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The maximal degree in a Poisson–Delaunay graph

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We investigate the maximal degree in a Poisson–Delaunay graph in $\mathbf{R}^d$, $d \geq 2$, over all nodes in the window $\mathbf{W}_\rho := \rho^{1/d}[0,1]^d$ as ρ goes to infinity. The exact order of this maximum is provided in any dimension. In the particular setting $d = 2$, we show that this quantity is concentrated on two consecutive integers with high probability. A weaker version of this result is discussed when $d \geq 3$.

Keywords: degree; Delaunay graph; extreme values; Poisson point process

1. Introduction

Let χ be a locally finite subset in $\mathbf{R}^d$, $d \geq 2$, endowed with its Euclidean norm $\|\cdot\|$, such that each subset of size $n < d+1$ is affinely independent and no $d+2$ points lie on a sphere. The Delaunay triangulation of χ is the unique triangulation with vertices in χ such that the circumball of each simplex contains no point of χ in its interior. The set of its edges is denoted by $\mathrm{Del}(\chi)$ and the graph $(\chi, \mathrm{Del}(\chi))$ is the so-called *Delaunay graph* associated with χ [28], page 478. Delaunay graphs are a very popular structure in computational geometry [2] and are extensively used in many areas such as surface reconstruction, mesh generation, molecular modeling, and medical image segmentation, see, for example, [11,15]. The book by Okabe et al. [26] gives a taste of the richness of the theory of these graphs and of the variety of their applications. In this paper, we consider a Poisson–Delaunay graph that is a random Delaunay graph based on a stationary Poisson point process in $\mathbf{R}^d$.

Recently, extremes of various quantities associated with Poisson–Delaunay graphs have been investigated by Chenavier, Devillers and Robert. In [13], the length of the shortest path between two distant vertices is considered. In [12,14], the extremes studied are the largest or smallest values of a given geometric characteristic, such as the volume or the circumradius, over all simplices in the Poisson–Delaunay graph with incenter in a large window. For a broad panorama of extreme values arising from construction based on a Poisson point process, we refer the reader to [29].

In this paper, we deal with the case of a discrete random variable, namely the maximal degree. More precisely, let η be a stationary Poisson point process in $\mathbf{R}^d$. Without restriction, we assume that the intensity of η equals 1. Let $\mathbf{W}_\rho = \rho^{1/d}[0,1]^d$, where ρ is a positive real number. We

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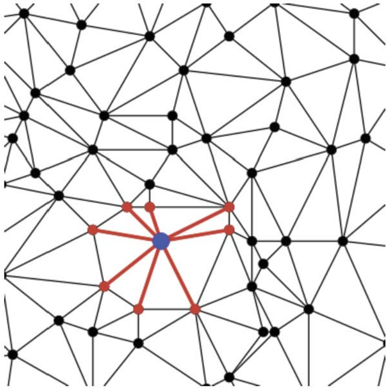

Figure 1. A node (blue) with its neighbors (red) which maximizes the degree in a planar Delaunay graph observed in a window.

investigate the asymptotic behaviour, as ρ goes to infinity, of the following random variable:

$$\Delta_\rho = \max_{x \in \eta \cap \mathbf{W}_\rho} d_\eta(x),$$

where $d_\eta(x)$ denotes the degree of any node $x \in \eta$ in the Poisson–Delaunay graph associated with η, i.e. the number of (non-oriented) edges passing through x (see Figure 1). The maximal degree of random combinatorial graphs has been extensively investigated, see, for example, [5, 10,16–18,24,25]. Much less has been done when the vertices are given by a point process and the edges are built according to geometric constraints. To the best of the authors' knowledge, one of the first results on the maximal degree in a Poisson–Delaunay graph was due to Bern *et al.* (see Theorem 7 in [4]) who showed that

$$\mathbb{E}[\Delta_\rho] = \Theta\left(\frac{\log \rho}{\log\log \rho}\right) \tag{1.1}$$

in any dimension $d \geq 2$. Broutin *et al.* [8] went on to provide a new bound for Δ_ρ in the following sense: when $d = 2$, with probability tending to 1, the maximal degree Δ_ρ is less than $(\log \rho)^{2+\xi}$, for any fixed $\xi > 0$. Our main theorem significantly improves these two results in dimension two.

Theorem 1. *Let Δ_ρ be the maximal degree in a planar Poisson–Delaunay graph over all nodes in $\mathbf{W}_\rho = \rho^{1/2}[0,1]^2$. Then there exists a deterministic function $\rho \mapsto I_\rho$, $\rho > 0$, with values in $\mathbf{N} = \{1, 2, \ldots\}$, such that*

(i) $\mathbb{P}(\Delta_\rho \in \{I_\rho, I_\rho + 1\}) \underset{\rho\to\infty}{\longrightarrow} 1$;

(ii) $I_\rho \underset{\rho\to\infty}{\sim} \frac{1}{2} \cdot \frac{\log \rho}{\log\log \rho}$.

In the above theorem, the notation $f(\rho) \underset{\rho\to\infty}{\sim} g(\rho)$ means that $\frac{f(\rho)}{g(\rho)}$ converges to 1 as ρ goes to infinity, for any functions $f, g : \mathbf{R} \to \mathbf{R}$, such that $f(\rho)$ and $g(\rho)$ differ from 0 for ρ large enough.

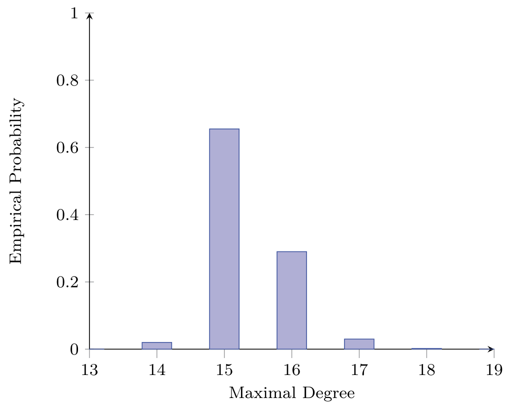

Figure 2. Empirical distribution of Δ_ρ, based on 75,000 simulations, of the maximal degree in a planar Poisson–Delaunay graph observed in the window $\mathbf{W}_{10^6} = 10^3[0, 1]^2$.

In particular, our result provides the exact order of the maximal degree and claims that, with high probability, the maximal degree is concentrated on two consecutive values. As observed in Figure 2, the concentration is already visible for $\rho = 10^6$. On the other hand, the estimate of I_ρ is good only for much larger values of ρ because of the extremely slow growth of the logarithm. This will be discussed further at the end of Section 2.2.

Our theorem is rather classical in the sense that similar results have already been established in the context of random combinatorial graphs [5,10,16–18,24,25]. Penrose [27], Th 6.6, established the same type of result as Theorem 1 for a Gilbert graph. More precisely, he proved that for this random geometric graph, the maximal degree and the maximal clique number are also concentrated on two consecutive values with high probability. To the best of the authors' knowledge, Anderson [1] was the first one, in the context of classical Extreme Value Theory, to prove that the maximum of n independent and identically distributed random variables is concentrated, with high probability as n goes to infinity, on two consecutive integers for a wide class of discrete random variables. Kimber [21] provided rates of convergence in the particular case where the random variables are Poisson distributed. However, two difficulties are added in the context of Poisson–Delaunay graphs. The first one is that the distribution of the typical degree cannot be made explicit. The second one, which constitutes the main difficulty, comes from the dependence between the degrees of the nodes and the geometric constraints in the Poisson–Delaunay graph.

As a corollary of Theorem 1, we can find arbitrary large windows for which the maximal degree is concentrated on only one integer with high probability.

Corollary 2. *Let Δ_ρ be the maximal degree in a planar Poisson–Delaunay graph over all nodes in $\mathbf{W}_\rho = \rho^{1/2}[0, 1]^2$. Then there exists an increasing sequence (ρ_i) converging to infinity such that*

$$\mathbb{P}(\Delta_{\rho_i} = i) \underset{i\to\infty}{\longrightarrow} 1.$$

The above corollary implies that, with probability 1, the event $\{\Delta_{\rho_i} = i\}$ occurs infinitely many times. A weaker version of Theorem 1, which deals with the general case $d \geq 2$, is stated below.

Theorem 3. *Let Δ_ρ be the maximal degree in a Poisson–Delaunay graph over all nodes in $\mathbf{W}_\rho = \rho^{1/d}[0,1]^d$, $d \geq 2$. Then there exists a deterministic function $\rho \mapsto J_\rho$, $\rho > 0$, with values in $\mathbf{N} = \{1, 2, \dots\}$, such that*

(i) $\mathbb{P}(\Delta_\rho \in \{J_\rho, J_\rho + 1, \dots, J_\rho + \ell_d\}) \underset{\rho\to\infty}{\longrightarrow} 1$, *where* $\ell_d = \lfloor \frac{d+3}{2} \rfloor$;

(ii) $J_\rho \underset{\rho\to\infty}{\sim} \frac{d-1}{2} \cdot \frac{\log \rho}{\log\log \rho}$.

In particular, when $d = 2$, the above result claims that the maximal degree is concentrated on three consecutive values, which is less accurate than Theorem 1. When $d = 3$ and $d = 4$, this also shows that the maximal degree is concentrated on four consecutive values.

Although Theorem 1 only deals with the two dimensional case, its proof is significantly more difficult than the proof of Theorem 3. Indeed, as opposed to Theorem 1, we think that Theorem 3 is not optimal in the sense that the maximal degree should also be concentrated on two consecutive integers, and not only on $\ell_d + 1$ integers. The proof of Theorem 1 extensively uses the fact that the graph is planar. In particular, as an intermediate result to derive Theorem 1, we prove that, with high probability, there is no family of five nodes in the Poisson–Delaunay graph which are close to each other, such that their degrees simultaneously exceed I_ρ. Such a result is essential in our proof and is specific to the two dimensional case.

As a consequence of Theorem 3, the following corollary improves the estimate (1.1).

Corollary 4. *Let Δ_ρ be the maximal degree in a Poisson–Delaunay graph over all nodes in $\mathbf{W}_\rho = \rho^{1/d}[0,1]^d$, $d \geq 2$. Then* $\mathbb{E}[\Delta_\rho] \underset{\rho\to\infty}{\sim} \frac{d-1}{2} \cdot \frac{\log \rho}{\log\log \rho}$.

All our results can be written in the context of the dual Poisson–Voronoi tessellation. More precisely, given a stationary Poisson point process η in $\mathbf{R}^d$, and $x \in \eta$, the set

$$C_\eta(x) = \{ y \in \mathbf{R}^d : \|x - y\| \leq \|x' - y\|, x' \in \eta \}$$

is called the *Voronoi cell* with nucleus x. The *Poisson–Voronoi tessellation* associated with η is defined as the family of the cells $C_\eta(x)$, $x \in \eta$. For a complete account on Poisson–Voronoi tessellations, we refer to [26], ch. 5. This model corresponds to the dual graph of Delaunay graph in the following way: there exists an edge between two points $x, x' \in \eta$ in the Delaunay graph if and only if they are Voronoi neighbors, that is,

$$C_\eta(x) \cap C_\eta(x') \neq \varnothing.$$

In particular the maximal degree Δ_ρ is also the maximal number of $(d-1)$-dimensional facets of Voronoi cells with nucleus in $\mathbf{W}_\rho$. Thus Theorems 1, 3 and Corollaries 2, 4 also provide estimates of the maximal number of facets of a Poisson–Voronoi tessellation as ρ goes to infinity.

The paper is organized as follows. In Section 2, we give several preliminaries by introducing some notation and by recalling a few known results. In Section 3, we present technical lemmas which will be used to derive Theorems 1 and 3. In Section 4, we prove our main theorems and their corollaries. The proofs of the technical lemmas are given in Section 5.

2. Preliminaries

2.1. Notation

Here we summarize the notation used throughout the text.

General notation. We denote by $\mathbf{N} = \{1, 2, \dots\}$ and $\mathbf{R}_+ = [0, \infty)$ the sets of positive integers and non-negative numbers, respectively. The d-dimensional Euclidean space $\mathbf{R}^d$ is endowed with the Euclidean norm $\|\cdot\|$ and with its d-dimensional Lebesgue measure $\mathrm{V}_d(\cdot)$. We denote by $\mathcal{B}_+^d$ the set of Borel sets $B \subset \mathbf{R}^d$ such that $0 < \mathrm{V}_d(B) < \infty$. The unit sphere with dimension $d-1$ is denoted by $\mathbf{S}^{d-1}$.

Now, let $k \in \mathbf{N}$ be fixed. We use the short notation $x_{1:k} = (x_1, \dots, x_k) \in (\mathbf{R}^d)^k$, and for such a k-tuple of points we write $sx_{1:k} + t = (sx_1 + t, \dots, sx_k + t)$ for any $s \in \mathbf{R}$ and $t \in \mathbf{R}^d$. We also consider concatenation of such vectors, for example, we write $(x_{1:k}, y_{1:\ell}) = (x_1, \dots, x_k, y_1, \dots, y_\ell)$. For any set S, we denote by $S^k_{\neq}$ the family of vectors $(s_1, \dots, s_k) \in S^k$ such that $s_i \neq s_j$ for any $i \neq j$. If χ is a finite set, we also denote by $\#\chi$ its cardinality.

Let $f, g : \mathbf{R} \to \mathbf{R}$ be two functions such that $f(\rho)$ and $g(\rho)$ differ from 0 for ρ large enough. Recall that the notation $g(x) \underset{x\to\infty}{\sim} f(x)$ means that f and g are asymptotically equivalent, that is, $\frac{g(x)}{f(x)} \underset{x\to\infty}{\longrightarrow} 1$. Moreover, we write $g(x) = O(f(x))$ if and only if there exists a positive number M and a real number x_0 such that $|g(x)| \leq M|f(x)|$ for any $x \geq x_0$. When $\frac{g(x)}{f(x)} \underset{x\to\infty}{\longrightarrow} 0$ we write $g(x) = o(f(x))$.

The quantity c denotes a generic constant which depends only on the dimension d. We occasionally index the constants when the distinction between several of them need to be made explicit, e.g. when two or more constants appear in a single equation.

Delaunay graph. We recall that a (undirected) graph $G = (V, E)$ is a set V of vertices together with a set E of edges with no orientation. Given a graph G, we denote the set of neighbors of a vertex v by $\mathcal{N}_G(v)$, that is the set of vertices $w \in V$ such that $\{v, w\} \in E$.

Let χ be a locally finite subset of $\mathbf{R}^d$ in general position, i.e. such that each subset of size $n \leq d$ is affinely independent and no $d+2$ points lie on a sphere. For a $(d+1)$-tuple of points $x_1, \dots, x_{d+1} \in \chi$, we denote by $B(x_{1:d+1})$ the unique open circumball associated with these points. We recall that we define a Delaunay edge between x_i and x_j for each $1 \leq i, j \leq d+1$, $i \neq j$, when $\chi \cap B(x_{1:d+1}) = \varnothing$, and denote by $\mathrm{Del}(\chi)$ the set of these edges. The graph $(\chi, \mathrm{Del}(\chi))$ is the so-called Delaunay graph associated with χ, see, for example, [28], page 478.

Let $x_0 \in \chi$. With a slight abuse of notation, we denote by $\mathcal{N}_\chi(x_0) = \mathcal{N}_{(\chi, \mathrm{Del}(\chi))}(x_0)$ the set of neighbors of x_0 in the Delaunay graph associated with χ. In particular, the degree of x_0 is $d_\chi(x_0) = \#\mathcal{N}_\chi(x_0)$. We also denote by $F_\chi(x_0)$ the *Voronoi flower at* x_0, defined as the union of all open balls which do not contain any point of χ and which are circumscribed around x_0 and d

other points of χ, that is,

$$F_\chi(x_0) = \bigcup_{\substack{x_{1:d}\in\chi^d_{\neq} \\ B(x_{0:d})\cap\chi=\varnothing}} B(x_{0:d}).$$

The Voronoi flower at x_0 only depends on its neighbors in the corresponding Delaunay graph. Reciprocally the Voronoi flower at x_0 determines its set of neighbors. We call Φ-*content* of x_0 the volume of its Voronoi flower and denote it by

$$\Phi_\chi(x_0) = \mathrm{V}_d\big(F_\chi(x_0)\big).$$

If χ is a finite subset $\{x_0, x_1, \dots, x_k\}$ of $\mathbf{R}^d$, with $k \geq d$, we use the shorter notation:

$$F_{x_{1:k}}(x_0) = F_{\{x_0,\dots,x_k\}}(x_0) \quad \text{and} \quad \Phi_{x_{1:k}}(x_0) = \mathrm{V}_d\big(F_{x_{1:k}}(x_0)\big).$$

Finally, for each $B \in \mathcal{B}^d_+$ and $k \in \mathbf{N}$, we let

$$M^B_\chi = \max_{x\in\chi\cap B} d_\chi(x) \quad \text{and} \quad N^B_\chi[k] = \sum_{x\in\chi\cap B} \mathbb{1}_{\{d_\chi(x)\geq k\}}. \tag{2.1}$$

If $\chi \cap B = \varnothing$, we take $M^B_\chi = -\infty$. In particular, if $\chi = \eta$ and $B = \mathbf{W}_\rho$, we have $M^{\mathbf{W}_\rho}_\eta = \Delta_\rho$.

2.2. The typical degree

Recall that η denotes a stationary Poisson point process of intensity 1 in $\mathbf{R}^d$, see [23], page 19, for the definition of a Poisson point process. To describe the mean behaviour of the Poisson–Delaunay graph, the notion of typical degree is introduced as follows. Let $B \in \mathcal{B}^d_+$ be fixed. The *typical degree* is defined as the discrete random variable $\mathcal{D}^0$ with distribution given by

$$\mathbb{P}\big(\mathcal{D}^0 = k\big) = \frac{1}{\mathrm{V}_d(B)}\,\mathbb{E}\bigg[\sum_{x\in\eta\cap B} \mathbb{1}_{\{d_\eta(x)=k\}}\bigg], \tag{2.2}$$

for any integer k. It is clear that $\mathbb{P}(\mathcal{D}^0 = k) = 0$ for any $k \leq d$. A consequence of the Mecke equation (e.g., Theorem 4.1 in [23]) shows that

$$\mathcal{D}^0 \overset{\mathrm{d}}{=} d_{\eta\cup\{0\}}(0), \tag{2.3}$$

where $\overset{\mathrm{d}}{=}$ denotes the equality in distribution. In particular, the right-hand side in (2.2) does not depend on B. As another consequence of the Mecke equation, notice that

$$\mathbb{E}\big[N^B_\eta[k]\big] = \mathrm{V}_d(B)\,\mathbb{P}\big(\mathcal{D}^0 \geq k\big) \tag{2.4}$$

for any $B \in \mathcal{B}^d_+$, and for any integer k.

Integral representation for the distribution of the typical degree. Let $x_{1:k} \in (\mathbf{R}^d)^k_{\neq}$ be a k-tuple of distinct points, with $k \geq d$. We say that $x_{1:k}$ is in *convex position* if $\{0, x_1, \dots, x_k\}$ is in general position, 0 is connected to all the x_i, $i = 1, \dots, k$, in the Delaunay graph associated with $\{0, x_1, \dots, x_k\}$ and 0 is in the interior of the closure of the Voronoi flower $F_{\{0,x_1,\dots,x_k\}}(0)$. We denote by C_k the set of all k-tuples of points in $\mathbf{R}^d$ which are in convex position. Note that C_k is also the set of k-tuples of points $x_{1:k}$ such that $\{0, x_1, \dots, x_k\}$ is in general position and such that the Voronoi cell with nucleus 0 associated with $\{0, x_1, \dots, x_k\}$, i.e. $C_{\{0,x_1,\dots,x_k\}}(0)$, is a (bounded) convex polytope with k facets. The set C_k is measurable and classical in the setting of Voronoi tessellations, see, for example, Equation (6) in [9]. Note also that C_k is stable under permutations, meaning that for any $x_{1:k} \in C_k$ and any permutation σ of the set $\{1, \dots, k\}$, we have $x_{\sigma(1:k)} = (x_{\sigma(1)}, \dots, x_{\sigma(k)}) \in C_k$. We shall now derive an integral representation of the distribution of the typical degree $\mathcal{D}^0$.

Lemma 5. *For each $k \geq 1$, we have*

$$\mathbb{P}(\mathcal{D}^0 = k) = \int_{C_k} \mathbb{1}_{\{\Phi_{y_{1:k}}(0) \leq 1\}} \,\mathrm{d}y_{1:k}.$$

Proof. Using the above notation and Equation (2.3), we write

$$\mathbb{P}(\mathcal{D}^0 = k) = \frac{1}{k!} \mathbb{E}\Bigg[\sum_{x_{1:k} \in \eta^k_{\neq}} \mathbb{1}_{\{x_{1:k} \in C_k\}} \mathbb{1}_{\{F_{x_{1:k}}(0) \cap \eta = \varnothing\}}\Bigg].$$

The multivariate Mecke equation (e.g., Theorem 4.4 in [23]) allows us to rewrite the expectation of a sum over k-tuples of points in a Poisson point process as an integral over k-tuples of points in $\mathbf{R}^d$. Thanks to this formula, this gives

$$\begin{aligned}
\mathbb{P}(\mathcal{D}^0 = k) &= \frac{1}{k!} \int_{C_k} \mathbb{P}\big(F_{x_{1:k}}(0) \cap \big(\eta \cup \{x_1, \dots, x_k\}\big) = \varnothing\big) \,\mathrm{d}x_{1:k} \\
&= \frac{1}{k!} \int_{C_k} \mathbb{P}\big(F_{x_{1:k}}(0) \cap \eta = \varnothing\big) \,\mathrm{d}x_{1:k} \\
&= \frac{1}{k!} \int_{C_k} e^{-\Phi_{x_{1:k}}(0)} \,\mathrm{d}x_{1:k},
\end{aligned}$$

where the third line is also a consequence of the fact that η is a Poisson point process. Using the fact that $e^{-t} = \int_t^\infty e^{-s} \,\mathrm{d}s$, we get

$$\mathbb{P}(\mathcal{D}^0 = k) = \frac{1}{k!} \int_{C_k} \int_0^\infty \mathbb{1}_{\{\Phi_{x_{1:k}}(0) \leq s\}} e^{-s} \,\mathrm{d}s \,\mathrm{d}x_{1:k}.$$

Now since being in convex position is invariant under rescaling and since $\Phi_{s^{1/d} y_{1:k}}(0) = s\Phi_{y_{1:k}}(0)$, the change of variables $x_{1:k} = s^{1/d} y_{1:k}$ gives

$$\mathbb{P}(\mathcal{D}^0 = k) = \frac{1}{k!} \int_0^\infty e^{-s} \int_{C_k} \mathbb{1}_{\{\Phi_{y_{1:k}}(0) \leq 1\}} s^k \,\mathrm{d}y_{1:k} \,\mathrm{d}s = \int_{C_k} \mathbb{1}_{\{\Phi y_{1:k}(0) \leq 1\}} \,\mathrm{d}y_{1:k},$$

where the last equality comes from the fact that $\int_0^\infty e^{-s}s^k\,\mathrm{d}s = \Gamma(k+1) = k!$. This concludes the proof of Lemma 5. □

Estimates for the distribution of the typical degree. The following result provides bounds for the distribution of the typical degree in $\mathbf{R}^d$, $d \geq 2$.

Proposition 6. *There exist two constants* $c_1, c_2 > 0$ *only depending on* d *such that, for any integer* $k \geq d+2$ *we have*

(i) $\mathbb{P}(\mathcal{D}^0 = k) \leq c_2 k^{-\frac{2}{d-1}}\,\mathbb{P}(\mathcal{D}^0 = k-1)$,
(ii) $\mathbb{P}(\mathcal{D}^0 = k) \geq c_1^k k^{\frac{-2}{d-1}k}$.

In particular, for some constant c_3, we have

$$c_1^k k^{\frac{-2}{d-1}k} \leq \mathbb{P}\big(\mathcal{D}^0 = k\big) \leq c_3^k k^{-\frac{2}{d-1}k} \tag{2.5}$$

and

$$\mathbb{P}\big(\mathcal{D}^0 \geq k\big) \underset{k\to\infty}{\sim} \mathbb{P}\big(\mathcal{D}^0 = k\big). \tag{2.6}$$

The last equation comes from the fact that, for $k \geq d+1$, we have $\mathbb{P}(\mathcal{D}^0 \geq k+1) = \sum_{\ell=1}^{\infty}\mathbb{P}(\mathcal{D}^0 = k+\ell) \leq \sum_{\ell=1}^{\infty} c_2^{\ell}(k+1)^{-\frac{2\ell}{d-1}}\,\mathbb{P}(\mathcal{D}^0 = k)$, which is negligible compared to $\mathbb{P}(\mathcal{D}^0 = k)$ as k goes to infinity.

Proposition 6 is very similar to two results in [7] (Theorem 1.1 and Theorem 1.2) in which estimates for the distribution of the typical number of facets in a Poisson hyperplane tessellation are given. We only give a sketch of proof because it relies on a simple adaptation of several arguments included in [7] to our setting. However, for a complete proof of Proposition 6, we refer the reader to Chapter 5 in [6] (Theorems 5.1.7 and 5.5.1).

Sketch of proof. Note that it is enough to prove that the proposition holds for $k \geq k_0$ for some arbitrarily large fixed k_0. Indeed the result extends to $k \geq d+2$ simply by decreasing the value of c_1 and increasing the value of c_2.

From Lemma 5, we know that

$$\mathbb{P}\big(\mathcal{D}^0 = k\big) = \mathrm{V}_{d\times k}(C_{k,1}),$$

where

$$C_{k,t} := \left\{y_{1:k} \in \left(\mathbf{R}^d\right)^k : y_{1:k} \in C_k,\, \Phi_{y_{1:k}}(0) \leq t\right\}$$

for any $t > 0$. Unfortunately the geometry of this set is too complex and a precise evaluation of its Lebesgue measure seems out of reach (except for low values of k and d). We present below the heart of the proof to get the lower and upper bounds of Proposition 6.

Upper bound: The key observation comes when one considers $y_{1:k} \in C_k$ and looks at what happens when we remove one point y_i of the collection $\{y_1, \ldots, y_k\}$. More specifically we are interested in how the Voronoi flowers $F_{y_{1:k}}(0)$ and $F_{y_{1:k-1}}(0)$ differ. There can be some bad configurations where removing one specific point will change dramatically the picture but there is always a good proportion of the points y_i which does not significantly affect the Voronoi flower. Indeed, by considering the associated Voronoi cell and using arguments from polytope approximation theory, it can be shown (see Lemma 4.3.2 of [6]) that for any $y_{1:k} \in C_{k,1}$, for at least a quarter of the indices $i \in \{1, \ldots, k\}$, the following two properties hold simultaneously:

(P_i) $(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_k) \in C_{k-1,1+\epsilon_k}$,
(P'_i) y_i is at distance at most ϵ'_k from the boundary of $F_{\{0,y_1,\ldots,y_k\}\setminus\{y_i\}}(0)$,

where $\epsilon_k = O(k^{-(d+1)/(d-1)})$ and $\epsilon'_k = O(k^{-2/(d-1)})$. Recall that elements $y_{1:k} \in C_{k,1}$ are stable under permutation of the points $y_1, \ldots, y_k$. Thus, if one picks a random point uniformly in $C_{k,1}$ the probability that (P_k) and (P'_k) are satisfied is at least $1/4$. This can also be written as

$$\begin{aligned} V_{d\times k}(C_{k,1}) &\leq 4 \int_{C_{k,1}} \mathbb{1}_{\{(P_k) \text{ is satisfied}\}} \mathbb{1}_{\{(P'_k) \text{ is satisfied}\}} \,\mathrm{d}y_{1:k} \\ &= 4 \int_{C_{k-1,1+\epsilon_k}} \int_{\mathbf{R}^d} \mathbb{1}_{\{d(y_k, \partial F_{y_{1:k-1}}(0)) \leq \epsilon'_k\}} \,\mathrm{d}y_k \,\mathrm{d}y_{1:k-1}. \end{aligned}$$

Now, it is easy to see that the inner integral is of order ϵ'_k and using also some homogeneity properties, we get

$$V_{d\times k}(C_{k,1}) \leq c \cdot \epsilon'_k V_{d\times(k-1)}(C_{k-1,1+\epsilon_k}) \leq c' \cdot \epsilon'_k (1 + \epsilon_k)^{k-1} V_{d\times(k-1)}(C_{k-1,1}),$$

for some positive constants c, c'. The factor $(1 + \epsilon_k)^{k-1}$ tends to 1 because $k\epsilon_k$ converges to 0 as k goes to infinity. Lemma 6(i) follows from the last upper bound and the fact that $V_{d\times(k-1)}(C_{k-1,1}) = \mathbb{P}(\mathcal{D}^0 = k - 1)$.

Lower bound: The idea for the lower bound is to build a subset $E_k \subset C_{k,1}$ with a sufficiently simple geometry such that we can easily estimate its volume. We start by picking a "nice" configuration $x_{1:k}$ in the interior of $C_{k,1}$. By "nice" we mean that $\|x_1\| = \cdots = \|x_k\|$, the vectors are well spread, i.e. $\min \|x_i - x_j\|$ is maximal, and $\Phi_{x_{1:k}}(0) = 1 - \delta$, where $\delta > 0$ is a fixed, arbitrarily small number. Note that in dimension 2 the points $x_1, \ldots, x_k$ are the vertices of a regular polygon centered at the origin. In higher dimension they do not have such a nice property, but by using usual arguments of minimal cap covering (see, e.g., Lemma 4.1.2. in [6]), it is easy to see that $\min \|x_i - x_j\|$ is of order $k^{-1/(d-1)}$ in any dimension.

Now we consider the following neighborhood of $x_{1:k}$,

$$D_k = \left\{y_{1:k} \in \left(\mathbf{R}^d\right)^k : \left(1 - \delta' k^{-2/(d-1)}\right)\|x_i\| \leq \|y_i\| \leq \|x_i\|, \|x_i - y_i\| \leq \delta' k^{-1/(d-1)} \ \forall i\right\},$$

where $\delta' > 0$ is a fixed number, arbitrarily small compare to δ. Basic classical geometry computation (essentially Pythagorean theorem) show that this is a subset of $C_{k,1}$. Also, it is easy to see that D_k has a volume of order $((\delta')^d k^{-(d+1)/(d-1)})^k$.

By considering permutation of the k coordinates for each element in D_k, we build the set E_k with a volume $k!$ bigger. Formally the set E_k is defined as

$$E_k = \left\{(y_{\sigma(1)}, \ldots, y_{\sigma(k)}) : y_{1:k} \in D_k, \sigma \text{ is a permutation of } \{1, \ldots, k\}\right\}.$$

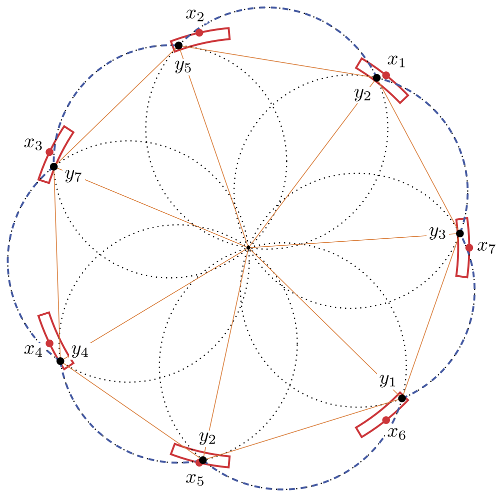

Figure 3. The set E_k with $k=7$ in the planar case. The black central dot is the origin. Elements of E_7 are 7-tuple of points such that there is exactly one of them in each of the "round" red rectangle. For each of the 7! possible indexing $y_1, \ldots, y_7$ of the 7 black dots, the 7-tuple $y_{1:7}$ belongs to E_7. The lines segments form the Delaunay triangulation $\mathrm{Del}(\{0, y_1, \ldots, y_7\})$. The thin black doted circles are the circumscribed circle around the different Delaunay triangles. The thick blue dashed curve is the boundary of the Voronoi flower $F_{y_{1:7}}(0)$.

An illustration of this set in the planar case with $k=7$ is provided in Figure 3. Using the lower bound $k! \geq (k/e)^k$ we get that E_k has a volume bigger than $c^k k^{-2/(d-1)}$ for some positive constant c. This is precisely the lower bound of the proposition. □

According to Proposition 6, the distribution of the typical degree belongs to the class of discrete distributions considered by Anderson [1]. Roughly speaking, this explains why the maximal degree belongs to two consecutive integers when the size of the window goes to infinity.

In the particular setting $d=2$, a more precise estimate of the distribution of the typical degree is established by Hilhorst (see Equation (1.2) in [19]):

$$\mathbb{P}\big(\mathcal{D}^0 = k\big) = \frac{C}{4\pi^2} \cdot \frac{(8\pi^2)^k}{(2k)!}\big(1 + O\big(k^{-\frac{1}{2}}\big)\big), \tag{2.7}$$

where $C \simeq 0.34$. The above result is extended by Calka and Hilhorst for a larger class of random polygons in $\mathbf{R}^2$ (see Equation (1.5) in [20]). Notice that (2.7) implies Proposition 6 in the particular case $d=2$. The following remark presents a heuristic argument suggesting that, in the case $d=2$, a careful study based on (2.7) should refine the estimate $\mathbb{E}[\Delta_\rho]$.

Remark. The estimate $\mathbb{E}[\Delta_\rho] \simeq \frac{d-1}{2} \cdot \frac{\log \rho}{\log \log \rho}$ seems to be accurate only for extremely high values of ρ. Indeed, for $d=2$, Figure 2 illustrates that the empirical distribution of Δ_{10^6} concentrates on 15 and 16 rather than around $\frac{1}{2} \cdot \frac{\log 10^6}{\log\log 10^6} \simeq 2.6$. This is not surprising because of the extremely slow growth of the logarithm.

Nevertheless, the following heuristic argument provides a much closer estimate of Δ_{10^6}. Thanks to (2.7) and because of the extremely fast decay of this expression as k grows, we have

$$\mathbb{P}(\mathcal{D}^0 \leq 13) \simeq 1 - \mathbb{P}(\mathcal{D}^0 = 14) \simeq 1 - 10^{-5}$$

and

$$\mathbb{P}(\mathcal{D}^0 \leq 15) \simeq 1 - \mathbb{P}(\mathcal{D}^0 = 16) \simeq 1 - 7.6 \cdot 10^{-8}.$$

Assuming that the maximal degree has the same behaviour as the maximum of 10^6 independent random variables with the same distribution as the typical degree,

$$\mathbb{P}(\Delta_{10^6} \leq 13) \simeq 4 \cdot 10^{-5} \quad \text{and} \quad \mathbb{P}(\Delta_{10^6} \leq 15) \simeq 0.93.$$

This suggests that $\Delta_{10^6} \in \{14, 15\}$ with high probability, which is almost what we observe in Figure 2. In the setting $d = 2$, a careful study based on (2.7) should provide an estimate of $\mathbb{E}[\Delta_\rho]$ which fits the correct value faster than ours.

2.3. The function $\rho \mapsto I_\rho$

In this section, we define a function $\rho \mapsto I_\rho$, $\rho > 0$, with values in $\mathbf{N}$, and which depends on the dimension d. When $d = 2$, this is the function appearing in Theorem 1. To define I_ρ for any $d \geq 2$, our approach is mainly inspired from [1]. For any $k \geq d + 1$, define $G(k) = \mathbb{P}(\mathcal{D}^0 > k)$, where $\mathcal{D}^0$ is the typical degree in $\mathbf{R}^d$. We extend G as a continuous function as follows. For any $k \geq d + 1$, we let $h(k) = -\log G(k)$. We consider an auxiliary function h_c defined as the extension of h obtained by linear interpolation, that is, for any $x \geq d + 1$,

$$h_c(x) = h(\lfloor x \rfloor) + (x - \lfloor x \rfloor)\big(h(\lfloor x + 1 \rfloor) - h(\lfloor x \rfloor)\big).$$

The function h_c is continuous, strictly increasing and $\lim_{x\to\infty} h_c(x) = \infty$. Then we extend G as the continuous function $G_c(x) = e^{-h_c(x)}$ for each $x \geq d + 1$. In particular, G_c is a continuous strictly decreasing function. Now, we define the function $\rho \mapsto I_\rho$, $\rho > 0$, by

$$I_\rho = \left\lfloor G_c^{-1}\left(\frac{1}{\rho}\right) + \frac{1}{2} \right\rfloor. \tag{2.8}$$

3. Intermediate results

In this section, we establish intermediate results which will be used in the proofs of Theorems 1 and 3. The proofs of these lemmas are given in Section 5.

3.1. Technical results

We will use several times the following trivial lemma.

Lemma 7. *Let $B \in \mathcal{B}_+^d$ and $k \in \mathbf{N}$. Then* $\mathbb{P}(M_\eta^B \geq k) \leq \mathrm{V}_d(B)\,\mathbb{P}(\mathcal{D}^0 \geq k)$.

The following lemma provides the exact order of I_ρ.

Lemma 8. *Let $d \geq 2$ and let I_ρ be as in* (2.8). *Then*

$$I_\rho \underset{\rho\to\infty}{\sim} \frac{d-1}{2} \cdot \frac{\log \rho}{\log\log \rho}.$$

The next lemma deals with the probability that the typical degree is larger than I_ρ up to an additive constant.

Lemma 9. *Let $d \geq 2$ and let I_ρ be as in* (2.8). *Then*

(i) $\rho\,\mathbb{P}(\mathcal{D}^0 \geq I_\rho + 2) \underset{\rho\to\infty}{\longrightarrow} 0$;

(ii) $(\frac{\log\log \rho}{\log \rho})^{\frac{2\ell}{d-1}} \rho\,\mathbb{P}(\mathcal{D}^0 \geq I_\rho - \ell) \underset{\rho\to\infty}{\longrightarrow} \infty$ *for each* $\ell \geq 0$.

In particular, when $\ell = 0$, Lemma 9(ii) means that $\rho\,\mathbb{P}(\mathcal{D}^0 \geq I_\rho)$ goes to infinity. By adapting the proof of Lemma 9, it can also be shown that $\rho\,\mathbb{P}(\mathcal{D}^0 \geq I_\rho + 1)$ does not converge as ρ goes to infinity because its infimum and supremum limits equal 0 and ∞ respectively.

As a consequence of Lemma 9, we could show that if $X_1, \ldots, X_n$ is a sequence of n independent and identically distributed random variables, with the same distribution as the typical degree, then the maximum of $X_1, \ldots, X_n$ belongs to $\{I_n, I_n + 1\}$ with probability tending to 1 as n goes to infinity. Even if the independency is lost, as it is the case with the vertices' degrees, the maximum remains upper bounded with high probability by $I_n + 1$, see Section 4.1, page 962. On the other hand, if the dependency is too strong it is much more delicate to give a non-trivial lower bound. Therefore the proofs of Theorems 1 and 3 rely on a quantification of the dependencies between the vertices' degrees. In Section 3.2 we show that vertices which are far enough have almost independent degrees. This is sufficient to derive Theorem 3. In Section 3.3, at the cost of reducing the setting to $d = 2$, we deal with a more local scale by showing that there are no 5 close nodes, such that their degrees are simultaneously larger than I_ρ. This is one of the greatest difficulties treated in this paper and one of the key arguments to prove Theorem 1.

3.2. A subdivision of the window $\mathbf{W}_\rho$

It is well-known that a Poisson–Delaunay graph in $\mathbf{R}^d$ has good mixing properties. To capture this property, we proceed as follows. We partition $\mathbf{W}_\rho = \rho^{1/d}[0, 1]^d$ into a set $\mathcal{V}_\rho$ of N_ρ^d closed sub-cubes of equal size, where

$$N_\rho := \left\lfloor \left(\frac{\rho}{\alpha \log \rho} \right)^{1/d} \right\rfloor, \tag{3.1}$$

for some $\alpha > 2$, with ρ large enough. The volume of each sub-cube is approximately $\alpha \log \rho$ as ρ goes to infinity. The sub-cubes are indexed by the set of $\mathbf{i} := (i_1, \ldots, i_d) \in \{1, \ldots, N_\rho\}^d$, analogously to the order of indexing the elements of a matrix. With a slight abuse of notation, we identify a cube with its index. We denote by $\mathbf{i}_0$ the unique sub-cube in $\mathcal{V}_\rho$ which contains the origin. We now introduce a distance between sub-cubes $\mathbf{i}$ and $\mathbf{j}$ as $d(\mathbf{i}, \mathbf{j}) := \max_{1 \leq s \leq d} |i_s - j_s|$. If $\mathcal{I}$ and $\mathcal{J}$ are two sets of sub-cubes, we let

$$d(\mathcal{I}, \mathcal{J}) := \min_{\mathbf{i} \in \mathcal{I}, \mathbf{j} \in \mathcal{J}} d(\mathbf{i}, \mathbf{j}).$$

For any $A \subset \mathbf{R}^d$, we define

$$\mathcal{I}(A) := \left\{ \mathbf{i} \in \mathcal{V}_\rho : \operatorname{int}(\mathbf{i} \cap A) \neq \varnothing \right\}.$$

Finally, to ensure several independence properties, we introduce the event that each sub-cube contains at least one point of η, that is

$$\mathcal{E}_\rho := \bigcap_{\mathbf{i} \in \mathcal{V}_\rho} \{\eta \cap \mathbf{i} \neq \varnothing\}.$$

The event $\mathcal{E}_\rho$ is extensively used in Stochastic Geometry to derive central limit theorems or limit theorems in Extreme Value Theory (see, e.g., [3,12]). It will play a crucial role in the rest of the paper. The following lemma captures the idea of “local dependence”.

Lemma 10. *Let $A, B \subset \mathbf{W}_\rho$ and let N_ρ as in (3.1), with $\alpha > 2$. Then*

(i) *conditioned on the event $\mathcal{E}_\rho$, the random variables M_η^A and M_η^B are independent when $d(\mathcal{I}(A), \mathcal{I}(B)) > D$, where $D := 4(\lfloor \sqrt{d} \rfloor + 1)$;*
(ii) *$\mathbb{P}(\mathcal{E}_\rho^c) = O(\rho^{-(\alpha-1)})$, where $\mathcal{E}_\rho^c$ denotes the complement of the event $\mathcal{E}_\rho$.*

The above lemma has been used in various papers (e.g., Lemma 5 in [12] and Lemma 1 in [14]). We refer the reader to these papers for a proof.

3.3. Family of five nodes with large degrees when $d = 2$

In this section, we only deal with the case $d = 2$. Recall that $\mathbf{i}_0$ is defined as the unique square in $\mathcal{V}_\rho$ which contains the origin (see Section 3.2). When ρ goes to infinity, the order of the area of this square is $\alpha \log \rho$.

The following result shows that, with high probability, there are no 5 close nodes, such that their degrees are simultaneously larger than I_ρ. Recall that the random variable $N_\eta^B[k] = \sum_{x \in \eta \cap B} \mathbb{1}_{\{d_\eta(x) \geq k\}}$, as introduced in (2.1), denotes the number of nodes in B with degree larger than k. By a convex and symmetric Borel subset B in $\mathbf{R}^2$, we mean a convex subset such that a point x lies in B if and only if its antipode, $-x$, also lies in B.

Proposition 11. *Let $B \in \mathcal{B}_+^2$ be a convex and symmetric Borel subset in $\mathbf{R}^2$. Then there exist two positive constants c_4, c_5 (independent of B) such, that for any $k \geq 1$, we have*

$$\mathbb{P}\big(N_\eta^B[k] \geq 5\big) \leq \big(c_4 \mathrm{V}_2(B) k^{-2} + c_5^k \mathrm{V}_2(B)^2 k^{-k/2}\big)\,\mathbb{P}\big(\mathcal{D}^0 = k\big).$$

The above result is the key ingredient to derive Theorem 1 and contains the main difficulty of our problem. It extensively uses the fact that the Delaunay graph is planar.

3.4. A lower bound for the distribution's tail of the maximal degree in a block

The following results deal with the reciprocal of Lemma 7. The first one only concerns the case $d = 2$ and will be used to prove Theorem 1.

Proposition 12. *Let $B \in \mathcal{B}_+^2$ be a convex and symmetric Borel subset in $\mathbf{R}^2$. Then there exists an integer k_0 depending on B such that*

$$\mathbb{P}\big(M_\eta^B \geq k\big) \geq \frac{\mathrm{V}_2(B)}{5}\,\mathbb{P}\big(\mathcal{D}^0 \geq k\big),$$

for any $k \geq k_0$.

The following result provides a lower bound which is less accurate than the one of Proposition 12, but deals with the general case $d \geq 2$. It will be used to prove Theorem 3.

Proposition 13. *Let $B \in \mathcal{B}_+^d$, $d \geq 2$. Then, for any $k \in \mathbf{N}$ and $h \geq 1$, we have*

$$\mathbb{P}\big(M_\eta^B \geq k\big) \geq \frac{\mathrm{V}_d(B)}{h+1}\left(\mathbb{P}\big(\mathcal{D}^0 \geq k\big) - \exp\left(-\mathrm{V}_d(B)\left(1 - e + \frac{h}{\mathrm{V}_d(B)}\right)\right)\right).$$

3.5. A bound for the probability of a finite union of events

Lemma 14. *Fix $K \geq 1$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $B^{(1)}, \dots, B^{(K)}$, be a collection of events such that $\mathbb{P}(\bigcap_{j \leq k+1} B^{(i_j)}) = 0$, for any $1 \leq i_1 < \cdots < i_{k+1} \leq K$. Then*

$$\mathbb{P}\left(\bigcup_{i=1}^K B^{(i)}\right) \geq \frac{1}{k}\sum_{i=1}^K \mathbb{P}\big(B^{(i)}\big).$$

Notice that when $k = 1$, the inequality is actually an equality.

4. Proofs of theorems and corollaries

4.1. Proof of Theorem 1

Let I_ρ be as in (2.8). According to Lemma 8, we have $I_\rho \underset{\rho\to\infty}{\sim} \frac{1}{2}\cdot\frac{\log\rho}{\log\log\rho}$. Now, we have to show that $\mathbb{P}(\Delta_\rho \in \{I_\rho, I_\rho+1\})$ converges to 1 as ρ goes to infinity. It is clear that $\mathbb{P}(\Delta_\rho \geq I_\rho+2)$ converges to 0 according to Lemma 7 (applied to $B=\mathbf{W}_\rho$ and $k=I_\rho+2$) and Lemma 9. To prove that $\mathbb{P}(\Delta_\rho \leq I_\rho-1)$ converges to 0, we use all intermediate results presented in the previous section. To do it, we subdivide the window $\mathbf{W}_\rho$ into N_ρ^2 sub-squares of equal size as described in Section 3.2, where N_ρ is defined in (3.1) with $d=2$, $\alpha>2$. This gives

$$\mathbb{P}(\Delta_\rho \leq I_\rho-1) \leq \mathbb{P}(\Delta_\rho \leq I_\rho-1 \mid \mathcal{E}_\rho) + \mathbb{P}\big(\mathcal{E}_\rho^c\big) \tag{4.1}$$

$$= \mathbb{P}\Bigg(\bigcap_{\mathbf{i}\in\mathcal{V}_\rho}\big\{M_\eta^{\mathbf{i}} \leq I_\rho-1\big\} \,\Big|\, \mathcal{E}_\rho\Bigg) + \mathbb{P}\big(\mathcal{E}_\rho^c\big), \tag{4.2}$$

where $M_\eta^{\mathbf{i}}$ is defined in Section 2.1 and where $\mathcal{E}_\rho$ and $\mathcal{V}_\rho$ are defined in Section 3.2 respectively. Let $\mathcal{V}'_\rho$ be the family of sub-cubes $\mathbf{i}$ for which each coordinate of its index (i_1,i_2) is a multiple of 9. Note that for any pair of distinct sub-cubes $\mathbf{i},\mathbf{j}\in\mathcal{V}'_\rho$, we have $d(\mathbf{i},\mathbf{j}) \geq 9 > 8 = 4(\lfloor\sqrt{2}\rfloor+1)$. According to Lemma 10(i), conditioned on $\mathcal{E}_\rho$, we know that the events $\{M_\eta^{\mathbf{i}} \leq I_\rho-1\}$ and $\{M_\eta^{\mathbf{j}} \leq I_\rho-1\}$ are independent for each $\mathbf{i}\neq\mathbf{j}\in\mathcal{V}'_\rho$. Using the fact that $\mathbb{P}(M_\eta^{\mathbf{i}} \leq I_\rho-1 \mid \mathcal{E}_\rho) \leq \mathbb{P}(M_\eta^{\mathbf{i}_0} \leq I_\rho-1)/\mathbb{P}(\mathcal{E}_\rho)$ for each $\mathbf{i}$, we get

$$\begin{aligned}
\mathbb{P}\Bigg(\bigcap_{\mathbf{i}\in\mathcal{V}_\rho}\big\{M_\eta^{\mathbf{i}} \leq I_\rho-1\big\} \mid \mathcal{E}_\rho\Bigg) &\leq \prod_{\mathbf{i}\in\mathcal{V}'_\rho}\mathbb{P}\big(M_\eta^{\mathbf{i}} \leq I_\rho-1 \mid \mathcal{E}_\rho\big)\\
&\leq \left(\frac{\mathbb{P}(M_\eta^{\mathbf{i}_0} \leq I_\rho-1)}{\mathbb{P}(\mathcal{E}_\rho)}\right)^{\#\mathcal{V}'_\rho}\\
&= \exp\Big(\#\mathcal{V}'_\rho\big(\log\big(1-\mathbb{P}\big(M_\eta^{\mathbf{i}_0} \geq I_\rho\big)\big) - \log\big(\mathbb{P}(\mathcal{E}_\rho)\big)\big)\Big).
\end{aligned}$$

It is clear that $\#\mathcal{V}'_\rho \geq \lfloor N_\rho/9\rfloor^2 \geq \frac{N_\rho^2}{82}$ for ρ large enough. Using this, and applying the standard inequality $\log(1-u) \leq -u$ by taking $u=\mathbb{P}(M_\eta^{\mathbf{i}_0} \geq I_\rho)$, we get

$$\mathbb{P}\Bigg(\bigcap_{\mathbf{i}\in\mathcal{V}_\rho}\big\{M_\eta^{\mathbf{i}} \leq I_\rho-1\big\} \mid \mathcal{E}_\rho\Bigg) \leq \exp\Bigg(-\frac{N_\rho}{82}\big(\mathbb{P}\big(M_\eta^{\mathbf{i}_0} \geq I_\rho\big) + \log\big(\mathbb{P}(\mathcal{E}_\rho)\big)\big)\Bigg).$$

Besides, according to Lemma 10(ii), we know that $\mathbb{P}(\mathcal{E}_\rho) = 1 - O(\rho^{-(\alpha-1)})$ with $\alpha>2$. This together with the above equation and Equation (4.1) gives, for ρ large enough,

$$\mathbb{P}(\Delta_\rho \leq I_\rho-1) \leq \exp\Bigg(-\frac{N_\rho^2}{82}\big(\mathbb{P}\big(M_\eta^{\mathbf{i}_0} \geq I_\rho\big) + \log\big(\mathbb{P}(\mathcal{E}_\rho)\big)\big)\Bigg) + O\big(\rho^{-(\alpha-1)}\big).$$

Since $\alpha > 2$, we have $N_\rho^2 \log(\mathbb{P}(\mathcal{E}_\rho)) \underset{\rho\to\infty}{\longrightarrow} 0$. Thus,

$$\mathbb{P}(\Delta_\rho \leq I_\rho - 1) \leq \exp\left(-\frac{N_\rho^2}{82}\mathbb{P}\big(M_\eta^{\mathbf{i}_0} \geq I_\rho\big) + o(1)\right) + O\big(\rho^{-(\alpha-1)}\big). \tag{4.3}$$

To prove that the right-hand side of the above equation converges to 0, we have to show that $N_\rho^2\,\mathbb{P}(M_\eta^{\mathbf{i}_0} \geq I_\rho)$ converges to infinity. Notice that we cannot directly apply Proposition 12 to the block $B = \mathbf{i}_0$ and to the integer $k = I_\rho$ because both quantities depend on ρ. To deal with $\mathbb{P}(M_\eta^{\mathbf{i}_0} \geq I_\rho)$, we sub-divide the square $\mathbf{i}_0$ into K_ρ^2 sub-squares of equal size, say $S_1, \dots, S_{K_\rho^2}$, with $K_\rho = \lfloor(\log\rho)^{1/2}\rfloor$. The area of each sub-square S_i is larger than α, and converges to α as ρ goes to infinity.

Now, we define a finite collection of events $B_\rho^{(1)}, \dots, B_\rho^{(K_\rho^2)}$ as follows. For each $1 \leq i \leq K_\rho^2$, we let

$$B_\rho^{(i)} = \big\{M_\eta^{S_i} \geq I_\rho\big\} \setminus \big\{N_\eta^{\mathbf{i}_0}[I_\rho] \geq 5\big\},$$

where we recall that $N_\eta^{\mathbf{i}_0}[I_\rho]$, as defined in (2.1), denotes the number of nodes with degree larger than I_ρ. In particular, we have

$$\mathbb{P}\big(M_\eta^{\mathbf{i}_0} \geq I_\rho\big) \geq \mathbb{P}\left(\bigcup_{i=1}^{K_\rho^2} B_\rho^{(i)}\right).$$

Moreover, we know that $\mathbb{P}(\bigcap_{j\leq 5} B_\rho^{(i_j)}) = 0$, for any $1 \leq i_1 < \dots < i_5 \leq K_\rho^2$. It follows from Lemma 14 that

$$\mathbb{P}\big(M_\eta^{\mathbf{i}_0} \geq I_\rho\big) \geq \frac{1}{4}\sum_{i=1}^{K_\rho^2}\mathbb{P}\big(B_\rho^{(i)}\big) \geq \frac{1}{4}\sum_{i=1}^{K_\rho^2}\mathbb{P}\big(M_\eta^{S_i} \geq I_\rho\big) - \frac{K_\rho^2}{4}\mathbb{P}\big(N_\eta^{\mathbf{i}_0}[I_\rho] \geq 5\big).$$

First, we provide an estimate of $\mathbb{P}(N_\eta^{\mathbf{i}_0}[I_\rho] \geq 5)$. According to Proposition 11, we know that

$$\mathbb{P}\big(N_\eta^{\mathbf{i}_0}[I_\rho] \geq 5\big) \leq \big(c_4 \mathrm{V}_2(\mathbf{i}_0) I_\rho^{-2} + c_5^{I_\rho}\mathrm{V}_2(\mathbf{i}_0)^2 I_\rho^{-I_\rho/2}\big)\,\mathbb{P}\big(\mathcal{D}^0 = I_\rho\big).$$

Since $\mathrm{V}_2(\mathbf{i}_0) = O(\log\rho)$ and $I_\rho \underset{\rho\to\infty}{\sim} \frac{1}{2}\cdot\frac{\log\rho}{\log\log\rho}$, it is clear that $\mathrm{V}_2(\mathbf{i}_0)I_\rho^{-2} = O(\frac{(\log\log\rho)^2}{\log\rho})$. Moreover, since $\log(\mathrm{V}_2(\mathbf{i}_0)) = O(I_\rho)$, we have for some positive constant c,

$$c_5^{I_\rho}\mathrm{V}_2(\mathbf{i}_0)^2 I_\rho^{-I_\rho/2} \leq \exp\left(-\frac{I_\rho}{2}\log I_\rho + c\cdot I_\rho\right).$$

The last term converges to 0 as ρ goes to infinity. Therefore

$$\mathbb{P}\big(N_\eta^{\mathbf{i}_0}[I_\rho] \geq 5\big) = o\big(\mathbb{P}\big(\mathcal{D}^0 = I_\rho\big)\big).$$

To deal with $\mathbb{P}(M_\eta^{S_i} \geq I_\rho)$, recall that the area of S_i is larger than α so that, up to a translation, the square S_i contains the square $S' = [-\frac{\sqrt{\alpha}}{2}, \frac{\sqrt{\alpha}}{2}]^2$. Due to the stationarity of η, we have $\mathbb{P}(M_\eta^{S_i} \geq I_\rho) \geq \mathbb{P}(M_\eta^{S'} \geq I_\rho)$. According to Proposition 12 applied to $B = S'$, with $\mathrm{V}_2(S') = \alpha$, we obtain for ρ large enough and for each $1 \leq i \leq K_\rho^2$,

$$\mathbb{P}\big(M_\eta^{S_i} \geq I_\rho\big) \geq \frac{\alpha}{5}\,\mathbb{P}\big(\mathcal{D}^0 \geq I_\rho\big).$$

Summing over $i = 1, \ldots, K_\rho^2$, we deduce for ρ large enough that

$$\mathbb{P}\big(M_\eta^{\mathbf{i}_0} \geq I_\rho\big) \geq \frac{K_\rho^2 \alpha}{21}\,\mathbb{P}\big(\mathcal{D}^0 \geq I_\rho\big).$$

Now, we can conclude the proof of Theorem 1. Indeed, it follows from the above inequality and Equation (4.3) that

$$\mathbb{P}(\Delta_\rho \leq I_\rho - 1) \leq \exp\big(-cN_\rho^2 K_\rho^2\,\mathbb{P}\big(\mathcal{D}^0 \geq I_\rho\big) + o(1)\big) + O\big(\rho^{-(\alpha-1)}\big),$$

for some positive constant c. This proves Theorem 1 thanks to Lemma 9(ii) and the fact that $N_\rho^2 \underset{\rho\to\infty}{\sim} \frac{\rho}{\alpha \log \rho}$ and $K_\rho^2 \underset{\rho\to\infty}{\sim} \log \rho$.

4.2. Proof of Corollary 2

To define the sequence (ρ_i), we first introduce for each $i \geq 1$ the set $D_i = \{\rho \in \mathbf{R}_+ : I_\rho = i\}$, where I_ρ is as in (2.8). Let $i \geq 1$ be fixed. The set D_i is non-empty since it contains the number $m_i = (G_c(i - \frac{1}{2}))^{-1}$. Because $\rho \mapsto I_\rho$ is increasing, D_i is an interval. Moreover, this interval is bounded since D_j is non-empty for each $j \geq 1$. Thus the family (D_i) is a partition of $\mathbf{R}_+$ into bounded intervals. We can easily show that these intervals are left-closed and right-open, respectively.

Now, we define the sequence (ρ_i) as follows. For each $i \geq 2$, we let $\rho_i = \sup D_{i-1} = \min D_i$. In particular, we have $I_{\rho_i - 1} \leq I_{\rho_i} - 1 = i - 1$. The sequence (ρ_i) is increasing and converges to infinity. According to Theorem 1, we have

$$\mathbb{P}\big(\Delta_{\rho_i} \in \{I_{\rho_i}, I_{\rho_i} + 1\}\big) \underset{i\to\infty}{\longrightarrow} 1. \tag{4.4}$$

Moreover, according to Lemma 9(i), we know that $(\rho_i - 1)\,\mathbb{P}(\mathcal{D}^0 \geq I_{\rho_i - 1} + 2)$ converges to 0 as i goes to infinity. Since $I_{\rho_i - 1} \leq I_{\rho_i} - 1$, this implies that

$$\rho_i\,\mathbb{P}\big(\mathcal{D}^0 \geq I_{\rho_i} + 1\big) \underset{i\to\infty}{\longrightarrow} 0.$$

Bounding $\mathbb{P}(\Delta_{\rho_i} \geq I_{\rho_i} + 1)$ by $\rho_i\,\mathbb{P}(\mathcal{D}^0 \geq I_{\rho_i} + 1)$ as before, we deduce that the probability $\mathbb{P}(\Delta_{\rho_i} \geq I_{\rho_i} + 1)$ converges to 0. This together with (4.4) and the fact that $I_{\rho_i} = i$ concludes the proof of Corollary 2.

4.3. Proof of Theorem 3

Let $J_\rho = I_\rho + 1 - \ell_d$, where I_ρ is defined in (2.8). According to Lemma 8, we know that $J_\rho \underset{\rho\to\infty}{\sim} \frac{d-1}{2} \cdot \frac{\log\rho}{\log\log\rho}$. Now, we have to show that $\mathbb{P}(\Delta_\rho \in \{J_\rho, J_\rho+1, \dots, J_\rho+\ell_d\}) \underset{\rho\to\infty}{\longrightarrow} 1$.

As in the proof of Theorem 1, we easily show that $\mathbb{P}(\Delta_\rho \geq I_\rho + 2)$ converges to 0 as ρ goes to infinity. It remains to prove that $\mathbb{P}(\Delta_\rho \leq I_\rho - \ell_d)$ also converges to 0. To do it, we proceed at this step in the same spirit as in the case $d = 2$. We divide $\mathbf{W}_\rho$ into N_ρ^d sub-cubes of equal size, where N_ρ is given in (3.1), for some $\alpha > 2$. For some positive constant c this gives (see Equation (4.3))

$$\mathbb{P}(\Delta_\rho \leq I_\rho - \ell_d) \leq \exp\bigl(-cN_\rho^d\,\mathbb{P}\bigl(M_\eta^{\mathbf{i}_0} \geq I_\rho - \ell_d + 1\bigr) + o(1)\bigr) + O\bigl(\rho^{-(\alpha-1)}\bigr). \tag{4.5}$$

Now we have to show that $N_\rho^d\,\mathbb{P}(M_\eta^{\mathbf{i}_0} \geq I_\rho - \ell_d + 1)$ converges to infinity. This time we apply Proposition 13 by taking $B = \mathbf{i}_0$, $k = I_\rho - \ell_d + 1$, and $h = \beta \mathrm{V}_d(\mathbf{i}_0)$ for some $\beta > 0$. This gives

$$\begin{aligned}
&\mathbb{P}\bigl(M_\eta^{\mathbf{i}_0} \geq I_\rho - \ell_d + 1\bigr)\\
&\quad\geq \frac{\mathrm{V}_d(\mathbf{i}_0)}{\beta\mathrm{V}_d(\mathbf{i}_0)+1}\bigl(\mathbb{P}\bigl(\mathcal{D}^0 \geq I_\rho - \ell_d + 1\bigr) - \exp\bigl(-(1-e+\beta)\mathrm{V}_d(\mathbf{i}_0)\bigr)\bigr).
\end{aligned}$$

Notice that the fraction $\frac{\mathrm{V}_d(\mathbf{i}_0)}{\beta\mathrm{V}_d(\mathbf{i}_0)+1}$ is bounded. To deal with the right-hand side, we recall that $N_\rho^d \underset{\rho\to\infty}{\sim} \frac{\rho}{\alpha\log\rho}$ and that $\mathrm{V}_d(\mathbf{i}_0) \underset{\rho\to\infty}{\sim} \alpha\log\rho$, with $\alpha > 2$. This gives

$$N_\rho^d \exp\bigl(-(1-e+\beta)\mathrm{V}_d(\mathbf{i}_0)\bigr) = o\bigl(\rho^{-(\alpha(1-e+\beta)-1)}\bigr).$$

Taking β in such a way that $\alpha(1-e+\beta) > 1$, we obtain that $N_\rho^d \exp(-(1-e+\beta)\mathrm{V}_d(\mathbf{i}_0))$ converges to 0. Moreover, it follows from Lemma 9(ii) that

$$\left(\frac{\log\log\rho}{\log\rho}\right)^{\frac{2(\ell_d-1)}{d-1}} \log(\rho) N_\rho^d\,\mathbb{P}\bigl(\mathcal{D}^0 \geq I_\rho - \ell_d + 1\bigr) \underset{\rho\to\infty}{\longrightarrow} \infty.$$

Combining all together we get

$$\left(\frac{\log\log\rho}{\log\rho}\right)^{\frac{2(\ell_d-1)}{d-1}} \log(\rho) N_\rho^d\,\mathbb{P}\bigl(M_\eta^{\mathbf{i}_0} \geq I_\rho - \ell_d + 1\bigr) \underset{\rho\to\infty}{\longrightarrow} \infty. \tag{4.6}$$

Since $\frac{2(\ell_d-1)}{d-1} > 1$, we deduce that $N_\rho^d\,\mathbb{P}(M_\eta^{\mathbf{i}_0} \geq I_\rho - \ell_d + 1)$ converges to infinity. This concludes the proof of Theorem 3.

4.4. Proof of Corollary 4

First, we write the expectation of the maximal degree as follows:

$$\mathbb{E}[\Delta_\rho] = \sum_{k=1}^{I_\rho-\ell_d} k\,\mathbb{P}(\Delta_\rho = k) + \sum_{k=I_\rho+1-\ell_d}^{I_\rho+d} k\,\mathbb{P}(\Delta_\rho = k) + \sum_{k=I_\rho+d+1}^{\infty} k\,\mathbb{P}(\Delta_\rho = k).$$

For the first term, we notice that

$$\sum_{k=1}^{I_\rho-\ell_d} k\,\mathbb{P}(\Delta_\rho = k) \leq \left(\sum_{k=1}^{I_\rho-\ell_d} k\right) \mathbb{P}(\Delta_\rho \leq I_\rho - \ell_d) \underset{\rho\to\infty}{\sim} \frac{I_\rho^2}{2}\,\mathbb{P}(\Delta_\rho \leq I_\rho - \ell_d).$$

Using (4.5), (4.6), Lemma 8 and the fact that $\frac{2(\ell_d-1)}{d-1} > 1$, it follows from basic computations that the right-hand side converges to 0. This proves that $\sum_{k=1}^{I_\rho-\ell_d} k\,\mathbb{P}(\Delta_\rho = k)$ also converges to 0. Moreover, as a consequence of Theorem 3, we know that $\sum_{k=I_\rho+1-\ell_d}^{I_\rho+d} k\,\mathbb{P}(\Delta_\rho = k)$ is asymptotically equivalent to $\frac{d-1}{2} \cdot \frac{\log\rho}{\log\log\rho}$. For the third term, we have

$$\sum_{k=I_\rho+d+1}^{\infty} k\,\mathbb{P}(\Delta_\rho = k) = (I_\rho + d)\,\mathbb{P}(\Delta_\rho \geq I_\rho + d + 1) + \sum_{k=I_\rho+d+1}^{\infty} \mathbb{P}(\Delta_\rho \geq k).$$

According to Lemma 7 (applied to $B = \mathbf{W}_\rho$ and $k = I_\rho + d + 1$), the first term of the right-hand side can be bounded as follows:

$$(I_\rho + d)\,\mathbb{P}(\Delta_\rho \geq I_\rho + d + 1) \leq (I_\rho + d)\rho\,\mathbb{P}\big(\mathcal{D}^0 \geq I_\rho + d + 1\big).$$

According to Proposition 6, there exists a positive constant c such that

$$(I_\rho + d)\,\mathbb{P}(\Delta_\rho \geq I_\rho + d + 1) \leq c I_\rho^{-1} \rho\,\mathbb{P}\big(\mathcal{D}^0 = I_\rho + 2\big).$$

The last term converges to 0 according to Lemma 9(i). Moreover, thanks again to Proposition 6, we can also show that the series $\sum_{k=I_\rho+d+1}^{\infty} \mathbb{P}(\Delta_\rho \geq k)$ is asymptotically less than $\rho\,\mathbb{P}(\mathcal{D}^0 \geq I_\rho + d + 1)$. Since this quantity converges to 0 by Lemma 9, this shows that $\sum_{k=I_\rho+d+1}^{\infty} k\,\mathbb{P}(\Delta_\rho = k) \underset{\rho\to\infty}{\longrightarrow} 0$. Consequently, we have $\mathbb{E}[\Delta_\rho] \underset{\rho\to\infty}{\sim} \frac{d-1}{2} \cdot \frac{\log\rho}{\log\log\rho}$.

5. Proofs of technical results

5.1. Proof of Lemma 7

Let $B \in \mathcal{B}_+^d$ and $k \in \mathbf{N}$. It is clear that $\mathbb{P}(M_\eta^B \geq k) = \mathbb{P}(N_\eta^B[k] \geq 1)$. According to the Markov's inequality, this gives

$$\mathbb{P}\big(M_\eta^B \geq k\big) \leq \mathbb{E}\big[N_\eta^B[k]\big] = \mathrm{V}_d(B)\,\mathbb{P}\big(\mathcal{D}^0 \geq k\big),$$

where the last equality comes from (2.4).

5.2. Proof of Lemma 8

Let

$$A_\rho = G_c^{-1}\left(\frac{1}{\rho}\right), \tag{5.1}$$

so that $I_\rho = \lfloor A_\rho + \frac{1}{2} \rfloor$. Since G_c is a continuous strictly decreasing function, the term A_ρ can be written as

$$A_\rho = \inf\left\{x \in \mathbf{R}_+ : G_c(x) \le \frac{1}{\rho}\right\} = \sup\left\{x \in \mathbf{R}_+ : G_c(x) \ge \frac{1}{\rho}\right\}.$$

It is enough to prove that $x_\rho - 2 \le A_\rho \le y_\rho$, where

$$x_\rho := \frac{d-1}{2} \cdot \frac{\log \rho}{\log\log \rho} \quad \text{and} \quad y_\rho := \frac{d-1}{2}\left(\frac{\log\rho}{\log\log\rho} + 2\frac{\log\rho}{(\log\log\rho)^2}\log\log\log\rho\right).$$

To prove that $A_\rho \ge x_\rho - 2$ for ρ large enough, we notice that

$$A_\rho \ge \inf\left\{k \in \mathbf{N} : G(k-1) \le \frac{1}{\rho}\right\} - 2.$$

Besides, according to Proposition 6 and the fact that $G(k-1) = \mathbb{P}(\mathcal{D}^0 \ge k)$ is larger than $\mathbb{P}(\mathcal{D}^0 = k)$, we have $G(k-1) \ge c_1^k k^{-\frac{2}{d-1}k}$. Thus

$$A_\rho \ge \inf\left\{k \in \mathbf{N} : c_1^k k^{\frac{-2}{d-1}k} \le \frac{1}{\rho}\right\} - 2 \ge \inf\left\{x \in \mathbf{R}_+ : c_1^x x^{\frac{-2}{d-1}x} \le \frac{1}{\rho}\right\} - 2.$$

Moreover, for ρ large enough, we have

$$c_1^{x_\rho} x_\rho^{\frac{-2}{d-1}x_\rho} = \exp\left(-\log\rho + \frac{\log\rho}{\log\log\rho}\big(\log\log\log\rho + O(1)\big)\right) > \frac{1}{\rho}.$$

In particular, we have $x_\rho \le \inf\{x \in \mathbf{R}_+ : c_1^{-x} x^{\frac{-2}{d-1}x} \le \frac{1}{\rho}\}$, which proves that $A_\rho \ge x_\rho - 2$.

To prove that $A_\rho \le y_\rho$, we proceed along the same lines as above. Indeed,

$$A_\rho \le \sup\left\{k \in \mathbf{N} : G(k-1) \ge \frac{1}{\rho}\right\}.$$

According to (2.5) and (2.6), there exists a constant $c_6 > 0$ such that, for each $k \in \mathbf{N}$, we have $G(k-1) \le c_6^k k^{-\frac{2}{d-1}k}$. Thus

$$A_\rho \le \sup\left\{k \in \mathbf{N} : c_6^k k^{\frac{-2}{d-1}k} \ge \frac{1}{\rho}\right\} \le \sup\left\{y \in \mathbf{R}_+ : c_6^y y^{\frac{-2}{d-1}y} \ge \frac{1}{\rho}\right\}.$$

Moreover, with standard computations, we can easily show that

$$c_6^{y_\rho} y_\rho^{\frac{-2}{d-1} y_\rho} = \exp\left(-\log\rho - \frac{\log\rho}{\log\log\rho}\big(\log\log\log\rho + O(1)\big)\right) < \frac{1}{\rho}.$$

In particular, for ρ large enough, we have $y_\rho \geq \sup\{y \in \mathbf{R}_+ : c_6^y y^{\frac{-2}{d-1}y} \geq \frac{1}{\rho}\}$, which proves that $A_\rho \leq y_\rho$.

5.3. Proof of Lemma 9

Proof of (i). First, we notice that for each $k \in \mathbf{N}$, we have $\frac{G(k+1)}{G(k)} \underset{k\to\infty}{\longrightarrow} 0$ by Proposition 6. With standard computations (see, e.g., the proof of Theorem 1 in [1]), we easily show that

$$\frac{G_c(x+y)}{G_c(x)} \underset{x\to\infty}{\longrightarrow} 0, \tag{5.2}$$

for each $x, y \in \mathbf{R}_+$. In particular, we get

$$\rho\,\mathbb{P}\big(\mathcal{D}^0 \geq I_\rho + 2\big) = \rho G_c(I_\rho + 1) \leq \rho G_c\left(A_\rho + \frac{1}{2}\right) = o\big(\rho G_c(A_\rho)\big),$$

where A_ρ is defined in (5.1). Since $G_c(A_\rho) = \frac{1}{\rho}$, we have $\rho\,\mathbb{P}(\mathcal{D}^0 \geq I_\rho + 2) = o(1)$. □

Proof of (ii). First, we deal with the case $\ell = 0$. Proceeding in the same spirit as above, Equation (5.2) gives

$$\rho\,\mathbb{P}\big(\mathcal{D}^0 \geq I_\rho\big) = \rho G_c(I_\rho - 1) \geq \rho G_c\left(A_\rho - \frac{1}{2}\right) = \frac{G_c(A_\rho - \frac{1}{2})}{G_c(A_\rho)} \underset{\rho\to\infty}{\longrightarrow} \infty.$$

The general case follows from an induction on ℓ and from the following lines:

$$\begin{aligned} \rho\,\mathbb{P}\big(\mathcal{D}^0 \geq I_\rho - \ell\big) &\geq \rho\,\mathbb{P}\big(\mathcal{D}^0 = I_\rho - \ell\big) \\ &\geq \rho c_2^{-1}(I_\rho - \ell + 1)^{\frac{2}{d-1}}\,\mathbb{P}\big(\mathcal{D}^0 = I_\rho - \ell + 1\big) \\ &\underset{\rho\to\infty}{\sim} c\rho\left(\frac{\log\rho}{\log\log\rho}\right)^{\frac{2}{d-1}} \mathbb{P}\big(\mathcal{D}^0 \geq I_\rho - \ell + 1\big), \end{aligned}$$

where the second inequality is a consequence of Proposition 6 and where the third line comes from (2.6) and the fact that $I_\rho \underset{\rho\to\infty}{\sim} \frac{d-1}{2} \cdot \frac{\log\rho}{\log\log\rho}$. □

5.4. Proof of Proposition 11

First, we show that if $N_\eta^B[k] \geq 5$, then almost surely there exists at least one pair of nodes in B, with degree larger than k but with few vertices in common. Then we show that such an event cannot occur with high probability. To do it, we begin with a result on deterministic geometric graphs, established in the following paragraph.

A bound for the number of common vertices in a deterministic geometric graph.

Proposition 15. *Let $G = (V, E)$ be a simple planar graph in $\mathbf{R}^2$ and let $S = \{s_1, s_2, s_3, s_4, s_5\} \subset V$ be a set of five distinct vertices. Then there exist two vertices $s_i, s_j \in S$ such that $\#(\mathcal{N}_G(s_i) \cap \mathcal{N}_G(s_j)) \leq 4$.*

Proof. Assume for the sake of contradiction that every pair of distinct vertices of S has at least five common neighbors and fix a planar drawing of G. We show below that G contains a subdivision of the complete graph with five vertices K_5, which is not planar. To do it, it is enough to prove that for every distinct vertices $u, v \in S$, there exists a common neighbor $p_{uv} \notin S$ that is adjacent to no vertex of S except u and v.

Let $u, v \in S$ be two distinct vertices and let $w_1, \ldots, w_5$ be the common neighbors of u and v. The edges of G between the vertices u, v and the vertices $w_1, \ldots, w_5$ partitions the plane into five disjoint open regions $R_1, \ldots, R_5$. Up to a reordering of $w_1, \ldots, w_5$, we may assume that the boundary of each region R_i is the quadrilateral uw_ivw_{i+1}, where $w_6 := w_1$ (see Figure 4).

We prove below that for some $1 \leq i \leq 5$ the closure $\overline{R_i}$ of R_i contains S. To do it, we observe that there exists a region R_i which contains at least a vertex in S. Indeed, if not, S is included in $\{u, v, w_1, w_2, w_3, w_4, w_5\}$ and contains a pair of the form w_j, w_{j+2}, for some $1 \leq j \leq 7$, with $w_6 := w_1$ and $w_7 := w_2$. Since the graph G is planar, the vertices w_j and w_{j+2} have at most three common neighbors, namely u, v and w_{j+1}, which contradicts our assumption. Therefore, there exists a vertex $x \in S \cap R_i$ for some $1 \leq i \leq 5$. To prove that S is included in $\overline{R_i}$, we assume by contradiction that there exists a vertex $y \in S \cap R_j$, with $j \neq i$. In the same spirit as above, the vertices x and y have at most three common neighbors, which also contradicts our assumption.

We can now conclude the proof of Proposition 15. Indeed, since $S \subset \overline{R_i}$, there exists a common neighbor $p_{uv} \notin S$ of u, v which is not a neighbor of any vertex in $S \setminus \{u, v\}$: for $i = 1, 2, 3, 4, 5$, we can take $p_{uv} = w_4, w_5, w_1, w_2, w_3$ respectively. This proves the claimed property. $\square$

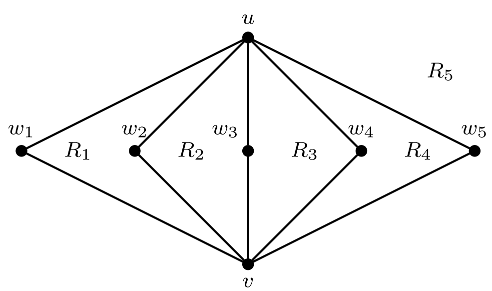

Figure 4. Five common neighbors between the vertices u and v.

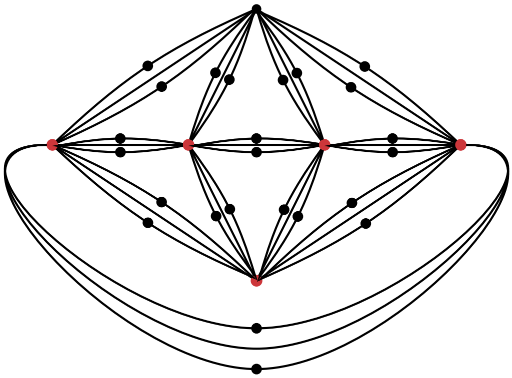

Figure 5. A planar graph with a set S of five vertices (red points) such that each pair of vertices of S has four common neighbors.

Proposition 15 is tight in the sense that the number 4 appearing in the upper bound cannot be replaced by 3 (see Figure 5).

Proof of Proposition 11. Let E_k be the following event:

$$E_k = \left\{ \sum_{x \in \eta \cap B} \mathbb{1}_{\{d_\eta(x)=k\}} \geq 5 \right\}.$$

Then

$$\begin{aligned}\mathbb{P}\big(N_\eta^B[k] \geq 5\big) &\leq \mathbb{P}(E_k) + \mathbb{P}\big(N_\eta^B[k+1] \geq 1\big)\\ &\leq \mathbb{P}(E_k) + \mathbb{E}\big[N_\eta^B[k+1]\big]\\ &= \mathbb{P}(E_k) + \mathrm{V}_2(B)\,\mathbb{P}\big(\mathcal{D}^0 \geq k+1\big),\end{aligned}$$

where the last equality comes from (2.4). According to Proposition 6, we know that

$$\mathbb{P}\big(\mathcal{D}^0 \geq k+1\big) \underset{\rho\to\infty}{\sim} \mathbb{P}\big(\mathcal{D}^0 = k+1\big) \leq ck^{-2}\,\mathbb{P}\big(\mathcal{D}^0 = k\big).$$

Now, we have to show that $\mathbb{P}(E_k) \leq c^k \mathrm{V}_2(B)^2 k^{-k/2}\,\mathbb{P}(\mathcal{D}^0 = k)$, which constitutes the main difficulty of the proof of Proposition 11. To do it, we apply Proposition 15: if the event E_k occurs then there exist five nodes $x_1, \ldots, x_5$ in $\eta \cap B$ with degree k such that $\#(\mathcal{N}_\eta(x_1) \cap \mathcal{N}_\eta(x_2)) \leq 4$. Thus

$$\begin{aligned}\mathbb{P}(E_k) \leq 2\,\mathbb{P}\big(&\exists (x_1, x_2) \in (\eta \cap B)^2 : d_\eta(x_1) = d_\eta(x_2) = k, \#\big(\mathcal{N}_\eta(x_1) \cap \mathcal{N}_\eta(x_2)\big) \leq 4,\\ &\text{and } \mathrm{V}_2\big(F_\eta(x_1)\big) \leq \mathrm{V}_2\big(F_\eta(x_2)\big)\big).\end{aligned}$$

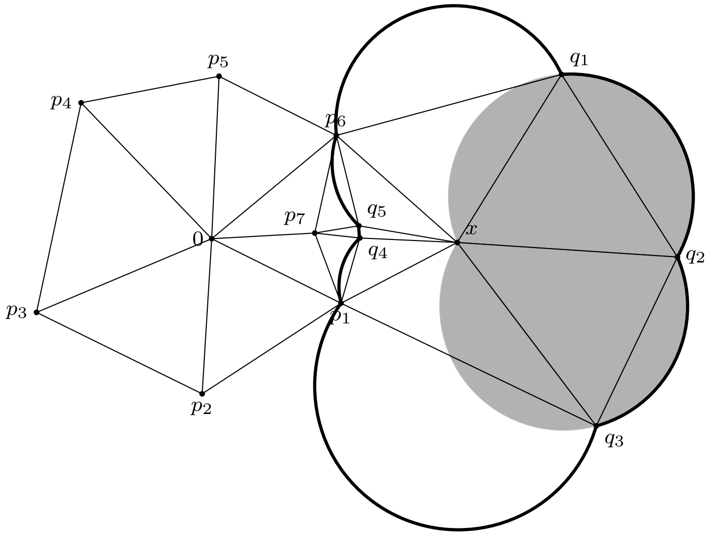

Figure 6. Part of the Delaunay graph $\mathrm{Del}(\eta \cup \{0, x\})$ around the points 0 and x.

The factor 2 comes from the fact that $\mathrm{V}_2(F_\eta(x_1))$ is assumed to be lower than $\mathrm{V}_2(F_\eta(x_2))$. It follows from the multivariate Mecke equation and the fact that η is stationary that

$$\mathbb{P}(E_k) \leq 2\mathrm{V}_2(B) \int_{2B} \mathbb{P}\big(\big\{d_{\eta\cup\{0,x\}}(0) = d_{\eta\cup\{0,x\}}(x) = k\big\} \\ \cap \big\{\#\big(\mathcal{N}_{\eta\cup\{0,x\}}(0) \cap \mathcal{N}_{\eta\cup\{0,x\}}(x)\big) \leq 4\big\} \cap \big\{\mathrm{V}_2\big(F_\eta(x)\big) \leq \mathrm{V}_2\big(F_\eta(0)\big)\big\}\big)\, \mathrm{d}x.$$

Note that the integration domain is $2B$ since, because of the symmetry of B, this is precisely the set of all differences $x_2 - x_1$ for $x_1, x_2 \in B$. To bound the right-hand side, we introduce for any $\ell \in \mathbf{N}$, $x \in \mathbf{R}^2$, $s \in [0, \infty]$ the set $D_{\ell,x,s} \subset \mathbf{R}^{2\ell}$ which consists of the family of (ordered) ℓ-tuples of points $q_{1:\ell} = (q_1, \dots, q_\ell)$ in $\mathbf{R}^2$ such that the following properties hold simultaneously:

$$\mathcal{P} : \begin{cases} q_1, \dots, q_\ell \text{ are clockwise ordered around } x; \\ q_j \notin B(x, q_i, q_{i+1}) \quad \text{for any } i < \ell \text{ and } j \leq \ell; \\ \mathrm{V}_2\Big(\bigcup_{i<\ell} B(x, q_i, q_{i+1})\Big) \leq s. \end{cases}$$

Here "clockwise ordered around x" means that the points appear in order when viewed from x and turning clockwise. These properties are illustrated by Figure 6. In this figure, the points q_1, q_2 and q_3 are three consecutive neighbors of x (clockwise ordered around x). The circumscribed disks of $\{x, q_1, q_2\}$ and $\{x, q_2, q_3\}$ are petals of the Voronoi flower centered at x, and therefore the area of their union is less then $\Phi_\eta(x)$. These facts imply that $q_{1:3}$ is an element of $D_{x,3,\Phi_\eta(x)}$.

Note that, contrary to the set C_k introduced just before Lemma 5, the set $D_{\ell,x,s}$ is not stable under coordinates permutation. This is due to the clockwise orientation restriction. We will also use several times the following homogeneity properties which hold for any $\ell \in \mathbf{N}$, $x \in \mathbf{R}^2$ and $0 < s < t$,

$$D_{\ell,x,s} = x + D_{\ell,0,s}, \qquad D_{\ell,x,s} \subset D_{\ell,x,t}, \qquad V_{2\ell}(D_{\ell,x,s}) = s^{\ell} V_{2\ell}(D_{\ell,x,1}). \tag{5.3}$$

Now, let $x \in B$ be fixed. Assume that the following events $\{d_{\eta\cup\{0,x\}}(0) = d_{\eta\cup\{0,x\}}(x) = k\}$, $\{\#(\mathcal{N}_{\eta\cup\{0,x\}}(0) \cap \mathcal{N}_{\eta\cup\{0,x\}}(x)) \leq 4\}$ and $\{\mathrm{V}_2(F_{\eta\cup\{0,x\}}(x)) \leq \mathrm{V}_2(F_{\eta\cup\{0,x\}}(0))\}$ hold simultaneously. In particular, there exist at least $k-4$ neighbors of x which do not belong to the Voronoi flower $F_{\eta\cup\{0,x\}}(0)$. Thus there exist at least $k' = \lfloor \frac{k-4}{4} \rfloor$ *consecutive* (clockwise ordered around x) neighbors of x, which are not neighbors of 0. Thus there exists a k'-tuple of points $p_{1:k'} \in \eta^{k'}$ such that the family of properties $\mathscr{P}$ holds, with $\ell = k'$ and $s = \mathrm{V}_2(F_{\eta\cup\{0,x\}}(0))$. Therefore

$$\begin{aligned}\mathbb{P}(E_k) \leq 2V_2(B) \int_{2B} \mathbb{P}\big(&\big\{d_{\eta\cup\{0,x\}}(0) = k\big\} \\ &\cap \big\{\big(\eta \setminus \mathcal{N}_{\eta\cup\{0,x\}}(0)\big)^{k'}_{\neq} \cap D_{k',x,\mathrm{V}_2(F_{\eta\cup\{0,x\}}(0))} \neq \varnothing\big\}\big)\,\mathrm{d}x.\end{aligned}$$

Now, we discuss two cases: the first one is when x and 0 are not neighbors and the second one deals with the complement event.

Case 1. *The nodes x and 0 are not neighbors.* In this case, we bound for any $x \in 2B$, the following probability:

$$\begin{aligned}P_1(x) = \mathbb{P}\big(&\big\{d_{\eta\cup\{0,x\}}(0) = k\big\} \cap \big\{\big(\eta \setminus \mathcal{N}_{\eta\cup\{0,x\}}(0)\big)^{k'}_{\neq} \cap D_{k',x,\mathrm{V}_2(F_{\eta\cup\{0,x\}}(0))} \neq \varnothing\big\} \\ &\cap \big\{x \notin \mathcal{N}_{\eta\cup\{0,x\}}(0)\big\}\big).\end{aligned}$$

To do it, we write

$$P_1(x) \leq \frac{1}{k!} \mathbb{E}\Bigg[\sum_{(p_{1:k}, q_{1:k'}) \in \eta^{k+k'}_{\neq}} \mathbb{1}_{\{F_{p_{1:k}}(0) \cap \eta = \varnothing\}} \mathbb{1}_{\{p_{1:k} \in C_k\}} \mathbb{1}_{\{q_{1:k'} \in D_{k',x,\Phi_{p_{1:k}}(0)}\}}\Bigg],$$

where we recall that $F_{p_{1:k}}(0)$ is the Voronoi flower with nucleus 0 induced by the set of points $\{0, p_1, \dots, p_k\}$. Notice that we have divided by $k!$ because C_k is stable under permutations which is not the case for $D_{\ell,x,s}$. It follows from the multivariate Mecke equation that

$$P_1(x) \leq \frac{1}{k!} \int_{\mathbf{R}^{2k}} \int_{\mathbf{R}^{2k'}} \mathbb{P}\big(F_{p_{1:k}}(0) \cap \eta = \varnothing\big) \mathbb{1}_{\{p_{1:k} \in C_k\}} \mathbb{1}_{\{q_{1:k'} \in D_{k',x,\Phi_{p_{1:k}}(0)}\}}\,\mathrm{d}q_{1:k'}\,\mathrm{d}p_{1:k}. \tag{5.4}$$

Integrating over $q_{1:k'}$, it follows from Fubini's theorem and the fact that η is a Poisson point process, that

$$P_1(x) \leq \frac{1}{k!} \int_{C_k} e^{-\Phi_{p_{1:k}}(0)} \mathrm{V}_{2k'}(D_{k',x,\Phi_{p_{1:k}}(0)})\,\mathrm{d}p_{1:k}.$$

As in the proof of Lemma 5, we use the fact that $e^{-\Phi_{p_{1:k}}(0)} = \int_0^\infty e^{-s} \mathbb{1}_{\{\Phi_{p_{1:k}}(0)\le s\}} \, \mathrm{d}s$. The change of variables $p_{1:k} = s^{1/2} y_{1:k}$, the properties (5.3), and the fact that $\Phi_{s^{1/2} y_{1:k}}(0) = s\Phi_{y_{1:k}}(0)$, give

$$\frac{1}{k!} \int_{C_k} e^{-\Phi_{p_{1:k}}(0)} \mathrm{V}_{2k'}(D_{k',x,\Phi_{p_{1:k}}(0)}) \, \mathrm{d}p_{1:k}$$
$$= \frac{1}{k!} \int_0^\infty e^{-s} \int_{C_k} \mathbb{1}_{\{\Phi_{y_{1:k}}(0)\le 1\}} \big(s\Phi_{y_{1:k}}(0)\big)^{k'} \mathrm{V}_{2k'}(D_{k',x,1}) s^k \, \mathrm{d}y_{1:k} \, \mathrm{d}s.$$

Bounding $\Phi_{y_{1:k}}(0)$ by 1 in the integrand, we get

$$P_1(x) \le \frac{1}{k!} \int_0^\infty e^{-s} \int_{C_k} \mathbb{1}_{\{\Phi_{y_{1:k}}(0)\le 1\}} s^{k+k'} \mathrm{V}_{2k'}(D_{k',x,1}) \, \mathrm{d}y_{1:k} \, \mathrm{d}s.$$

Integrating over s, we deduce from Lemma 5 that

$$P_1(x) \le \frac{(k+k')!}{k!} \mathbb{P}\big(\mathcal{D}^0 = k\big) \mathrm{V}_{2k'}(D_{k',x,1}).$$

The next lemma provides an upper bound for $\mathrm{V}_{2k'}(D_{k',x,1})$.

Lemma 16. *There exists a constant $c > 0$ such that, for any $j \in \mathbf{N}$ and $x \in \mathbf{R}^2$,*

$$\mathrm{V}_{2j}(D_{j,x,1}) \le \frac{c}{(j-1)!} \mathbb{P}\big(\mathcal{D}^0 = j\big).$$

Proof. First, we notice that this term actually does not depend on x since $D_{j,x,s} = x + D_{j,0,s}$ for any j, x, s. In the proof of this lemma we will use the notation $\tilde{\Phi}_{p_{1:j}}(x) := \mathrm{V}_2(\bigcup_{i\le j-1} B(x, p_i, p_{i+1}))$ for any $p_{1:j} \in D_{j,x,\infty}$. Similarly as above, we combine the substitution $p_{1:j} = s^{1/2} y_{1:j}$ with the observation that $\int_0^\infty e^{-s} \mathbb{1}_{\{\tilde{\Phi}_{p_{1:j}}(0)\le s\}} \, \mathrm{d}s = e^{-\tilde{\Phi}_{p_{1:j}}(0)}$. This gives

$$\mathrm{V}_{2j}(D_{j,0,1}) = \frac{1}{j!} \int_0^\infty e^{-s} \int_{D_{j,0,\infty}} \mathbb{1}_{\{\tilde{\Phi}_{y_{1:j}}(0)\le 1\}} s^j \, \mathrm{d}y_{1:j} \, \mathrm{d}s = \frac{1}{j!} \int_{D_{j,0,\infty}} e^{-\tilde{\Phi}_{p_{1:j}}(0)} \, \mathrm{d}p_{1:j}.$$

Using the fact that $e^{-\tilde{\Phi}_{p_{1:j}}(0)} = \mathbb{E}[\mathbb{1}_{\{(\eta\cup\{p_1,\dots,p_j\})\cap(\bigcup_{i\le j-1} B(0,p_i,p_{i+1}))=\varnothing\}}]$, the multivariate Mecke equation implies that

$$\mathrm{V}_{2j}(D_{j,0,1}) = \frac{1}{j!} \mathbb{E}\Bigg[\sum_{p_{1:j}\in\eta_{\neq}^j} \mathbb{1}_{\{p_{1:j}\in D_{j,0,\infty}\}} \mathbb{1}_{\{\eta\cap(\bigcup_{i\le j-1} B(0,p_i,p_{i+1}))=\varnothing\}}\Bigg].$$

Note that a.s. the random variable $d_{\eta\cup\{0\}}(0)$ is larger than j whenever the indicator functions above are equal to one. Moreover if $d_{\eta\cup\{0\}}(0) = \ell \ge j$, then there exist exactly ℓ tuples of points $p_{1:\ell}$ in η such that the corresponding events hold. In fact p_1 must be a neighbor of 0 in $\mathrm{Del}(\eta)$, and picking it arbitrarily implies that $p_2, p_3, \ldots$ are the (clockwise ordered around 0) neighbors of 0. Thus, according to (2.3), we can write

$$\mathrm{V}_{2j}(D_{j,0,1}) = \frac{1}{j!}\mathbb{E}\Big[\sum_{\ell\geq j}\ell\mathbb{1}_{\{\mathcal{D}^0=\ell\}}\Big] \tag{5.5}$$

Moreover, we know that

$$\mathbb{E}\Big[\sum_{\ell\geq j}\ell\mathbb{1}_{\{\mathcal{D}^0=\ell\}}\Big] = \sum_{\ell=1}^{\infty}(j+\ell)\,\mathbb{P}\big(\mathcal{D}^0 = j+\ell\big) \leq c'\cdot\mathbb{P}\big(\mathcal{D}^0 = j\big),$$

for some constant c', where the inequality relies on an adaptation of the proof of (2.6), see page 955. This together with (5.5) concludes the proof of Lemma 16. □

According to Lemma 16, for any $x \in \mathbf{R}^2$, we have

$$P_1(x) \leq c\frac{(k+k')!}{k!(k'-1)!}\,\mathbb{P}\big(\mathcal{D}^0 = k\big)\,\mathbb{P}\big(\mathcal{D}^0 = k'\big).$$

To provide an upper bound for the the right-hand side, we first we give an estimate of the logarithm of $\frac{(k+k')!}{k!(k'-1)!}\mathbb{P}(\mathcal{D}^0 = k')$. To do it, we use the fact that $\log(k!) = k\log k + O(k)$. Since $k' = \lfloor\frac{k-4}{4}\rfloor$, we have $\log((k+k')!) = \frac{5}{4}k\log k + O(k)$ and $\log((k'-1)!) = \frac{1}{4}k\log k + O(k)$. Moreover, according to (2.5), we also know that $\log(\mathbb{P}(\mathcal{D}^0 = k')) = -\frac{1}{2}k\log k + O(k)$. Therefore,

$$\log\bigg(\frac{(k+k')!}{k!(k'-1)!}\,\mathbb{P}\big(\mathcal{D}^0 = k'\big)\bigg) = -\frac{1}{2}k\log k + O(k).$$

Now we can provide an upper bound for $P_1(x)$. Indeed, it follows from the above equality that $\frac{(k+k')!}{k!(k'-1)!}\mathbb{P}(\mathcal{D}^0 = k')$ is lower than $c^k k^{-k/2}$ for some positive constant c, which implies

$$P_1(x) \leq c^k k^{-k/2}\,\mathbb{P}\big(\mathcal{D}^0 = k\big).$$

Integrating over $x \in 2B$, we obtain $\int_{2B} P_1(x)\,\mathrm{d}x \leq c^k\mathrm{V}_2(B)k^{-k/2}\,\mathbb{P}(\mathcal{D}^0 = k)$.

Case 2. *The nodes x and* 0 *are neighbors.* In this case, for any $x \in B$, we deal with the following probability:

$$\begin{aligned}P_2(x) = \mathbb{P}\Big(&\big\{d_{\eta\cup\{0,x\}}(0) = k\big\} \cap \big(\eta\setminus\mathcal{N}_{\eta\cup\{0,x\}}(0)\big)^{k'}_{\neq} \cap D_{k',x,\mathrm{V}_2(F_{\eta\cup\{0,x\}}(0))} \neq \varnothing\}\\ &\cap\big\{x \in \mathcal{N}_{\eta\cup\{0,x\}}(0)\big\}\Big).\end{aligned}$$

Since we now consider situations where x is one of the k neighbors of 0, it will be practical in the following lines to set $p_k = x$ in order to keep relatively short notation. This time we write

$$\begin{aligned}P_2(x) &= P_2(p_k)\\ &\leq \frac{1}{(k-1)!}\mathbb{E}\bigg[\sum_{(p_{1:k-1},q_{1:k'})\in\eta_{\neq}^{k-1+k'}} \mathbb{1}_{\{F_{p_{1:k}}(0)\cap\eta=\varnothing\}}\mathbb{1}_{\{p_{1:k}\in C_k\}}\mathbb{1}_{\{q_{1:k'}\in D_{k',p_k,\Phi_{p_{1:k}}(0)}\}}\bigg].\end{aligned}$$

Integrating over $x \in B$ and applying the multivariate Mecke equation as in the first case, we have

$$\begin{aligned}\int_B P_2(x)\,\mathrm{d}x \leq \frac{1}{(k-1)!}&\int_{\mathbf{R}^{2(k-1)}\times B}\int_{\mathbf{R}^{2k'}} \mathbb{P}\big(F_{p_{1:k}}(0)\cap\eta=\varnothing\big)\\ &\times \mathbb{1}_{\{p_{1:k}\in C_k\}}\mathbb{1}_{\{q_{1:k'}\in D_{k',p_k,\Phi_{p_{1:k}}(0)}\}}\,\mathrm{d}q_{1:k'}\,\mathrm{d}p_{1:k}.\end{aligned}$$

The right-hand side is very similar to the upper bound in (5.4). There are only two differences between these upper bounds. The first one is that we integrate over $\mathbf{R}^{2(k-1)}\times B$ instead of $\mathbf{R}^{2k}$. The second one is that we consider the ratio $\frac{1}{(k-1)!}$ instead of $\frac{1}{k!}$. However, proceeding exactly along the same lines as in the first case, we obtain that $\int_{2B} P_2(x)\,\mathrm{d}x \leq c^k k^{-k/2}\,\mathbb{P}(\mathcal{D}^0 = k)$.

Since $\mathbb{P}(E_k) \leq 2\mathrm{V}_2(B)\int_{2B}(P_1(x)+P_2(x))\,\mathrm{d}x$, it follows from the two cases discussed above that $\mathbb{P}(E_k) \leq c^k\mathrm{V}_2(B)^2 k^{-k/2}\,\mathbb{P}(\mathcal{D}^0 = k)$. This concludes the proof of Proposition 11.

5.5. Proof of Proposition 12

Recall that $M_\eta^B = \max_{x\in\eta\cap B} d_\eta(x)$ and $N_\eta^B[k] = \sum_{x\in\eta\cap B}\mathbb{1}_{\{d_\eta(x)\geq k\}}$ denote the maximum degree and the number of exceedances in the set B, respectively. This gives

$$\begin{aligned}\mathbb{P}\big(\mathcal{D}^0 \geq k\big) &= \frac{1}{\mathrm{V}_2(B)}\mathbb{E}\bigg[\sum_{x\in\eta\cap B}\mathbb{1}_{\{d_\eta(x)\geq k\}}\bigg]\\ &= \frac{1}{\mathrm{V}_2(B)}\mathbb{E}\big[N_\eta^B[k]\mathbb{1}_{\{N_\eta^B[k]\leq 4\}}\mathbb{1}_{\{M_\eta^B\geq k\}}\big] + \frac{1}{\mathrm{V}_2(B)}\mathbb{E}\big[N_\eta^B[k]\mathbb{1}_{\{N_\eta^B[k]\geq 5\}}\big].\end{aligned}$$

We bound $N_\eta^B[k]\mathbb{1}_{\{N_\eta^B[k]\leq 4\}}$ by 4 in the first expectation and $N_\eta^B[k]$ by $\#(\eta\cap B)$ in the second one. We get

$$\mathbb{P}\big(\mathcal{D}^0 \geq k\big) \leq \frac{4}{\mathrm{V}_2(B)}\mathbb{P}\big(M_\eta^B \geq k\big) + \frac{1}{\mathrm{V}_2(B)}\mathbb{E}\big[\#(\eta\cap B)\mathbb{1}_{\{N_\eta^B[k]\geq 5\}}\big]. \tag{5.6}$$

We show below that the second term of the right-hand side equals $o(\mathbb{P}(\mathcal{D}^0 = k))$. To do it, we write

$$\begin{aligned}&\mathbb{E}\big[\#(\eta\cap B)\mathbb{1}_{\{N_\eta^B[k]\geq 5\}}\big]\\ &\quad\leq k^{1+\epsilon}\mathrm{V}_2(B)\,\mathbb{P}\big(N_\eta^B[k]\geq 5\big) + \mathbb{E}\big[\#(\eta\cap B)\mathbb{1}_{\{\#(\eta\cap B)\geq k^{1+\epsilon}\mathrm{V}_2(B)\}}\big],\end{aligned}$$

for some $\epsilon \in (0, 1)$. According to Proposition 11, since $\epsilon < 1$, we have

$$k^{1+\epsilon}\mathrm{V}_2(B)\,\mathbb{P}\big(N_B^{\eta}[k] \geq 5\big) = o\big(\mathbb{P}\big(\mathcal{D}^0 = k\big)\big)$$

as k goes to infinity. Besides, we can easily show that if X is a Poisson random variable with parameter λ, then $\mathbb{E}[X\mathbb{1}_{\{X\geq x\}}] \underset{x\to\infty}{\sim} x\,\mathbb{P}(X \geq x)$. Since $\#(\eta \cap B)$ is a Poisson random variable with parameter $\mathrm{V}_2(B)$, this gives

$$\begin{aligned}\mathbb{E}\big[\#(\eta \cap B)\mathbb{1}_{\{\#(\eta\cap B)\geq k^{1+\epsilon}\mathrm{V}_2(B)\}}\big] &\underset{k\to\infty}{\sim} k^{1+\epsilon}\mathrm{V}_2(B)\,\mathbb{P}\big(\#(\eta \cap B) \geq k^{1+\epsilon}\mathrm{V}_2(B)\big)\\ &\leq k^{1+\epsilon}\mathrm{V}_2(B)\exp\big(-k^{1+\epsilon}\mathrm{V}_2(B)\big)\,\mathbb{E}\big[e^{\#(\eta\cap B)}\big],\end{aligned}$$

where the second line is a consequence of the Markov's inequality. Besides, according to Proposition 6, we have $k^{1+\epsilon}\exp(-k^{1+\epsilon}\mathrm{V}_2(B)) = o(\mathbb{P}(\mathcal{D}^0 = k))$. This implies that

$$\mathbb{E}\big[\#(\eta \cap B)\mathbb{1}_{\{N_\eta^B[k]\geq 5\}}\big] = o\big(\mathbb{P}\big(\mathcal{D}^0 = k\big)\big).$$

This together with (5.6) concludes the proof of Proposition 12.

5.6. Proof of Proposition 13

Proceeding along the same lines as in the proof of Proposition 12 (see Equation (5.6)), we obtain for any $k \in \mathbf{N}$, $h \geq 1$ that

$$\mathbb{P}\big(\mathcal{D}^0 \geq k\big) \leq \frac{h+1}{\mathrm{V}_d(B)}\,\mathbb{P}\big(M_\eta^B \geq k\big) + \frac{1}{\mathrm{V}_d(B)}\,\mathbb{E}\big[N_\eta^B[k]\mathbb{1}_{\{N_\eta^B[k]\geq h+1\}}\big].$$

To deal with the second term of the right-hand side, we apply the Mecke equation. This gives

$$\begin{aligned}\mathbb{E}\big[N_\eta^B[k]\mathbb{1}_{\{N_\eta^B[k]\geq h+1\}}\big] &= \mathbb{E}\Bigg[\sum_{x\in\eta\cap B}\mathbb{1}_{\{d_\eta(x)\geq k\}}\mathbb{1}_{\{N_\eta^B[k]\geq h+1\}}\Bigg]\\ &= \int_B \mathbb{P}\big(d_{\eta\cup\{x\}}(x) \geq k,\, N_{\eta\cup\{x\}}^B[k] \geq h+1\big)\,\mathrm{d}x.\end{aligned}$$

Bounding the integrand by the probability of the event $\{\#((\eta \cup \{x\}) \cap B) \geq h+1\}$, which equals $\{\#\eta \cap B \geq h\}$ for almost all $x \in B$, we obtain

$$\mathbb{E}\big[N_\eta^B[k]\mathbb{1}_{\{N_\eta^B[k]\geq h+1\}}\big] \leq \mathrm{V}_d(B)\,\mathbb{P}(\#\eta \cap B \geq h).$$

Thus

$$\mathbb{P}\big(\mathcal{D}^0 \geq k\big) \leq \frac{h}{\mathrm{V}_d(B)}\,\mathbb{P}\big(M_\eta^B \geq k\big) + \mathbb{P}(\#\eta \cap B \geq h). \tag{5.7}$$

Since the random variable $\#\eta \cap B$ is Poisson distributed with parameter $\mathrm{V}_d(B)$, it follows from the Markov's inequality that

$$\mathbb{P}(\#\eta \cap B \geq h) \leq e^{-h}\,\mathbb{E}\big[e^{\#\eta\cap B}\big] = \exp\biggl(-\mathrm{V}_d(B)\biggl(1-e+\frac{h}{\mathrm{V}_d(B)}\biggr)\biggr).$$

This together with (5.7) concludes the proof of Proposition 13.

5.7. Proof of Lemma 14

For any $\omega \in \Omega$, let $d(\omega)$ be the number of $B^{(i)}$'s which contain ω. Let $1 \leq i, \ell \leq K$. It is clear that the set $\{\omega \in B^{(i)} : d(\omega) = \ell\}$ is measurable because it can be written as a union, intersection and complement of events $B^{(i)}$. Now, let $a_i(\ell) = \mathbb{P}(\{\omega \in B^{(i)} : d(\omega) = \ell\})$. According to Lemma 1 in [22], we know that

$$\mathbb{P}\Biggl(\bigcup_{i=1}^{K} B^{(i)}\Biggr) = \sum_{i=1}^{K}\sum_{\ell=1}^{K}\frac{a_i(\ell)}{\ell}$$

if Ω is finite. As mentioned in a footnote of [22], page 147, the above equality remains true for a general probability space. Moreover, according to the assumption, we have $a_i(\ell) = 0$ for any $i \leq K$ and $\ell > k$. Thus

$$\mathbb{P}\Biggl(\bigcup_{i=1}^{K} B^{(i)}\Biggr) \geq \frac{1}{k}\sum_{i=1}^{K}\sum_{\ell=1}^{k} a_i(\ell) = \frac{1}{k}\sum_{i=1}^{K}\mathbb{P}\bigl(B^{(i)}\bigr).$$

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References

[1] Anderson, C.W. (1970). Extreme value theory for a class of discrete distributions with applications to some stochastic processes. *J. Appl. Probab.* **7** 99–113. MR0256441 https://doi.org/10.1017/s0021900200026978

[2] Aurenhammer, F., Klein, R. and Lee, D.-T. (2013). *Voronoi Diagrams and Delaunay Triangulations*. Hackensack, NJ: World Scientific Co. Pte. Ltd. MR3186045 https://doi.org/10.1142/8685

[3] Avram, F. and Bertsimas, D. (1993). On central limit theorems in geometrical probability. *Ann. Appl. Probab.* **3** 1033–1046. MR1241033

[4] Bern, M., Eppstein, D. and Yao, F. (1991). The expected extremes in a Delaunay triangulation. In *Automata*, *Languages and Programming* (*Madrid*, 1991). *Lecture Notes in Computer Science* **510** 674–685. Berlin: Springer. MR1129945 https://doi.org/10.1007/3-540-54233-7_173

[5] Bollobás, B. (2001). *Random Graphs*, 2nd ed. *Cambridge Studies in Advanced Mathematics* **73**. Cambridge: Cambridge Univ. Press. MR1864966 https://doi.org/10.1017/CBO9780511814068

[6] Bonnet, G. (2016). Poisson hyperplane tessellation: Asymptotic probabilities of the zero and typical cells. Ph.D. thesis, Univ. of Osnabrück.

[7] Bonnet, G., Calka, P. and Reitzner, M. (2018). Cells with many facets in a Poisson hyperplane tessellation. *Adv. Math.* **324** 203–240. MR3733885 https://doi.org/10.1016/j.aim.2017.11.016

[8] Broutin, N., Devillers, O. and Hemsley, R. (2014). The maximum degree of a random Delaunay triangulation in a smooth convex. In *AofA – 25th International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of Algorithms*.

[9] Calka, P. (2003). An explicit expression for the distribution of the number of sides of the typical Poisson–Voronoi cell. *Adv. in Appl. Probab.* **35** 863–870. MR2014258 https://doi.org/10.1239/aap/1067436323

[10] Carr, R., Goh, W.M.Y. and Schmutz, E. (1994). The maximum degree in a random tree and related problems. *Random Structures Algorithms* **5** 13–24. MR1248172 https://doi.org/10.1002/rsa.3240050104

[11] Cazals, F. and Giesen, J. (2006). Delaunay triangulation based surface reconstruction. In *Effective Computational Geometry for Curves and Surfaces* 231–276. Springer.

[12] Chenavier, N. (2014). A general study of extremes of stationary tessellations with examples. *Stochastic Process. Appl.* **124** 2917–2953. MR3217429 https://doi.org/10.1016/j.spa.2014.04.009

[13] Chenavier, N. and Devillers, O. (2018). Stretch factor in a planar Poisson–Delaunay triangulation with a large intensity. *Adv. in Appl. Probab.* **50** 35–56. MR3781976 https://doi.org/10.1017/apr.2018.3

[14] Chenavier, N. and Robert, C.Y. (2018). Cluster size distributions of extreme values for the Poisson–Voronoi tessellation. *Ann. Appl. Probab.* **28** 3291–3323. MR3861814 https://doi.org/10.1214/17-AAP1345

[15] Cheng, S.-W., Dey, T.K. and Shewchuk, J.R. (2012). *Delaunay Mesh Generation*. *Chapman & Hall/CRC Computer and Information Science Series*. Boca Raton, FL: CRC Press/CRC. MR3156288

[16] Drmota, M., Giménez, O., Noy, M., Panagiotou, K. and Steger, A. (2014). The maximum degree of random planar graphs. *Proc. Lond. Math. Soc.* (3) **109** 892–920. MR3273487 https://doi.org/10.1112/plms/pdu024

[17] Gao, Z. and Wormald, N.C. (2000). The distribution of the maximum vertex degree in random planar maps. *J. Combin. Theory Ser. A* **89** 201–230. MR1741015 https://doi.org/10.1006/jcta.1999.3006

[18] Giménez, O., Mitsche, D. and Noy, M. (2016). Maximum degree in minor-closed classes of graphs. *European J. Combin.* **55** 41–61. MR3474791 https://doi.org/10.1016/j.ejc.2016.02.001

[19] Hilhorst, H.J. (2005). Asymptotic statistics of the n-sided planar Poisson–Voronoi cell. I. Exact results. *J. Stat. Mech. Theory Exp.* **9** P09005, 45. MR2174088

[20] Hilhorst, H.J. and Calka, P. (2008). Random line tessellations of the plane: Statistical properties of many-sided cells. *J. Stat. Phys.* **132** 627–647. MR2429698 https://doi.org/10.1007/s10955-008-9577-0

[21] Kimber, A.C. (1983). A note on Poisson maxima. *Z. Wahrsch. Verw. Gebiete* **63** 551–552. MR0705624 https://doi.org/10.1007/BF00533727

[22] Kuai, H., Alajaji, F. and Takahara, G. (2000). A lower bound on the probability of a finite union of events. *Discrete Math.* **215** 147–158. MR1746455 https://doi.org/10.1016/S0012-365X(99)00246-0

[23] Last, G. and Penrose, M. (2017). *Lectures on the Poisson Process*. *Institute of Mathematical Statistics Textbooks* **7**. Cambridge: Cambridge Univ. Press. MR3791470

[24] McDiarmid, C. and Reed, B. (2008). On the maximum degree of a random planar graph. *Combin. Probab. Comput.* **17** 591–601. MR2433943 https://doi.org/10.1017/S0963548308009097

[25] McDiarmid, C., Steger, A. and Welsh, D.J.A. (2006). Random graphs from planar and other addable classes. In *Topics in Discrete Mathematics*. *Algorithms Combin.* **26** 231–246. Berlin: Springer. MR2249274 https://doi.org/10.1007/3-540-33700-8_15

[26] Okabe, A., Boots, B., Sugihara, K. and Chiu, S.N. (2000). *Spatial Tessellations*: *Concepts and Applications of Voronoi Diagrams*, 2nd ed. *Wiley Series in Probability and Statistics*. Chichester: Wiley. With a foreword by D.G. Kendall. MR1770006 https://doi.org/10.1002/9780470317013

[27] Penrose, M. (2003). *Random Geometric Graphs*. *Oxford Studies in Probability* **5**. Oxford: Oxford Univ. Press. MR1986198 https://doi.org/10.1093/acprof:oso/9780198506263.001.0001

[28] Schneider, R. and Weil, W. (2008). *Stochastic and Integral Geometry*. *Probability and Its Applications* (*New York*). Berlin: Springer. MR2455326 https://doi.org/10.1007/978-3-540-78859-1

[29] Schulte, M. and Thäle, C. (2016). Poisson point process convergence and extreme values in stochastic geometry. In *Stochastic Analysis for Poisson Point Processes*. *Bocconi Springer Ser.* **7** 255–294. Bocconi Univ. Press. MR3585403 https://doi.org/10.1007/978-3-319-05233-5_8

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