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Optimizing a portfolio of power-producing plants

JURI HINZ

Mathematisches Institut, Universität Tübingen, 72076 Tübingen, Germany.
E-mail: juri.hinz@uni-tuebingen.de

We address the question of the optimal allocation of electricity production resources among several production technologies. We consider the effect of competition among different power generators on the equilibrium of the real-time electricity market. For this equilibrium optimal strategies of electricity producers are obtained, yielding for each technology the production capacity to be installed.

Keywords: auction; electricity risk; equilibrium model.

1. Introduction

The recent world-wide introduction of competition to electricity production and trading raises a number of interesting problems concerning market design, the estimation of risk, and the question of optimal strategies for power producers. This paper addresses the last problem. The basic point is that after liberalization of the electricity market, production capacities must be optimized with respect to new requirements: each market participant has to take into account the randomness of electricity demand and the impact of his competitors on the electricity price to decide how much production capacity to install and how to allocate it among different types of generators (typically, to find the right proportion of base-load and peak-load generators). Moreover, the electricity producer has to establish an optimal production plan for each generator type, depending on the electricity price. We consider the optimization of a portfolio consisting of diverse electricity production units which are utilized in the *real-time market* for electricity.

We suppose that the trading rules obey the requirements of the *pay-as-bid auction*. Let us explain how this works. A real-time electricity market is different from a commodity spot market in that it must match demand and supply continuously to maintain the electrical equilibrium in the network. The electricity *system price* is adjusted hourly by the *system operator* by the following procedure. Each electricity producer submits a schedule for each hour of the next day consisting of a bid quantity and a bid price for power which he is willing to sell at that price. The system operator arranges the bids for each hour in increasing price order. The system price set for the current hour equals the bid price of the last generator needed to meet the demand. Those producers who are *in merit* (i.e. whose bid price was below or equal to the system price) supply power and obtain their bid price. Other producers suffer a loss since the fixed costs for their idle production capacities have to be met. Qualitatively, each producer individually has to solve the optimization problem of

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submitting a bid price in such a way that the potential gain of production is optimally balanced against the possible loss of being idle. This decision depends on the cumulative bid quantity submitted by competitors below his bid price, on the distribution of the electricity demand, but also on the production technology: for a base-load generator (high fixed, low variable costs) the optimal bid price is lower than for a peak-load generator (low fixed, high variable cost). Experience with competitive electricity markets shows that simultaneous individual bid optimization drives the market to an equilibrium such that each production technology achieves a price interval where bids exclusively from this technology are placed. Moreover, within each price interval there is a fixed total bid quantity. This issue indicates that each electricity production technology possesses a uniquely determined total capacity to be optimally installed in the market. If so and if such an equilibrium is unique, then a producer may optimize the allocation of his production capacities among different technologies as follows. Calculate for each technology the total equilibrium capacity to be optimally installed and compare it to the actually existing capacity. Increase one's capacity for those technologies where a shortage is determined until the optimal equilibrium amount is reached in the market. At the same time, reduce one's capacity for those technologies where there is an excess.

Among other work on the modelling of deregulated electricity markets, Eydeland and Geman (1998) is devoted to the question of pricing electricity derivatives. An economical mechanism of electricity price formation is discussed in Bessembinder and Lemmon (2000), and the statistical properties of real-world prices are considered in Schwartz and Lucia (2002). Hinz (2003a) discusses system price distribution for a single-technology real-time market. Hinz (2003b) proves the existence and the uniqueness of market equilibrium for the *system-marginal-price* electricity auction, where, in contrast to the *pay-as-bid auction*, each producer who is in merit sells electricity at the system price. An economic comparison of both auction types is given in Federico and Rahman (2001).

2. Minimal-saturated installed capacities

Our concept of equilibrium is based on the following consideration: after all producers have submitted their schedules, the system operator determines the production capacity $\mathcal{I}(p)$ installed at the price $p \geqslant 0$ by summing all bids at bid price less than or equal to p. The non-decreasing installed capacity $\mathcal{I}(\cdot)$ is saturated if renting a small production unit of some technology to submit a schedule at a price p is not better than doing nothing. The idea here is that in the real market, the installed capacity *must always be saturated* since otherwise additional producers will submit schedules until saturation is reached. Moreover, the list of bids must be minimal-saturated in the sense that removing an arbitrary bid leads to loss of saturation.

Let us make the notion of *minimal-saturated installed capacity* mathematically precise. Denote by Q the electricity demand within one hour, a non-negative random variable on a probability space $(\Omega, \mathcal{F}, P)$ with distribution function F such that

$$[0, \infty[\to \mathbb{R}, \quad q \mapsto F(q) := P(Q \leqslant q) \qquad \textit{is strictly increasing and continuous}. \tag{1}$$

Suppose that the distribution of Q is known to all producers. Assume that $N \in \mathbb{N}$ different technologies are used in the electricity market, and that, for all $i = 1, \ldots, N$, p_i^{f} and p_i^{v} are respectively fixed and variable costs of the ith technology, expressed in US dollars per megawatt-hour, such that

$$p_i^{\mathrm{f}} > 0, \; p_i^{\mathrm{v}} \geqslant 0, \qquad i = 1, \ldots, N,$$
$$(p_i^{\mathrm{f}}, p_i^{\mathrm{v}}) \neq (p_j^{\mathrm{f}}, p_j^{\mathrm{v}}) \qquad \text{for all } i, j \in \{1, \ldots, N\}, \, i \neq j. \tag{2}$$

Denote by $p_i^{\mathrm{fv}} = p_i^{\mathrm{f}} + p_i^{\mathrm{v}}$ the full costs for all $i = 1, \ldots, N$, and set $\underline{p}^{\mathrm{fv}} := \min_{i=1}^{N} p_i^{\mathrm{fv}}$. Furthermore, write $J \subset [0, \infty[$ for the set of all bid prices which the system operator can accept. Let us suppose that J is a discrete set, since the money unit is not infinitely divisible. Note that J may be bounded due to a price cap. Let $c > 0$ be the capacity (in megawatt-hours) of a small rentable production unit belonging to technology $i = 1, \ldots, N$. Then, for the non-decreasing installed capacity $\mathcal{I}$: $J \to [0, \infty[$, the income of an additional producer depends on his strategy of renting the unit of the technology $i \in \{1, \ldots, N\}$ and submitting a schedule at the price $p \in J$, which yields a random gain

$$G^{\mathcal{I}}(p, i) := c(p - p_i^{\mathrm{v}})1_{\{Q > \mathcal{I}(p)\}} - cp_i^{\mathrm{f}}, \qquad \text{for all } p \in J, \, i = 1, \ldots, N,$$

or remaining idle, which gives a non-random gain of zero: $G^{\mathcal{I}}(\text{idle}) = 0$.

Remark 1. Note that the gain $G^{\mathcal{I}}(p, i)$ can be modelled in this way only if c is small in the sense that one can neglect the power surplus which may occur after the unit is switched on. This ensures that the entire production is sold as the unit runs. We suppose that, for each technology, there exists a rentable unit of such small capacity $c > 0$. This assumption is justified by the observation that, in reality, there exists a market for diverse agreements on free electricity production capacity and the corresponding contracts are not constrained to a physical production unit. Hence, the ability to rent an (abstract) unit of capacity $c > 0$ of arbitrary technology does not seem to be a strong assumption. However, the critical point here is to assume that the market price for renting the unit will be equal to the unit's fixed costs, which is economically justified in the long run.

Suppose that the risk aversion of the additional producer is described by the strictly increasing concave utility function $U \in C(\mathbb{R})$, giving the utility functional $\mathcal{U}^{\mathcal{I}}$ which evaluates his strategies by

$$\mathcal{U}^{\mathcal{I}}(p, i) = \mathrm{E}(U(G^{\mathcal{I}}(p, i))), \qquad \text{for all } p \in J, \, i = 1, \ldots, N,$$

for the case where a schedule is submitted, and $\mathcal{U}^{\mathcal{I}}(\text{idle}) = \mathrm{E}(U(G^{\mathcal{I}}(\text{idle}))) = U(0)$ otherwise.

Definition 1. *A non-decreasing installed capacity $\mathcal{I}$: $J \to [0, \infty[$ is called minimal-saturated for the additional producer, with c, $(p_i^{\mathrm{f}}, p_i^{\mathrm{v}})_{i=1}^{N}$, U as above, if the following conditions are fulfilled:*

(a) For all $p \in J$ and $i \in \{1, \ldots, N\}$, $\mathcal{U}^{\mathcal{I}}(p, i) \leqslant U(0)$.

(b) If $p \in J$ satisfies $\mathcal{I}(p) > 0$, then there exists $i \in \{1, \ldots, N\}$ with $\mathcal{U}^{\mathcal{I}}(p, i) = U(0)$.

Intuitively, (a) says that for minimal-saturated installed capacity, the best strategy is to remain idle, since either the price is too low to cover production costs or there is already sufficient capacity at this price. However, (b) says that if there is capacity $\mathcal{I}(p) > 0$ at price p, then by removing an arbitrarily small quantity of bids below p we obtain an installed capacity $\tilde{\mathcal{I}}$ with $0 < \tilde{\mathcal{I}}(p) < \mathcal{I}(p)$ such that there exists a technology $i \in \{1, \ldots, N\}$ which could be profitably placed at p since

$$\mathcal{U}^{\tilde{\mathcal{I}}}(p, i) > U(0) \qquad \text{for all } i \in \{1, \ldots \mathrm{N}\} \text{ with } \mathcal{U}^{\mathcal{I}}(p, i) = U(0).$$

Hence, (a) ensures that $\mathcal{I}$ is saturated while (b) serves that it is in fact minimal-saturated.

Remark 2. As pointed out by a referee, Definition 1 implies that all market participants estimate their risk in terms of 'marginal utility'. In fact, (a) means that no electricity producer adds the position $G(p, i)$ to his own portfolio if the additional producer (without any position) prefers to be idle rather than to follow (p, i). Clearly, this point of view neglects all existing portfolios, since otherwise one would postulate instead of (a) that, for each market participant $k = 1, \ldots, K$, at equilibrium

$$\mathrm{E}(U_k(X_k + G^{\mathcal{I}}(p, i))) \leqslant \mathrm{E}(U_k(X_k)),$$

where X_k denotes the equilibrium revenue and U_k the utility function of the agent k. However, the above equation follows from Definition 1(a) by neglecting the revenue $X_k := 0$ to consider merely the marginal income $G^{\mathcal{I}}(p, i)$ and by supposing that $U_k - U_k(0) \leqslant U - U(0)$ for all $k = 1, \ldots, K$, meaning that the additional producer is the least risk-averse market player. In this sense, our model is not completely compatible with the utility-based point of view. Still, the minimal-saturated installed capacity reflects the equilibrium of a market where all producers estimate their risk separately for each submitted megawatt-hour using the strategy of the additional producer as a margin for the maximal risk which can be incurred.

The minimal-saturated installed capacity is unique and is obtained explicitly. Let us introduce the functions

$$f_i\colon\ [0, \infty[\to \mathbb{R}, \qquad p \mapsto \frac{U(cp - cp_i^{\mathrm{fv}}) - U(0)}{U(cp - cp_i^{\mathrm{fv}}) - U(-cp_i^{\mathrm{f}})} 1_{]p_i^{\mathrm{fv}}, \infty[}(p), \qquad i = 1, \ldots, N. \tag{3}$$

The intuition behind these functions comes from the observation that, for $p > p_i^{\mathrm{fv}}$,

$$U(cp - cp_i^{\mathrm{fv}})P(Q > \mathcal{I}(p)) + U(-cp^f)P(Q \leqslant \mathcal{I}(p)) = \mathcal{U}^{\mathcal{I}}(p, i) \leqslant U(0), \tag{4}$$

(recall Definition 1(a)) is equivalently transformed to

$$\mathcal{I}(p) \geqslant F^{-1}\left(\frac{U(cp - cp_i^{\mathrm{fv}}) - U(0)}{U(cp - cp_i^{\mathrm{fv}}) - U(-cp_i^{\mathrm{f}})}\right). \tag{5}$$

That is, a minimal-saturated installed capacity $\mathcal{I}$ should satisfy (5) for all $p > p_i^{\mathrm{fv}}$ for arbitrary $i \in \{1, \ldots, N\}$. Moreover, since (4) and (5) do hold with equality simultaneously, we ensure Definition 1(b) by postulating assertion (5) for $\mathcal{I}(p) > 0$, with equality for some $i \in \{1, \ldots N\}$. Let us make this concept more precise.

Proposition 1. *Under assumptions* (1) *and* (2) *for an additional producer, with c,* $(p_i^{\mathrm{f}}, p_i^{\mathrm{v}})_{i=1}^N$, *U as above, there exists a unique minimal-saturated installed capacity* $\mathcal{I}^*\colon J \to [0, \infty[$ *with*

$$\mathcal{I}^*(p) = F^{-1}\left(\max_{i=1}^{N} f_i(p)\right), \qquad \text{for all } p \in J. \tag{6}$$

Proof. Let us show that formula (6) does indeed define a minimal-saturated installed capacity. First, we observe that $\mathcal{I}^*(p) > 0$ if and only if $p \in]\underline{p}^{\mathrm{fv}}, \infty[\cap J$. If $p \in]\underline{p}^{\mathrm{fv}}, \infty[\cap J$, then $p > p_i^{\mathrm{fv}}$ for some $i \in \{1, \ldots, N\}$ and we have $f_i(p) > 0$ since $U(cp - cp_i^{\mathrm{fv}}) > U(0) > U(-cp_i^{\mathrm{f}})$ because U is strictly increasing. This implies that $\mathcal{I}^*(p) > 0$ since F^{-1} is strictly increasing on $[0, \infty[$ with $F^{-1}(0) = 0$. On the other hand, if $p \in [0, \underline{p}^{\mathrm{fv}}] \cap J$, then $\mathcal{I}^*(p) = 0$ by (3) and (6).

Now we shall see that Definition 1 in fact applies to $\mathcal{I}^*$. Let us show Definition 1(a). If $p \leqslant p_i^{\mathrm{fv}}$, then we have

$$G^{\mathcal{I}^*}(p, i) = c(p - p_i^{\mathrm{fv}})1_{\{Q > \mathcal{I}^*(p)\}} - cp_i^{\mathrm{f}} 1_{\{Q \leqslant \mathcal{I}^*(p)\}} \leqslant 0, \tag{7}$$

which yields $\mathcal{U}^{\mathcal{I}^*}(p, i) = \mathrm{E}(U(G^{\mathcal{I}^*}(p, i))) \leqslant U(0)$. If $p > p_i^{\mathrm{fv}}$, then

$$f_i(p) = \frac{U(cp - cp_i^{\mathrm{fv}}) - U(0)}{U(cp - cp_i^{\mathrm{fv}}) - U(-cp_i^{\mathrm{f}})},$$

and from (6) we have $\mathcal{I}^*(p) \geqslant F^{-1}(f_i(p))$ which implies $\mathcal{U}^{\mathcal{I}^*}(p, i) \leqslant U(0)$. Turning to Definition 1(b), if $\mathcal{I}^*(p) > 0$, then choose $j \in \{1, \ldots, N\}$ so as to attain the maximum in (6),

$$0 < \mathcal{I}^*(p) = F^{-1}(f_j(p)) = F^{-1}\left(\frac{U(cp - cp_j^{\mathrm{fv}}) - U(0)}{U(cp - cp_j^{\mathrm{fv}}) - U(-cp_j^{\mathrm{f}})}\right),$$

giving $\mathcal{U}^{\mathcal{I}^*}(p, j) = U(0)$.

Let us now show uniqueness. Suppose that $\tilde{\mathcal{I}}$ is some minimal-saturated installed capacity and $p \in [0, \underline{p}^{\mathrm{fv}}] \cap J$. Then $\tilde{\mathcal{I}}(p) = 0$ since otherwise $\tilde{\mathcal{I}}(p) > 0$ would yield

$$\mathcal{U}^{\tilde{\mathcal{I}}}(p, i) = U(cp - cp_i^{\mathrm{fv}})P(Q > \tilde{\mathcal{I}} p)) + U(-cp_i^{\mathrm{f}})P(Q \leqslant \tilde{\mathcal{I}} p)) < U(0), \qquad \text{for all } i \in \{1, \ldots, N\},$$

due to $p \leqslant p_i^{\mathrm{fv}}$, $p_i^{\mathrm{f}} > 0$ for all $i = 1, \ldots, N$, which contradicts Definition 1(b), and we obtain $\mathcal{I}^*(p) = 0 = \tilde{\mathcal{I}}(p)$ for all $p \in [0, \underline{p}^{\mathrm{fv}}] \cap J$. For $p \in]\underline{p}^{\mathrm{fv}}, \infty[\cap J$ we have by Definition 1(a) that $\mathcal{U}^{\tilde{\mathcal{I}}}(p, i) \leqslant U(0)$ for all $i \in \{1, \ldots, N\}$. In particular, for $i \in \{1, \ldots, N\}$, with $p_i^{\mathrm{fv}} < p$ we have

$$\mathcal{U}^{\tilde{\mathcal{I}}}(p, i) = U(cp - cp_i^{\mathrm{fv}})P(Q > \tilde{\mathcal{I}}(p)) + U(-cp_i^{\mathrm{f}})P(Q \leqslant \tilde{\mathcal{I}}(p)) \leqslant U(0),$$

which we observe is equivalent to

$$\tilde{\mathcal{I}}(p) \geqslant F^{-1}\left(\frac{U(cp - cp_i^{\text{fv}}) - U(0)}{U(cp - cp_i^{\text{fv}}) - U(-cp_i^{\text{f}})}\right) > 0 \qquad \text{for all } i \in \{1, \ldots, N\} \text{ with } p_i^{\text{fv}} < p,$$

and which implies that

$$\tilde{\mathcal{I}}(p) \geqslant F^{-1}\left(\max_{i=1}^{N} f_i(p)\right) > 0 \qquad \text{for all } p \in]\underline{p}^{\text{fv}}, \infty[\cap J, \tag{8}$$

since $f_i(p) = 0$ for all $i \in \{1, \ldots, N\}$ with $p_i^{\text{fv}} \geqslant p$. Thus, for $p \in]\underline{p}^{\text{fv}}, \infty[\cap J$ the installed capacity $\tilde{\mathcal{I}}(p)$ is positive and we have by Definition 1(b) that there exists $j \in \{1, \ldots, N\}$ with

$$\mathcal{U}^{\tilde{\mathcal{I}}}(p, j) = U(cp - cp_j^{\text{fv}})P(Q > \tilde{\mathcal{I}}(p)) + U(-cp_j^{\text{f}})P(Q \leqslant \tilde{\mathcal{I}}(p)) = U(0),$$

which gives

$$\tilde{\mathcal{I}}(p) = F^{-1}\left(\frac{U(cp - cp_j^{\text{fv}}) - U(0)}{U(cp - cp_j^{\text{fv}}) - U(-cp_j^{\text{f}})}\right)$$

and equivalently,

$$\text{for each } p \in]\underline{p}^{\text{fv}}, \infty[\cap J, \qquad \text{there exists } j := j(p) \in \{1, \ldots, N\} \text{ with } \tilde{\mathcal{I}}(p) = F^{-1}(f_j(p)). \tag{9}$$

Combining (8) and (9), we see that $\tilde{\mathcal{I}}(p) = \mathcal{I}^*(p)$ for all $p \in]\underline{p}^{\text{fv}}, \infty[\cap J$. □

3. Optimal allocation of production capacity

Let us now turn to the question of how much capacity must be installed for each technology to reach equilibrium. We have to determine those areas in J where all bids from technology $i \in \{1, \ldots N\}$ are placed.

Definition 2. *We say that $p \in]\underline{p}^{\text{fv}}, \infty[\cap J$ is occupied by technology $j \in \{1, \ldots, N\}$ if*

$$U(0) = \mathcal{U}^{\mathcal{I}^*}(p, j). \tag{10}$$

Futhermore, we say that $p \in]\underline{p}^{\text{fv}}, \infty[\cap J$ is strictly occupied by (the uniquely determined) technology $j \in \{1, \ldots, N\}$ if

$$U(0) = \mathcal{U}^{\mathcal{I}^*}(p, j) > \mathcal{U}^{\mathcal{I}^*}(p, i), \qquad \textit{for all } i \in \{1, \ldots, N\} \backslash \{j\}. \tag{11}$$

Again, this definition is justified by the observation that if p is occupied by j, then by removing a small quantity of bids below p from $\mathcal{I}^*$, we obtain an installed capacity $\tilde{\mathcal{I}}$ with $0 < \tilde{\mathcal{I}}(p) < \mathcal{I}^*(p)$ where it is still worth offering a bid at p for technology j as $\mathcal{U}^{\tilde{\mathcal{I}}}(p, j) > \mathcal{U}^{\mathcal{I}^*}(p, j) = U(0)$. Similarly, if p is strictly occupied by $j \in \{1, \ldots, N\}$ then we see that at p, only technology j can survive since by removing a sufficiently small quantity of bids below p from $\mathcal{I}^*$, we obtain $\tilde{\mathcal{I}}$ with

$$F^{-1}\Big(\max_{i=1\ldots N, i\neq j} f_i(p)\Big) < \tilde{\mathcal{I}}(p) < \mathcal{I}^*(p),$$

to see that only j can be placed profitably at p:

$$\mathcal{U}^{\tilde{\mathcal{I}}}(p, i) < U(0) < \mathcal{U}^{\tilde{\mathcal{I}}}(p, j), \qquad i \in \{1, \ldots, N\}\backslash\{j\}.$$

For each $J' \subset J$ we denote by $\mu^*(J')$ the total capacity of J', where μ^* is the measure on $\mathcal{P}(J)$ given by $\mu^*(\{p\}) = \mathcal{I}^*(p) - \max\{\mathcal{I}^*(p')\colon\ p' \in J,\ p' < p\}$ for all $p \in J$. The total capacities $(C_i^*)_{i=1}^N$ to be installed at the equilibrium for technologies $i = 1, \ldots, N$ are not uniquely determined, since $\mu^*(\{p\})$ may come from each technology which occupies p. However, this uncertainty is not crucial and we shall adopt the following definition.

Definition 3. *$(C_i^*)_{i=1}^N \in \mathbb{R}^N$ is called an optimal capacity allocation if there exists a partition $(Z_i)_{i=1}^N$ of J such that, for all $i = 1, \ldots, N$:*

(a) $\{p \in J\colon\ p$ is strictly occupied by $i\} \subseteq Z_i \subseteq \{p \in J\colon p$ is occupied by $i\}$;
(b) $C_i^* = \mu^*(Z_i)$.

It turns out that the prices occupied by a technology form an interval. Consequently, we find an optimal capacity allocation by calculating increments of $\mathcal{I}^*$ on the corresponding intervals.

Proposition 2. *Under assumptions* (1), (2) *there exist $\{i_1, \ldots, i_k\} \subset \{1, \ldots, N\}$ and a set $\{\underline{p}^{\mathrm{fv}} = a_0 < \ldots < a_k\} \subset [0, \infty]$ such that, for $j = 1, \ldots, N$,*

$$C_j^* = \begin{cases} \mathcal{I}^*(a_l) - \mathcal{I}^*(a_{l-1}), & \text{if } j = i_l \in \{i_1, \ldots, i_k\}, \\ 0, & \text{if } j \notin \{i_1, \ldots, i_k\}, \end{cases} \tag{12}$$

defines an optimal capacity allocation.

The proof is based on the fact that two functions f_i, f_j, $i \neq j$, intersect at most once in the sense of (ii) from the following lemma:

Lemma 1. *(i) Define $R(x, h) := (U(x+h) - U(0))(U(x) - U(0))^{-1}$ for all $x > 0$, $h \geqslant 0$. Then, for each $h > 0$, $R(\cdot, h)$ is strictly decreasing, with* $\lim_{x\to 0} R(x, h) = \infty$, $\lim_{x\to\infty} R(x, h) = 1$.

(ii) If $p_i^{\mathrm{fv}} \leqslant p_j^{\mathrm{fv}}$ then there exists $p^* \in\,]p_j^{\mathrm{fv}}, \infty[$ with $f_i(p^*) = f_j(p^*)$ if and only if $p_i^{\mathrm{fv}} < p_j^{\mathrm{fv}}$ and $p_i^{\mathrm{f}} > p_j^{\mathrm{f}}$. In this case $f_i(p) > f_j(p)$ holds for all $p \in\,]p_i^{\mathrm{fv}}, p^*[$, and $f_i(p) < f_j(p)$ is satisfied for all $p \in\,]p^*, \infty[$.

(iii) For all $j = 1, \ldots, N$, the set $T_j := \{p > \underline{p}^{\mathrm{fv}}\colon f_i(p) \leqslant f_j(p)$ for all $i = 1, \ldots, N\}$ forms an interval, and the interior T_j° of T_j satisfies $T_j^\circ = \{p > \underline{p}^{\mathrm{fv}}\colon f_i(p) < f_j(p)$ for all $i \neq j\}$.

(iv) $\cup_{i=1}^N T_i = \,]\underline{p}^{\mathrm{fv}}, \infty[$ and $T_i^\circ \cap T_j^\circ = \varnothing$ for $i, j \in \{1, \ldots, N\}$, $i \neq j$.

Proof. (i) For $h > 0$ and $0 < x_1 < x_2$, define $\alpha_1 := R(x_1, h)$, $\alpha_2 := R(x_2, h)$. Then

$$U(x_2 + h) - U(0) = \alpha_2(U(x_2) - U(0)), \tag{13}$$

$$U(x_1 + h) - U(0) = \alpha_1(U(x_1) - U(0)). \tag{14}$$

If $\alpha_1 \leqslant \alpha_2$, subtracting (14) from (13) gives

$$U(x_2 + h) - U(x_1 + h) \geqslant \alpha_1(U(x_2) - U(x_1)),$$

where $\alpha_1 > 1$ since U is strictly increasing. Hence, we obtain

$$U(x_2 + h) - U(x_1 + h) > U(x_2) - U(x_1) \tag{15}$$

contradicting $\alpha_1 \leqslant \alpha_2$ since

$$]0, \infty[\to \mathbb{R}, \qquad x \mapsto U(x + h) - U(x)$$

is decreasing for $h > 0$ due to concavity of U. The limit $\lim_{x \to 0} R(x, h) = \infty$ is obvious. Let us show that $\lim_{x\to\infty} R(x, h) = 1$ by supposing on the contrary that $R(x, h) \geqslant \beta$ for all $x > 0$ with some $\beta > 1$. Then

$$R(kh, h) = \frac{U((k+1)h) - U(0)}{U(kh) - U(0)} \geqslant \beta, \qquad \text{for all } k \in \mathbb{N},$$

implies the exponential growth $U((k + 1)h) - U(0) \geqslant \beta^k(U(h) - U(0))$ for $k \in \mathbb{N}$, which is impossible due to concavity of U:

$$U((k+1)h) - U(0) \leqslant (k+1)h\frac{U(h) - U(0)}{h}, \qquad \text{for all } k \in \mathbb{N}.$$

(ii) Suppose $p_i^{\mathrm{fv}} \leqslant p_j^{\mathrm{fv}}$. Then $f_i(p) < f_j(p)$ for $p \in]p_j^{\mathrm{fv}}, \infty[$ if and only if $f_i(p)^{-1} > f_j(p)^{-1}$, which is written as

$$\frac{U(cp - cp_i^{\mathrm{fv}}) - U(0) + U(0) - U(-cp_i^{\mathrm{f}})}{U(cp - cp_i^{\mathrm{fv}}) - U(0)} > \frac{U(cp - cp_j^{\mathrm{fv}}) - U(0) + U(0) - U(-cp_j^{\mathrm{f}})}{U(cp - cp_j^{\mathrm{fv}}) - U(0)}$$

and is equivalent to

$$\frac{U(cp - cp_i^{\mathrm{fv}}) - U(0)}{U(cp - cp_j^{\mathrm{fv}}) - U(0)} < \frac{U(0) - U(-cp_i^{\mathrm{f}})}{U(0) - U(-cp_j^{\mathrm{f}})}. \tag{16}$$

The left-hand side of inequality (16) equals $R(cp - cp_j^{\mathrm{fv}}, cp_j^{\mathrm{fv}} - cp_i^{\mathrm{fv}})$ with R from (i). Let us denote the right-hand side of (16) by $\alpha(p_i^{\mathrm{f}}, p_j^{\mathrm{f}})$, so that $R(cp - cp_j^{\mathrm{fv}}, cp_j^{\mathrm{fv}} - cp_i^{\mathrm{fv}}) < \alpha(p_i^{\mathrm{f}}, p_j^{\mathrm{f}})$. Hence, we obtain, for $p \in]p_j^{\mathrm{fv}}, \infty[$,

$$f_i(p) < f_j(p) \iff R(cp - cp_j^{\mathrm{fv}}, cp_j^{\mathrm{fv}} - cp_i^{\mathrm{fv}}) < \alpha(p_i^{\mathrm{f}}, p_j^{\mathrm{f}}), \tag{17}$$

$$f_i(p) = f_j(p) \iff R(cp - cp_j^{\mathrm{fv}}, cp_j^{\mathrm{fv}} - cp_i^{\mathrm{fv}}) = \alpha(p_i^{\mathrm{f}}, p_j^{\mathrm{f}}), \tag{18}$$

$$f_i(p) > f_j(p) \iff R(cp - cp_j^{\mathrm{fv}}, cp_j^{\mathrm{fv}} - cp_i^{\mathrm{fv}}) > \alpha(p_i^{\mathrm{f}}, p_j^{\mathrm{f}}), \tag{19}$$

where the equivalences (18), (19) are obtained by the same arguments as (17). Now if $p \in]p_j^{\mathrm{fv}}, \infty[$ satisfies $f_i(p) = f_j(p)$, then p fulfils (18). Here $p_i^{\mathrm{fv}} = p_j^{\mathrm{fv}}$ is impossible since otherwise

$$R(cp - cp_j^{\mathrm{fv}},\, cp_j^{\mathrm{fv}} - cp_i^{\mathrm{fv}}) = 1 = \alpha(p_i^{\mathrm{f}},\, p_j^{\mathrm{f}})$$

giving $(p_i^{\mathrm{fv}}, p_i^{\mathrm{f}}) = (p_j^{\mathrm{fv}}, p_j^{\mathrm{f}})$, which is excluded by (2). Consequently, we have $p_i^{\mathrm{fv}} < p_j^{\mathrm{fv}}$, hence $\alpha(p_i^{\mathrm{f}}, p_j^{\mathrm{f}}) > 1$, which yields $p_i^{\mathrm{f}} > p_j^{\mathrm{f}}$.

Let us now show the reverse direction. Suppose that $p_i^{\mathrm{fv}} < p_j^{\mathrm{fv}}$ and $p_i^{\mathrm{f}} > p_j^{\mathrm{f}}$. Then $\alpha(p_i^{\mathrm{f}}, p_j^{\mathrm{f}}) > 1$, and from (i) it follows that there exists $p \in \,]p_j^{\mathrm{fv}}, \infty[$ with

$$R(cp - cp_j^{\mathrm{fv}},\, cp_j^{\mathrm{fv}} - cp_i^{\mathrm{fv}}) = \alpha(p_i^{\mathrm{f}},\, p_j^{\mathrm{f}}),$$

which is equivalent by (18) to $f_i(p) = f_j(p)$.

Consider now, for the case $p_i^{\mathrm{fv}} < p_j^{\mathrm{fv}}$ and $p_i^{\mathrm{f}} > p_j^{\mathrm{f}}$, a solution $p^* \in \,]p_j^{\mathrm{fv}}, \infty[$ of

$$f_i(p) = f_j(p), \qquad p \in \,]p_j^{\mathrm{fv}}, \infty[$$

which satisfies $R(cp^* - cp_j^{\mathrm{fv}}, cp_j^{\mathrm{fv}} - cp_i^{\mathrm{fv}}) = \alpha(p_i^{\mathrm{f}}, p_j^{\mathrm{f}})$ by (18). Using (i), we conclude that

$$R(cp - cp_j^{\mathrm{fv}},\, cp_j^{\mathrm{fv}} - cp_i^{\mathrm{fv}}) > \alpha(p_i^{\mathrm{f}},\, p_j^{\mathrm{f}}), \qquad \text{for } p \in \,]p_j^{\mathrm{fv}}, p^*[,$$

$$R(cp - cp_j^{\mathrm{fv}},\, cp_j^{\mathrm{fv}} - cp_i^{\mathrm{fv}}) < \alpha(p_i^{\mathrm{f}},\, p_j^{\mathrm{f}}), \qquad \text{for } p \in \,]p^*, \infty[,$$

which implies, due to (17) and (19), the inequalities

$$f_i(p) > f_j(p), \qquad \text{for all } p \in \,]p_j^{\mathrm{fv}}, p^*[,$$

$$f_i(p) < f_j(p), \qquad \text{for all } p \in \,]p^*, \infty[.$$

(iii) Let $j \in \{1, \ldots, N\}$ and p, p'' satisfy $\underline{p}^{\mathrm{fv}} < p < p''$ and

$$f_i(p) \leqslant f_j(p),\; f_i(p'') \leqslant f_j(p''), \qquad \text{for all } i = \{1, \ldots, N\}. \tag{20}$$

Then both values $f_j(p)$, $f_j(p'')$ are positive, hence $p, p'' \in \,]p_j^{\mathrm{fv}}, \infty[$. For $p' \in \,]p, p''[$ it follows that $f_i(p') < f_j(p')$ for all $i \neq j$, since otherwise $f_i(p') \geqslant f_j(p')$ for some $i \neq j$ would imply by the intermediate value theorem that there exists $p^* \in [p, p']$ with $f_i(p^*) = f_j(p^*)$, so that $f_i(p'') > f_j(p'')$ holds by (ii), contradicting (20).

(iv) The equation $\cup_{i=1}^N T_i = \,]\underline{p}^{\mathrm{fv}}, \infty[$ holds by the definition of T_i, $i = 1, \ldots, N$, whereas $T_i^\circ \cap T_j^\circ = \varnothing$ for $i \neq j$ follows immediately from the representation of the sets $(T_i^\circ)_{i=1}^N$ from (iii). □

Proof of Proposition 2. Using Lemma 1(iv), we arrange the non-intersecting open intervals $(T_i^\circ)_{i=1}^N$ in increasing order as

$$T_{i_1}^\circ < T_{i_2}^\circ < \ldots < T_{i_k}^\circ,$$

where $\{i_1, \ldots, i_k\} = \{j \in \{1, \ldots, N\} : T_j^\circ \neq \varnothing\}$. By $\cup_{i=1}^N T_i = \,]\underline{p}^f, \infty[$ from Lemma 1(iv), we have

$$\inf T_{i_1}^\circ = \underline{p}^{\mathrm{fv}}, \quad \sup T_{i_k}^\circ = \infty, \qquad \inf T_{i_l}^\circ = \sup T_{i_{l-1}}^\circ, \quad \text{for all } l = 1, \ldots, k.$$

By definition, the sets $T_{i_l} \cap J$, $T_{i_l}^\circ \cap J$ are exactly those prices which are occupied and strictly occupied by the technology i_l, hence, for $j = 1, \ldots, N$,

$$Z_j = \begin{cases}]\inf T^{\circ}_{i_l}, \sup T^{\circ}_{i_l}] \cap J, & \text{if } j = i_l \in \{i_1, \ldots, i_k\}, \\ \varnothing, & \text{if } j \notin \{i_1, \ldots, i_k\}, \end{cases}$$

is a partition of J matching Definition 3, and an optimal capacity allocation $(C^*_i)_{i=1}^N$ is calculated for $j = 1, \ldots, N$ as

$$C^*_j := \mu(Z_j) = \begin{cases} \mu^*(]\inf T^{\circ}_{i_l}, \sup T^{\circ}_{i_l}] \cap J), & \text{if } j = i_l \in \{i_1, \ldots, i_k\}, \\ 0, & \text{if } j \notin \{i_1, \ldots, i_k\}, \end{cases} \tag{21}$$

which gives (12) with $a_0 := \underline{p}^{\mathrm{fv}}$ and $a_l = \sup\big(]\inf T^{\circ}_{i_l}, \sup T^{\circ}_{i_l}] \cap J\big)$ for $l = 1, \ldots, k$. □

Example. Consider a market where two different production technologies ($N = 2$) compete in a pay-as-bid auction with price cap $\sup J < \infty$. Suppose that the first technology (base-load generators) produces electricity at the price $p_1^{\mathrm{fv}} > 0$ and the second technology (peak-load generators) yields electricity at $p_2^{\mathrm{fv}} > 0$ with $p_1^{\mathrm{fv}} \leqslant p_2^{\mathrm{fv}}$. First, we calculate the intervals

$$T^{\circ}_1 := \{p > p_1^{\mathrm{fv}} : f_1(p) < f_2(p)\}, \qquad T^{\circ}_2 := \{p > p_1^{\mathrm{fv}} : f_2(p) < f_1(p)\}.$$

According to Lemma 1(ii), two alternative situations are possible:

1. There exists $p^* \in]p_2^{\mathrm{fv}}, \infty[$ with $f_1(p^*) = f_2(p^*)$, in which case $T^{\circ}_1 :=]p_1^{\mathrm{fv}}, p^*[$, $T^{\circ}_2 :=]p^*, \infty[$.
2. For all $p \in]p_2^{\mathrm{fv}}, \infty[$ we have $f_1(p) > f_2(p)$, in which case $T^{\circ}_1 :=]p_1^{\mathrm{fv}}, \infty[$, $T^{\circ}_2 := \varnothing$.

In case 1, which occurs by Lemma 1 if and only if $p_1^{\mathrm{fv}} < p_2^{\mathrm{fv}}$ and $p_1^f > p_2^f$, the peak-load generators strictly occupy prices $]p^*, \infty[\cap J$, while in case 2 they could be replaced by those of the base-load technology. For the affine-linear utility function $U: x \mapsto ax + b$ with $a > 0$, $b \in \mathbb{R}$, in case 1 we obtain $p^* = (p_2^{\mathrm{fv}} p_1^f - p_1^{\mathrm{fv}} p_2^f)(p_1^f - p_2^f)^{-1}$. According to (12), we have $a_0 := p_1^{\mathrm{fv}}$, $a_1 := \max\{p \in J : p \leqslant p^*\}$, $a_2 := \max\{p \in J : p > p^*\}$. An optimal capacity allocation is

$$C^*_1 := \mathcal{I}^*(a_1) - \mathcal{I}^*(a_0) = F^{-1}(f_1(a_1)) - F^{-1}(f_1(a_0)) = F^{-1}(f_1(a_1)),$$

$$C^*_2 := \mathcal{I}^*(a_2) - \mathcal{I}^*(a_1) = F^{-1}(f_2(a_2)) - F^{-1}(f_1(a_0)).$$

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