Bernoulli **9**(4), 2003, 705–723

Directed polymers in a random environment: path localization and strong disorder

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We consider directed polymers in a random environment. Under some mild assumptions on the environment, we prove equivalence between the decay rate of the partition function and some natural localization properties of the path; some quantitative estimates of the decay of the partition function in one or two dimensions, or at sufficiently low temperature; and the existence of quenched free energy. In particular, we generalize to general environments the results recently obtained by Carmona and Hu for a Gaussian environment. Our approach is based on martingale decomposition and martingale analysis. It leads to a natural, asymptotic relation between the partition function, and the probability that two polymers in the same environment, but otherwise independent, end up at the same point.

Keywords: directed polymers; martingales; random environment

1. Introduction

The models we consider in this paper are defined in terms of a random walk and of a random environment. The process $(\{S_n\}_{n\geqslant 0}, \{P^x\}_{x\in\mathbb{Z}^d})$ is a simple *random walk* on the d-dimensional integer lattice $\mathbb{Z}^d$. More precisely, let Ω be the path space $\Omega = \{\omega = (\omega_n)_{n\geqslant 0}; \omega_n \in \mathbb{Z}^d, n \geqslant 0\}$, let $\mathcal{F}$ be the cylindrical σ-field on Ω, and, for all $n \geqslant 0$, let $S_n : \omega \mapsto \omega_n$ be the projection map. For any $x \in \mathbb{Z}^d$ we consider the unique probability measure P^x on $(\Omega, \mathcal{F})$ such that $S_1 - S_0, \ldots, S_n - S_{n-1}$ are independent and

$$P^x\{S_0 = x\} = 1,$$

$$P^x\{S_n - S_{n-1} = \pm\delta_j\} = (2d)^{-1}, \qquad j = 1, 2, \ldots, d,$$

where $\delta_j = (\delta_{kj})_{k=1}^d$ is the jth vector of the canonical basis of $\mathbb{Z}^d$. For $x = 0$ we will write P instead of P^0.

The *random environment* $\xi = \{\xi(x, n) : x \in \mathbb{Z}^d, n \geqslant 1\}$ is an independent and identically distributed (i.i.d.) sequence of random variables which are real-valued, non-constant, and defined on a probability space $(\Xi, \mathcal{E}, Q)$ such that

1350–7265

$$Q[\exp(\beta\xi(x, n))] < \infty, \qquad \text{for all } \beta \in \mathbb{R}. \tag{1.1}$$

(Throughout, $Q[Y]$ denotes the Q-expectation of a random variable Y.) Let $\lambda(\beta)$ be the logarithmic moment generating function of $\xi(x, n)$,

$$\lambda(\beta) = \ln Q[\exp(\beta\xi(x, n))], \qquad \beta \in \mathbb{R}. \tag{1.2}$$

For any $n > 0$, define the probability measure μ_n on the path space $(\Omega, \mathcal{F})$,

$$\mu_n(\mathrm{d}\omega) = P[e_n]^{-1} e_n P(\mathrm{d}\omega), \tag{1.3}$$

where

$$e_n = e_n(\xi, S) = \exp\left(\sum_{1 \leqslant j \leqslant n} (\beta\xi(S_j, j) - \lambda(\beta))\right) \tag{1.4}$$

with a parameter $\beta \in \mathbb{R}$. Here, the graph $\{(S_j, j)\}_{j \geqslant 0}$ may be interpreted as a polymer chain living in $(d+1)$-dimensional space, constrained to stretch in the $(d+1)$th direction, and governed by the Hamiltonian

$$-\beta \sum_{j \geqslant 1} \xi(S_j, j),$$

that is, the so-called directed polymer in the environment ξ. Note that the term $\lambda(\beta)$, from the exponent in (1.4), cancels out in definition (1.3). The reason for including it in (1.4) is to normalize $P[e_n]$, which now has expectation equal to 1. If $\beta > 0$, then the parameter $\beta > 0$ plays the role of the inverse temperature in this interpretation. Since this Hamiltonian is parametrized by ξ, the polymer measure μ_n is random. The polymer is attracted to sites where the random environment is large and positive, and repelled by sites where the environment is large and negative. Here are two standard choices for ξ.

Example 1.1 *Gaussian environment*. This is the case in which $\xi(x, n)$ is a standard normal random variable, so that $\lambda(\beta) = \frac{1}{2}\beta^2$ (Carmona and Hu 2002).

Example 1.2 *Bernoulli environment*. This is the case in which $\xi(x, n)$ takes two different values a and b with probability $p > 0$ and $1 - p > 0$, respectively, so that $\lambda(\beta) = \ln(p\mathrm{e}^{\beta a} + (1-p)\mathrm{e}^{\beta b})$ (Bolthausen 1989; Imbrie and Spencer 1988; Song and Zhou 1996). As discussed by Johansson (2000, Remark 1.8), directed percolation can be understood as the case of $0 = a > b$ and zero temperature ($\beta \to \infty$), which, however, is outside the scope of this paper.

We are interested in the behaviour of the path $\{S_k\}_{k=1}^n$ for large n under the (sequence of) polymer measures μ_n. As is the case in many other models in statistical mechanics, one of the fundamental questions is the asymptotic behaviour of the partition function

$$Z_n = Z_n(\xi) = P[e_n]. \tag{1.5}$$

Since Z_n is a positive martingale on $(\Xi, \mathcal{E}, Q)$, the following limit exists Q-almost surely:

$$Z_\infty \stackrel{\text{def}}{=} \lim_{n \nearrow \infty} Z_n. \tag{1.6}$$

The event $\{Z_\infty = 0\}$ is measurable with respect to the tail σ-field

$$\bigcap_{n \geqslant 1} \sigma[\xi(x, j); j \geqslant n, x \in \mathbb{Z}^d],$$

and therefore, by Kolmogorov's 0–1 law,

$$Q\{Z_\infty = 0\} = 0 \text{ or } 1. \tag{1.7}$$

We refer to the former case as *weak disorder* and the latter as *strong disorder*. It is known (see, for instance, Song and Zhou 1996) that, for $d \geqslant 3$,

$$Q\{Z_\infty = 0\} = 0 \text{ if } \gamma_1(\beta) \stackrel{\text{def}}{=} \lambda(2\beta) - 2\lambda(\beta) < -\ln(1 - q), \tag{1.8}$$

where $q = P\{S_n \neq 0 \text{ for all } n \geqslant 1\}$; similar results for weak disorder were obtained by Bolthausen (1989) and Sinai (1995). Note that $\gamma_1(\beta)$ is decreasing on $(-\infty, 0]$, increasing on $[0, \infty)$ and $\gamma_1(0) = 0$ so that the condition in (1.8) does hold if $|\beta|$ is small. In dimension $d \geqslant 3$, this condition amounts to L^2-convergence in (1.6), and allows the so-called second moment method to be used: for small β and $d \geqslant 3$, first Imbrie and Spencer (1988), and then Bolthausen (1989) with martingale techniques, proved that the polymer is diffusive, that is, $\mu_n[S_n^2] \sim n$ as $n \nearrow \infty$; more recently, Albeverio and Zhou (1996) showed that the invariance principle holds for almost every environment. On the other hand, for strong disorder, it can be seen that

$$Q\{Z_\infty = 0\} = 1 \text{ if } \gamma_2(\beta) \stackrel{\text{def}}{=} \beta\lambda'(\beta) - \lambda(\beta) \geqslant \ln(2d). \tag{1.9}$$

This was shown by Kahane and Peyrière (1976) for a different model called the Mandelbrot martingale (or, equivalently, multiplicative chaos), where graphs $\{(S_j, j)\}_{j \geqslant 0}$ are replaced by infinite paths, without loops and starting from the root, on the d-ary tree. Although the directed polymer we are considering here is more intricate due to correlations, the same argument applies for deducing (1.9). Note that $\gamma_2(\beta)$ is decreasing on $(-\infty, 0]$, increasing on $[0, \infty)$ and $\gamma_2(0) = 0$ so that the condition in (1.9) roughly says that $|\beta|$ is large enough. Recently, Carmona and Hu (2002) proved for the Gaussian environment that, for all $\beta \neq 0$,

$$Q\{Z_\infty = 0\} = 1, \qquad d = 1, 2, \tag{1.10}$$

which, together with (1.8) and (1.9), displays a non-trivial dependence on the dimension.

In the present paper, we consider general environments and present some results mainly for the strong disorder case, $Q\{Z_\infty = 0\} = 1$, including the extension of (1.10) to the non-Gaussian case. Using martingale analysis, we also obtain natural localization properties which characterize the strong disorder regime. More precisely, the decay of the partition function is equivalent to concentration of the path on favourite sites. All the proofs presented in this paper are self-contained, except for that of Proposition 2.4(b).

Among other interesting subjects related to directed polymers are superdiffusivity and critical exponents. We do not discuss these here, referring instead to Johansson (2000), Licea *et al*. (1996), Petermann (2000) and Piza (1997) for relevant rigorous results.

2. Results

On the product space $(\Omega^2, \mathcal{F}^{\otimes 2})$, we consider the probability measure $\mu_n^{\otimes 2} = \mu_n^{\otimes 2}(\mathrm{d}\omega, \mathrm{d}\tilde{\omega})$, which we will view as the distribution of the couple $(S, \tilde{S})$ with $\tilde{S} = \{\tilde{S}_k\}_{k\geqslant 0}$ an independent copy of $S = \{S_k\}_{k\geqslant 0}$ with law μ_n. An important role in the analysis is played by the random sequence

$$I_n = \mu_{n-1}^{\otimes 2}(S_n = \tilde{S}_n), \tag{2.1}$$

which conveys some information on the localization of paths under μ_n; see (2.8) below. Roughly, large values of $I_n \in (0, 1]$ indicate that the polymer concentrates, at time n, on a few significant sites, though small values indicate that it spreads out on a large number of sites. Our basic result relates the partition function Z_n and the expected intersection time $\sum_{j\leqslant n} I_j$ of two independent polymers in the same environment.

Theorem 2.1. *Let* $\beta \neq 0$. *Then*

$$\{Z_\infty = 0\} = \left\{\sum_{n\geqslant 1} I_n = \infty\right\}, \qquad Q\text{-a.s.} \tag{2.2}$$

Moreover, if $Q\{Z_\infty = 0\} = 1$, *then there exist* $c_1, c_2 \in (0, \infty)$ *such that*

$$-c_1 \ln Z_n \leqslant \sum_{1\leqslant j\leqslant n} I_j \leqslant -c_2 \ln Z_n \qquad \textit{for large enough } n, \ Q\text{-a.s.} \tag{2.3}$$

We make a brief comment on the result. On the one hand, we recall the definition of weak and strong disorder (see (1.7)), which is natural in view of the high-temperature behaviour (in higher dimensions) (1.8) and the low-temperature behaviour (1.9). On the other hand, when the polymer is strongly influenced by the environment, it is strongly attracted to sites with favourable environment and it follows from definition (2.1) that I_n takes large values. Our result is a rigourous statement of equivalence of these two properties. The decay property of Z_n is reflected in some specific localization property of the path $\{S_n\}_{n\geqslant 1}$ under the random measure (1.3). The proof of Theorem 2.1 is based on a general estimate for the summation of i.i.d. random variables (Lemma 3.1 below) and martingale analysis.

The most interesting case relative to the following, straightforward corollary is $a_n = n$, $n \geqslant 1$.

Corollary 2.2. *For* $\beta \neq 0$ *and a sequence* $a_n \nearrow \infty$ *of positive numbers, the following properties are equivalent:*

(Z1) *There exists* $c > 0$ *such that*

$$Q\left\{\varliminf_{n\nearrow\infty} -\frac{1}{a_n} \ln Z_n \geqslant c\right\} = 1. \tag{2.4}$$

(I1) *There exists* $c > 0$ *such that*

$$Q\left\{ \varliminf_{n\nearrow\infty} \frac{1}{a_n} \sum_{1\leqslant j\leqslant n} I_j \geqslant c \right\} = 1. \tag{2.5}$$

Remark 2.1. The equivalence presented in Theorem 2.1 was first shown by Carmona and Hu (2002, Theorem 1.1 and Proposition 5.1) in the Gaussian case.

Some sufficient conditions for (Z1) and (I1) are provided by the following result.

Theorem 2.3. (a) (Z1) *in Corollary 2.2 holds for* $a_n = n$ *if* $\gamma_2(\beta) > \ln(2d)$; *cf.* (1.9).
(b) *If* $\beta \neq 0$, as $n \nearrow \infty$, *then Q-a.s.,*

$$Z_n \begin{cases} = \mathcal{O}(\exp(-c_1 n^{1/3})), & \text{if } d = 1, \\ \to 0, & \text{if } d = 2, \end{cases}$$

where c_1 *is a positive constant.*

Theorem 2.3 is proved by estimating the fractional moment $Q[Z_n^\theta]$, $0 < \theta < 1$; see Lemma 4.1 below. Besides the quantitative bound for the rate of decay for $d = 1$ presented above, we also give the quantitative bound for the fractional moment for $d = 1, 2$ in the course of the proof.

Remark 2.2. Theorem 2.3(b) generalizes Theorem 1.1 in Carmona and Hu (2002) to non-Gaussian environments. Moreover, the proof in this paper sheds more light on the decay rate.

We now go on to discuss sufficient conditions for another localization property of the polymer chain, described in terms of I_n.

Proposition 2.4. *Consider the following property:*

(I2) *There is a constant* $c \in (0, \infty)$ *such that*

$$\varlimsup_{n\nearrow\infty} I_n \geqslant c, \qquad Q\text{-a.s.} \tag{2.6}$$

Then we have:

(a) (I2) *holds if* (Z1) *holds with* $a_n = n$, *in particular if* $\gamma_2(\beta) > \ln(2d)$; *cf.* (1.9) *and Theorem* 2.3(a).
(b) (I2) *holds if* $d = 1, 2$.
(c) *If* $Q\{Z_\infty > 0\} = 1$, *then, in contrast to* (I2),

$$\lim_{n\nearrow\infty} I_n = 0, \qquad Q\text{-a.s.}$$

Assume, moreover that, $\gamma_1(\beta) < -\ln(1-q)$; *cf.* (1.8). *Then there is a constant* $c > 0$ *such that*

$$I_n = O(n^{-c}) \qquad \text{in } Q\text{-probability.} \tag{2.7}$$

A natural quantity of interest here, related to localization phenomenon, is the favourite site for the path at time n. First observe that

$$\max_{x\in\mathbb{Z}^d} \mu_{n-1}(S_n = x)^2 \leqslant I_n \leqslant \max_{x\in\mathbb{Z}^d} \mu_{n-1}(S_n = x). \tag{2.8}$$

Therefore, all statements obtained for I_n can be translated into those for $\max_{x\in\mathbb{Z}^d} \mu_{n-1}(S_n = x)$. In particular, we showed in Proposition 2.4 that the probability of the favourite site vanishes for weak disorder, but not for strong disorder. In the latter case the polymer localizes (in a set of lattice points depending on the environment), though in the former it spreads out somewhat similarly to the usual simple random walk.

Remark 2.3. Proposition 2.4(b) generalizes Theorem 1.2 in Carmona and Hu (2002) to non-Gaussian environments. To prove this, we refer the readers to some of the arguments in Carmona and Hu (2002).

Finally, we remark that the 'quenched free energy'

$$\lim_{n\nearrow\infty} \frac{1}{n} \ln Z_n$$

exists Q-a.s. under our assumption (1.1).

Proposition 2.5. *The limit*

$$\psi(\beta) = \lim_{n\nearrow\infty} \frac{1}{n} Q[\ln Z_n] \in (-\infty, 0]$$

exists. Moreover, for any $\varepsilon > 0$, there is an $n_0 = n_0(\beta, \varepsilon) < \infty$ such that

$$Q\left\{\left|\frac{1}{n}\ln Z_n - Q\left[\frac{1}{n}\ln Z_n\right]\right| > \varepsilon\right\} \leqslant \exp\left(-\frac{\varepsilon^{2/3} n^{1/3}}{4}\right), \qquad n \geqslant n_0. \tag{2.9}$$

As a consequence,

$$\lim_{n\nearrow\infty} \frac{1}{n} \ln Z_n = \psi(\beta), \qquad Q\text{-a.s.}$$

Remark 2.4. Inequality (2.9) is a concentration inequality with stretched exponential decay rate. An inspection of our proof reveals that an exponential concentration result can be obtained by a slightly stronger assumption. In fact, if we assume that there is a $\delta > 0$ such that

$$Q[\exp(\delta|\xi(x, n)|^2)] < \infty, \tag{2.10}$$

then we obtain the following: for any $\varepsilon > 0$, there is an $n_0 = n_0(\beta, \varepsilon) < \infty$ such that

$$Q\left\{\left|\frac{1}{n}\ln Z_n - \psi(\beta)\right| > \varepsilon\right\} \leqslant \exp\left(-\frac{\varepsilon^2 n}{c}\right), \qquad n \geqslant n_0, \tag{2.11}$$

where $c = c(\beta) > 0$. See Remark 6.1 below for the proof. Note also that (2.10) is true if $\xi(x, n)$ is a Gaussian or Bernoulli random variable as in Example 1.1 or Example 1.2.

Remark 2.5. We can define a similar model by considering a Markov chain $(\{S_n\}_{n\geqslant 0}, \{P^x\}_{x\in\Gamma})$ on a certain state space Γ instead of the random walk on $\mathbb{Z}^d$. The proofs presented in this paper can be adapted to this generalization.

3. Proof of Theorem 2.1

We first state some technical estimates.

Lemma 3.1. *Let η_i, $1 \leqslant i \leqslant m$, be positive, non-constant i.i.d. random variables on a probability space $(\Xi, \mathcal{E}, Q)$ such that*

$$Q[\eta_1] = 1, \qquad Q[\eta_1^3 + \ln^2 \eta_1] < \infty.$$

For $\{\alpha_i\}_{1\leqslant i\leqslant m} \subset [0, \infty)$ such that $\sum_{1\leqslant i\leqslant m} \alpha_i = 1$, define a centred random variable $U > -1$ by $U = \sum_{1\leqslant i\leqslant m} \alpha_i\eta_i - 1$. Then there exists a constant $c \in (0, \infty)$, independent of $\{\alpha_i\}_{1\leqslant i\leqslant m}$, such that

$$\frac{1}{c}\sum_{1\leqslant i\leqslant m} \alpha_i^2 \leqslant Q\left[\frac{U^2}{2+U}\right], \tag{3.1}$$

$$\frac{1}{c}\sum_{1\leqslant i\leqslant m} \alpha_i^2 \leqslant -Q[\ln(1+U)] \leqslant c\sum_{1\leqslant i\leqslant m} \alpha_i^2, \tag{3.2}$$

$$Q[\ln^2(1+U)] \leqslant c\sum_{1\leqslant i\leqslant m} \alpha_i^2. \tag{3.3}$$

Remark 3.1. These estimates are proved in Carmona and Hu (2002) for the Gaussian case with the help of Brownian motion and making use of Itô's formula. Here, we present a simple argument which works in the general case.

We postpone the proof of Lemma 3.1 to the end of the section, and, assuming the lemma, we begin our proof of Theorem 2.1.

For (2.2) and (2.3) to hold, it is enough to show that

$$\{Z_\infty = 0\} \subset \left\{\sum_{n\geqslant 1} I_n = \infty\right\}, \qquad Q\text{-a.s.}, \tag{3.4}$$

and that there exist $c_1, c_2 \in (0, \infty)$ such that

$$\left\{\sum_{n\geqslant 1} I_n = \infty\right\} \subset \left\{-c_1 \ln Z_n \leqslant \sum_{1\leqslant j\leqslant n} I_j \leqslant -c_2 \ln Z_n \text{ for large enough } n\right\}, \qquad Q\text{-a.s.} \tag{3.5}$$

The proof of (3.4) and (3.5) is based on Doob's decomposition for the process $-\ln Z_n$.

It is convenient to introduce some more notation. For a sequence $(a_n)_{n\geqslant 0}$ (random or non-random), we set $\Delta a_n = a_n - a_{n-1}$ for $n \geqslant 1$. We denote by $\mathcal{E}_n$ the σ-field generated by $\{\xi(x, j);\ 1 \leqslant j \leqslant n,\ x \in \mathbb{Z}^d\}$, and we denote by Q_n^ξ the conditional expectation with respect to Q given $\mathcal{E}_n$.

Let us now recall Doob's decomposition in this context; any $(\mathcal{E}_n)$-adapted process $X = \{X_n\}_{n\geqslant 0} \subset L^1(Q)$ can be uniquely decomposed as

$$X_n = M_n(X) + A_n(X), \qquad n \geqslant 1,$$

where $M(X)$ is an $(\mathcal{E}_n)$-martingale and

$$A_0 = 0,$$

$$\Delta A_n = Q_{n-1}^\xi[\Delta X_n], \qquad n \geqslant 1.$$

$M_n(X)$ and $A_n(X)$ are called the martingale part and compensator of the process X, respectively. If X is a square-integrable martingale, then the compensator $A_n(X^2)$ of the process $X^2 = \{(X_n)^2\}_{n\geqslant 0} \subset L^1(Q)$ is denoted by $\langle X\rangle_n$ and is given by the formula

$$\Delta\langle X\rangle_n = Q_{n-1}^\xi[(\Delta X_n)^2].$$

Here, we are interested in Doob's decomposition of $X_n = -\ln Z_n$, whose martingale part and compensator will be henceforth denoted M_n and A_n, respectively:

$$-\ln Z_n = M_n + A_n. \tag{3.6}$$

To compute M_n and A_n, we introduce

$$U_n = \mu_{n-1}[\exp(\beta\xi(S_n, n) - \lambda(\beta))] - 1.$$

It is then clear that

$$Z_n/Z_{n-1} = 1 + U_n \tag{3.7}$$

and hence that

$$\Delta A_n = -Q_{n-1}^\xi \ln(1 + U_n), \qquad \Delta M_n = -\ln(1 + U_n) + Q_{n-1}^\xi \ln(1 + U_n). \tag{3.8}$$

In particular,

$$\Delta\langle M\rangle_n \leqslant Q_{n-1}^\xi \ln^2(1 + U_n). \tag{3.9}$$

On the other hand, we have that

$$I_n = \sum_{|z|_1\leqslant n} \mu_{n-1}(S_n = z)^2.$$

We now claim that there is a constant $c \in (0, \infty)$ such that

$$\frac{1}{c} I_n \leqslant \Delta A_n \leqslant c I_n, \tag{3.10}$$

$$\Delta \langle M \rangle_n \leqslant c I_n. \tag{3.11}$$

Indeed, both follow from (3.8), (3.9) and Lemma 3.1; $\{\eta_i\}$, $\{\alpha_i\}$ and Q in the lemma play the roles of $\{\exp(\beta\xi(z, n) - \lambda(\beta))\}_{|z|_1 \leqslant n}$, $\{\mu_{n-1}(S_n = z)\}_{|z|_1 \leqslant n}$ and Q^{ξ}_{n-1}.

We now derive (3.4) from (3.10), (3.11) as follows (the equalities and the inclusions here being understood as holding Q-a.s.):

$$\begin{aligned} \left\{ \sum_{n \geqslant 1} I_n < \infty \right\} &= \{A_\infty < \infty\} \\ &= \{A_\infty < \infty, \langle M \rangle_\infty < \infty\} \\ &\subset \{A_\infty < \infty, \lim_{n \nearrow \infty} M_n \text{ exists and is finite}\} \\ &\subset \{Z_\infty > 0\}. \end{aligned}$$

Here, on the third line, we have used a well-known property for martingales; see Durrett (1995, (4.9), p. 255) or Neveu (1975).

Finally, we prove (3.5). By (3.10), it is enough to show that

$$\{A_\infty = \infty\} \subset \left\{ \lim_{n \nearrow \infty} -\frac{\ln Z_n}{A_n} = 1 \right\}, \qquad Q\text{-a.s.} \tag{3.12}$$

Thus, let us suppose that $A_\infty = \infty$. If $\langle M \rangle_\infty < \infty$, then, again by Durrett (1995, (4.9), p. 255) or Neveu (1975), $\lim_{n \nearrow \infty} M_n$ exists and is finite and therefore (3.12) holds. If, on the other hand, $\langle M \rangle_\infty = \infty$, then

$$-\frac{\ln Z_n}{A_n} = \frac{M_n}{\langle M \rangle_n} \frac{\langle M \rangle_n}{A_n} + 1 \to 1, \qquad Q\text{-a.s.},$$

by (3.10), (3.11) and the law of large numbers for martingales; see Durrett (1995, (4.10), p. 255) or Neveu (1975). This completes the proof of Theorem 2.1. □

Proof of Lemma 3.1. In this proof, we let $c_1, c_2, \ldots$ stand for constants which are independent of $\{\alpha_i\}_{1 \leqslant i \leqslant m}$. We have, by direct computation, that

$$Q[U^2] = c_1 \sum_{1 \leqslant i \leqslant m} \alpha_i^2, \qquad Q[U^3] \leqslant c_2 \sum_{1 \leqslant i \leqslant m} \alpha_i^2.$$

Then (3.1) is obtained as follows:

$$c_1 \sum_{1 \leqslant i \leqslant m} \alpha_i^2 = Q\left[\frac{U}{\sqrt{2+U}} U\sqrt{2+U}\right]$$

$$\leqslant Q\left[\frac{U^2}{2+U}\right]^{1/2} Q[2U^2 + U^3]^{1/2}$$

$$\leqslant c_3 Q\left[\frac{U^2}{2+U}\right]^{1/2} \left(\sum_{1 \leqslant i \leqslant m} \alpha_i^2\right)^{1/2}.$$

To prove the other inequalities, it is convenient to define a function $\varphi : (-1, \infty) \to [0, \infty)$ by $\varphi(u) = u - \ln(1+u)$, so that

$$-Q[\ln(1+U)] = Q[\varphi(U)].$$

Since

$$\frac{1}{4}\frac{u^2}{2+u} \leqslant \varphi(u), \qquad u > -1,$$

the left-hand inequality of (3.2) follows from (3.1). The right-hand inequality can be seen as follows. We have, for any $\varepsilon \in (0, 1)$,

$$Q[\varphi(U)] = Q[\varphi(U) : 1 + U \geqslant \varepsilon] + Q[\varphi(U) : 1 + U \leqslant \varepsilon]$$

$$\leqslant Q[\varphi(U) : 1 + U \geqslant \varepsilon] - Q[\ln(1+U) : 1 + U \leqslant \varepsilon].$$

Since $\varphi(u) \leqslant \frac{1}{2}(u/\varepsilon)^2$ if $1 + u \geqslant \varepsilon$,

$$Q[\varphi(U) : 1 + U \geqslant \varepsilon] \leqslant \tfrac{1}{2}\varepsilon^{-2} Q[U^2]$$

$$= \tfrac{1}{2}\varepsilon^{-2} c_1 \sum_{1 \leqslant i \leqslant m} \alpha_i^2. \tag{3.13}$$

We now set $\gamma = -Q[\ln \eta_1] \geqslant 0$ and choose $\varepsilon > 0$ so small that $\ln(1/\varepsilon) - \gamma \geqslant 1$. We introduce another centred random variable $V = \sum_{1 \leqslant i \leqslant m} \alpha_i(\ln \eta_i + \gamma)$. We then see from Jensen's inequality that

$$\{1 + U \leqslant \varepsilon\} = \{V - \gamma \leqslant \ln(1+U) \leqslant \ln \varepsilon\}$$

$$\subset \{-\ln(1+U) \leqslant -V + \gamma\} \cap \{1 \leqslant -V\}.$$

Hence, we have

$$-Q[\ln(1+U) : 1 + U \leqslant \varepsilon] \leqslant Q[-V : 1 \leqslant -V] + \gamma Q\{1 \leqslant -V\}$$

$$\leqslant (1+\gamma) Q[V^2]$$

$$= c_4 \sum_{1 \leqslant i \leqslant m} \alpha_i^2.$$

This, together with (3.13), proves the right-hand inequality of (3.2).

The proof of (3.3) is similar. Indeed, since $|\ln(1+u)| \leqslant \varepsilon^{-1}\ln(\varepsilon^{-1})|u|$ if $\varepsilon \leqslant 1+u$, we have that

$$Q[\ln^2(1+U) : \varepsilon \leqslant 1+U] \leqslant \varepsilon^{-2}\ln^2(\varepsilon^{-1})Q[U^2].$$

On the other hand,

$$\begin{aligned}\{1+U \leqslant \varepsilon\} &= \{V-\gamma \leqslant \ln(1+U) \leqslant \ln\varepsilon\}\\ &\subset \{\ln^2(1+U) \leqslant 2V^2+2\gamma^2\} \cap \{1 \leqslant -V\}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}Q[\ln^2(1+U) : 1+U \leqslant \varepsilon] &\leqslant 2Q[V^2] + 2\gamma^2 Q\{1 \leqslant -V\}\\ &\leqslant c_5 \sum_{1\leqslant i\leqslant m} \alpha_i^2.\end{aligned}$$

□

4. Proof of Theorem 2.3

4.1. A sufficient condition for (Z1) via fractional moments

Lemma 4.1. *Suppose that there exist constants* $c \in (0, \infty)$, $\theta \in (0, 1)$ *and a sequence* $a_n \nearrow \infty$ *such that*

$$Q[Z_n^\theta] \leqslant c\exp(-a_n), \qquad n \geqslant 1. \tag{4.1}$$

Then $Q\{Z_\infty = 0\} = 1$. *If, moreover,*

$$\sum_{n\geqslant 1} \exp(-\delta a_n) < \infty \qquad \text{for some } \delta \in (0, 1),$$

then (Z1) *holds*.

Proof. The first statement follows from Fatou's lemma and the second from the Borel–Cantelli lemma. □

4.2. Proof of part (a)

We will check (4.1) with $a_n = cn$ for some $c > 0$. Set $\eta(x, j) = \exp(\beta\xi(x, j) - \lambda(\beta))$ and

$$Z^x_{n,m} = P^x\left[\exp\left(\sum_{1\leqslant j\leqslant m} (\beta\xi(S_j, j+n) - \lambda(\beta))\right)\right], \qquad n, m \geqslant 1. \tag{4.2}$$

For $\theta \in (0, 1)$, by the subadditive estimate $(u+v)^\theta \leqslant u^\theta + v^\theta$, $u, v > 0$, we obtain

$$Z_n^\theta \leqslant (2d)^{-\theta} \sum_{x,|x|_1=1} (\eta(x,1) Z_{1,n-1}^x)^\theta.$$

Since $Z_{1,n-1}^x$ has the same law as Z_{n-1},

$$Q[Z_n^\theta] \leqslant r(\theta) Q[Z_{n-1}^\theta],$$

where $r(\theta) = (2\mathrm{d})^{1-\theta} Q[\eta(x,1)^\theta]$. Note that $\theta \mapsto \ln r(\theta)$ is convex and continuously differentiable, and that $\ln(2d) = \ln r(0) > \ln r(1) = 0$. Therefore $r(\theta) < 1$ for some $\theta \in (0, 1)$ if and only if $0 < \mathrm{d} \ln r(\theta)/\mathrm{d}\theta|_{\theta=1}$, which is equivalent to $\gamma_2(\beta) > \ln(2\mathrm{d})$.

4.3. Proof of part (b)

We will check (4.1) with

$$a_n = \begin{cases} c_1 n^{1/3}, & \text{if } d = 1, \\ c_2 \sqrt{\ln n}, & \text{if } d = 2, \end{cases} \tag{4.3}$$

for constants $c_1, c_2 \in (0, \infty)$. With this in mind, we first prove an auxiliary lemma.

Lemma 4.2. *For* $\theta \in [0, 1]$ *and* $\Lambda \subset \mathbb{Z}^d$,

$$Q[Z_{n-1}^\theta I_n] \geqslant \frac{1}{|\Lambda|} Q[Z_{n-1}^\theta] - \frac{2}{|\Lambda|} P(S_n \notin \Lambda)^\theta. \tag{4.4}$$

Proof. Repeating the argument in Liggett (1985, p. 453), we have that

$$\begin{aligned} I_n &\geqslant \sum_{z \in \Lambda} \mu_{n-1}(S_n = z)^2 \\ &\geqslant \frac{1}{|\Lambda|} \mu_{n-1}(S_n \in \Lambda)^2 \\ &= \frac{1}{|\Lambda|} (1 - \mu_{n-1}(S_n \notin \Lambda))^2 \\ &\geqslant \frac{1}{|\Lambda|} (1 - 2\mu_{n-1}(S_n \notin \Lambda)) \\ &\geqslant \frac{1}{|\Lambda|} (1 - 2\mu_{n-1}(S_n \notin \Lambda)^\theta). \end{aligned}$$

Note also that

$$\begin{aligned} Q[Z_{n-1}^\theta \mu_{n-1}(S_n \notin \Lambda)^\theta] &\leqslant Q[Z_{n-1} \mu_{n-1}(S_n \notin \Lambda)]^\theta \\ &= P(S_n \notin \Lambda)^\theta. \end{aligned}$$

Therefore,

$$Q[Z_{n-1}^{\theta} I_n] \geqslant \frac{1}{|\Lambda|} Q[Z_{n-1}^{\theta}] - \frac{2}{|\Lambda|} Q[Z_{n-1}^{\theta} \mu_{n-1}(S_n \notin \Lambda)^{\theta}]$$
$$\geqslant \frac{1}{|\Lambda|} Q[Z_{n-1}^{\theta}] - \frac{2}{|\Lambda|} P(S_n \notin \Lambda)^{\theta}.$$

□

Assume now that $\theta \in (0, 1)$, and define a function $f : (-1, \infty) \to [0, \infty)$ by

$$f(u) = 1 + \theta u - (1 + u)^{\theta}.$$

It is then clear that there are constants $c_1, c_2 \in (0, \infty)$ such that

$$\frac{c_1 u^2}{2 + u} \leqslant f(u) \leqslant c_2 u^2 \qquad \text{for all } u \in (-1, \infty). \tag{4.5}$$

From (3.7), (4.5) and (3.1), we have that

$$\begin{aligned} Q_{n-1}^{\xi} \Delta Z_n^{\theta} &= Z_{n-1}^{\theta} Q_{n-1}^{\xi}((1 + U_n)^{\theta} - 1) \\ &= -Z_{n-1}^{\theta} Q_{n-1}^{\xi} f(U_n) \\ &\leqslant -c_3 Z_{n-1}^{\theta} I_n. \end{aligned}$$

Therefore, by (4.4),

$$QZ_n^{\theta} \leqslant \left(1 - \frac{c_3}{|\Lambda|}\right) Q[Z_{n-1}^{\theta}] + \frac{2c_3}{|\Lambda|} P(S_n \notin \Lambda)^{\theta}. \tag{4.6}$$

For $d = 1$, set $\Lambda = (-n^{2/3}, n^{2/3}]$. Then,

$$P(S_n \notin \Lambda) = P\left(\left|\frac{S_n}{n^{1/2}}\right| \geqslant n^{1/6}\right) \leqslant 2 \exp\left(-\frac{n^{1/3}}{2}\right),$$

so that (4.6) reads

$$QZ_n^{\theta} \leqslant \left(1 - \frac{c_3}{2n^{2/3}}\right) Q[Z_{n-1}^{\theta}] + 4c_3 \exp\left(-\frac{n^{1/3}}{2}\right).$$

It is not difficult to conclude (4.1) with $a_n = c_1 n^{1/3}$ from the above.

For $d = 2$, we set

$$\Lambda = (-n^{1/2} \ln^{1/4} n, n^{1/2} \ln^{1/4} n]^2$$

to obtain (4.1) with $a_n = c_2 \sqrt{\ln n}$ in a similar way.

5. Proof of Proposition 2.4

5.1. Proof of part (a)

This follows directly from (2.5).

5.2. Proof of part (b)

We now state the following lemma which corresponds to Lemma 2.2 in Carmona and Hu (2002).

Lemma 5.1. *Let η_i, $1 \leqslant i \leqslant m$, be positive, non-constant i.i.d. random variables on a probability space $(\Xi, \mathcal{E}, Q)$ such that*

$$m_\theta \stackrel{\text{def}}{=} Q[\eta_1^\theta] < \infty, \qquad \text{for } \theta = \pm 4 \text{ and } m_1 = 1.$$

For $\{\alpha_i\}_{1\leqslant i\leqslant m} \subset [0, \infty)^m$ such that $\sum_{1\leqslant i\leqslant m}\alpha_i = 1$, define a centred random variable $U > -1$ by $U = \sum_{1\leqslant i\leqslant m}\alpha_i\eta_i - 1$. Then

$$1 - 2(m_2 - 1)(\alpha_1 + \alpha_2) + \frac{1}{C}\sum_{1\leqslant i\leqslant m}\alpha_i^2 \leqslant Q\left[\frac{\eta_1\eta_2}{(1+U)^2}\right] \leqslant m_2\sqrt{m_{-4}}, \tag{5.1}$$

$$m_2 - 2(m_3 - m_2)\alpha_1 + \frac{1}{C}\sum_{1\leqslant i\leqslant m}\alpha_i^2 \leqslant Q\left[\frac{\eta_1^2}{(1+U)^2}\right] \leqslant \sqrt{m_4 m_{-4}}, \tag{5.2}$$

where $C > 0$ is a constant which depends only on m_4.

Proof. Since the proofs of (5.1) and (5.2) are similar, we present that of (5.1) only.

$$\begin{aligned} Q[\eta_1\eta_2(1+U)^{-2}]^2 &\leqslant m_2^2 Q[(1+U)^{-4}] \\ &\leqslant m_2^2 Q\left[\sum_{1\leqslant i\leqslant m}\alpha_i\eta_i^{-4}\right] \\ &= m_2^2 m_{-4}, \end{aligned}$$

where, on the second line, we have used the Jensen inequality for the measure $\{\alpha_i\}$.

To prove the other inequalities, it is convenient to define a function $\varphi : (-1, \infty) \to [0, \infty)$ by $\varphi(u) = (1+u)^{-2} - 1 + 2u$. By an elementary inequality, $cu^2/(2+u) \leqslant \varphi(u)$, $u > -1$, we have

$$Q[\eta_1\eta_2(1+U)^{-2}] \geqslant 1 - 2Q[\eta_1\eta_2 U] + c_1 Q\left[\frac{\eta_1\eta_2 U^2}{2+U}\right]. \tag{5.3}$$

On the other hand, we have by direct computation that

$$Q[\eta_1\eta_2 U] = (m_2 - 1)(\alpha_1 + \alpha_2),$$

$$\frac{1}{c}\sum_{1\leqslant i\leqslant m}\alpha_i^2 \leqslant Q[\eta_1\eta_2 U^2] \leqslant c\sum_{1\leqslant i\leqslant m}\alpha_i^2,$$

$$Q[\eta_1\eta_2 U^3] \leqslant c\sum_{1\leqslant i\leqslant m}\alpha_i^2.$$

Therefore,

$$\begin{aligned}\frac{1}{c}\sum_{1\leqslant i\leqslant m}\alpha_i^2 &\leqslant Q[\eta_1\eta_2 U^2]\\ &= Q\left[\frac{\sqrt{\eta_1\eta_2}U}{\sqrt{2+U}}\sqrt{\eta_1\eta_2}U\sqrt{2+U}\right]\\ &\leqslant Q\left[\frac{\eta_1\eta_2 U^2}{2+U}\right]^{1/2} Q[2\eta_1\eta_2 U^2 + \eta_1\eta_2 U^3]^{1/2}\\ &\leqslant cQ\left[\frac{\eta_1\eta_2 U^2}{2+U}\right]^{1/2}\left(\sum_{1\leqslant i\leqslant m}\alpha_i^2\right)^{1/2}.\end{aligned}$$

Drawing it all together, we obtain (5.1). □

With Lemma 5.1 established, Proposition 2.4(b) is obtained merely by following the argument in Carmona and Hu (2002, Section 6), using our Lemma 5.1 in place of their Lemma 2.2. In fact, Carmona and Hu used the specific properties of the Gaussian random variable only in the proof of Lemma 2.2.

5.3. Proof of part (c)

The first statement is derived using the convergence of I_n to 0. We now prove (2.7). Since $Q\{Z_\infty > 0\} = 1$ in the present case – see (1.8) – it is enough to show that

$$Z_{n-1}^2 I_n = O(n^{-c}) \tag{5.4}$$

in Q-probability. With $\gamma = \lambda(2\beta) - 2\lambda(\beta) < -\ln(1-q)$, we compute

$$
\begin{aligned}
Q[Z_{n-1}^2 I_n] &= Q[P^{\otimes 2}(e_{n-1}(\xi, S)e_{n-1}(\xi, \tilde{S}) : S_n = \tilde{S}_n)] \\
&= P^{\otimes 2}(Q[e_{n-1}(\xi, S)e_{n-1}(\xi, \tilde{S})] : S_n = \tilde{S}_n) \\
&= P^{\otimes 2}\left(\exp\left\{\gamma \sum_{j=1}^{n-1} \mathbf{1}_{S_j=\tilde{S}_j}\right\} : S_n = \tilde{S}_n\right) \\
&\leqslant P^{\otimes 2}\left(\exp\left\{\alpha\gamma \sum_{j=1}^{n-1} \mathbf{1}_{S_j=\tilde{S}_j}\right\}\right)^{1/\alpha} P^{\otimes 2}(S_n = \tilde{S}_n)^{1/\alpha'},
\end{aligned}
$$

using Hölder's inequality with the conjugate exponents α, α'. Since $\sum_{j\geqslant 1}\mathbf{1}_{S_j=\tilde{S}_j}$ is geometrically distributed with failure probability $1-q \in (0, 1)$ with q as in (1.8), the first factor on the right-hand side is bounded for $\alpha\gamma < -\ln(1-q)$. The second factor is $O(n^{-d/(2\alpha')})$. From this we obtain (2.7) for arbitrary $c < d[1+\gamma/\ln(1-q)]/2$.

6. Proof of Proposition 2.5

Though the first statement is well known, we give a proof here for completeness. Recall the notation $Z_{n,m}^x$ introduced by (4.2) and note that, for $m, n \geqslant 1$,

$$
Z_{n+m} = Z_n \sum_x \mu_n\{S_n = x\} Z_{n,m}^x.
$$

Since $Z_{n,m}^x$ has the same law as Z_m, we have by Jensen's inequality that

$$
\ln Z_{n+m} \geqslant \ln Z_n + \sum_x \mu_n\{S_n = x\} \ln Z_{n,m}^x.
$$

Recall also the notation $\mathcal{E}_n$ and Q_n^ξ introduced in the proof of Theorem 2.1. Taking expectations and using independence, we obtain

$$
Q[\ln Z_{n+m}] \geqslant Q[\ln Z_n] + Q\left[\sum_x \mu_n\{S_n = x\} Q_n^\xi[\ln Z_{n,m}^x]\right] = Q[\ln Z_n] + Q[\ln Z_m],
$$

that is, $Q[\ln Z_n]$ is superadditive. From the superadditive lemma we see that

$$
\lim_{n\nearrow\infty} \frac{1}{n} Q[\ln Z_n] = \sup_n \frac{1}{n} Q[\ln Z_n] = \psi(\beta).
$$

In order to prove (2.9), we write $\ln Z_n - Q[\ln Z_n]$ as a sum of $(\mathcal{E}_j)_{1\leqslant j\leqslant n}$-martingale differences,

$$
\ln Z_n - Q[\ln Z_n] = \sum_{j=1}^n V_{n,j},
$$

with $V_{n,j} = Q_j^\xi[\ln Z_n] - Q_{j-1}^\xi[\ln Z_n]$. Set

$$\hat{e}_{n,j} = \exp\left(\sum_{1\leq k\leq n, k\neq j} (\beta\xi(S_k, k) - \lambda(\beta))\right), \qquad \hat{Z}_{n,j} = P[\hat{e}_{n,j}].$$

Clearly $Q_j^\xi[\ln \hat{Z}_{n,j}] = Q_{j-1}^\xi[\ln \hat{Z}_{n,j}]$, and hence,

$$V_{n,j} = Q_j^\xi\left[\ln \frac{Z_n}{\hat{Z}_{n,j}}\right] - Q_{j-1}^\xi\left[\ln \frac{Z_n}{\hat{Z}_{n,j}}\right].$$

By (3.2) in Lemma 3.1 with $\eta_\cdot = \eta(\cdot, j) = \exp(\beta\xi(\cdot, j) - \lambda(\beta))$ and $\alpha_\cdot = P[\hat{e}_{n,j} : S_j = \cdot]/\hat{Z}_{n,j}$, we have that

$$-Q_{j-1}^\xi\left[\ln \frac{Z_n}{\hat{Z}_{n,j}}\right] = -Q_{j-1}^\xi\left[Q\left[\ln\left(\sum_x \alpha_x \eta(x, j)\right)\middle|\mathcal{E}_{n,j}\right]\right]$$
$$\in [0, c],$$

where $\mathcal{E}_{n,j} = \sigma[\xi(\cdot, k); 1 \leq k \leq n, k \neq j]$; note that $\eta(x, j)$ is independent of $\mathcal{E}_{n,j}$ and that $\alpha_\cdot$ is $\mathcal{E}_{n,j}$-measurable. Therefore, for $\theta \in \mathbb{R}$, we have by Jensen's inequality that

$$\begin{aligned}
Q[\exp \theta V_{n,j}|\mathcal{E}_{n,j}] &\leq e^{c\theta^-} Q\left[\exp \theta Q_j^\xi\left[\ln \frac{Z_n}{\hat{Z}_{n,j}}\right]\middle|\mathcal{E}_{n,j}\right] \\
&\leq e^{c\theta^-} Q\left[\left(\frac{Z_n}{\hat{Z}_{n,j}}\right)^\theta\middle|\mathcal{E}_{n,j}\right] \\
&= e^{c\theta^-} Q\left[\left(\sum_x \alpha_x \eta(x, j)\right)^\theta\middle|\mathcal{E}_{n,j}\right] \\
&\leq e^{c\theta^-} \sum_x \alpha_x Q[\eta(x, j)^\theta|\mathcal{E}_{n,j}] \\
&= e^{c\theta^-} \sum_x \alpha_x Q[\eta(x, 1)^\theta] \\
&= \exp\{c\theta^- + \lambda(\theta\beta) - \theta\lambda(\beta)\},
\end{aligned}$$

where $\theta^- = \max\{0, -\theta\}$. Finally, we conclude that

$$Q[\exp|V_{n,j}|] \leq Q[\exp(V_{n,j}) + \exp(-V_{n,j})] \leq c_1 < \infty.$$

Therefore, the large deviation estimate for sums of martingale differences due to Lesigne and Volný (2001, Theorem 3.2) applies to our case, yielding (2.9). The final statement in Proposition 2.5 is now a simple consequence of the Borel–Cantelli lemma.

Remark 6.1. The stronger assumption (2.10) implies the exponential concentration (2.11). In what follows, $c_i = c_i(\delta)$ $(i = 1, 2)$ and $c_i = c_i(\beta, \delta)$ $(i = 3, 4, \ldots)$ are positive constants. Note first that (2.10) implies that $\lambda(\beta) \leqslant c_1 + c_2\beta^2$ for all $\beta \in \mathbb{R}$, and hence that

$$Q[\exp(\theta V_{n,j})|\mathcal{E}_{n,j}] \leqslant \exp(c_3 + c_4\theta^2), \qquad \text{for all } \theta \in \mathbb{R}, \tag{6.1}$$

by the computations in our proof of Proposition 2.5. By expanding the exponential and using the fact $Q[V_{n,j}|\mathcal{E}_{n,j}] = 0$, we can improve (6.1) into the stronger form

$$Q[\exp(\theta V_{n,j})|\mathcal{E}_{n,j}] \leqslant \exp(c_5\theta^2), \qquad \text{for all } \theta \in \mathbb{R}. \tag{6.2}$$

It is now straightforward to show that (2.11) follows from (6.2) and a standard Gaussian estimate for a martingale.

Acknowledgements

We would like to thank Philippe Carmona and Yueyun Hu for sending us the preprint of their paper prior to its publication. FC wishes to thank the Graduate School of Science of Kyoto University for its hospitality. FC is partially supported by CNRS UMR 7599, Laboratoire de Probabilités et Modèles Aléatoires. TS is partially supported by Japan Society for the Promotion of Science (JSPS) Grant-in-Aid for Scientific Research, Kiban(C) 13640103. NY is partially supported by JSPS Grant-in-Aid for Scientific Research, Shorei(A) 40240303.

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Received January 2002 and revised August 2002