^{-\beta}L_W\left(\frac{N(T)}{|\phi_i|u}\right)P(U=u).$$

By fixing $u_0 > 0$, we can then bound $|J_1 - C_\beta\sigma^\beta|$ by the sum

$$\begin{aligned}
&\frac{\mu^{-1}}{T}\sum_{i=1}^{d}|\phi_i|^\beta\sum_{x=1}^{T_i-T_{i-1}-1}\sum_{u=1}^{x}u^\beta F_{i,u}(T)P(U=u)1_{\{u>u_0\}}\\
&\quad+\frac{\mu^{-1}}{T}\sum_{i=1}^{d}|\phi_i|^\beta\sum_{x=1}^{T_i-T_{i-1}-1}\sum_{u=1}^{x}u^\beta|F_{i,u}(T)-1|P(U=u)1_{\{u\leqslant u_0\}}\\
&\quad+\frac{\mu^{-1}}{T}\sum_{i=1}^{d}|\phi_i|^\beta\sum_{x=1}^{T_i-T_{i-1}-1}\sum_{u=1}^{x}u^\beta P(U=u)1_{\{u>u_0\}}\\
&\quad+\left|\frac{\mu^{-1}}{T}\sum_{i=1}^{d}|\phi_i|^\beta\sum_{x=1}^{T_i-T_{i-1}-1}\sum_{u=1}^{x}u^\beta P(U=u)-C_\beta\sigma^\beta\right|,
\end{aligned}$$

where

$$F_{i,u}(T)=(L_W^*(MT))^{-\beta}L_W\left(\frac{M^{1/\beta}T^{1/\beta}L_W^*(MT)}{|\phi_i|u}\right).$$

Denote the four terms in the bound by $J_{1,1}$, $J_{1,2}$, $J_{1,3}$ and $J_{1,4}$, respectively. Then, using (1.5) and Potter's bounds as before,

$$J_{1,1}\leqslant\frac{C}{T}\sum_{i=1}^{d}|\phi_i|^{\beta\pm\delta}\sum_{x=1}^{T_i-T_{i-1}-1}\sum_{u=u_0+1}^{\infty}u^{\beta\pm\delta}P(U=u)\leqslant C\,\mathrm{E}U^{\beta\pm\delta}1_{\{U>u_0\}}.$$

By taking $\delta > 0$ such that $\beta\pm\delta<\alpha$, $J_{1,1}$ can be made arbitrarily small for big enough u_0. The same conclusion is true for $J_{1,3}$. As for $J_{1,2}$, its convergence to 0 will follow from that of $J_{1,4}$ because, assuming a fixed u_0, $F_{i,u}(T)\to 1$ uniformly for $u\leqslant u_0$ and $i=1,\ldots,d$ (see (1.5)). Finally, the convergence $J_{1,4}\to 0$ is established in Levy and Taqqu (2000) or can be obtained directly.

Turning to the case of asymmetric rewards, observe first that the β-stable Lévy motion Λ_β in (1.16) is now characterized by its characteristic function with (1.17) and (1.18). Using the ideas of the proof of Theorem 2.2 in the case of asymmetric rewards and also the proof in the case of symmetric rewards above, it is enough to show (3.28) and (3.29), where $\sigma^\beta=\sigma^\beta(\boldsymbol{\theta}, \boldsymbol{y})$ and $\zeta=\zeta(\boldsymbol{\theta}, \boldsymbol{y})$ are now defined by (1.17) and (1.18), respectively. The convergence (3.28) has been established in the case of symmetric rewards above. The proof of (3.29) is similar to that of (3.28).

4.2. The case $1 \leqslant \beta < \alpha$

We cannot take advantage here of the relation (4.2) since $1 \leqslant \beta$. We will prove the convergence (2.12) by considering two cases: for large enough T,

$$M^{1/\beta} T^{1/\beta - 1} L_W^*(MT) \geqslant T^\rho \tag{4.6}$$

and

$$M^{1/\beta} T^{1/\beta - 1} L_W^*(MT) \leqslant T^\rho, \tag{4.7}$$

where $\rho > 0$ is small and will be chosen below. This will establish the convergence (2.12) in general as $T \to \infty$.

Suppose that $\rho > 0$ in (4.6) is fixed. Then the proof is analogous to the case $0 < \beta < 1 < \alpha$. Suppose, for simplicity, that the rewards are symmetric. Observe that, with ϑ_k defined in (3.13), we still have, by (4.6),

$$|\vartheta_k| \leqslant C(M^{1/\beta} T^{1/\beta - 1} L_W^*(MT))^{-1} \leqslant CT^{-\rho} \to 0,$$

as $T \to \infty$. Moreover, with F defined in (3.17), we have, by (4.4),

$$F \leqslant \frac{C}{MT} \sum_{k=0}^{\infty} (T \wedge S_k - S_{k-1})_+^{\beta+\delta},$$

where $\delta > 0$ can be taken arbitrarily small, and by writing $(T \wedge S_k - S_{k-1})_+^{\beta+\delta} \leqslant (T \wedge S_k - S_{k-1})_+ T^{\beta+\delta-1}$ and using $\sum_k (T \wedge S_k - S_{k-1})_+ = T$ we conclude that $F \leqslant CM^{-1} T^{\beta+\delta-1}$. It follows from (4.6) that, for any $\epsilon > 0$ and large enough T, $M^{1/\beta} T^{1/\beta-1} (MT)^\epsilon \geqslant M^{1/\beta} T^{1/\beta-1} L_W^*(MT) \geqslant T^\rho$ or, after elementary calculations, $MT^{1-\beta} \geqslant T^{\beta(\rho-\epsilon\beta)/(1+\epsilon\beta)}$. Then $F \leqslant CT^{\delta-\beta(\rho-\epsilon\beta)/(1+\epsilon\beta)}$ and, by taking small enough $\delta, \epsilon > 0$, we obtain $F \to 0$ as $T \to \infty$. Therefore it is enough to show the convergence (4.3). This can be done as in the case $0 < \beta < 1 < \alpha$ by using results of Levy and Taqqu (2000).

We turn now to the situation in which (4.7) holds; $\rho > 0$ will be chosen below. The proof uses ideas from Section 3.3. Set $N(T) = T^{1/\beta} M^{1/\beta} L_W^*(MT)$ as in (4.1). Let $a(t) = (1/G_W)^{\leftarrow}(t)$, where $G_W(w) = w^{-\beta} L_W(w)$. Then, as in (2.11),

$$a(MT) \sim T^{1/\beta} M^{1/\beta} L_W^*(MT), \tag{4.8}$$

as $T \to \infty$. The function a should not be confused with the function b defined in Section 2, which satisfies $b(MT) \sim T^{1/\alpha} M^{1/\alpha} L_U^*(MT)$.

We first prove the convergence (2.12) at time $y = 1$. Using (3.30), we need to show the three steps of Section 3.3, where $\Lambda_\alpha(1)$ is replaced by $\Lambda_\beta(1)$.

1. To show $N(T)^{-1} A_1(T) \to 0$ in probability, it is enough to prove that, for some $p \in (0, 1)$, $N(T)^{-p} \mathrm{E}|A_1(T)|^p \to 0$. Using the inequality $(\sum_m c_m)^p \leqslant \sum_m c_m^p$ valid for $p \in (0, 1)$, $c_m > 0$, and $\mathrm{E}\min(T, S_0)^p \leqslant CT^{p-\alpha+1} L_U(T)$ (argue as in (3.31)), we have

$$\frac{E|A_1(T)|^p}{N(T)^p} \leqslant \frac{1}{N(T)^p} E\left(\sum_{m=1}^{M} |W_0^m| \min(T, S_0^m)\right)^p \leqslant \frac{1}{N(T)^p} E \sum_{m=1}^{M} |W_0^m|^p \min(T, S_0^m)^p$$

$$= \frac{M E \min(T, S_0)^p E|W|^p}{M^{p/\beta} T^{p/\beta} L_W^*(MT)^p} \leqslant C \frac{M T^{p-\alpha+1} L_U(T)}{M^{p/\beta} T^{p/\beta} L_W^*(MT)^p} \leqslant C M^{1-p/\beta+\delta_1} T^{p-\alpha+1-p/\beta+\delta_2},$$

where $\delta_1, \delta_2 > 0$ can be taken arbitrarily small. (One cannot take $p = 1$ above because, when $\beta = 1$, it may happen that $E|W| = \infty$.) By assumption (4.7), there exists $\epsilon > 0$ such that $M \leqslant T^\epsilon$ for large enough T. Then $N(T)^{-p} E|A_1(T)|^p \to 0$ as long as $\epsilon(1 - p/\beta + \delta_1) + (p - \alpha + 1 - p/\beta + \delta_2) < 0$. This last condition is clearly satisfied by taking small enough $\delta_1, \delta_2 > 0$ and p close to 1.

2. As in step 2 of Section 3.3, we need to verify conditions (i), (ii) and (iii) of Petrov (1975), where EW_+^α and EW_-^α are now replaced by $c^+ EU^\beta$ and $c^- EU^\beta$, respectively. The key observation in proving (i) is as follows: using (2.11) and (4.7), for large enough T,

$$\frac{b(MT)}{T} \sim M^{1/\alpha} T^{1/\alpha - 1} L_U^*(MT) = M^{1/\beta} T^{1/\beta - 1} L_W^*(MT)(MT)^{1/\alpha - 1/\beta} \frac{L_U^*(MT)}{L_W^*(MT)}$$

$$\leqslant T^\rho (MT)^{1/\alpha - 1/\beta} \frac{L_U^*(MT)}{L_W^*(MT)} \leqslant (MT)^{\rho + 1/\alpha - 1/\beta} \frac{L_U^*(MT)}{L_W^*(MT)} \to 0, \tag{4.9}$$

as long as $\rho < (\alpha - \beta)/\alpha\beta$. This means that the slow-growth condition (2.2) is satisfied. Consequently, by Lemma 4 in Mikosch *et al.* (2002), there exists $\epsilon_T \to 0$ satisfying (3.32) such that $MP(|\xi_T - \mu_T| > \epsilon_T \mu_T) = o(1)$ as $T \to \infty$, where ξ_T is the total number of renewals in [0, T] and $\mu_T = E\xi_T$, as in Section 3.3. Then, arguing as in Section 3.3, one can show that, as $T \to \infty$,

$$MP(S(\xi_T) > xa(MT)) = MP\left(\sum_{k=1}^{\xi_T} Y_k > xa(MT)\right) \sim MP\left(\sum_{k=1}^{[\mu_T]} Y_k > xa(MT)\right),$$

that is, ξ_T can be replaced by its mean μ_T in the limit. Recall that the Y_k above are independent and have the same distribution as $Y = WU$. By Lemma 1.1, we have

$$\bar{F}_Y(y) = P(Y > y) \sim c^+ EU^\beta y^{-\beta} L_W(y) = c^+ EU^\beta G_W(y), \tag{4.10}$$

and similarly $1 - \bar{F}_Y(-y) = P(Y \leqslant -y) \sim c^- EU^\beta G_W(y)$, as $y \to \infty$. Then, by applying Corollary A.1 in Appendix A, we obtain that, as $T \to \infty$,

$$MP\left(\sum_{k=1}^{[\mu_T]} Y_k > xa(MT)\right) \sim M[\mu_T]\bar{F}_Y(xa(MT)) \sim c^+ \mu^{-1} EU^\beta x^{-\beta} MT G_W(a(MT)). \tag{4.11}$$

The part (i) then follows because, by Theorem 1.5.12 in Bingham *et al.* (1987),

$$MT G_W(a(MT)) \sim 1. \tag{4.12}$$

The proof of (ii) is similar to that of (i). Part (iii) can be proved similarly to part (iii) in Section 3.3 by using arguments of type (4.11).

3. We will show that $A_3(T)$ in (3.30) tends to zero in probability. Using (3.35), it is

enough to show that $a(MT)^{-p}\mathrm{E}|A_{3,1}(T)|^p \to 0$ for some $p \in (0, 1)$ and $a(MT)^{-2}\mathrm{E}|A_{3,2}(T)|^2 \to 0$, as $T \to \infty$. It follows from (3.35) that

$$a(MT)^{-p}\mathrm{E}|A_{3,1}(T)|^p \leqslant Ma(MT)^{-p}\mathrm{E}|Y_{\xi_T}|^p 1_{\{|Y_{\xi_T}|>a(MT)\}}1_{\{\xi_T\geqslant 1\}},$$

where $Y_{\xi_T} = U_{\xi_T}W_{\xi_T}$. To show that the bound tends to 0, we again modify the arguments of the proof of Lemma 5 in Mikosch *et al.* (2002). First, using Karamata's theorem,

$$\frac{M}{a(MT)^p}\int_{a(MT)}^{\infty} x^{p-1}P(|Y_{\xi_T}| > x, |\xi_T - \mu_T| \leqslant \epsilon_T\mu_T, \xi_T \geqslant 1)\mathrm{d}x$$

$$\leqslant \frac{2M\epsilon_T\mu_T}{a(MT)^p}\int_{a(MT)}^{\infty} x^{p-1}P(|Y| > x)\mathrm{d}x \leqslant \frac{CMT\epsilon_T}{a(MT)^p}a(MT)(a(MT))^{p-1}\overline{F}_{|Y|}(a(MT)),$$

where $\overline{F}_{|Y|}(y) = P(|Y| > y)$ is a regularly varying function at infinity. Then, using $\overline{F}_{|Y|} \sim CG_W$ (see (4.10)) and (4.12), the bound above behaves (up to a constant) like ϵ_T and hence tends to 0, as $T \to \infty$. One then needs to show that

$$\frac{M}{a(MT)^p}I := \frac{M}{a(MT)^p}\int_{a(MT)}^{\infty} x^{p-1}P(|Y_{\xi_T}| > x, |\xi_T - \mu_T| > \epsilon_T\mu_T, \xi_T \geqslant 1)\mathrm{d}x \to 0. \quad (4.13)$$

Since the slow-growth condition (4.9) holds, one may choose (see the proofs of Lemmas 4 and 5 in Mikosch *et al.* 2002) $c_T \to \infty$ such that $b(MT)/c_T^{-1}\epsilon_T T \to 0$ and $MP(|\xi_T - \mu_T| > \epsilon_T\mu_T) = o(c_T^{-\alpha})$. Then there also exists a big enough $K > 0$ such that $T^K > c_T a(MT)$ for large T. This can be seen from

$$c_T a(MT) = \frac{b(MT)}{c_T^{-1}\epsilon_T T}\frac{a(MT)}{b(MT)}\epsilon_T T \leqslant C\frac{a(MT)}{b(MT)}T \leqslant C(MT)^{\delta_1}T \leqslant CT^{\delta_2}$$

for some $\delta_1, \delta_2 > 0$ (the last inequality here follows from $MT \leqslant T^p$ for some $p > 0$, which is a consequence of assumption (4.7)). Then one can bound the integral I in (4.13) as in Lemma 5 in Mikosch *et al.* (2002), by

$$I \leqslant \int_{a(MT)}^{c_T a(MT)} P(|\xi_T - \mu_T| > \epsilon_T\mu_T)\mathrm{d}x + \int_{c_T a(MT)}^{\infty} x^{p-1}P(|Y_{\xi_T}| > x, 1 \leqslant \xi_T < (1-\epsilon_T)\mu_T)\mathrm{d}x$$

$$+ \int_{c_T a(MT)}^{T^K} P(\xi_T > (1+\epsilon_T)\mu_T)\mathrm{d}x + \int_{T^K}^{\infty} x^{p-1}P(|Y_{\xi_T}| > x, \xi_T \geqslant 1)\mathrm{d}x =: I_1 + I_2 + I_3 + I_4.$$

The convergence $Ma(MT)^{-p}I_k$, $k = 1, 3$, can be shown as in Mikosch *et al.* (2002). For I_2, by Karamata's theorem, the relation $MT \sim G_W(a(MT)) \sim C\overline{F}_{|Y|}(a(MT))$ and Potter's bounds, we have

$$\frac{M}{a(MT)^p} I_2 = \frac{M}{a(MT)^p} \int_{c_T a(MT)}^{\infty} x^{p-1} P(|Y_{\xi_T}| > x, \, \xi_T < (1-\epsilon_T)\mu_T, \, \xi_T \geqslant 1) \mathrm{d}x$$

$$\leqslant \frac{C\, MT}{a(MT)^p} \int_{c_T a(MT)}^{\infty} x^{p-1} P(|Y| > x) \mathrm{d}x \leqslant \frac{C\, MT}{a(MT)^p} (c_T a(MT))^p \overline{F}_{|Y|}(c_T a(MT))$$

$$\leqslant C\, c_T^p \frac{\overline{F}_{|Y|}(c_T a(MT))}{\overline{F}_{|Y|}(a(MT))} \leqslant C\, c_T^p c_T^{-\beta+\epsilon} \to 0,$$

for small $\epsilon > 0$, since $p < 1$ (taking $p = 1$ would not be enough when $\beta = 1$). As for I_4, we obtain, for small enough $\epsilon > 0$,

$$\frac{M}{a(MT)^p} I_4 = \frac{M}{a(MT)^p} \int_{T^K}^{\infty} x^{p-1} P(|Y_{\xi_T}|^{\beta-\epsilon} > x^{\beta-\epsilon}, \, \xi_T \geqslant 1) \mathrm{d}x$$

$$\leqslant \frac{M}{a(MT)^p} \int_{T^K}^{\infty} x^{p-1} P(\tilde{S}_{\xi_T} > x^{\beta-\epsilon}) \mathrm{d}x \leqslant \frac{M}{a(MT)^p} \mathrm{E}\tilde{S}_{\xi_T} \int_{T^K}^{\infty} x^{p-1} x^{\epsilon-\beta} \, \mathrm{d}x,$$

where $\tilde{S}_0 = 0$, $\tilde{S}_n = |Y_1|^{\beta-\epsilon} + \ldots + |Y_n|^{\beta-\epsilon}$, $n \geqslant 1$, by Markov's inequality. The last integral is finite since $p < 1$. Since $\{\tilde{S}_n - nE|Y_1|^{\beta-\epsilon}\}_{n \geqslant 0}$ is a martingale and ξ_T is a stopping time with respect to the filtration $\mathcal{F}_0 = \{\varnothing, \Omega\}$, $\mathcal{F}_n = \sigma\{U_0, U_1, W_1, \ldots, U_n, W_n\}$, $n \geqslant 1$, it follows from the optional sampling theorem (apply, for example, Theorem 8 and Proposition 10 in Section 24.5 of Fristedt and Gray 1997) that

$$\mathrm{E}\tilde{S}_{\xi_T} = \mathrm{E}\xi_T \mathrm{E}|Y_1|^{\beta-\epsilon} = (T+1)\mu^{-1}\mathrm{E}|Y_1|^{\beta-\epsilon}.$$

Then, for some $\delta > 0$,

$$\frac{M}{a(MT)^p} I_4 \leqslant C \frac{MT}{a(MT)^p} T^{-K(\beta-p-\epsilon)} \leqslant C (MT)^{\delta} T^{-K(\beta-p-\epsilon)} \leqslant C T^{(\delta_0+1)\delta - K(\beta-p-\epsilon)}$$

since, by assumption (4.6), there exists $\delta_0 > 0$ such that $M \leqslant T^{\delta_0}$ for large enough T. Since $p < 1$, by taking K large enough, we obtain $Ma(MT)^{-p} I_4 \to 0$. The convergence $a(MT)^{-2}\mathrm{E}|A_{3,2}(T)|^2 \to 0$ can be proved as in Section 3.3.

Finally, to prove the convergence of the finite-dimensional distributions, proceed as in Lemmas 11 and 12 in Mikosch *et al.* (2002) (see also the end of Section 3.3).

Appendix A. Large deviations of heavy-tailed sums

We provide here the result on large deviations of heavy-tailed sums which was used earlier in this work. The presentation below expands on that of Appendix A in Mikosch *et al.* (2002). Consider a sequence of i.i.d. random variables Z, Z_n, $n \geqslant 1$, such that, as $z \to \infty$,

$$F(-z) = P(Z \leqslant -z) \sim c_1 z^{-\alpha} L(z), \qquad \overline{F}(z) = P(Z > z) \sim c_2 z^{-\alpha} L(z), \qquad \text{(A.1)}$$

with $c_1 + c_2 > 0$ ($c_1, c_2 \geqslant 0$), $\alpha \in (0, 2)$ and a slowly varying (at infinity) function L. Observe that in (A.1) the left and right tails involve the same value of α. We will treat the cases $c_2 = 0$ and $c_2 \neq 0$. Set

$$S_n = Z_1 + \ldots + Z_n, \qquad n \geqslant 1,$$

and

$$\mu_2(z) = z^{-2}\mathrm{E}Z^2 1_{\{|Z|\leqslant z\}} = z^{-2}\int_{|u|\leqslant z} u^2\,\mathrm{d}F(u).$$

The following large-deviation result is proved in Theorem 2.1 of Cline and Hsing (1989).

Theorem A.1. *Let $\beta_n \to \infty$ be such that $S_n/\beta_n \to_P 0$. Suppose $B_n \subset [\beta_n, \infty)$. If $c_2 \neq 0$ in* (A.1) *and the condition*

$$\lim_{n\to\infty}\sup_{z\in B_n}\left| n\mu_2(z)\ln(n\overline{F}(z))\right| = 0 \tag{A.2}$$

holds, then

$$\lim_{n\to\infty}\sup_{z\in B_n}\left|\frac{P(S_n > z)}{n\overline{F}(z)} - 1\right| = 0. \tag{A.3}$$

Corollary A.1 below is first used in (3.34).

Corollary A.1. *Let $\alpha \in (0, 2)$ and Z_n, $n \geqslant 1$, be a sequence of i.i.d. random variables with a common distribution satisfying* (A.1). *When $\alpha = 1$ assume that Z is symmetric, and when $\alpha \in (1, 2)$ suppose that $\mathrm{E}Z = 0$. Then:*

(i) *if $c_2 \neq 0$, relation* (A.2) *holds with $\beta_n = a_n h_n$, where $h_n \to \infty$ and a sequence (a_n) satisfies $n\overline{F}(a_n) \sim 1$;*
(ii) *if $c_2 = 0$,*

$$\lim_{n\to\infty}\sup_{z\in B_n}\frac{P(S_n > z)}{nz^{-\alpha}L(z)} = 0 \tag{A.4}$$

with $\beta_n = a_n h_n$, where $h_n \to \infty$ and the sequence a_n satisfies $na_n^{-\alpha}L(a_n) \sim 1$.

Proof. (i) $c_2 \neq 0$. Since $a_n^{-1}S_n$ converges to an α-stable random variable and $h_n \to \infty$, we have $S_n/\beta_n = (S_n/a_n)\,h_n^{-1} \to_P 0$. Now, by writing $\mu_2(z) = 2z^{-2}\int_0^z uP(|Z| > u)\mathrm{d}u$ and using Karamata's theorem, we have

$$\mu_2(z) \leqslant CP(|Z| > z), \qquad z > 0.$$

Moreover, using Potter's bounds and $n\overline{F}(a_n) \sim 1$, we have, for small enough $\epsilon > 0$,

$$nP(|Z| > \beta_n) \leqslant Cn\beta_n^{-\alpha}L(\beta_n) \sim Cn\overline{F}(a_n)h_n^{-\alpha}\frac{L(a_n h_n)}{L(a_n)} \leqslant Ch_n^{-\alpha+\epsilon} \to 0.$$

It follows that (A.2) is satisfied because, for $z \in B_n$,

$$n\mu_2(z)\ln(n\overline{F}(z)) \leqslant CnP(|Z| > \beta_n)\ln(nP(|Z| > \beta_n)) \to 0.$$

(ii) $c_2 = 0$. Since $\lim_{z\to\infty}\overline{F}(z)/(z^{-\alpha}L(z)) = 0$, there is a sequence of i.i.d. random variables $\tilde{Z}$, $\tilde{Z}_n$, $n \geqslant 1$, such that

$$P(Z > z) \leqslant P(\tilde{Z} > z), \qquad \text{for all } z \in \mathbb{R},$$

and $P(\tilde{Z} \leqslant -z) \sim c_1 z^{-\alpha}L(z)$, $P(\tilde{Z} > z) \sim \tilde{c}_2 z^{-\alpha}L(z)$, as $z \to \infty$, where $\tilde{c}_2 > 0$ is arbitrary small. In the case $\alpha \in (1, 2)$, we may also choose $\tilde{Z}$ such that $\mathrm{E}\tilde{Z} = 0$. Setting $\tilde{S}_n = \tilde{Z}_1 + \ldots + \tilde{Z}_n$, $n \geqslant 1$, and using stochastic domination (see, for example, Corollary 3.1 of Chapter 1 in Thorisson 2000), we obtain $P(S_n > z) \leqslant P(\tilde{S}_n > z)$ for all $n \in \mathbb{N}$, $z \in \mathbb{R}$, and hence

$$\frac{P(S_n > z)}{nz^{-\alpha}L(z)} \leqslant \frac{P(\tilde{S}_n > z)}{nz^{-\alpha}L(z)} \sim \tilde{c}_2 \frac{P(\tilde{S}_n > z)}{nP(\tilde{Z} > z)}. \tag{A.5}$$

Applying the proof in the case $c_2 \neq 0$, we conclude that the right-hand side of (A.5) tends (uniformly for $z \in B_n$) to $\tilde{c}_2$. Since $\tilde{c}_2$ can be taken arbitrarily small, we obtain (A.4). □

Appendix B. Weak convergence

We show here the weak convergence of the total reward process in the space of functions $D[0, 1]$ when the limit process is either fractional Brownian motion B_H (Theorem 2.1) or the stable self-similar process with stationary dependent increments Z_β (Theorem 2.2). Recall that $D[0, 1]$ is the space of right-continuous functions defined on $[0, 1]$ which have limits from the left. We will suppose that $D[0, 1]$ is equipped with the usual Skorokhod J_1 topology. See Billingsley (1968) for more information on this function space and the J_1 topology.

Theorem B.1. *The convergence* (2.3) *and* (2.4) *of the normalized total reward process* $W^*(T\cdot, M)$ *in Theorems* 2.1 *and* 2.2, *respectively, extends to the weak convergence in the space* $D[0, 1]$ *equipped with the* J_1 *topology.*

Remark. Since the limiting processes in Theorem B.1 are almost surely continuous, the convergence in the Skorokhod J_1 topology topology can be replaced by convergence in the uniform topology generated by the uniform metric $\rho(f, g) = \sup_{y\in[0,1]}|f(y) - g(y)|$ (see Billingsley 1968, p. 151).

Proof. We first give a proof of the weak convergence in (2.4), which is slightly more involved.

Weak convergence for (2.4). Let $N(T)$ be the normalization (3.8) used in Theorem 2.2. Since $P(Z_\beta(1) \neq Z_\beta(1-)) = 0$ (the process Z_β has continuous paths; see Pipiras and Taqqu 2000) and since one already has the convergence (2.4) of $W^*(T\cdot, M)/N(T)$ to Z_β in the sense of the finite-dimensional distributions, by Theorem 15.6 in Billingsley (1968), it is enough to show that there exist $c, \epsilon, \gamma > 0$ and $T_0 \geqslant 1$ such that

$$P\left(\left|\frac{W^*(Ty, M)}{N(T)} - \frac{W^*(Ty_1, M)}{N(T)}\right| \geqslant \lambda, \left|\frac{W^*(Ty_2, M)}{N(T)} - \frac{W^*(Ty, M)}{N(T)}\right| \geqslant \lambda\right) \leqslant \frac{c}{\lambda^\gamma}|y_2 - y_1|^{1+\epsilon}, \tag{B.1}$$

for all $T \geqslant T_0$ and $0 \leqslant y_1 < y \leqslant y_2 \leqslant 1$. Using the inequality $P(A \cap B) \leqslant P(A)^{1/2}P(B)^{1/2}$, the stationarity of the increments of $W^*(T\cdot, M)$ and the inequality $4ab \leqslant (a+b)^2$, one can see that (B.1) follows from

$$P\left(\left|\frac{W^*(Ty, M)}{N(T)}\right| \geqslant \lambda\right) \leqslant \frac{c}{\lambda^\gamma}\left(\frac{[Ty]}{T}\right)^{1+\epsilon}, \tag{B.2}$$

for all $T \geqslant T_0$, $0 \leqslant y \leqslant 1$. Observe next that, by (7.15) in Billingsley (1968, p. 47),

$$P\left(\left|\frac{W^*(Ty, M)}{N(T)}\right| \geqslant \lambda\right) \leqslant \frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda}|1 - \mathrm{E}\exp\{\mathrm{i}\theta W^*(Ty, M)/N(T)\}|\mathrm{d}\theta. \tag{B.3}$$

As in the proof of Theorem 2, one can bound the integrand in (B.3) by

$$|1 - \mathrm{E}\exp\{\mathrm{i}\theta W^*(Ty, M)/N(T)\}| \leqslant M\sum_{k=0}^{\infty}\mathrm{E}\left|1 - \phi_W\left(\frac{\theta([Ty] \wedge S_k - S_{k-1})_+}{N(T)}\right)\right|, \tag{B.4}$$

where $\phi_W(u) = \mathrm{E}\exp\{\mathrm{i}uW\}$ is the characteristic function of W. Since W is in the domain of attraction of a β-stable random variable, one can show that, for all $u \in \mathbb{R}$ and some constant $c > 0$,

$$|1 - \phi_W(u)| \leqslant c|u|^\beta L_W(|u|^{-1}). \tag{B.5}$$

By applying (B.5), one can bound (B.4) by

$$c|\theta|^\beta M\sum_{k=0}^{\infty}\mathrm{E}\left(\frac{([Ty] \wedge S_k - S_{k-1})_+^\beta}{N(T)^\beta}L_W\left(\frac{N(T)}{|\theta|([Ty] \wedge S_k - S_{k-1})_+}\right)\right). \tag{B.6}$$

With the notation of (3.9) and in view of (3.8), the term in the expectation in (B.6) becomes

$$\frac{([Ty] \wedge S_k - S_{k-1})_+^\beta}{T^\beta Q(T)}L_W^*(Q(T))^{-\beta}L_W\left(Q(T)^{1/\beta}L_W^*(Q(T))\frac{T}{|\theta|([Ty] \wedge S_k - S_{k-1})_+}\right). \tag{B.7}$$

Since, by (3.9), $Q(T) \to \infty$ as $T \to \infty$, we obtain from (1.5), for large enough T,

$$L_W^*(Q(T))^{-\beta}L_W(Q(T)^{1/\beta}L_W^*(Q(T))) \leqslant 2. \tag{B.8}$$

Since $0 \leqslant y \leqslant 1$ and $|\theta| \leqslant 2/\lambda$ in (B.3), we obtain

$$\frac{T}{|\theta|([Ty] \wedge S_k - S_{k-1})_+} \geqslant \frac{T}{|\theta|[Ty]} \geqslant \frac{\lambda}{2}.$$

Then, since $Q(T) \to \infty$, by applying Potter's bounds, we obtain, for large enough T,

$$\left\{L_W(Q(T)^{1/\beta}L_W^*(Q(T)))\right\}^{-1}L_W\left(Q(T)^{1/\beta}L_W^*(Q(T))\frac{T}{|\theta|([Ty]\wedge S_k-S_{k-1})_+}\right)$$
$$\leqslant 2\left(\frac{|\theta|([Ty]\wedge S_k-S_{k-1})_+}{T}\right)^{-\delta}, \tag{B.9}$$

where $\delta > 0$ is fixed. Using (B.6)–(B.9) to bound (B.4), we obtain that there exists $T_0 \geqslant 1$ such that, for all $T \geqslant T_0$,

$$|1-\mathrm{E}\exp\{\mathrm{i}\theta W^*(Ty, M)/N(T)\}| \leqslant \text{ const.}|\theta|^{\beta-\delta}\sum_{k=0}^{\infty}\frac{\mathrm{E}([Ty]\wedge S_k-S_{k-1})_+^{\beta-\delta}}{T^{\beta-\delta-\alpha+1}l_U(T)}. \tag{B.10}$$

Now let $\delta > 0$ be such that $\alpha < \beta - \delta$. By (3.6) and (3.7), the function $f(u) = \sum_{k=0}^{\infty}\mathrm{E}(u\wedge S_k - S_{k-1})_+^{\beta-\delta}$, $u > 0$, is regularly varying with index $\beta-\delta-\alpha+1$ and the slowly varying function l_U. Hence, using Potter's bounds again, there exists u_0 such that, for $u, v \geqslant u_0$,

$$\frac{f(u)}{f(v)} \leqslant 2\max\left\{\left(\frac{u}{v}\right)^{\beta-\delta-\alpha+1-\epsilon}, \left(\frac{u}{v}\right)^{\beta-\delta-\alpha+1+\epsilon}\right\},$$

where $\epsilon > 0$ is fixed. It follows that, if $[Ty] \geqslant u_0$ (and $T \geqslant [Ty] \geqslant u_0$ as well), then

$$\sum_{k=0}^{\infty}\frac{\mathrm{E}([Ty]\wedge S_k-S_{k-1})_+^{\beta-\delta}}{T^{\beta-\delta-\alpha+1}l_U(T)} \leqslant \text{ const. }\frac{f([Ty])}{f(T)} \leqslant \text{ const.}\left(\frac{[Ty]}{T}\right)^{\beta-\delta-\alpha+1-\epsilon}. \tag{B.11}$$

If $[Ty] \leqslant u_0$ and $\epsilon > 0$ is such that $\beta-\delta-\alpha-\epsilon > 0$, then, since one can have at most $[Ty]$ renewals in the time interval $[0, [Ty]]$,

$$\sum_{k=0}^{\infty}\frac{\mathrm{E}([Ty]\wedge S_k-S_{k-1})_+^{\beta-\delta}}{T^{\beta-\delta-\alpha+1}l_U(T)} \leqslant \frac{[Ty][Ty]^{\beta-\delta}}{T^{\beta-\delta-\alpha+1}l_U(T)} = \frac{T^{1+\epsilon}[Ty]^{\beta-\delta-\epsilon}}{T^{\beta-\delta-\alpha+1}l_U(T)}\left(\frac{[Ty]}{T}\right)^{1+\epsilon}$$
$$\leqslant \frac{u_0^{\beta-\delta-\epsilon}}{T^{\beta-\delta-\alpha-\epsilon}l_U(T)}\left(\frac{[Ty]}{T}\right)^{1+\epsilon} \leqslant \text{ const.}\left(\frac{[Ty]}{T}\right)^{1+\epsilon}. \tag{B.12}$$

By taking $\epsilon > 0$ such that $1+\epsilon < \beta-\delta-\alpha+1-\epsilon$ (or $0 < \beta-\delta-\alpha-2\epsilon$) and bounding (B.10) by (B.11) and (B.12), we obtain, for $T \geqslant T_0$ and all $0 \leqslant y \leqslant 1$,

$$|1-\mathrm{E}\exp\{\mathrm{i}\theta W^*(Ty, M)/N(T)\}| \leqslant \text{ const.}|\theta|^{\beta-\delta}\left(\frac{[Ty]}{T}\right)^{1+\epsilon}. \tag{B.13}$$

By substituting (B.13) into (B.3), we obtain (B.2) with $\gamma = \beta - \delta > 0$.

Weak convergence for (2.3). Let $N(T) = T^{(3-\alpha)/2}M^{1/2}(L_U(T))^{1/2}$ be the normalization used in Theorem 2.1. By the same arguments as above, it is enough to show that, for all $T \geqslant 1$ and $0 \leqslant y \leqslant 1$,

$$\lambda^2 P\left(\left|\frac{W^*(Ty, M)}{N(T)}\right| \geqslant \lambda\right) \leqslant \mathrm{E}\left|\frac{W^*(Ty, M)}{N(T)}\right|^2 = \frac{\mathrm{E}|W^*([Ty])|^2}{T^{3-\alpha}L_U(T)} \leqslant c\left(\frac{[Ty]}{T}\right)^{1+\epsilon}. \tag{B.14}$$

Since the function $f(u) = \mathrm{E}|W^*([u])|^2$, $u > 0$, is regularly varying with index $3 - \alpha$ and the slowly varying function L_U (see Taqqu and Levy 1986, p. 87), the last bound in (B.14) can be obtained as in the proof of weak convergence for (2.4) above. □

Acknowledgements

This research was partially supported by National Science Foundation grants ANI-9805623 and DMS-0102410 at Boston University. We thank an anonymous referee whose suggestions and comments substantially improved the presentation.

References

Aaronson, J. and Denker, M. (1998) Characteristic functions of random variables attracted to 1-stable laws. *Ann. Probab.*, **26**(1), 399–415.

Billingsley, P. (1968) *Convergence of Probability Measures*. New York: Wiley.

Bingham, N.H., Goldie, C.M. and Teugels, J.L. (1987) *Regular Variation*. Cambridge: Cambridge University Press.

Breiman, L. (1965) On some limit theorems similar to the arc-sin law. *Theory Probab. Appl.*, **10**(2), 323–331.

Cline, D.B. (1986) Convolution tails, product tails and domains of attraction. *Probab. Theory Related Fields*, **72**, 529–557.

Cline, D.B.H. and Hsing, T. (1989) Large deviation probabilities for sums of random variables with heavy or subexponential tails. Technical report, Texas A&M University, Department of Statistics.

Fristedt, B. and Gray, L. (1997) *A Modern Approach to Probability Theory*. Boston: Birkhäuser.

Gaigalas, R. and Kaj, I. (2003) Convergence of scaled renewal processes and a packet arrival model. *Bernoulli*, **9**, 671–703.

Guerin, C.A., Nyberg, H., Perrin, O., Resnick, S., Rootzén, H. and Stărică, C. (2003) Empirical testing of the infinite source Poisson data traffic model. *Stochastic Models*, **19**, 151–200.

Ibragimov, I.A. and Linnik, Yu.V. (1971) *Independent and Stationary Sequences of Random Variables*. Wolters-Noordhoff, Groningen.

Leland, W.E., Taqqu, M.S., Willinger, W. and Wilson, D.V. (1994) On the self-similar nature of Ethernet traffic (Extended version). *IEEE/ACM Trans. Networking* **2**, 1–15.

Levy, J. and Taqqu, M.S. (1987) On renewal processes having stable inter-renewal intervals and stable rewards. *Ann. Sci. Math. Québec*, **11**, 95–110.

Levy, J.B. and Taqqu, M.S. (2000) Renewal reward processes with heavy-tailed inter-renewal times and heavy-tailed rewards. *Bernoulli*, **6**, 23–44.

Maulik, K., Resnick, S. and Rootzén, H. (2002) A network traffic model with random transmission rate. *J. Appl. Probab.* **39**, 671–699.

Mikosch, T., Resnick, S., Rootzén, H. and Stegeman, A. (2002) Is network traffic approximated by stable Lévy motion or fractional Brownian motion? *Ann. Appl. Probab.* **12**(1), 23–68.

Petrov, V.V. (1975) *Sums of Independent Random Variables*. New York: Springer-Verlag.

Petrov, V.V. (1995) *Limit Theorems of Probability Theory*. Oxford: Oxford University Press.

Pipiras, V. and Taqqu, M.S. (2000) The limit of a renewal reward process with heavy-tailed rewards is not a linear fractional stable motion. *Bernoulli*, **6**, 607–614.

Resnick, S. and van der Berg, E. (2000) Weak convergence of high-speed network traffic model. *J. Appl. Probab.* **37**(2), 575–597.

Royden, H.L. (1988) *Real Analysis* (3rd edition). New York: Macmillan.

Samorodnitsky, G. and Taqqu, M.S. (1994) *Stable Non-Gaussian Processes: Stochastic Models with Infinite Variance*. London: Chapman & Hall.

Taqqu, M.S. and Levy, J. (1986) Using renewal processes to generate long-range dependence and high variability. In E. Eberlein and M.S. Taqqu, (eds), *Dependence in Probability and Statistics*, pp. 73–89. Boston: Birkhäuser.

Taqqu, M.S., Willinger, W. and Sherman, R. (1997) Proof of a fundamental result in self-similar traffic modeling. *Comput. Commun. Rev.*, **27**(2), 5–23.

Thorisson, H. (2000) *Coupling, Stationarity and Regeneration*. New York: Springer-Verlag.

Willinger, W., Taqqu, M.S., Sherman, R. and Wilson, D.V. (1997) Self-similarity through high-variability: statistical analysis of Ethernet LAN traffic at the source level. *IEEE/ACM Trans. Networking* **5**(1), 71–86. Extended version of the paper with the same title that appeared in 1995 in *Comput. Commun. Rev.*, **25**, 100–113.

Received May 2002 and Revised July 2003