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Multidimensional backward stochastic differential equations with uniformly continuous coefficients

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In this paper we consider the problem of the existence of a solution for backward stochastic differential equations with uniformly continuous coefficients.

Keywords: backward stochastic differential equations; deterministic backward differential equations

1. Introduction

A solution of a backward stochastic differential equation (BSDE) associated with a coefficient f and a terminal value ξ on [0, 1] is an adapted process $(Y_t, Z_t)_{t\leqslant 1}$ such that

$$Y_t = \xi + \int_t^1 f(s, Y_s, Z_s)\mathrm{d}s - \int_t^1 Z_s\,\mathrm{d}B_s, \qquad t \leqslant 1.$$

This type of equation, at least in the nonlinear case, was first introduced by Pardoux and Peng (1990a), who proved the existence and uniqueness of a solution under suitable assumptions on f and ξ, the most important of which are the Lipschitz continuity of f and the square integrability of ξ. Their aim was to give a probabilistic interpretation of a solution to a second-order quasilinear partial differential equation. Since then, these equations have gradually become an important mathematical tool in many fields such as financial mathematics (see, for example, El-Karoui *et al*. 1997a; 1997b; Buckdahn and Hu 1998; Cvitanić and Karatzas 1996), stochastic games and optimal control (Hamadène and Lepeltier 1995a; 1995b; Hamadène *et al*. 1997; 1999; Cvitanić and Karatzas 1996; Dermoune *et al*. 1999), partial differential equations and homogenization (Pardoux and Peng 1990b; 1992; Pardoux 1999; Peng 1991; Darling and Pardoux 1997; Buckdahn and Peng 1999) and construction of Γ-martingales (Darling 1995).

A further problem under widespread discussion is how to improve the existence and uniqueness result of Pardoux and Peng (1990a) by weakening the Lipschitz continuity condition on f. Hamadène (1996), Kobylanski (2000) and Lepeltier and San Martín (1997; 1998) have dealt with the situation where Y is a unidimensional process. They obtained an existence result without assuming f Lipschitz continuous. However, the solution is not necessarily unique. Hamadène (1996) takes f to be just locally Lipschitz, while Kobylanski

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(2000) and Lepeltier and San Martín (1997; 1998) merely assume that f is continuous. In the proofs of these results, the comparison theorem for BSDE solutions plays a crucial role.

The situation where Y is a multidimensional process has also been studied, although the comparison theorem does not apply. In general, however, the existence and uniqueness results are obtained only under weaker regularity assumptions with respect to the component y of f. The improvements are not concerned with the regularity of f in z since it is usually supposed that f is Lipschitz with respect to that component (Pardoux 1999; Darling and Pardoux 1997; Briand and Carmona 2000; Mao 1995). Nevertheless, some work has been done where this latter condition is removed. Pardoux and Peng (1994) assume that f is deterministic and sufficiently regular, while Hamadène *et al*. (1997) suppose that the randomness stems from a Markov process. Finally, Zhou (1999) uses a generalization to the multidimensional case of the comparison theorem. However, he requires some monotonicity conditions and some form of pattern for the components f_i of f.

In this paper we mainly address the problem of the existence of a solution for BSDEs associated with coefficients which are not Lipschitz in z. In fact, we show that the multidimensional BSDE associated with (f, ξ) has a solution if the coefficient f of the BSDE satisfies the following conditions:

(i) $y \mapsto f(t, y, z)$ is uniformly continuous uniformly in (t, ω, z) and its modulus of continuity Φ satisfies Assumption 2 below.
(ii) The function $z \mapsto f(t, y, z)$ is uniformly continuous uniformly in (t, ω, y).
(iii) The ith component f_i of f depends only on the ith row of z.

We give further consideration to the problem of uniqueness, and a result in this direction is given.

Condition (i) is obviously satisfied when f is Lipschitz in y, with $\Phi(x) = kx$. But there are functions which satisfy (i) and are not Lipschitz in y (see Example 2 in Section 3).

Conditions (i) and (ii) imply that f is uniformly continuous with respect to (y, z). Therefore it can be approximated uniformly, on the whole space of (y, z), by a sequence of Lipschitz functions $(f_n)_{n\geqslant 0}$. Furthermore, we introduce a sequence of processes (Y^n, Z^n) which solve the BSDE associated with (f_n, ξ). Working on the components of Y^n, we show that the sequence $(Y^n)_n$ is of Cauchy type. For this purpose we make use of condition (iii) and Girsanov's theorem in order to cancel some troublesome terms by putting them in the martingale part. Assumption 2 on Φ enables us to conclude. The solution of the BSDE associated with (f, ξ) is constructed from the limit of $(Y^n)_n$ and that of $(Z^n)_n$.

This paper is organized as follows. In Section 2 we formulate the problem accurately and give some preliminary results. Section 3 is devoted to the proof of the main theorem and investigates the conditions under which Assumption 2 is satisfied. Finally, in Section 4 we consider the problem of uniqueness.

2. Formulation of the problem and preliminary results

Let $B = (B_t)_{t\leqslant 1}$ be an m-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$, whose natural filtration is denoted $(F_t)_{t\leqslant 1}$, where $F_t = \sigma\{B_s, s \leqslant t\}$. Let $\mathcal{P}$ be the σ-algebra on $[0, 1] \times \Omega$ of F_t-progressively measurable sets. For $k \geqslant 1$, let $H^{2,k}$ be the set of $\mathcal{P}$-measurable processes $V = (V_t)_{t\leqslant 1}$ with values in $\mathbb{R}^k$ such that $\mathrm{E}[\int_0^1 |V_s|^2 \, \mathrm{d}s] < \infty$, and let $S^{2,k}$ be the set of continuous $\mathcal{P}$-measurable processes $V = (V_t)_{t\leqslant 1}$ with values in $\mathbb{R}^k$ such that $\mathrm{E}[\sup_{t\leqslant 1}|V_t|^2] < \infty$.

We are now given two objects: a terminal value $\xi \in L^2(\Omega, F_1, P)$; a coefficient f which is a mapping $(t, \omega, y, z) \mapsto f(t, \omega, y, z)$ from $[0, 1] \times \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times m}$ (where $\mathbb{R}^{d\times m}$ is the space of real matrices with d rows and m columns) to $\mathbb{R}^d$ satisfying the following assumption:

Assumption 1. *The process $(f(t, \omega, 0, 0))_{t\leqslant 1}$ belongs to $H^{2,d}$ and, for any $(y, z) \in \mathbb{R}^d \times \mathbb{R}^{d\times m}$, $(f(t, \omega, y, z))_{t\leqslant 1}$ is $\mathcal{P}$-measurable.*

Let us now introduce the BSDE associated with (f, ξ). A solution of such an equation is a $\mathcal{P}$-measurable process $(Y, Z) = (Y_t, Z_t)_{t\leqslant 1}$ valued in $\mathbb{R}^d \times \mathbb{R}^{d\times m}$ such that:

$$\begin{aligned} &(Y, Z) \in S^{2,d} \times H^{2,d\times m} \\ &Y_t = \xi + \int_t^1 f(s, \omega, Y_s, Z_s)\mathrm{d}s - \int_t^1 Z_s \, \mathrm{d}B_s, \qquad \forall t \leqslant 1. \end{aligned} \tag{1}$$

The following theorem of Pardoux (1999) enables us to affirm the existence and uniqueness of a solution for a BSDE.

Theorem 2.1. *Let g be a mapping $(t, \omega, y, z) \mapsto g(t, \omega, y, z)$ from $[0, 1] \times \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times m}$ to $\mathbb{R}^d$ satisfying the following assumptions, along with Assumption 1 above:*

(i) $|g(t, y, 0)| \leqslant |g(t, 0, 0)| + \phi(|y|)$, *for all t, y, where ϕ is a continuous increasing function from $\mathbb{R}_+$ into itself.*
(ii) $|g(t, y, z) - g(t, y, z')| \leqslant K\|z - z'\|$, *for all t, y, z, z', where* $\|z\| = [\mathrm{tr}(zz^*)]^{1/2}$
(iii) $\langle y - y', g(t, y, z) - g(t, y', z)\rangle \leqslant \mu|y - y'|$, *for all t, y, y', z, where μ is a real number.*
(iv) $y \mapsto g(t, y, z)$ *is continuous, for all t, z.*

Then the BSDE (1) *associated with (g, ξ) has a unique solution (Y, Z).*

Now, let us give a result concerning the solutions of deterministic differential equations which will be useful later.

Proposition 2.2. *Let $\overline{\Phi}$ be a continuous function from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that $\overline{\Phi}(x) \leqslant ax + b$, for all $x \in \mathbb{R}^+$, where a and b are given non-negative constants. Then the deterministic backward differential equation (DBE)*

$$u(t) = \gamma + \int_t^1 \overline{\Phi}(u(s))\mathrm{d}s, \qquad t \leqslant 1, \ \gamma \in \mathbb{R}^+, \tag{2}$$

has a solution u^γ. *In addition, if* $\gamma > 0$ *and* $\overline{\Phi}(x) > 0$, *for all* $x > 0$, *then the solution is unique.*

Proof. For $n \geqslant 0$, let $\overline{\Phi}_n : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\overline{\Phi}_n(x) := \inf_{y\in\mathbb{R}^+}\{\overline{\Phi}(y) + n|x - y|\}$. Since the rate of growth of $\overline{\Phi}$ is at most linear, $\overline{\Phi}_n$ is defined and is Lipschitz. Moreover, the sequence $(\overline{\Phi}_n)_n$ is non-decreasing and converges to $\overline{\Phi}$. So let $u_n : \mathbb{R}^+ \to \mathbb{R}^+$ be the solution of the following DBE:

$$u_n(t) = \gamma + \int_t^1 \overline{\Phi}_n(u_n(s))\mathrm{d}s.$$

Since $\overline{\Phi}_n \leqslant \overline{\Phi}_{n+1}$, then $u_n \leqslant u_{n+1}$. Moreover, for all $n \geqslant 0$, $0 \leqslant u_n(t) \leqslant \tilde{u}(t)$, $t \in [0, 1]$, where $\tilde{u}$ is a function from $\mathbb{R}^+$ into itself such that $\tilde{u}(t) = \gamma + \int_t^1(a\tilde{u}(s) + b)\mathrm{d}s$, $t \leqslant 1$. It follows that, for any $t \in [0, 1]$, the sequence $(u_n(t))_n$ converges increasingly to a function $u^\gamma(t)$. On the other hand, since $\tilde{u}$ is bounded then there exists a constant k such that, for any $n \geqslant 0$, we have $u_n \in [0, k]$. Now on $[0, k]$, by virtue of Dini's theorem, the sequence $(\overline{\Phi}_n)_n$ converges uniformly towards Φ. Then, for any $\epsilon > 0$, there exists a rank n_0 such that for all $n \geqslant n_0$, $\Phi(x) - \epsilon \leqslant \overline{\Phi}_n(x) \leqslant \Phi(x) + \epsilon$. Therefore, for any $t \leqslant 1$, we have

$$\gamma + \int_t^1 \Phi(u_n(s))\mathrm{d}s - \epsilon \leqslant u_n(t) \leqslant \gamma + \int_t^1 \Phi(u_n(s))\mathrm{d}s + \epsilon.$$

Taking the limit as $n \to \infty$ yields u^γ as a solution of (2) since ϵ is arbitrary.

Let us now suppose that $\gamma > 0$ and $\overline{\Phi}(x) > 0$, for all $x > 0$. For $z > 0$, let us set $G(z) = \int_z^1 [\overline{\Phi}(x)]^{-1}\,\mathrm{d}x$ and let u be a solution of (2). It is obvious that $u \geqslant \gamma$ and $(G(u(t)))' = 1$, for all $t \leqslant 1$. Hence $G(u(1)) - G(u(t)) = 1 - t$, which implies that $u(t) = G^{-1}(G(\gamma) - 1 + t)$, $t \leqslant 1$, whence the desired result. □

Uniformly continuous functions can be approximated uniformly in the whole space of (y, z) by Lipschitz functions. To be precise, we have the following lemma.

Lemma 2.3. *Let* g *be a mapping* $(t, \omega, y, z) \mapsto g(t, \omega, y, z)$ *from* $[0, 1] \times \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times m}$ *to* $\mathbb{R}^d$ *satisfying Assumption* 1. *In addition, there exists a continuous function* φ *from* $\mathbb{R}^+ \times \mathbb{R}^+$ *into* $\mathbb{R}^+$ *such that* $\varphi(0, 0) = 0$ *and*

$$|g(t, \omega, y, z) - g(t, \omega, y', z')| \leqslant \varphi(|y - y'|, \|z - z'\|), \qquad \forall t, y, y', z, z', \text{a.s.} \tag{3}$$

Then there exists a sequence $(g_n)_{n\geqslant 0}$ such that:

(i) *For any* $n \geqslant 0$, g_n *is a mapping from* $[0, 1] \times \Omega \times \mathbb{R}^{d+d\times m}$ *into* $\mathbb{R}^d$ *satisfying Assumption* 1 *and which is Lipschitz with respect to* (y, z) *uniformly in* (t, ω).

(ii) For all $\epsilon > 0$, there is an $N_\epsilon \geqslant 0$ such that, for all $n \geqslant N_\epsilon$, $|g_n(t, \omega, y, z) - g(t, \omega, y, z)| \leqslant \epsilon$ for all t, y, z, a.s.

Proof. Let ψ be a function of $C^\infty(\mathbb{R}^{d+d\times m}, \mathbb{R}^+)$ with compact support and which satisfies $\int_{\mathbb{R}^{d+d\times m}} \psi(y, z)\mathrm{d}y\,\mathrm{d}z = 1$. For $n \geqslant 0$, let $\psi_n : (y, z) \in \mathbb{R}^{d+d\times m} \mapsto \psi_n(y, z) = n^2\psi(ny, nz)$ and $g_n := g{*}\psi_n$, the convolution product of g and ψ_n. Therefore,

$$\forall t,\ y,\ z, \qquad g_n(t,\ \omega,\ y,\ z) = \int_{\mathbb{R}^{d+d\times m}} g(t,\ \omega,\ u,\ v) n^2 \psi(n(y-u),\ n(z-v)) \mathrm{d}u\, \mathrm{d}v,$$
$$= \int_{\mathbb{R}^{d+d\times m}} g\left(t,\ \omega,\ y - \frac{u}{n},\ z - \frac{v}{n}\right) \psi(u,\ v) \mathrm{d}u\, \mathrm{d}v.$$

It is easily seen that the sequence $(g_n)_{n\geqslant 0}$ converges pointwise to g, and, for any $n \geqslant 0$, g_n satisfies Assumption 1 and is Lipschitz with respect to (y, z) uniformly in (t, ω). On the other hand, for any $n, m \geqslant 0$, we have

$$|g_n(t,\ \omega,\ y,\ z) - g_m(t,\ \omega,\ y,\ z)|$$
$$= \left| \int_{\mathbb{R}^{d+d\times m}} g\left(t,\ \omega,\ y - \frac{u}{n},\ z - \frac{v}{n}\right) - g\left(t,\ \omega,\ y - \frac{u}{m},\ z - \frac{v}{m}\right) \psi(u,\ v) \mathrm{d}u\, \mathrm{d}v \right|$$
$$\leqslant \int_{\mathbb{R}^{d+d\times m}} \left| g\left(t,\ \omega,\ y - \frac{u}{n},\ z - \frac{v}{n}\right) - g\left(t,\ \omega,\ y - \frac{u}{m},\ z - \frac{v}{m}\right) \right| \psi(u,\ v) \mathrm{d}u\, \mathrm{d}v$$
$$\leqslant \int_{\mathbb{R}^{d+d\times m}} \varphi\left(\left| \frac{u}{n} - \frac{u}{m} \right|,\ \left| \frac{u}{n} - \frac{v}{m} \right| \right) \psi(u,\ v) \mathrm{d}u\, \mathrm{d}v.$$

But since the function ψ is of compact support, then, by virtue of Lebesgue's theorem, the last integral tends to 0 as n, m tend to $+\infty$, whence the desired result. □

Remark. Functions g which are uniformly continuous in (y, z) uniformly in (t, ω) satisfy (3).

3. The main result

We begin with a couple of useful assumptions. Let $\overline{\Phi} : \mathbb{R}^+ \mapsto \mathbb{R}^+$ be a continuous function of at most linear growth such that, $\overline{\Phi}(0) = 0$ and $\overline{\Phi}(x) > 0$ for all $x > 0$. We would like $\overline{\Phi}$ to satisfy the following assumption:

Assumption 2. *$u^\gamma(0) \to 0$ as $\gamma \searrow 0$, where u^γ is the unique solution on* $[0, 1]$ *of the DBE*

$$u^\gamma(t) = \gamma + \int_t^1 \overline{\Phi}(d.u^\gamma(t))dt,\ t \leqslant 1;\ \gamma > 0.$$

This assumption implies, in particular, that $u^\gamma(t) \to 0$ as $\gamma \to 0$ for any $t \in [0, 1]$.

Suppose now that, besides Assumption 1, the mapping $(t, \omega, y, z) \mapsto f(t, \omega, y, z)$ satisfies the following:

Assumption 3.

(i) *f is uniformly continuous in y uniformly with respect to* (t, ω, z), *i.e., there exists a* *continuous non-decreasing function* Φ *from* $\mathbb{R}^+$ *into itself with at most linear growth and satisfying* $\Phi(0) = 0$ *and* $\Phi(x) > 0$ for all $x > 0$ *such that*:

$$|f(t, y, z) - f(t, y', z)| \leqslant \Phi(|y - y'|), \qquad \forall t, y, y', z \text{ a.s.}$$

Moreover, Φ *satisfies Assumption* 2.

(ii) f *is uniformly continuous in* z*, i.e., there exists a continuous function* Ψ *from* $\mathbb{R}^+$ *into itself with at most linear growth and satisfying* $\Psi(0) = 0$*, such that:*

$$|f(t, y, z) - f(t, y, z')| \leqslant \Psi(\|z - z'\|), \qquad \forall t, y, z, z' \text{ a.s.}$$

(iii) *For* $i = 1, \ldots, d$*, the ith component* f_i *of* f *depends only on the ith row of the matrix* z.

In Assumption 3(i) the case where $\Phi(x) = 0$ on some interval $[0, \delta]$ is irrelevant, since then $f(t, y, z) \equiv f(t, 0, z)$.

Example 1. If f is Lipschitz in (y, z) uniformly in (t, ω), then it satisfies Assumption 3 with $\Phi(x) = \Psi(x) = kx$.

Example 2. We now give a mapping f which satisfies Assumption 3 and which is not Lipschitz in (y, z). Let f be the function which with $(t, y, z) \in [0, 1] \times \mathbb{R}^{d+d\times m}$ associates $f(t, y, z) = [h(|y| + |z^1|), \ldots, h(|y| + |z^d|)]$, where z^i is the ith row of z and

$$h(x) = x \ln\frac{1}{x}.1_{[0\leqslant x\leqslant\delta]} + (h'(\delta-)(x - \delta) + h(\delta)).1_{[x>\delta]}$$

with δ small enough. Since $h(0) = 0$, and h is concave increasing, then for all $x, x' \in \mathbb{R}^+$, $h(x + x') \leqslant h(x) + h(x')$. This implies that $|h(x) - h(x')| \leqslant h(|x - x'|)$, $\forall x, x' \in \mathbb{R}^+$. Therefore f satisfies Assumption 3 with $\Phi = \Psi = d.h$ (see Proposition 3.2 below for the second part of Assumption 3(i)).

We shall now prove the main result of this section which provides, under Assumptions 1 and 3, a solution (Y, Z) for the BSDE associated with (f, ξ). The known existence results (Pardoux and Peng 1990a; Darling and Pardoux 1997; Pardoux 1999; Briand and Carmona 2000; Mao 1995) do not provide a solution for the BSDE associated with such a pair (f, ξ).

Theorem 3.1. *Suppose that Assumptions* 1 *and* 3 *hold. Then there exists a process* (Y, Z) *in* $S^{2,d} \times H^{2,d\times m}$ *such that*

$$Y_t = \xi + \int_t^1 f(s, Y_s, Z_s)\mathrm{d}s - \int_t^1 Z_s\,\mathrm{d}B_s, \qquad \forall t \leqslant 1. \tag{4}$$

Proof. Let $(f_n)_{n\geqslant 0}$ be a sequence of mappings from $[0, 1] \times \Omega \times \mathbb{R}^{d+d\times m}$ into $\mathbb{R}^d$ such that for all $n \geqslant 0$, $f_n = f * \psi_n$ (ψ_n is as in Lemma 2.3). This sequence converges uniformly to f and for any $n \geqslant 0$, f_n satisfies Assumption 1 and is Lipschitz with respect to (y, z) uniformly in (t, ω). In addition, we have

$$|f_n(t, y, z) - f_n(t, y', z)| \leqslant \Phi(|y - y'|), \qquad \forall t, y, y', z, \text{ a.s.}$$

For any $n \geqslant 0$, let (Y^n, Z^n) be the solution of the BSDE associated with (f_n, ξ) which, according to Theorem 2.1, exists. So, for any $n \geqslant 0$, we have

$$\begin{aligned} &(Y^n, Z^n) \in S^{2,d} \times H^{2,d\times m}, \\ &Y_t^n = \xi + \int_t^1 f_n(s, Y_s^n, Z_s^n)\mathrm{d}s - \int_t^1 Z_s^n \,\mathrm{d}B_s, \qquad \forall t \leqslant 1. \end{aligned} \tag{5}$$

The proof is based on the fact that $(Y^n)_{n\geqslant 0}$ and $(Z^n)_{n\geqslant 0}$ are Cauchy sequences in $S^{2,d}$ and $H^{2,d\times m}$, respectively.

Step 1. In this step we show that $(Y^n)_{n\geqslant 0}$ is a Cauchy sequence in $S^{2,d}$. For any $n \geqslant 0$, let $(Y^{n,k}, Z^{n,k})_{k\geqslant 0}$ be the sequence of processes of $S^{2,d} \times H^{2,d\times m}$ defined recursively as follows:

$$\begin{aligned} &(Y^{n,0}, Z^{n,0}) = (0, 0), \\ &\qquad Y_t^{n,k} = \xi + \int_t^1 f_n(s, Y_s^{n,k-1}, Z_s^{n,k})\mathrm{d}s - \int_t^1 Z_s^{n,k} \,\mathrm{d}B_s, \qquad \forall t \leqslant 1,\ k \geqslant 1. \end{aligned}$$

It is easily seen that $(Y^{n,k}, Z^{n,k})_{k\geqslant 0}$ converges, as $k \to +\infty$, to (Y^n, Z^n) in $S^{2,d} \times H^{2,d\times m}$. Now, for $i = 1, \ldots, d$, let ${}^iY^{n,k}$, ξ^i, f_n^i and ${}^iZ^{n,k}$, be the ith components and row of respectively $Y^{n,k}$, ξ, f_n and $Z^{n,k}$. On the other hand, for any $\epsilon > 0$, let $N_\epsilon \geqslant 0$ such that if $n, m \geqslant N_\epsilon$ then $|f_n(t, y, z) - f_m(t, y, z)| < \epsilon$.

(a) For all $n, m \geqslant N_\epsilon$, $k \geqslant 0$, $i = 1, \ldots, d$, $t \leqslant 1$, $|{}^iY_t^{n,k} - {}^iY_t^{m,k}| \leqslant C_{k,\epsilon}^{n,m}$, where $C_{k,\epsilon}^{n,m}$ is a constant which may depend on n, m, k, ϵ. Indeed, we have

$$\begin{aligned} &{}^iY_t^{n,k+1} - {}^iY_t^{m,k+1} \\ &\quad = \int_t^1 \{f_n^i(s, Y_s^{n,k}, {}^iZ_s^{n,k+1}) - f_m^i(s, Y_s^{m,k}, {}^iZ_s^{m,k+1})\}\mathrm{d}s - \int_t^1 ({}^iZ_s^{n,k+1} - {}^iZ_s^{m,k+1})\mathrm{d}B_s \\ &\quad = \int_t^1 \{f_n^i(s, Y_s^{n,k}, {}^iZ_s^{n,k+1}) - f_n^i(s, Y_s^{m,k}, {}^iZ_s^{n,k+1}) + f_n^i(s, Y_s^{m,k}, {}^iZ_s^{n,k+1}) \\ &\qquad - f_n^i(s, Y_s^{m,k}, {}^iZ_s^{m,k+1}) + f_n^i(s, Y_s^{m,k}, {}^iZ_s^{m,k+1}) - f_m^i(s, Y_s^{m,k}, {}^iZ_s^{m,k+1})\}\mathrm{d}s \\ &\qquad - \int_t^1 ({}^iZ_s^{n,k+1} - {}^iZ_s^{m,k+1})\mathrm{d}B_s. \end{aligned}$$

But since f_n^i is a Lipschitz mapping,

$$
\begin{aligned}
{}^iY_t^{n,k+1} - {}^iY_t^{m,k+1} = &\int_t^1 \{ {}^ib_{n,m}^k(s)(Y_s^{n,k} - Y_s^{m,k}) + {}^ia_{n,m}^k(s)({}^iZ_s^{n,k+1} - {}^iZ_s^{m,k+1}) \\
&+ (f_n^i(s, Y_s^{m,k}, {}^iZ_s^{m,k+1}) - f_m^i(s, Y_s^{m,k}, {}^iZ_s^{m,k+1})\}\mathrm{d}s \\
&- \int_t^1 ({}^iZ_s^{n,k+1} - {}^iZ_s^{m,k+1})\mathrm{d}B_s.
\end{aligned}
$$

Here $({}^ia_{n,m}^k(t))_{t\leqslant 1}$ and $({}^ib_{n,m}^k(t))_{t\leqslant 1}$ are bounded and F_t-adapted processes defined as:

$$
{}^ia_{n,m}^k(t) = \begin{cases} \dfrac{f_n^i(t, Y_t^{m,k}, {}^iZ_t^{n,k+1}) - f_n^i(t, Y_t^{m,k}, {}^iZ_t^{m,k+1})}{{}^iZ_t^{n,k+1} - {}^iZ_t^{m,k+1}} & \text{if } {}^iZ_t^{n,k+1} - {}^iZ_t^{m,k+1} \neq 0, \\ 0 & \text{otherwise,} \end{cases}
$$

and

$$
{}^ib_{n,m}^k(t) = \begin{cases} \dfrac{f_n^i(t, Y_t^{n,k}, {}^iZ_t^{n,k+1}) - f_n^i(t, Y_t^{m,k}, {}^iZ_t^{n,k+1})}{Y_t^{n,k} - Y_t^{m,k}} & \text{if } Y_t^{n,k} - Y_t^{m,k} \neq 0, \\ 0 & \text{otherwise.} \end{cases}
$$

Let $P_{n,m}^{i,k}$ be the probability on $(\Omega, \mathcal{F})$ which is equivalent to P and defined by:

$$
\frac{\mathrm{d}P_{n,m}^{i,k}}{\mathrm{d}P} = \mathcal{E}\left(\int_0^1 {}^ia_{n,m}^k(\mathrm{s})\mathrm{d}B_s\right) := \exp\left[\int_0^1 {}^ia_{n,m}^k(\mathrm{s})\mathrm{d}B_s - \frac{1}{2}\int_0^1 |{}^ia_{n,m}^k(s)|^2\,\mathrm{d}s\right].
$$

From Girsanov's theorem (Karatzas and Shreve 1991; Revuz and Yor 1991), under $P_{n,m}^{i,k}$ the process

$$
{}^iB_{n,m}^k(t) = B_t - \int_0^t {}^ia_{n,m}^k(s)\mathrm{d}s, \qquad t \leqslant 1,
$$

is an $(F_t, P_{n,m}^{i,k})$-Brownian motion. Moreover,

$$
\left(\int_0^t ({}^iZ_s^{n,k+1} - {}^iZ_s^{m,k+1})\mathrm{d}^iB_{n,m}^k(s)\right)_{t\leqslant 1}
$$

is an $(F_t, P_{n,m}^{i,k})$-martingale. Indeed,

$$
\begin{aligned}
\mathrm{E}_{n,m}^{i,k}\left[\sup_{t\leqslant 1}\left|\int_0^t ({}^iZ_s^{n,k+1} - {}^iZ_s^{m,k+1})\mathrm{d}^iB_{n,m}^k(s)\right|\right] &\leqslant C\mathrm{E}_{n,m}^{i,k}\left[\sqrt{\int_0^1 |{}^iZ_s^{n,k+1} - {}^iZ_s^{m,k+1}|^2\,\mathrm{d}s}\right] \\
&\leqslant C\sqrt{\mathrm{E}\left[\left(\frac{\mathrm{d}P_{n,m}^{i,k}}{\mathrm{d}P}\right)^2\right]}\sqrt{\mathrm{E}\left[\int_0^1 |{}^iZ_s^{n,k+1} - {}^iZ_s^{m,k+1}|^2\,\mathrm{d}s\right]} < \infty,
\end{aligned}
$$

and the result follows. The first inequality stems from that of Burkholder, Davis and Gundy (Karatzas and Shreve 1991; Revuz and Yor 1991).

On the other hand, let $\beta_{n,m}$ be a real number such that

$$\beta_{n,m} \geqslant \sum_{i=1,d} |{}^i b^k_{n,m}(t)|, \qquad \forall t \leqslant 1, \text{ a.s.}$$

Since $n,\ m \geqslant N_\epsilon$ then, after writing ${}^iY^{n,k+1}_t - {}^iY^{m,k+1}_t$ with the Brownian motion ${}^iB^k_{n,m}$ instead of B and taking the conditional expectation, we obtain

$$|{}^iY^{n,k+1}_t - {}^iY^{m,k+1}_t| \leqslant \mathrm{E}^{i,k}_{n,m}\left[\int_t^1 \{\beta_{n,m}|Y^{n,k}_s - Y^{m,k}_s| + \epsilon\}\mathrm{d}s|F_t\right].$$

Now it is easily seen by induction on k, that for all $k \geqslant 0$, for all $i = 1, \ldots, d$, and $t \leqslant 1$, $|{}^iY^{n,k}_t - {}^iY^{m,k}_t| \leqslant u_{n,m}(t) \leqslant u_{n,m}(0)$. Here $u_{n,m}$ is the deterministic function on [0, 1] which satisfies $u_{n,m}(t) = \int_t^1 \{d\beta_{n,m}u_{n,m}(s) + \epsilon\}\mathrm{d}s$, $t \leqslant 1$ (this exists according to Proposition 2.2). Whence the desired result.

(b) We now need the fact that $(Y^n)_{n\geqslant 0}$ is a Cauchy sequence in $\mathcal{S}^{2,d}$. Tanaka's formula implies that

$$|{}^iY^{n,k}_t - {}^iY^{m,k}_t| + 2({}^i\Lambda^{nmk}_1(0) - {}^i\Lambda^{nmk}_t(0)) = \int_t^1 \mathrm{sgn}({}^iY^{n,k}_s - {}^iY^{m,k}_s)(f^i_n(s,\ Y^{n,k-1}_s,\ {}^iZ^{n,k}_s)$$
$$- f^i_m(s,\ Y^{m,k-1}_s,\ {}^iZ^{m,k}_s))\mathrm{d}s - \int_t^1 \mathrm{sgn}({}^iY^{n,k}_s - {}^iY^{m,k}_s)({}^iZ^{n,k}_s - {}^iZ^{m,k}_s)\mathrm{d}B_s,$$

where $({}^i\Lambda^{nmk}_t(0))_{t\leqslant 1}$ is the local time of ${}^iY^{n,k} - {}^iY^{m,k}$ at 0. It follows that

$$|{}^iY^{n,k}_t - {}^iY^{m,k}_t| \leqslant \int_t^1 \mathrm{sgn}({}^iY^{n,k}_s - {}^iY^{m,k}_s)\{f^i_n(s,\ Y^{n,k-1}_s,\ {}^iZ^{n,k}_s) - f^i_n(s,\ Y^{m,k-1}_s,\ {}^iZ^{n,k}_s)$$
$$+ f^i_n(s,\ Y^{m,k-1}_s,\ {}^iZ^{n,k}_s) - f^i_n(s,\ Y^{m,k-1}_s,\ {}^iZ^{m,k}_s) + f^i_n(s,\ Y^{m,k-1}_s,\ {}^iZ^{m,k}_s)$$
$$- f^i_m(s,\ Y^{m,k-1}_s,\ {}^iZ^{m,k}_s)\}\mathrm{d}s - \int_t^1 \mathrm{sgn}({}^iY^{n,k}_s - {}^iY^{m,k}_s)({}^iZ^{n,k}_s - {}^iZ^{m,k}_s)\mathrm{d}B_s.$$

So since $n,\ m \geqslant N_\epsilon$ then

$$|{}^iY^{n,k}_t - {}^iY^{m,k}_t| \leqslant -\int_t^1 \mathrm{sgn}({}^iY^{n,k}_s - {}^iY^{m,k}_s)({}^iZ^{n,k}_s - {}^iZ^{n,k}_s)\mathrm{d}^iB^{k-1}_{n,m}(s)$$
$$+ \int_t^1 \{\epsilon + \Phi(|Y^{n,k-1}_s - Y^{m,k-1}_s|)\mathrm{d}s.$$

On the other hand, let u^ϵ be the solution on [0, 1] of the DBE

$$u^\epsilon(t) = \epsilon + \int_t^1 \Phi(d.u^\epsilon(s))\mathrm{d}s, \qquad t \leqslant 1$$

(cf. Proposition 2.2). Since

$$\left(\int_0^t \mathrm{sgn}({}^iY^{n,k}_s - {}^iY^{m,k}_s)({}^iZ^{n,k}_s - {}^iZ^{m,k}_s)\mathrm{d}^iB^{k-1}_{n,m}(s)\right)_{t\leqslant 1}$$

is an $(F_t,\ P^{i,k-1}_{n,m})$-martingale, we have

$$|{}^iY_t^{n,k} - {}^iY_t^{m,k}| - u^\epsilon(t) \leqslant \mathrm{E}_{n,m}^{i,k}\left[\int_t^1 \{\Phi(|Y_s^{n,k-1} - Y_s^{m,k-1}|) - \Phi(d.u^\epsilon(s))\}\mathrm{d}s|F_t\right], \qquad t \leqslant 1. \tag{6}$$

Now by induction on k we have, for all $k \geqslant 0$, $|{}^iY_t^{n,k} - {}^iY_t^{m,k}| \leqslant u^\epsilon(t)$ for any $t \leqslant 1$ and $i = 1, \ldots, d$. Indeed, for $k = 0$ the property holds. Suppose it also holds for some $k-1$, i.e., $|{}^iY_t^{n,k-1} - {}^iY_t^{m,k-1}| \leqslant u^\epsilon(t)$, $\forall t \leqslant 1$ and $i = 1, \ldots, d$, then $|Y_t^{n,k-1} - Y_t^{m,k-1}| \leqslant d.u^\epsilon(t)$, $\forall t \leqslant 1$. Now combining that with the fact that Φ is non-decreasing and taking into account (6) yields $|{}^iY_t^{n,k} - {}^iY_t^{m,k}| \leqslant u^\epsilon(t)$ for all $t \leqslant 1$, $i = 1, \ldots, d$.

Taking the limit as $k \to \infty$ implies that $|{}^iY_t^n - {}^iY_t^m| \leqslant u^\epsilon(t)$, for all $t \leqslant 1$, $i = 1, \ldots, d$. Hence for all $\epsilon > 0$ there exists N_ϵ such that, for all n, $m \geqslant N_\epsilon$, we have $\sup_{t\leqslant 1}|Y_t^n - Y_t^m| \leqslant d.u^\epsilon(0)$. As $u^\epsilon(0) \to 0$ as $\epsilon \to 0$ (according to Assumption 3(i)), then $(Y_n)_{n\geqslant 0}$ is a Cauchy sequence in $S^{2,d}$ and converges to a process which we denote by Y.

Step 2. We now show that $(Z_n)_{n\geqslant 0}$ is a Cauchy sequence in $H^{2,d\times m}$.

(a) There exists a constant C such that $E[\int_0^1 \|Z_s^n\|^2\,\mathrm{d}s] \leqslant C$. Indeed, since the rate of growth of the functions Ψ and Φ is at most linear, there exists a constant α such that, for any $n \geqslant 0$, we have $|f_n(t, \omega, y, z)| \leqslant |f(t, \omega, 0, 0)| + \alpha(1 + |y| + \|z\|)$, for all t, y, z, a.s. Now using Itô's formula with the process Y^n defined in (5), we obtain

$$|Y_t^n|^2 + \int_t^1 \|Z_s^n\|^2\,\mathrm{d}s = |\xi|^2 + 2\int_t^1 Y_s^n f_n(s, Y_s^n, Z_s^n)\mathrm{d}s - 2\int_t^1 Y_s^n Z_s^n\,\mathrm{d}B_s.$$

Then, for all $t \leqslant 1$ and $\nu > 0$,

$$\begin{aligned}
|Y_t^n|^2 + \int_t^1 \|Z_s^n\|^2\,\mathrm{d}s &\leqslant |\xi|^2 + \int_t^1 |Y_s^n|\{|f(s, 0, 0)| + \alpha(1 + |Y_s^n| + \|Z_s^n\|)\}\mathrm{d}s - 2\int_t^1 Y_s^n Z_s^n\,\mathrm{d}B_s \\
&\leqslant |\xi|^2 + \frac{1}{\nu}\int_t^1 |Y_s^n|^2\,\mathrm{d}s + \nu\int_t^1 \{|f(s, 0, 0)| + \alpha(1 + |Y_s^n| + \|Z_s^n\|)\}^2\,\mathrm{d}s \\
&\quad - 2\int_t^1 Y_s^n Z_s^n\,\mathrm{d}B_s \\
&\leqslant |\xi|^2 + \frac{1}{\nu}\int_t^1 |Y_s^n|^2\,\mathrm{d}s + C\nu\int_t^1 \{1 + |f(s, 0, 0)|^2 + |Y_s^n|^2 + \|Z_s^n\|^2\}\mathrm{d}s \\
&\quad - 2\int_t^1 Y_s^n Z_s^n\,\mathrm{d}B_s \\
&\leqslant |\xi|^2 + \left(\frac{1}{\nu} + \nu C\right)\int_t^1 |Y_s^n|^2\,\mathrm{d}s + C\nu\int_t^1 (1 + |f(s, 0, 0)|^2)\mathrm{d}s + C\nu\int_t^1 \|Z_s^n\|^2\,\mathrm{d}s \\
&\quad - 2\int_t^1 Y_s^n Z_s^n\,\mathrm{d}B_s.
\end{aligned}$$

Through the convergence of $(Y^n)_{n\geqslant 0}$ in $S^{2,d}$ we have $\sup_{n\geqslant 0} \mathrm{E}[\sup_{t\leqslant 1}|Y_t^n|^2] \leqslant C$, and then, once again using the Burkholder–Davis–Gundy inequality, we deduce that $(\int_0^t Y_s^n Z_s^n \,\mathrm{d}B_s)_{t\leqslant 1}$ is an (F_t, P)-martingale. Now choosing $\nu = 1/4C$, we obtain

$$E\left[\int_0^1 \| Z_s^n\|^2 \,\mathrm{d}s\right] \leqslant 4\{E[\xi^2] + (4C + \tfrac{1}{4})\mathrm{E}\left[\int_0^1 |Y_s^n|^2 \,\mathrm{d}s\right] + E\left[\int_0^1 (1 + |f(s, 0, 0)|^2)\mathrm{d}s\right] \leqslant C,$$

whence the desired result. Here C is a constant which may change from one line to another.

(b) We show that $(Z_n)_{n\geqslant 0}$ is a Cauchy sequence. Indeed, for any $n, m \geqslant 0$, we have

$$\mathrm{E}\left[\int_0^1 \| Z_s^n - Z_s^m\|^2 \,\mathrm{d}s\right] \leqslant 2\mathrm{E}\left[\int_0^1 (Y_s^n - Y_s^m)(f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m))\mathrm{d}s\right]$$

since the process $(\int_0^t (Y_s^n - Y_s^m)(Z_s^n - Z_s^m)\mathrm{d}B_s)_{t\leqslant 1}$ is an (F_t, P)-martingale. It follows that

$$\mathrm{E}\left[\int_0^1 \| Z_s^n - Z_s^m\|^2 \,\mathrm{d}s\right] \leqslant 2\sqrt{\mathrm{E}\left[\sup_{t\leqslant 1} |Y_t^n - Y_t^m|^2\right]}\sqrt{\mathrm{E}\left[\int_0^1 |f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m)|^2 \,\mathrm{d}s\right]}.$$

But there exists a constant $C \geqslant 0$ such that, for all $n \geqslant 0$,

$$\mathrm{E}\left[\int_0^1 |f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m)|^2 \,\mathrm{d}s\right] \leqslant C.$$

Then the sequence $(Z_n)_{n\geqslant 0}$ is a Cauchy sequence in $H^{2,d\times m}$ whose limit will be denoted by Z.

Step 3. We now show that the process (Y, Z) is a solution of the BSDE associated with (f, ξ). For any $n \geqslant 0$ and $t \leqslant 1$, we know from (5) that

$$Y_t^n = \xi + \int_t^1 f_n(s, Y_s^n, Z_s^n)\mathrm{d}s - \int_t^1 Z_s^n \,\mathrm{d}B_s.$$

Now, for a fixed $t \in [0, 1]$, the sequences $(Y_t^n)_{n\geqslant 0}$ and $(\int_t^1 Z_s^n \,\mathrm{d}B_s)_{n\geqslant 0}$ converge in $L^2(\Omega, \mathrm{d}P)$ towards Y_t and $\int_t^1 Z_s \,\mathrm{d}B_s$, respectively. On the other hand,

$$\begin{aligned}\mathrm{E}\left[\left|\int_t^1 f_n(s, Y_s^n, Z_s^n)\mathrm{d}s - \int_t^1 f(s, Y_s, Z_s)\mathrm{d}s\right|\right] &\leqslant \mathrm{E}\left[\int_0^1 |f_n(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)|\mathrm{d}s\right]\\ &\leqslant \mathrm{E}\left[\int_0^1 |f_n(s, Y_s^n, Z_s^n) - f(s, Y_s^n, Z_s^n)|\mathrm{d}s\right] + \mathrm{E}\left[\int_0^1 |f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)|\mathrm{d}s\right].\end{aligned}$$

The first term converges to 0 as $n \to +\infty$ since $(f_n)_{n\geqslant 0}$ converges uniformly to f (cf. Lemma 2.3(ii)). In addition, for any $\beta \geqslant 0$, we have

$$
\begin{aligned}
&\mathrm{E}\left[\int_0^1 |f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)| \mathrm{d}s\right] \\
&\qquad \leqslant \mathrm{E}\left[\int_0^1 \{\Phi(|Y_s^n - Y_s|)1_{[|Y_s^n - Y_s| \leqslant \beta]} + \Psi(\|Z_s^n - Z_s\|)1_{[\|Z_s^n - Z_s\| \leqslant \beta]}\} \mathrm{d}s\right] \\
&\qquad\quad + \mathrm{E}\left[\int_0^1 |f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)| 1_{[|Y_s^n - Y_s| \geqslant \beta]} \mathrm{d}s\right] \\
&\qquad\quad + \mathrm{E}\left[\int_0^1 |f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)| 1_{[\|Z_s^n - Z_s\| \geqslant \beta]} \mathrm{d}s\right].
\end{aligned}
$$

Then, after extracting a subsequence, the first term converges to 0 as $n \to +\infty$. Moreover,

$$
\begin{aligned}
&\mathrm{E}\left[\int_0^1 |f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)| 1_{[|Y_s^n - Y_s| \geqslant \beta]} \mathrm{d}s\right] \\
&\quad \leqslant \left(\mathrm{E}\left[\int_0^1 |f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)|^2 \,\mathrm{d}s\right]\right)^{1/2} \left(\mathrm{E}\left[\int_0^1 1_{[|Y_s^n - Y_s| \geqslant \beta]} \mathrm{d}s\right]\right)^{1/2} \\
&\quad \leqslant \beta^{-1} \left(\mathrm{E}\left[\int_0^1 |f(s, Y_s^n, Z_s^n) - f(s, Y_s, Z_s)|^2 \,\mathrm{d}s\right]\right)^{1/2} . \left(\mathrm{E}\left[\int_0^1 |Y_s^n - Y_s|^2 \,\mathrm{d}s\right]\right)^{1/2} \\
&\quad \leqslant C\beta^{-1} \sqrt{\mathrm{E}\left[\int_0^1 |Y_s^n - Y_s|^2 \,\mathrm{d}s\right]}.
\end{aligned}
$$

The last inequality follows from the convergence of $(Y_n)_{n \geqslant 0}$ $((Z_n)_{n \geqslant 0})$ in $S^{2,d}$ $(H^{2,d \times m})$ and the linear growth of f, i.e., $|f(t, \omega, y, z)| \leqslant |f(t, \omega, 0, 0)| + \alpha(1 + |y| + \|z\|)$ a.s. Therefore, the second term converges to 0 as $n \to +\infty$. In the same way, it is easily seen that the third term also converges to 0 as $n \to +\infty$. Consequently, since Y is a continuous process, we have

$$
Y_t = \xi + \int_t^1 f(s, Y_s, Z_s) \mathrm{d}s - \int_t^1 Z_s \,\mathrm{d}B_s, \qquad \forall t \leqslant 1, \text{ a.s.,}
$$

i.e. (Y, Z) is a solution for the BSDE associated with (f, ξ). The proof is now complete. $\square$

We now focus on the conditions under which condition the function Φ of Assumption 3(i) satisfies Assumption 2. Here is a result in this direction.

Proposition 3.2. (i) *Assume the DBE* $u(t) = \int_t^1 \Phi(d.u(s))\mathrm{d}s$, $t \leqslant 1$, *has a unique solution* $u \equiv 0$. *Then* Φ *satisfies Assumption* 2.

(ii) *The deterministic backward differential equation* $u(t) = \int_t^1 \Phi(d.u(s))ds$, $t \leqslant 1$, *has a unique solution if and only if* $\int_{0+}[\Phi(x)]^{-1}\,\mathrm{d}x = \infty$.

Proof. First let us emphasize that Φ is supposed to be continuous, non-decreasing, with at most linear growth, and satisfies $\Phi(0) = 0$ and $\Phi(x) > 0$ for all $x > 0$.

(i) For any $\epsilon > 0$, let u^ϵ be the solution of the DBE

$$u^\epsilon(t) = \epsilon + \int_t^1 \Phi(d.u^\epsilon(s))\mathrm{d}s, \qquad t \leqslant 1.$$

If $\delta > \epsilon$ then $0 \leqslant u^\epsilon \leqslant u^\delta$, and it follows that, for all $t \leqslant 1$, $u^\epsilon(t) \searrow \tilde{u}(t)$ as $\epsilon \searrow 0$. In addition, $\tilde{u}$ satisfies $\tilde{u}(t) = \int_t^1 \Phi(d.\tilde{u}(s))\mathrm{d}s$, $t \leqslant 1$. As the solution of this latter equation is unique then $\tilde{u}(t) = 0$, for all $t \leqslant 1$, and so $u^\epsilon(0) \to 0$ as $\epsilon \searrow 0$.

(ii) The condition is sufficient. For $z > 0$, let $G(z) = \int_z^1 [\Phi(d.x)]^{-1}\,\mathrm{d}x$. If $u \neq 0$ then there exists some $t_0 \in]0, 1]$ such that $u(t_0) = 0$ and $u(t) > 0$ for any $t < t_0$, and then $(G(u(t))' = 1$ for all $t < t_0$. This implies that $u(t) = G^{-1}(G(u(t_1) - t_1 + t)$ for any $t \leqslant t_1 < t_0$. Now taking the limit as $t_1 \nearrow t_0$ yields $G(u(t_1)) \to +\infty$ and $G^{-1}(+\infty) = 0$, whence $u(t) = 0$ for all $t \leqslant t_0$ which is a contradiction.

The condition is also necessary. Let us suppose $\int_{0+}[\Phi(x)]^{-1}\,\mathrm{d}x < \infty$. For $\epsilon > 0$, let $G_\epsilon(z) = \int_z^1 [\epsilon + \Phi(d.x)]^{-1}\,\mathrm{d}x$, $z \geqslant 0$, and let u^ϵ be a function such that $u^\epsilon(t) = \int_t^1 [\epsilon + \Phi(d.u^\epsilon(s))]\mathrm{d}s$, $t \leqslant 1$. u^ϵ is unique since $G_\epsilon(u^\epsilon(t))' = 1$ for all $t \leqslant 1$, and then $G_\epsilon(u^\epsilon(1)) - G_\epsilon(u^\epsilon(t)) = 1 - t$, $t \leqslant 1$, which implies $u^\epsilon(t) = G_\epsilon{}^{-1}(G_\epsilon(0) - 1 + t)$, $t \leqslant 1$. Now if $\delta > \epsilon$ then $u^\delta \geqslant u^\epsilon$, hence $u^\epsilon \searrow \overline{v}$ pointwise as $\epsilon \searrow 0$. But $\overline{v}$ satisfies $\overline{v}(t) = \int_t^1 \Phi(d.\overline{v}(s))\mathrm{d}s$, and then $\overline{v} = 0$, since the solution of this latter equation is unique. It follows that $u^\epsilon \searrow 0$ as $\epsilon \searrow 0$ pointwise and uniformly by virtue of Dini's theorem. Henceforth, for all $t \leqslant 1$, $G_\epsilon(u^\epsilon(t)) \to \int_0^1 [\Phi(x)]^{-1}\,\mathrm{d}x < \infty$ as $\epsilon \to 0$. Now since $G_\epsilon(u^\epsilon(1)) - G_\epsilon(u^\epsilon(t)) = 1 - t$, for all $t \leqslant 1$, taking the limit as $\epsilon \to 0$, we obtain $0 = 1 - t$, for all $t \leqslant 1$ which is a contradiction. $\square$

We are now ready to give the following result whose proof is a direct consequence of Theorem 3.1 and Proposition 3.2.

Theorem 3.3. *Assume the mapping* $(t, \omega, y, z) \mapsto f(t, \omega, y, z)$ *satisfies Assumptions 1, 3(ii) and 3(iii), and* $|f(t, y, z) - f(t, y', z)| \leqslant \Phi(|y - y'|)$, *for all* t, z, y, y', *where* Φ *is a continuous non-decreasing function from* $\mathbb{R}^+$ *into itself with at most linear growth and such that* $\Phi(0) = 0$, $\int_{0+}[\Phi(x)]^{-1}\mathrm{d}x = +\infty$ *and* $\Phi(x) > 0$ *for all* $x > 0$. *Then the BSDE associated with* (f, ξ) *has a solution.*

According to this theorem, if f is as in Example 2 above then the BSDE associated with (f, ξ) has a solution.

4. Uniqueness

In this section we deal with the issue of the uniqueness of the solution for the BSDE (4) associated with (f, ξ). Let us assume that the mapping $(t, \omega, y, z) \mapsto f(t, \omega, y, z)$ satisfies the following assumption.

Assumption 4.

(i) *For all t, y, y', z, $|f(t, y, z) - f(t, y', z)| \leqslant \Phi(|y - y'|)$, where Φ is continuous and non-decreasing, grows at most linearly and satisfies $\Phi(0) = 0$, $\Phi(x) > 0$, for all $x > 0$, and $\int_{0^+} [\Phi(x)]^{-1} \mathrm{d}x = \infty$.*

(ii) *The function $z \mapsto f(t, y, z)$ is Lipschitz uniformly with respect to (t, ω, y). In addition, for any $i = 1, \ldots, d$, $f^i(t, y, z)$, the ith component of f, depends only on the ith row of z.*

Then we have the following result.

Theorem 4.1. *Under Assumptions* 1 *and* 4*, the solution (Y, Z) of the BSDE* (4) *associated with (f, ξ) is unique.*

Proof. The existence follows from Theorem 3.3. Let us focus on the uniqueness. Let $(Y', Z') \in S^{2,d} \times H^{2,d\times m}$ be another solution of the BSDE associated with (f, ξ), i.e.

$$Y'_t = \xi + \int_t^1 f(s, Y'_s, Z'_s)\mathrm{d}s - \int_t^1 Z'_s\, \mathrm{d}B_s, \qquad \forall t \leqslant 1, \text{ a.s.}$$

(i) The process $Y - Y'$ is uniformly bounded, i.e. there exists a constant $\tilde{C}$ such that $|Y_t - Y'_t| \leqslant \tilde{C}$, for all $t \leqslant 1$. Indeed, using Itô's formula we arrive, for all $t \leqslant 1$, at

$$\begin{aligned}
|Y_t - Y'_t|^2 + \int_t^1 \|Z'_s - Z_s\|^2\, \mathrm{d}s &= 2\int_t^1 (Y_s - Y'_s)(f(s, Y_s, Z_s) - f(s, Y'_s, Z'_s))\mathrm{d}s \\
&\quad - 2\int_t^1 (Y_s - Y'_s)(Z_s - Z'_s)\mathrm{d}B_s \\
&\leqslant 2\int_t^1 |Y_s - Y'_s|\{\Phi(|Y_s - Y'_s|) + k\|Z_s - Z'_s\|\}\mathrm{d}s \\
&\quad - 2\int_t^1 (Y_s - Y'_s)(Z_s - Z'_s)\mathrm{d}B_s \\
&\leqslant C\int_t^1 |Y_s - Y'_s|^2\, \mathrm{d}s + \int_t^1 \Phi(|Y_s - Y'_s|)^2\, \mathrm{d}s + \frac{1}{2}\int_t^1 \|Z_s - Z'_s\|^2\, \mathrm{d}s \\
&\quad - 2\int_t^1 (Y_s - Y'_s)(Z_s - Z'_s)\mathrm{d}B_s
\end{aligned} \tag{7}$$

since, for all $a, b \in \mathbb{R}$ and $\epsilon > 0$, $|ab| \leqslant \epsilon a^2 + \epsilon^{-1} b^2$. The growth of Φ is at most linear, then $\Phi(|y|)^2 \leqslant C(1+|y|^2)$ for all $y \in \mathbb{R}$. On the other hand, since (Y, Z) and (Y', Z') belong to $S^{2,d} \times H^{2,d\times m}$ then, using the Burkholder–Davis–Gundy inequality, we deduce that $(\int_0^t (Y_s - Y'_s)(Z_s - Z'_s)\mathrm{d}B_s)_{t\leqslant 1}$ is an (F_t, P)-martingale. We thus have

$$|Y_t - Y'_t|^2 \leqslant C\left\{1 + \int_t^1 |Y_s - Y'_s|^2 \,\mathrm{d}s\right\} - \int_t^1 (Y_s - Y'_s)(Z_s - Z'_s)\,\mathrm{d}B_s, \qquad t \leqslant 1.$$

Then, for any $s \geqslant t \geqslant 0$, we have

$$\mathrm{E}[|Y_s - Y'_s|^2 | F_t] \leqslant C\left\{1 + \int_s^1 \mathrm{E}[|Y_u - Y'_u|^2 | F_t]\mathrm{d}u\right\}.$$

Now by Gronwall's inequality we obtain $\mathrm{E}[|Y_s - Y'_s|^2|F_t] \leqslant \tilde{C}$, which yields the desired result after taking $s = t$.

(ii) We show that the solution of the BSDE associated with (f, ξ) is unique. For any $i = 1, \ldots, d$ and $t \leqslant 1$, we have

$${}^iY_t - {}^iY'_t = \int_t^1 (f^i(s, Y_s, {}^iZ_s) - f^i(s, Y'_s, {}^iZ'_s))\mathrm{d}s - \int_t^1 ({}^iZ_s - {}^iZ'_s)\mathrm{d}B_s$$

where, once again, ${}^iY$, ${}^iY'$, f^i, ${}^iZ$ and ${}^iZ'$ are the ith components and rows of respectively Y, Y', f, Z and Z'. Then, using Tanaka's formula, we obtain

$$\begin{aligned}|{}^iY_t - {}^iY'_t| + 2(\Lambda^i_1(0) - \Lambda^i_t(0)) &= \int_t^1 \mathrm{sgn}({}^iY_s - {}^iY'_s)(f^i(s, Y_s, {}^iZ_s) - f^i(s, Y'_s, {}^iZ'_s))\mathrm{d}s \\ &\quad - \int_t^1 \mathrm{sgn}({}^iY_s - {}^iY'_s)({}^iZ_s - {}^iZ'_s)\mathrm{d}B_s \qquad t \leqslant 1,\end{aligned}$$

where $(\Lambda^i_t(0))_{t\leqslant 1}$ is the local time of ${}^iY - {}^iY'$ at 0. Now let $(a^i_t)_{t\leqslant 1}$ be the following bounded and F_t-adapted process:

$$a^i_t = \begin{cases} \dfrac{f^i(t, Y'_t, {}^iZ_t) - f^i(t, Y'_t, {}^iZ'_t)}{{}^iZ_t - {}^iZ'_t} & \text{if } {}^iZ_t - {}^iZ'_t \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned}|{}^iY_t - {}^iY'_t| &\leqslant \int_t^1 \mathrm{sgn}({}^iY_s - {}^iY'_s)(f^i(s, Y_s, {}^iZ_s) - f^i(s, Y'_s, {}^iZ_s))\mathrm{d}s \\ &\quad - \int_t^1 \mathrm{sgn}({}^iY_s - {}^iY'_s)({}^iZ_s - {}^iZ'_s)\mathrm{d}\bar{B}_s, \qquad t \leqslant 1\end{aligned}$$

where $\bar{B}_t = B_t - \int_0^t a^i_s\,\mathrm{d}s$, $t \leqslant 1$, is a Brownian motion under the probability $\bar{P}^i$ on $(\Omega, \mathcal{F})$ defined by $\mathrm{d}\bar{P}^i/\mathrm{d}P = \mathcal{E}(\int_0^1 a^i_s\,\mathrm{d}B_s)$. We thus have

$$|^iY_t - {}^iY'_t| \leqslant \int_t^1 \Phi(|Y_s - Y'_s|)\mathrm{d}s - \int_t^1 \mathrm{sgn}({}^iY_s - {}^iY'_s)({}^iZ_s - {}^iZ'_s)\mathrm{d}\overline{B}_s, \qquad t \leqslant 1.$$

Then, for all $s \geqslant t$,

$$\overline{\mathrm{E}}^i[|{}^iY_s - {}^iY'_s||F_t] \leqslant \int_s^1 \overline{\mathrm{E}}^i[\Phi(|Y_u - Y'_u|)|F_t]\mathrm{d}u \tag{8}$$

since, as in step 1(a) of the proof of Theorem 3.1, $\int_0^t \mathrm{sgn}({}^iY_u - {}^iY'_u)({}^iZ_u - {}^iZ'_u)\mathrm{d}\overline{B}_u$, $t \leqslant 1$, is an $(F_t, \overline{P}^i)$-martingale.

Now for $n \geqslant 0$, let Φ_n be a Lipschitz function from $\mathbb{R}^+$ into itself such that, for all $x \in \mathbb{R}^+$, $\Phi_n(x) \searrow \Phi(x)$ as $n \to \infty$ (see the proof of Proposition 2.2 for the existence of Φ_n).

For $n \geqslant 0$ and $\epsilon > 0$, let v_ϵ^n be the function such that

$$v_\epsilon^n(t) = \epsilon + \int_t^1 \Phi_n(\mathrm{d}.v_\epsilon^n(s))\mathrm{d}s, \qquad t \leqslant 1.$$

Since $(\Phi_n)_n$ is a non-increasing sequence, $v_\epsilon^{n+1} \leqslant v_\epsilon^n$ for any $n \geqslant 0$. This implies that the sequence $(v_\epsilon^n)_{n\geqslant 0}$ converges pointwise to a function $v_\epsilon : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfies

$$v_\epsilon(t) = \epsilon + \int_t^1 \Phi(d.v_\epsilon(s))\mathrm{d}s, \qquad t \leqslant 1.$$

Now if $\epsilon \leqslant \delta$ then $v_\epsilon^n \leqslant v_\delta^n$ for any $n \geqslant 0$, and then $v_\epsilon \leqslant v_\delta$. It follows that $v_\epsilon \searrow \overline{v}$ as $\epsilon \searrow 0$ where, for any $t \leqslant 1$, $\overline{v}_t = \int_t^1 \Phi(d.\overline{v}(s))\mathrm{d}s$, so that $\overline{v} \equiv 0$ (according to Proposition 3.2). Therefore, we have $v_\epsilon(0) \searrow 0$ as $\epsilon \searrow 0$.

Now for ϵ, n and $k \geqslant 0$, let $v_\epsilon{}^{n,k}$ be the function defined recursively as follows:

$$v_\epsilon^{n,0} = \tilde{C},$$

$$v_\epsilon^{n,k}(t) = \epsilon + \int_t^1 \Phi_n(d.v_\epsilon^{n,k-1}(s))\mathrm{d}s, \qquad k \geqslant 1,\ t \leqslant 1. \tag{9}$$

Since Φ_n is Lipschitz, $v_\epsilon^{n,k} \to v_\epsilon^n$ as $k \to +\infty$. On the other hand, it is easily seen by induction that for all $k \geqslant 0$, $|^iY_t - {}^iY'_t| \leqslant v_\epsilon^{n,k}(t)$, $t \leqslant 1$, $i = 1, \ldots, d$. Indeed, for $k = 0$ the formula holds. Suppose it also holds for some $k - 1$, then $\Phi(|Y_t - Y'_t|) \leqslant \Phi(d.v_\epsilon^{n,k-1}(t)) \leqslant \Phi_n(d.v_\epsilon^{n,k-1}(t))$ for all $t \leqslant 1$. Now, using (8) and (9), we have $|^iY_t - {}^iY'_t| \leqslant v_\epsilon^{n,k}(t)$ for all $t \leqslant 1$, $i = 1, \ldots, d$. Taking the limit as first $k \to \infty$, then $n \to \infty$, and finally $\epsilon \to 0$, we obtain $|^iY_t - {}^iY'_t| = 0$ for all $t \leqslant 1$. Therefore the solution is unique. □

Finally a word about the work of Mao (1995) on the same subject. He shows that if the coefficient f satisfies $|f(t, \omega, y, z) - f(t, \omega, y', z')|^2 \leqslant \Phi(|y - y'|^2) + k|z - z'|^2$, where k is constant and Φ satisfies Assumption 4(i) and (importantly) is concave, then the BSDE associated with (f, ξ) has a unique solution. He does not require the second part of Assumption 4(ii). In his proof he uses Bihari's inequality.

Using our approach and under the same hypotheses as in Mao (1995), but without requiring the concavity of Φ, it could be possible to obtain the same result as he does.

The issue of the existence and uniqueness of the solution for the BSDE associated with (f, ξ) when f satisfies Assumption 4(i) and the mapping $z \mapsto f(t, \omega, y, z)$ is uniformly Lipschitz is still open.

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