

On squared Bessel particle systems

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We study the existence and uniqueness of solutions of SDEs describing squared Bessel particle systems in full generality. We define nonnegative and non-colliding squared Bessel particle systems and we study their properties. Particle systems dissatisfying non-colliding and unicity properties are pointed out. The structure of squared Bessel particle systems is described.

Keywords: non-colliding solution; particle system; squared Bessel process; stochastic differential equation; Wishart process

1. Introduction

The main objective of the paper is to study in details the following system of stochastic differential equations

$$dX_i = 2\sqrt{|X_i|}dB_i + \left(\alpha + \sum_{j \neq i} \frac{|X_i| + |X_j|}{X_i - X_j} \mathbf{1}_{\{X_i \neq X_j\}} \right) dt, \quad i = 1, \dots, p, \quad (1.1)$$

$$X_1(t) \leq X_2(t) \leq \dots \leq X_p(t), \quad t \geq 0, \quad (1.2)$$

with the initial condition $X_i(0) = x_i$, $i = 1, \dots, p$ and the drift parameter $\alpha \in \mathbf{R}$. We provide results on the existence, unicity and properties of the solutions of the system (1.1) in the whole generality of its parameters and initial values. The system (1.1) is called *squared Bessel particle system* following the fact that for $p = 1$ it reduces to the classical squared Bessel stochastic differential equation

$$dX = 2\sqrt{|X|}dB + \alpha dt, \quad X(0) = x. \quad (1.3)$$

It follows from the Yamada–Watanabe theorem [14] that there exists a unique strong solution to (1.3) and the solution is called squared Bessel process of dimension α starting from x . It is usually denoted by $BESQ^{(\alpha)}(x)$. In the classical setting, the nonnegativity of α and x are assumed. However, Göing-Jaeschke and Yor studied squared Bessel processes starting from negative points as well as having negative dimensions (see [4]), that play an important role in the stochastic calculus in one dimension. The present paper generalizes the Göing-Jaeschke–Yor's description of squared Bessel processes to the multidimensional case.

A systematic study of non-colliding particle systems was initiated by Rogers and Shi in [12], for the Dyson Brownian Motion and for some more general interacting Brownian particles starting from non-colliding points $X_1(0) < \dots < X_p(0)$. The study was continued by Bru in [2], where the squared Bessel particle systems for $\alpha > p - 1$ were considered as eigenvalues of related Wishart processes. However, the methods of the papers [2,12] can not be used to deal with the general case (1.1). Considering initial conditions $X_1(0) \leq \dots \leq X_p(0)$, where some equalities hold true, requires a different approach. Consequently, our results are partially based on the theory built in [6], which allows to construct non-colliding solutions to general particle systems with colliding starting points. However, there are some special cases of α and starting points $X(0)$ in (1.1), for which the results of [6] cannot be applied directly. These cases require more in-depth analysis.

Although the coefficients of the equations are 1/2-Hölder continuous in the martingale part (as in the classical one-dimensional Yamada–Watanabe theorem) and the repulsive force comes from a drift term in (1.1), similar to logarithmic potential (as in the systems considered by Rogers and Shi in [12]), there are some special values of $\alpha \in \mathbf{R}$ and $X(0)$ (having collisions), for which there exist colliding solutions and consequently the unicity of solutions does not hold. It makes the study much more complicated than in the one dimensional case studied in [4]. Moreover, such phenomenon in the context of particle systems has not been noticed so far.

Note also a close relation of systems (1.1) to the random matrix theory. By [5], Theorem 3, the system (1.1) describes the ordered eigenvalues of the solution to the following matrix stochastic differential equation

$$d\mathbb{Y}_t = \sqrt{|\mathbb{Y}_t|} d\mathbb{W}_t + d\mathbb{W}_t^T \sqrt{|\mathbb{Y}_t|} + \alpha \mathbb{I} dt, \tag{1.4}$$

where $\mathbb{Y}_t \in \mathcal{S}_p$, the vector space of real symmetric matrices, \mathbb{W}_t is a Brownian $p \times p$ matrix and the eigenvalues of \mathbb{Y}_0 are all different. The equation (1.4) is usually considered with the additional assumption $\alpha \geq p - 1$ (which for $p = 1$ corresponds to the condition $\alpha \geq 0$), and then it is called *Wishart SDE*, which can be viewed as the matrix generalization of the squared Bessel SDE (1.3) (see [2,3,5]). If $\alpha \geq p - 1$ and the eigenvalues $X_1(0), \dots, X_p(0)$ of \mathbb{Y}_0 are supposed to be nonnegative, then the particles $X_i(t)$ (i.e., the eigenvalues of \mathbb{Y}_t) remain nonnegative (i.e., $X_i(t) \geq 0$ for every $t \geq 0$) and they never collide for $t > 0$. In fact in this case we can remove absolute values and the indicators from (1.1) and (1.4). However, the matrix equations (1.4) are also considered without any restrictions on α and the behaviour of their eigenvalues for $\alpha < p - 1$ is of great importance (see [7]).

Systems of stochastic differential equations with the indicators $\mathbf{1}_{\{X_i \neq X_j\}}$ in the drift part were introduced by Katori ([9], Theorem 1, [8]), but he uniquely considered cases when one can omit these indicators.

Note that the results obtained for the solutions of the system (1.1) may be generalized for the β -BESQ particle systems, obtained by multiplying the drift term in (1.1) by a $\beta > 1$, see [5], Section III.D. When $\beta = 2$, these are the SDEs for p independent BESQ processes on \mathbf{R}^+ , conditioned not to collide ([10]). Such β -generalization of the present study will be done in the upcoming paper.

The paper is organized as follows. In Section 2, we present all the main results of the paper. We begin with introducing definitions and notations for non-colliding and nonnegative solutions

of (1.1) together with the results on their existence and uniqueness (Theorems 1 and 3). In Theorem 2, we give necessary and sufficient conditions on parameters of a squared Bessel particle system to have a unique strong solution. In Section 2.2, we emphasize the examples dissatisfying the non-colliding property, which are also examples of non-uniqueness of strong solutions. In Theorem 4, we provide the stochastic description of the symmetric polynomials related to the nonnegative solutions of the system (1.1). Finally, in Theorem 5 we describe the structure of non-colliding solutions $X_i(t)$ which are negative or positive. In Sections 3 and 4, the proofs of the main results are provided.

2. Main results

2.1. Existence and uniqueness of solutions of BESQ particle system

We start our considerations with studying so-called non-colliding solutions.

Definition 1. A solution (X_1, \dots, X_p) of (1.1) is called *non-colliding* if there are no collisions between particles after the start, i.e.

$$T = \inf\{t > 0 : X_i(t) = X_j(t) \text{ for some } i \neq j\}$$

is infinite almost surely.

It appears that we can always build a non-colliding solution of (1.1) and uniqueness among non-colliding solutions holds, which is provided in the following theorem.

Theorem 1. For every $\alpha \in \mathbf{R}$ and $x_1 \leq \dots \leq x_p$ there exists a non-colliding solution to the system of stochastic differential equations

$$dX_i = 2\sqrt{|X_i|}dB_i + \left(\alpha + \sum_{j \neq i} \frac{|X_i| + |X_j|}{X_i - X_j} \mathbf{1}_{\{X_i \neq X_j\}} \right) dt, \quad i = 1, \dots, p, \quad (2.1)$$

$$X_1(t) \leq X_2(t) \leq \dots \leq X_p(t), \quad t \geq 0 \quad (2.2)$$

with the initial condition $X_i(0) = x_i$ for $i = 1, \dots, p$. Moreover, pathwise uniqueness among non-colliding solutions holds and there exists unique non-colliding strong solution.

The proof of Theorem 1 is postponed until Section 4. It requires some knowledge of elementary symmetric polynomials studied in details in Section 3.

Remark 1. Note that if we study non-colliding solutions we can remove the indicators from the drift parts of equations (2.1).

Theorem 1 enables us to introduce the following.

Definition 2. The unique strong solution to (2.1), which has no collisions after the start is called *non-colliding squared Bessel particle system* of dimension $\alpha \in \mathbf{R}$ starting from the point (x_1, \dots, x_p) , where $x_1 \leq x_2 \leq \dots \leq x_p$ and it will be denoted by $BESQ_{nc}^{(\alpha)}(x_1, \dots, x_p)$.

Since, by Theorem 1, there always exists a unique non-colliding solution, it is natural to ask if there are any other solutions. To formulate the result providing necessary and sufficient conditions for (1.1) to have unique strong solution, we have to introduce the following notation. For a fixed point $x = (x_1, \dots, x_p) \in \mathbf{R}^p$, $x_1 \leq \dots \leq x_p$, we define

$$rk^+(x) = \sum_{i=1}^p \mathbf{1}_{(0,\infty)}(x_i), \quad rk^-(x) = \sum_{i=1}^p \mathbf{1}_{(-\infty,0)}(x_i),$$

and set $rk(x) = rk^+(x) + rk^-(x)$, that is, $rk^+(x)$, $rk^-(x)$, $rk(x)$ is the number of strictly positive, strictly negative and all nonzero values among x_1, \dots, x_p .

Theorem 2. *Pathwise uniqueness for the system*

$$dX_i = 2\sqrt{|X_i|} dB_i + \left(\alpha + \sum_{j \neq i} \frac{|X_i| + |X_j|}{X_i - X_j} \mathbf{1}_{\{X_i \neq X_j\}} \right) dt, \quad i = 1, \dots, p,$$

$$X_1(t) \leq X_2(t) \leq \dots \leq X_p(t), \quad t \geq 0$$

with the initial condition $X(0) = x$, where $x = (x_1, \dots, x_p)$, holds if and only if one of the following conditions holds

- (a) $|\alpha| \notin \{0, 1, \dots, p - 2\}$,
- (b) $|\alpha| \in \{0, 1, \dots, p - 2\}$ and $(rk^+(x) \geq \frac{p+\alpha-1}{2}$ or $rk^-(x) \geq \frac{p-\alpha-1}{2})$.

Then there exists unique strong solution, which is non-colliding. If (a) and (b) are not satisfied, then there exist strong solutions, but neither pathwise uniqueness nor uniqueness in law hold.

Remark 2. Note that the numbers $(p + \alpha - 1)/2$ or $(p - \alpha - 1)/2$ may not be integers.

Obviously, the unique strong solution from Theorem 2 must be, by Theorem 1, non-colliding.

Next, we consider the problem of existence and uniqueness of non-negative solutions. The classical results related to $p = 1$ say that the squared Bessel process $BESQ^{(\alpha)}(x)$ is non-negative if and only if $x \geq 0$ and $\alpha \geq 0$. In the multidimensional case, we can ask analogous question introducing the following.

Definition 3. A solution (X_1, \dots, X_p) of (1.1) is called *non-negative* if $X_1(t) \geq 0$ for every $t > 0$ a.s.

Looking at the matrix interpretation of considered particle systems, nonnegativity of (X_1, \dots, X_p) is equivalent to the condition saying that the corresponding matrix value process stays in $\overline{\mathcal{S}_p^+}$, where \mathcal{S}_p^+ is the open cone of positive definite symmetric matrices. The multidimensional result is provided in the following theorem.

Theorem 3. *There exists a nonnegative solution to*

$$dX_i = 2\sqrt{|X_i|} dB_i + \left(\alpha + \sum_{j \neq i} \frac{|X_i| + |X_j|}{X_i - X_j} \mathbf{1}_{\{X_i \neq X_j\}} \right) dt, \quad i = 1, \dots, p,$$

$$X_1(t) \leq X_2(t) \leq \dots \leq X_p(t), \quad t \geq 0,$$

with the initial condition $X(0) = x$, where $x = (x_1, \dots, x_p)$ and $x_1 \geq 0$, if and only if one of the following conditions holds

- (a) $\alpha \geq p - 1$,
- (b) $\alpha \in \{0, 1, \dots, p - 2\}$ and $\text{rk}(x) \leq \alpha$.

Then pathwise uniqueness among non-negative solutions holds and there exists unique nonnegative strong solution.

Remark 3. Note that Theorem 3 is a spectral analogue of the characterization of the non-central Gindikin set proved in [7].

2.2. Examples dissatisfying non-colliding property

Theorem 2 implies that when $|\alpha| \in \{0, 1, \dots, p - 2\}$, $\text{rk}^+(x) < (p + \alpha - 1)/2$ and $\text{rk}^-(x) < (p - \alpha - 1)/2$, then the uniqueness of the strong solutions is violated, that is, there exist at least two solutions. On the other hand, we deduce from Theorem 1 that only one of these solutions is non-colliding, so they are all colliding, except one. Let us illustrate these new phenomena on simple examples.

Example 1. Consider $p = 2, \alpha = 0$. The system (1.1) is then reduced to

$$dX_1 = 2\sqrt{|X_1|} dB_1 + \frac{|X_1| + |X_2|}{X_1 - X_2} \mathbf{1}_{\{X_1 \neq X_2\}} dt,$$

$$dX_2 = 2\sqrt{|X_2|} dB_2 + \frac{|X_1| + |X_2|}{X_2 - X_1} \mathbf{1}_{\{X_1 \neq X_2\}} dt,$$

where $X_1(t) \leq X_2(t), t > 0$ and $X(0) = (x_1, x_2)$. Since $(p + \alpha - 1)/2 = (p - \alpha - 1)/2 = 1/2$, we consider $\text{rk}^+(x) = \text{rk}^-(x) = 0$, which means $x_1 = x_2 = 0$. This is the only possible choice of the starting point, where the unicity of the solution fails when $p = 2$. The process $X_1(t) = X_2(t) \equiv 0$ is a strong solution of the system and obviously it is colliding. Notice also that this is the unique nonnegative solution. On the other hand, consider the process $\tilde{X} = (\tilde{X}_1, \tilde{X}_2)$ such that

$$d\tilde{X}_1 = 2\sqrt{|\tilde{X}_1|} dB_1 - dt, \quad \tilde{X}_1(0) = 0,$$

$$d\tilde{X}_2 = 2\sqrt{|\tilde{X}_2|} dB_2 + dt, \quad \tilde{X}_2(0) = 0.$$

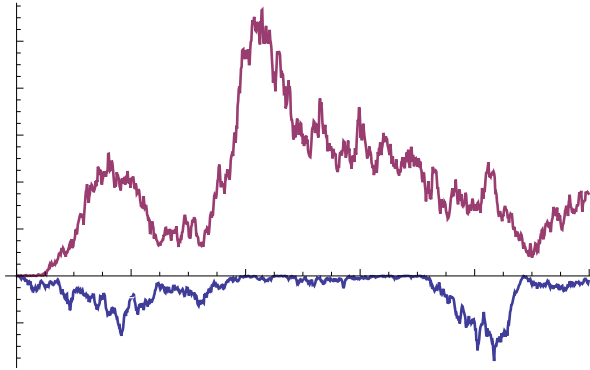


Figure 1. The unique non-colliding solution for $p = 2, \alpha = 0$, starting from $x = (0, 0)$.

The processes \tilde{X}_1, \tilde{X}_2 are two independent squared Bessel processes of dimension -1 and $+1$ respectively. Note that the first one immediately after the start becomes nonpositive and $\tilde{X}_2(t) \geq 0$ for every $t > 0$. By independence, these processes do not collide after the start. The fact that $(|x| + |y|)/(x - y)$ is equal -1 for $x \leq 0 \leq y$ and it is 1 for $x \geq 0 \geq y$ implies that \tilde{X} is another strong solution of the system, that is, the unique non-colliding strong solution (Figure 1). Thus, we have two strong solutions and neither uniqueness in law nor pathwise uniqueness hold.

Example 2. Consider $p = 5$ and $\alpha = 1$ and let us begin with the zero initial condition $x = (x_1, \dots, x_5) = 0$. The unique nonnegative solution is the process $X = (X_1, \dots, X_5)$, where $X_1(t) = X_2(t) = X_3(t) = X_4(t) \equiv 0$ and the last particle is $BESQ^{(5)}(0)$ described by

$$dX_5 = 2\sqrt{X_5} dB_5 + 5 dt.$$

One can easily check it using the fact that $(|x| + |y|)/(x - y)$ is equal -1 for $x \leq 0 \leq y$ and it is 1 for $x \geq 0 \geq y$. This solution is not non-colliding (Figure 3). However, Theorem 1 ensures existence of the unique non-colliding solution and such solution in the considered example is the process $\tilde{X} = (\tilde{X}_1, \dots, \tilde{X}_5)$, where $(\tilde{X}_1, \tilde{X}_2)$ is $BESQ_{nc}^{(-2)}(0, 0)$, which is independent from $(\tilde{X}_3, \tilde{X}_4, \tilde{X}_5)$ being $BESQ_{nc}^{(3)}(0, 0, 0)$ (Figure 2). Finally, taking $\bar{X} = (\bar{X}_1, \dots, \bar{X}_5)$, where \bar{X}_1 is $BESQ^{(-3)}(0)$, $\bar{X}_2(t) = \bar{X}_3(t) \equiv 0$ and (\bar{X}_4, \bar{X}_5) being independent $BESQ_{nc}^{(4)}(0, 0)$ we construct another solution which is neither non-collision nor nonnegative (Figure 4).

Notice that if we increase rank of the starting point assuming that $x_1 < 0 = x_2 = x_3 < x_4 \leq x_5$, we still have $rk^+(x) = 2 < 5/2 = (p + \alpha - 1)/2$ and $rk^-(x) = 1 < 3/2 = (p - \alpha - 1)/2$ and the unicity of the solution does not hold, although we obviously do not have non-negative solutions in this case. The non-colliding solution is once again obtained by gluing $BESQ_{nc}^{(-2)}(x_1, 0)$ with independent $BESQ_{nc}^{(3)}(0, x_4, x_5)$. A solution having collisions can be constructed as above by taking $BESQ^{(-3)}(x_1)$ to be the first particle, setting $X_2(t) = X_3(t) \equiv 0$ and the last two particles evolving independently as $BESQ_{nc}^{(4)}(x_4, x_5)$.

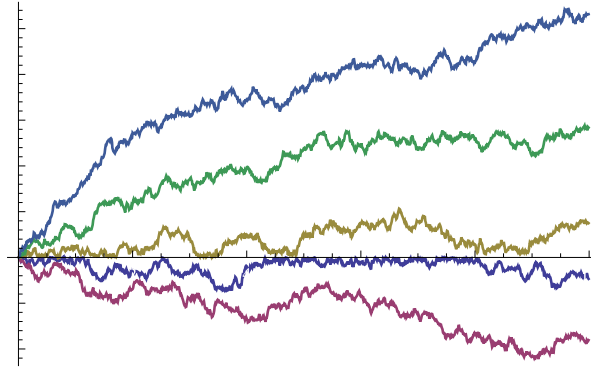


Figure 2. The unique non-colliding solution for $p = 5, \alpha = 1$, starting from $x = (0, \dots, 0)$.

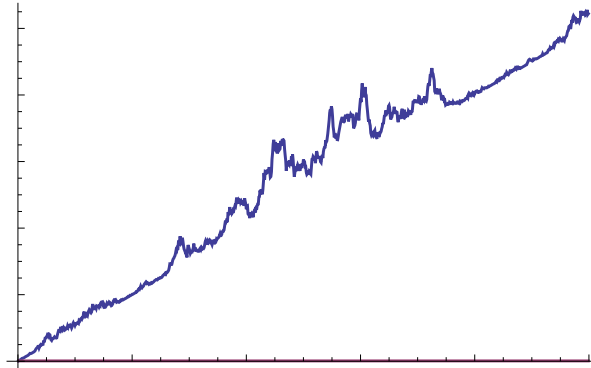


Figure 3. The unique non-negative solution for $p = 5$ and $\alpha = 1$ starting from $x = (0, \dots, 0)$, where only the last particle is not identically zero.

Note that if the condition $x_2 = x_3 = 0$ fails, only one solution exists and it is obtained by gluing $BESQ_{nc}^{(-2)}(x_1, x_2)$ with independent $BESQ_{nc}^{(3)}(x_3, x_4, x_5)$. This solution is non-colliding.

Remark 4. The general construction of non-unique and colliding solutions of the system (1.1) is described in the proof of the “only if” part of Theorem 2. Let us point out here that a colliding solution exists for each pair of nonnegative integers \tilde{n} and \tilde{l} such that

- (1) $\alpha + \tilde{l} - \tilde{n} = 0$,
- (2) $\text{rk}^+(x) \leq \tilde{n} < (p + \alpha - 1)/2$,
- (3) $\text{rk}^-(x) \leq \tilde{l} < (p - \alpha - 1)/2$.

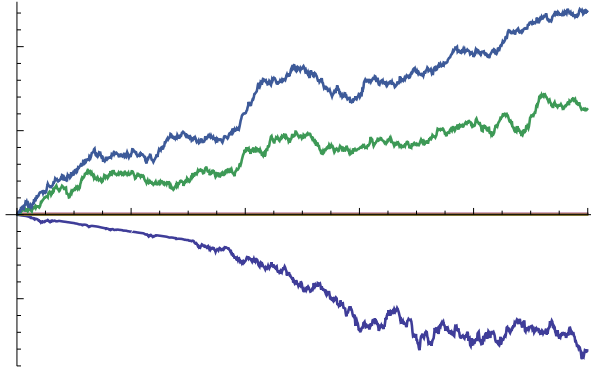


Figure 4. The solution for $p = 5$ and $\alpha = 1$, which is neither non-collision nor non-negative. Here the second and the third particles are identically zero.

It is constructed as a triple of independent processes

$$X = (-BESQ_{nc}^{(\alpha^-)}(-x_{\tilde{l}}, -x_{\tilde{l}-1}, \dots, -x_1), \mathbf{0}_{p-\tilde{l}-\tilde{n}}, BESQ_{nc}^{(\alpha^+)}(x_{p-\tilde{n}+1}, \dots, x_p)),$$

where $\alpha^- = p - \alpha - \tilde{l}$ and $\alpha^+ = p + \alpha - \tilde{n}$. Note that $p - \tilde{l} - \tilde{n} \geq 2$ and X is colliding. The pair $(\tilde{l}, \tilde{n}) = ([\frac{p-\alpha}{2}] - 1, [\frac{p+\alpha}{2}] - 1)$ always verifies conditions (1), (2) and (3), but there may be more of them. In Example 2, for $p = 5, \alpha = 1$ and $x = \mathbf{0}$ we have $(\tilde{l}, \tilde{n}) = (1, 2)$ or $(\tilde{l}, \tilde{n}) = (0, 1)$.

2.3. Properties and structure of BESQ particle systems

As noticed in [6] for general particle systems, the study of properties of BESQ particle systems will be greatly simplified if we control the symmetric polynomials of the particles $X_i(t)$.

The elementary symmetric polynomials of $X = (X_1, \dots, X_p)$ are defined by

$$e_n(X) = \sum_{i_1 < i_2 < \dots < i_n} X_{i_1} X_{i_2} \dots X_{i_n}$$

for every $n = 1, 2, \dots, p$. We use the convention that $e_0(X) \equiv 1$ and $e_n(X) \equiv 0$ for $n > p$.

The SDEs for symmetric polynomials of nonnegative BESQ particle systems are elementary and we give them in the following theorem. In order to shorten the formulas, we write e_n instead of $e_n(X)$ and we set $e_r \equiv 0$ if $r < 0$ or $r > p$.

Theorem 4. *The elementary symmetric polynomials of the non-colliding solution of (1.1) starting from $0 \leq x_1 \leq \dots \leq x_p$ are semi-martingales described up to the first exit time $T = \inf\{t > 0 : X_1(t) < 0\}$ by the following system of p SDEs*

$$de_n = 2 \left(\sum_{k=1}^p (2k-1) e_{n-k} e_{n+k-1} \right)^{1/2} dV_n + (p-n+1)(\alpha-n+1) e_{n-1} dt, \quad (2.3)$$

where V_n are one-dimensional Brownian motions such that

$$d\langle e_n, e_m \rangle = 4 \sum_{k=1}^p (m - n + 2k - 1) e_{n-k} e_{m+k-1} dt \tag{2.4}$$

for every $1 \leq n \leq m \leq p$.

Remark 5. The sum in formula (2.3) has nonzero terms for $k = 1, \dots, K = \min(n, p + 1 - n)$ and the sum in (2.4) for $k = 1, \dots, K = \min(n, p + 1 - m)$.

Göing-Jaesckke and Yor in [4] studied the structure of squared Bessel processes with negative dimensions. They showed that $BESQ^{(-\alpha)}(x)$ starting from positive x with $\alpha > 0$ hits zero almost surely and then behaves as $-BESQ^{(\alpha)}(0)$. We study the corresponding problem for non-colliding squared Bessel particle systems $BESQ_{nc}^{(\alpha)}(x_1, \dots, x_p)$. The negativity of the dimension in the classical case is translated to the condition $\alpha < p - 1$ and we assume that $0 \leq x_1 \leq \dots \leq x_p$. We define the family of first hitting times

$$T_0^{(i)} = \inf\{t \geq 0 : X_i(t) = 0\}, \quad i = 1, \dots, p$$

and the family of first entrance times

$$T_-^{(i)} = \inf\{t \geq 0 : X_i(t) < 0\}, \quad i = 1, \dots, p.$$

In the following theorem, we generalize the well-known fact saying that $BESQ^{(\alpha)}(x)$ hits zero whenever $\alpha \in [0, 2)$, visits negative half-line for $\alpha < 0$ and stays non-positive after first entrance to the negative half-line. We also describe the evolution of the solution between the moments when the succeeding particles become negative.

Theorem 5. Let $X = (X_1, \dots, X_p)$ be $BESQ_{nc}^{(\alpha)}(x_1, \dots, x_p)$, where $0 \leq x_1 \leq x_2 \leq \dots \leq x_p$ and $\alpha < p + 1$. Let $n = \lceil \frac{p-\alpha+1}{2} \rceil$. Then

$$T_0^{(1)} \leq T_0^{(2)} \leq \dots \leq T_0^{(n)} < \infty, \quad T_0^{(n+1)} = \dots = T_0^{(p)} = \infty$$

and

$$T_-^{(1)} \leq \dots \leq T_-^{(n-1)} < \infty, \quad T_-^{(n)} = \dots = T_-^{(p)} = \infty.$$

Moreover, for every $k = 1, \dots, n - 1$, on the interval $[T_-^{(k)}, T_-^{(k+1)})$ the subsystems of particles $\mathbf{Y}_k = (X_1, \dots, X_k)$ and $\mathbf{Z}_k = (X_{k+1}, \dots, X_p)$ are conditionally independent given $(\mathbf{Y}_k(T_-^{(k)}), \mathbf{Z}_k(T_-^{(k)}))$ and they evolve as $-BESQ_{nc}^{(p-\alpha-k)}$ on \mathbf{R}^k and $BESQ_{nc}^{(\alpha+k)}$ on \mathbf{R}^{p-k} respectively.

In particular, if $T_-^{(i)}$ is finite then $X_i(t) \leq 0$ for $t \geq T_-^{(i)}$, that is, the particles do not go back to the positive half-line after going below zero.

Remark 6. Note that for given p and $\alpha < p + 1$ the number $n = \lceil \frac{p-\alpha+1}{2} \rceil$ is 1 for $\alpha \in [p - 1, p + 1)$, $n = 2$ for $\alpha \in [p - 3, p - 1)$ and so on. Consequently, the above-given result states

that the i th particle $X_i(t)$ hits zero if and only if $p - \alpha + 3 > 2i$ and the i th particle visits negative half-line $(-\infty, 0)$ if and only if $p - \alpha + 1 > 2i$.

Remark 7. Since the system becomes non-colliding immediately, we can have $T_0^{(i)} = T_0^{(i+1)}$ or $T_-^{(i)} = T_-^{(i+1)}$ only if $x_i = x_{i+1} = 0$. Consequently, if $x_i > 0$ or $x_i < x_{i+1}$ then we have strict inequalities between times $T_0^{(i)}$ and $T_0^{(i+1)}$ (analogously $T_-^{(i)} < T_-^{(i+1)}$) in the above-given theorem.

3. Symmetric polynomials of squared Bessel particles

This section concerns the results announced in the first part of the Section 2.3 and contains the proof of Theorem 4.

We write $e_n^{\bar{j}_1, \bar{j}_2, \dots, \bar{j}_m}(X)$ for an incomplete elementary symmetric polynomial

$$e_n^{\bar{j}_1, \bar{j}_2, \dots, \bar{j}_m}(X) = \sum_{\substack{i_1 < i_2 < \dots < i_n \\ i_k \neq j_l}} X_{i_1} X_{i_2} \dots X_{i_n},$$

that is, the sum of all products of length n of different X_i 's, not including any of X_{j_1}, \dots, X_{j_m} .

Proposition 1. *If X is a non-colliding solution of (1.1), then (e_1, \dots, e_p) are semi-martingales described by*

$$\begin{aligned} de_n(X) &= \left(\sum_{i=1}^p |X_i| (e_{n-1}^{\bar{i}}(X))^2 \right)^{1/2} dV_n \\ &+ \left(\sum_{i=1}^p \alpha e_{n-1}^{\bar{i}}(X) - \sum_{i < j} (|X_i| + |X_j|) e_{n-2}^{\bar{i}, \bar{j}}(X) \right) dt \end{aligned} \tag{3.1}$$

for $n = 1, \dots, p$. Here (V_1, \dots, V_p) is a collection of one-dimensional Brownian motions such that

$$d\langle e_n(X), e_m(X) \rangle = 4 \sum_{i=1}^p |X_i| e_{n-1}^{\bar{i}}(X) e_{m-1}^{\bar{i}}(X) dt. \tag{3.2}$$

Proof. We apply [6], Proposition 3.1. □

The map $e = (e_1, \dots, e_p)$ is a diffeomorphism between $C_+ = \{(x_1, \dots, x_p) \in \mathbf{R}^p : x_1 < x_2 < \dots < x_p\}$ and $e(C_+)$. Following [6], Chapter 3, we denote by $f : e(C_+) \rightarrow C_+$ its inverse and note that f can be continuously extended to

$$f : \overline{e(C_+)} \xrightarrow{1-1} \overline{C_+}.$$

It implies that using the map f we can write SDEs (3.1) and (3.2) only in terms of e_1, \dots, e_p . The coefficients of those equations are continuous and the singularities of the form $(X_i - X_j)^{-1}$ disappear. In particular, there always exists a solution of those equations (see Proposition 3.2 in [6]).

In Theorem 4, we manage to write the coefficients of equations (3.1) and (3.2) in a transparent way in terms of e_1, \dots, e_p themselves (i.e., without incomplete polynomials and X).

Proof of Theorem 4. Since we consider only $t < T$, we remove all the absolute values from (3.1) and (3.2). We first compute the drift part in equation (3.1). It is easy to see that

$$\sum_{i=1}^p e_{n-1}^{\bar{i}}(X) = (p - n + 1)e_{n-1}(X),$$

since every product of length $n - 1$ appears $p - (n - 1)$ times in the last sum. Similarly, we have

$$\sum_{i < j} (X_i + X_j)e_{n-2}^{\bar{i}, \bar{j}}(X) = \sum_{i \neq j} X_i e_{n-2}^{\bar{i}, \bar{j}}(X) = (p - n + 1)(n - 1)e_{n-1}(X)$$

since the last sum consists of products of length $n - 1$ and every product appears $(p - n + 1)(n - 1)$ times. Indeed, if we fix a product $X_{i_1} X_{i_2} \dots X_{i_{n-1}}$ of length $n - 1$, it appears in $X_i e_{n-2}^{\bar{i}, \bar{j}}(X)$ if and only if $i \in \{i_1, i_2, \dots, i_{n-1}\}$ and $j \notin \{i_1, i_2, \dots, i_{n-1}\}$. Consequently, we can choose i on $n - 1$ ways and j on $p - (n - 1)$ ways. It implies that the drift part of $e_n(X)$ equals $(p - n + 1)(\alpha - n + 1)e_{n-1}(X) dt$. In order to show (2.3) and (2.4), it remains to show that

$$\sum_{i=1}^p X_i e_{n-1}^{\bar{i}}(X) e_{m-1}^{\bar{i}}(X) = \sum_{k=1}^p (m - n + 2k - 1) e_{n-k}(X) e_{m+k-1}(X) \tag{3.3}$$

for every $1 \leq n \leq m \leq p$ (recall the notation $e_r \equiv 0$ if $r < 0$ or $r > p$). Observe that both sides of (3.3) are symmetric polynomials of degree $m + n - 1$, where the variables X_1, \dots, X_p appear at most in power 2. Due to symmetry, it is enough to show that, for a fixed $l \geq 0$ and $j \geq 1$, the expression

$$X_1^2 \cdot \dots \cdot X_l^2 X_{l+1} \cdot \dots \cdot X_{l+j}$$

appears on both sides of (3.3) the same number of times. Here $2l + j = n + m - 1$. Moreover, by the form of the LHS of (3.3), we have $l \leq n - 1$ and, consequently, $l + j = n - 1 - l + m \geq m \geq n$. The quadratic expression $X_1^2 \cdot \dots \cdot X_l^2$ can only appear on the left-hand side of (3.3) from the multiplication of $e_{n-1}^{\bar{i}}(X)$ and $e_{m-1}^{\bar{i}}(X)$ and $X_1 \cdot \dots \cdot X_l$ must appear in both of them. Thus, it remains to count in how many terms of the LHS the factors X_{l+1}, \dots, X_{l+j} appear, so that the product $X_1^2 \cdot \dots \cdot X_l^2 X_{l+1} \cdot \dots \cdot X_{l+j}$ is obtained.

Let $s_i = X_i e_{n-1}^{\bar{i}}(X) e_{m-1}^{\bar{i}}(X)$ be a term of the left-hand side of (3.3). Observe that obligatorily $X_i \in \{X_{l+1}, \dots, X_{l+j}\}$. Thus, there are j possible choices of a term s_i . We fix such a choice and count the terms of the polynomial $e_{n-1}^{\bar{i}}(X)$, which contain the product $X_1 \cdot \dots \cdot X_l$ and

have remaining $n - 1 - l$ variables in the set $\{X_{l+1}, \dots, X_{l+j}\} \setminus \{X_i\}$. Equivalently, we count all choices of $n - 1 - l$ elements in a set with $j - 1$ elements. The remaining factors of $X_1^2 \cdot \dots \cdot X_l^2 X_{l+1} \cdot \dots \cdot X_{l+j}$ come from the polynomial $e^{\bar{i}}_{m-1}(X)$. Finally, the coefficient of $X_1^2 \cdot \dots \cdot X_l^2 X_{l+1} \cdot \dots \cdot X_{l+j}$ on the LHS of (3.3) is

$$j \binom{j-1}{n-1-l} = (n-l) \binom{j}{n-l}$$

(recall that $1 \leq n - l \leq j$). Similarly, the considered product $X_1^2 \cdot \dots \cdot X_l^2 X_{l+1} \cdot \dots \cdot X_{l+j}$ appears in $e_{n-k}(X)e_{m+k-1}(X)$ exactly $\binom{j}{n-k-l}$ times. Thus, it is enough to show that for j, l, m, n satisfying $1 \leq n - l \leq j$ and $2l + j = n + m - 1$, the following combinatorial identity holds:

$$(n-l) \binom{j}{n-l} = \sum_{k=1}^n (m-n+2k-1) \binom{j}{n-k-l}.$$

We use a convention that the Newton’s symbol $\binom{n}{r}$ is zero whenever $r > n$ or $r < 0$.

Using the relation $2l + j = m + n - 1$, we can rewrite the right-hand side as

$$\sum_{k=1}^n (m-n+2k-1) \binom{j}{n-k-l} = \sum_{k=1}^{n-l} (j-2(n-l-k)) \binom{j}{n-l-k}.$$

Substitutions $N = n - l - 1$ and $r = n - l - k$ together with reordering the sum lead to a combinatorial formula

$$\sum_{r=0}^N (j-2r) \binom{j}{r} = (N+1) \binom{j}{N+1}, \tag{3.4}$$

where $0 \leq N \leq j - 1$. Formula (3.4) is known (see, e.g., [13]) and can be easily proved by elementary induction on N . □

4. Proofs

In the sequel, we use the following corollary of results of [6]. It is contained in Corollaries 6.5 and 6.6 of [6] (in the statement of Corollary 6.6, \mathbf{R} should be \mathbf{R}^+). Recall that condition (A4) from [6] fails if $\alpha \in \{0, 1, \dots, p - 2\}$ and this case is not covered by the results of [6].

Corollary 1. *Let $\alpha \in \mathbf{R}^+ \setminus \{0, 1, \dots, p - 2\}$. Then the system*

$$dX_i = 2\sqrt{|X_i|} dB_i + \left(\alpha + \sum_{j \neq i} \frac{|X_i| + |X_j|}{X_i - X_j} \right) dt, \quad i = 1, \dots, p,$$

$$X_1(t) \leq X_2(t) \leq \dots \leq X_p(t), \quad t \geq 0,$$

has a unique non-colliding solution for $t > 0$. If $\alpha \geq p - 1$ and $X_1(0) \geq 0$, then the solution is non-negative, that is, $X_1(t) \geq 0$.

Proof of Theorem 1. Since we consider all possible starting points $x_1 \leq \dots \leq x_p$ (without restriction that x_1 must be nonnegative), we can and we do assume that $\alpha \geq 0$. The general case follows immediately by multiplying equations (1.1) by -1 and reordering the particles.

By Corollary 1, we focus on $\alpha \in \{0, 1, \dots, p - 2\}$ and consider a general starting point $x = (x_1, \dots, x_p)$. First, we note that the conditions (C1) and (A1) (or equivalently (A1')) from [6] hold for functions $\sigma(x) = 2\sqrt{|x|}$, $b(x) = \alpha$ and $H(x, y) = |x| + |y|$. For (A1), see the proof of [6], Corollary 6.5.

By Theorem 5.3 and Remark 2.4 in [6], we get the pathwise uniqueness for non-colliding solutions (the other assumptions in Theorem 5.3 of [6] were used to construct such non-colliding solution). Consequently, it is enough to prove the existence of a non-colliding solution.

For simplicity, we denote $\text{rk}^+(x) = n$, $\text{rk}^-(x) = l$ and $m = p - \text{rk}(x)$, i.e.

$$x_1 \leq \dots \leq x_l < 0 = x_{l+1} = x_{l+2} = \dots = x_{l+m} < x_{l+m+1} \leq \dots \leq x_p.$$

Now we consider two cases.

Case 1: $\text{rk}^+(x) < (p + \alpha - 1)/2$ and $\text{rk}^-(x) < (p - \alpha - 1)/2$. In this case, we construct a solution by gluing two independent processes. We define an integer number n^* by requesting

$$2n^* \in \{p + \alpha, p + \alpha + 1\}. \tag{4.1}$$

Note that n^* is uniquely determined, since exactly one of the consecutive integer numbers is even. Moreover, we have $\alpha < n^* < p$ since $\alpha \leq p - 2$. We set $p_- = p - n^* > 0$ and $\alpha_- = n^* - \alpha > 0$ and consider a system of p_- SDEs

$$dZ_i = 2\sqrt{|Z_i|} dB_{p-n^*-i+1} + \left(\alpha_- + \sum_{j=1, j \neq i}^{p_-} \frac{|Z_i| + |Z_j|}{Z_i - Z_j} \right) dt, \quad i = 1, \dots, p_-,$$

starting from $Z_i(0) = -x_{p-n^*-i+1}$ for $i = 1, \dots, p_-$.

Note that our assumption $n = \text{rk}^+(x) < (p + \alpha - 1)/2 < n^*$ implies $p - n^* < p - n$ and consequently $Z = (Z_1, \dots, Z_{p_-})$ starts from nonnegative point, that is, $Z_1(0) = -x_{p-n^*} \geq 0$. Moreover, we have $\alpha_- \geq p_-$, since $2n^* \geq p + \alpha$. It follows, by Corollary 1, that there exists a unique strong solution $Z(t)$ which is non-colliding and this solution is nonnegative. Then, we put $p_+ = n^*$ and $\alpha_+ = \alpha + p - n^*$ and consider a system of p_+ SDEs

$$dY_i = 2\sqrt{|Y_i|} dB_i + \left(\alpha_+ + \sum_{j=p-n^*+1, j \neq i}^p \frac{|Y_i| + |Y_j|}{Y_i - Y_j} \right) dt, \quad i = p - n^* + 1, \dots, p,$$

where $Y_i(0) = x_i$ for $i = p - n^* + 1, \dots, p$. Using the bounds on $\text{rk}^-(x)$ in the following way: $l = \text{rk}^-(x) < (p - \alpha - 1)/2 = p - (p + \alpha + 1)/2 \leq p - n^*$ we get $p - n^* + 1 > l + 1$ and consequently the considered starting point is nonnegative, that is, $x_{p-n^*+1} \geq 0$. Moreover, we have

$\alpha_+ \geq p_+ - 1$ since $2n^* \leq p + \alpha + 1$, which means, by Corollary 1, that there exists a unique strong non-colliding solution $Y(t)$ which is also nonnegative. Now we put

$$X_i(t) = \begin{cases} -Z_{p-n^*-i+1}(t), & i = 1, \dots, p - n^*, \\ Y_i(t), & i = p - n^* + 1, \dots, p \end{cases}$$

and obviously we have $X_i(0) = x_i$ for every $i = 1, \dots, p$. Moreover, for every $i = 1, \dots, p - n^*$ and $j = p - n^* + 1, \dots, p$ we have

$$\frac{|X_i| + |X_j|}{X_i - X_j} = -1, \quad \frac{|X_j| + |X_i|}{X_j - X_i} = 1$$

since $X_i(t) \leq 0$ and $X_j(t) \geq 0$. It implies that for $i = 1, \dots, p - n^*$ we can write

$$\begin{aligned} dX_i &= 2\sqrt{|X_i|} dB_i + \left(\alpha - n^* + \sum_{j=1, j \neq i}^{p-n^*} \frac{|X_i| + |X_j|}{X_i - X_j} \right) dt \\ &= 2\sqrt{|X_i|} dB_i + \left(\alpha + \sum_{j=1, j \neq i}^p \frac{|X_i| + |X_j|}{X_i - X_j} \right) dt \end{aligned}$$

and the analogous computations can be done for remaining $i = p - n^* + 1, \dots, p$. Note also that $X = (X_1, \dots, X_p)$ is non-colliding. Indeed, as we have seen, there are no collisions between X_1, \dots, X_{p-n^*} and separately between X_{p-n^*+1}, \dots, X_p . Moreover, the first particle system is non-positive and the other is non-negative, that is, $X_{p-n^*}(t) \leq 0 \leq X_{p-n^*+1}(t)$ for every $t > 0$ a.s. It remains to show that these two particles do not collide at zero. However, if $2n^* = p + \alpha + 1$, then $\alpha_- = n^* - \alpha = p - n^* + 1 = p_- + 1$ and consequently $X_{p-n^*}(t) < 0$ for every $t > 0$. If $2n^* = p + \alpha$ then we have $\alpha_- = p_-$ and $\alpha_+ = p_+$ which implies that particles X_{p-n^*} and X_{p-n^*+1} visit zero but the sets $\{t : X_{p-n^*}(t) = 0\}$ and $\{t : X_{p-n^*+1}(t) = 0\}$ are of Lebesgue measure zero (see Proposition 4 in [2]). In particular, there exists a sequence $t_i \searrow 0$ such that $X_{p-n^*}(t_i) > 0$ a.s. and consequently, there are no collisions at any t_i . By Proposition 4.2 in [6], we know that the particles will never collide after t_i and thus there are no collisions for any $t > 0$.

Case 2: $\text{rk}^+(x) \geq (p + \alpha - 1)/2$ or $\text{rk}^-(x) \geq (p - \alpha - 1)/2$. Following the main idea of [6], we get a solution, solving first the SDEs for the elementary symmetric polynomials, that is, we use a solution $e = (e_1, \dots, e_p)$ of (3.1). We set $(X_1, \dots, X_p) = f(e_1, \dots, e_p)$, where f is the diffeomorphism described in Section 3. It remains to show that (X_1, \dots, X_p) is non-colliding. If $m \leq 1$ ($\text{rk}(x) \geq p - 1$), that is, there is at most one particle starting from zero, the result follows directly from the argument in the first part of the proof of Proposition 4.3 in [6].

The same argument implies that it is enough to show that if $m > 1$ ($\text{rk}(x) < p - 1$), the m particles starting from zero will exit that point just after the start. Let $\tau_1 = \inf\{t > 0 : X_l(t) = 0\} \wedge \inf\{t > 0 : X_{l+m+1}(t) = 0\}$. By continuity of the paths, we have $\tau_1 > 0$ a.s., that is, we do not have any additional zero particle up to time τ_1 . Assume that all the m particles starting from zero remain at zero for some $\tau_2 > 0$ with positive probability and put $\tau = \tau_1 \wedge \tau_2$. Then it is clear

that $e_N(X) \equiv 0$ for $t < \tau$, where $N = l + n + 1$, since every product of length N contains at least one zero particle. In particular, the drift of $e_N(X)$ vanishes for $t < \tau$, but from the other side, by Proposition 1, it is equal to

$$\begin{aligned} \text{drift}[e_N] &= \sum_{i=1}^p \alpha e_{N-1}^{\bar{i}}(X) - \sum_{i < j} (|X_i| + |X_j|) e_{N-2}^{\bar{i}, \bar{j}}(X) \\ &= m e_{N-1}(X) (\alpha + l - n). \end{aligned}$$

Indeed, for $t < \tau$, we have $e_{N-1}^{\bar{i}}(X) \equiv 0$ if $X_i(t) \neq 0$ and $e_{N-1}^{\bar{i}}(X) = e_{N-1}(X)$ (the product of all nonzero particles) if $X_i(t) = 0$. Moreover, the expression $(|X_i| + |X_j|) e_{N-2}^{\bar{i}, \bar{j}}(X)$ is non-zero only if exactly one of particles X_i, X_j is zero and

$$\begin{aligned} \sum_{i < j} (|X_i| + |X_j|) e_{N-2}^{\bar{i}, \bar{j}}(X) &= \sum_{i=l+1}^{l+m} \sum_{j=1}^p |X_j| e_{N-2}^{\bar{i}, \bar{j}}(X) \mathbf{1}_{\{X_i \neq 0\}} = \sum_{i=l+1}^{l+m} \sum_{j=1}^p |X_j| \frac{e_{N-1}(X)}{X_j} \mathbf{1}_{\{X_j \neq 0\}} \\ &= m e_{N-1}(X) \sum_{j=1}^p \text{sgn}(X_j) = m e_{N-1}(X) (l - n). \end{aligned}$$

However, if $n \geq (p + \alpha - 1)/2$, then $\alpha + l - n \leq l + n - p + 1 = 1 - m < 0$. On the other hand, if $l \geq (p - \alpha - 1)/2$, then $\alpha + l - n \geq p - n - l - 1 = m - 1 > 0$. In both cases we have $\alpha + l - n \neq 0$. It leads to a contradiction since $e_{N-1}(X(t))$ does not vanish for $t < \tau$ as the product of non-zero particles. It means that at least one zero particle must become nonzero immediately. It will increase the number of nonzero particles on $\{t < \tau_1\}$ and consequently we will still have $n' \geq (p + \alpha - 1)/2$ or $l' \geq (p - \alpha - 1)/2$, where l' and n' are numbers of strictly negative and positive particles after instant exit from zero of some particles. Thus, we can proceed using Strong Markov property and inductively show that all the m particles must leave zero just after the start. This ends the proof. \square

In fact, the above-given proof leads directly to the result presented in Theorem 2.

Proof of Theorem 2. As in Theorem 1, we can suppose that $\alpha \geq 0$.

The “if” part. Existence of a solution was proved in Theorem 1. Thus, it is enough to show that under the hypotheses (a) or (b) of Theorem 2, any solution of (1.1) is non-colliding. Then, using uniqueness of non-colliding solutions proved in Theorem 1, we get the “if” part of Theorem 2. Thus let $X = (X_1, \dots, X_p)$ be a solution. Then by Itô formula and the computations provided in Proposition 3.1 in [6], we obtain that the SDEs for $e_n(X)$ are of the same form as (3.1), but with $|X_i| + |X_j|$ replaced by $(|X_i| + |X_j|) \mathbf{1}_{\{X_i \neq X_j\}}$. However, it does not affect the arguments presented above in the proof of Theorem 1, which say that whenever $\alpha \notin \{0, \dots, p - 2\}$ or $\alpha \in \{0, \dots, p - 2\}$ but $\text{rk}^+(x) \geq (p + \alpha - 1)/2$ or $\text{rk}^-(x) \geq (p - \alpha - 1)/2$, the particles become immediately distinct and never collide again. Note that adding the indicators $\mathbf{1}_{\{X_i \neq X_j\}}$ does not affect conditions (A1), (A3), (A4) and (A5) needed in [6] and used above. The condition (A2),

which here simplifies to

$$|x| + |y| \leq (|x| + |y|)\mathbf{1}_{\{x \neq y\}},$$

holds for every $x \neq y$, but it is enough for Theorem 4.4 from [6] to be true.

The “only if” part. We construct a solution for $\alpha \in \{0, \dots, p - 2\}$, starting from $x = (x_1, \dots, x_p)$ such that $\text{rk}^+(x) < (p + \alpha - 1)/2$ and $\text{rk}^-(x) < (p - \alpha - 1)/2$, which is not non-colliding, that is, the uniqueness of a solution does not hold. First, we note that there exist non-negative integers \tilde{n} and \tilde{l} such that

- (1) $\alpha + \tilde{l} - \tilde{n} = 0$,
- (2) $\text{rk}^+(x) \leq \tilde{n} < (p + \alpha - 1)/2$,
- (3) $\text{rk}^-(x) \leq \tilde{l} < (p - \alpha - 1)/2$.

Indeed, it is easy to check that $\tilde{l} = \lfloor \frac{p-\alpha}{2} \rfloor - 1$ and $\tilde{n} = \lfloor \frac{p+\alpha}{2} \rfloor - 1$ satisfy the conditions (1), (2) and (3). The choice of \tilde{n} and \tilde{l} is not unique in many cases, cf. Remark 4 and Example 2. Observe that $\tilde{l} + \tilde{n} \leq p - 2$ and consequently $\tilde{l} \leq p - \tilde{n} - 2 \leq p - \text{rk}^+(x) - 2 \leq p - \text{rk}^+(x)$. Similarly, $\tilde{n} \leq p - \text{rk}^-(x)$.

Let $Z = (Z_1, \dots, Z_{\tilde{l}})$ be the process $BESQ_{\text{nc}}^{(\alpha^-)}(-x_{\tilde{l}}, -x_{\tilde{l}-1}, \dots, -x_1)$, where $\alpha^- = p - \alpha - \tilde{l}$, described by

$$dZ_i = 2\sqrt{|Z_i|}dB_{\tilde{l}-i+1} + \left(\alpha^- + \sum_{j=1, j \neq i}^{\tilde{l}} \frac{|Z_i| + |Z_j|}{Z_i - Z_j} \right) dt, \quad i = 1, \dots, \tilde{l}.$$

Inequality $\tilde{l} < (p - \alpha - 1)/2$ implies $\alpha^- > \tilde{l} + 1$. Since $\tilde{l} \leq p - \text{rk}^+(x)$ we have $-x_{\tilde{l}} \geq 0$, we are in the classical setting of Corollary 1 and consequently the process Z is well-defined and non-negative. Moreover, we consider

$$dY_i = 2\sqrt{|Y_i|}dB_i + \left(\alpha^+ + \sum_{j \neq i, j=p-\tilde{n}+1}^p \frac{|Y_i| + |Y_j|}{Y_i - Y_j} \right) dt, \quad i = p - \tilde{n} + 1, \dots, p,$$

where $Y(0) = (x_{p-\tilde{n}+1}, \dots, x_p)$ and $\alpha^+ = \alpha + p - \tilde{n}$. Similarly, this process is $BESQ_{\text{nc}}^{(\alpha^+)}(x_{p-\tilde{n}+1}, \dots, x_p)$ with $\alpha^+ > \tilde{n} + 1$ and the starting point is non-negative.

Now we glue these solutions together with $p - \tilde{n} - \tilde{l} \geq 2$ particles constantly equal to zero, that is, we set

$$X_i = \begin{cases} -Z_{\tilde{l}-i+1}, & i = 1, \dots, \tilde{l}, \\ 0, & i = \tilde{l} + 1, \dots, p - \tilde{n}, \\ Y_i, & i = p - \tilde{n} + 1, \dots, p. \end{cases}$$

We can easily check that $X = (X_1, \dots, X_p)$ solves

$$dX_i = 2\sqrt{|X_i|}dB_i + \left(\alpha + \sum_{j \neq i} \frac{|X_i| + |X_j|}{X_i - X_j} \mathbf{1}_{\{X_i \neq X_j\}} \right) dt.$$

Indeed, since $X_1, \dots, X_{p-\tilde{n}}$ are non-positive and $X_{\tilde{l}+1}, \dots, X_p$ are nonnegative we have

$$\frac{|X_i| + |X_j|}{X_i - X_j} = -1 \quad \text{for } i = 1, \dots, \tilde{l}, j = \tilde{l} + 1, \dots, p,$$

$$\frac{|X_i| + |X_j|}{X_i - X_j} = 1 \quad \text{for } i = p - \tilde{n} + 1, \dots, p, j = 1, \dots, p - \tilde{n},$$

and the drift parts for $i = 1, \dots, \tilde{l}$ and $i = p - \tilde{n} + 1, \dots, p$ are reduced to those for Z and Y respectively. Moreover, for $i = \tilde{l} + 1, \dots, p - \tilde{n}$ the drift part equals $\alpha - \tilde{n} + \tilde{l}$ which is zero as we have assumed. Finally, it is obvious that $X = (X_1, \dots, X_p)$ has collisions after the start since there are at least two zero particles ($\tilde{l} + \tilde{n} \leq p - 2$), that is, they collide for every $t > 0$. \square

Proof of Theorem 3. If $\alpha \geq p - 1$ then, by Theorem 2, there exists unique strong solution $X(t)$, which is non-colliding (by Theorem 1). By Corollary 1, $X(t)$ is nonnegative.

In the case $\alpha \in \{0, 1, \dots, p - 2\}$ and $\text{rk}(x) \leq \alpha$, the nonnegative solution was constructed in [1,2], see also [7]. Note that one can construct such solution in the same way as in the proof of Theorem 2 by letting $\tilde{l} = 0$.

Assume that there exists a nonnegative solution (X_1, \dots, X_p) in one of the following cases:

- (i) $\alpha < p - 1$ but not in $\{0, 1, \dots, p - 2\}$ or
- (ii) $\alpha \in \{0, 1, \dots, p - 2\}$ but $\text{rk}(X(0)) > \alpha$.

Then there are at least $\alpha + 1$ particles different from X_1 on some positive time interval $[0, T]$, $T > 0$. Indeed, in the case (i) and in the case (ii) with $x_1 > 0$ we have only non-colliding solution, so all the particles are different (for the case (ii) with $x_1 > 0$ we apply the first part of the proof of Proposition 4.3 in [6], where the condition (A4) is not needed. The instant diffraction takes place, if the start is from a collision).

In the case (ii) with $x_1 = 0$ we just use the continuity of the paths. In both cases, the drift of X_1 can be estimated as follows

$$\text{drift}(X_1) = \alpha + \sum_{j=2}^p \frac{|X_1| + |X_j|}{X_1 - X_j} \mathbf{1}_{\{X_i \neq X_j\}} \leq \alpha - (\alpha + 1) \leq -1. \tag{4.2}$$

Here we used the simple inequality $(|x| + |y|)/(x - y) \leq -1$ valid for every $x < y$. Consequently, by the comparison theorem and the fact that $BESQ^{(-1)}(X_1(0))$ becomes strictly negative on every time interval with positive probability [4], we get a contradiction with our initial assumption that X_1 is nonnegative.

Thus, it remains to show that for $\alpha \in \{0, 1, \dots, p - 2\}$ and $\text{rk}(x) \leq \alpha$ the solution is unique among nonnegative solutions. We show that the first $p - \alpha$ particles of non-negative solutions must stay at zero. Indeed, if at any time there are more than α particles different from X_1 , then we go back to the above-described situation when (4.2) holds. Using Strong Markov property, we can conclude that the solution becomes negative with positive probability. Consequently $X_1(t) =$

$\dots = X_{p-\alpha}(t)$ for every $t \geq 0$. Moreover, if $X_1(t) > 0$ at some time $t > 0$, then by the first part of the proof of Proposition 4.3 in [6], the solution immediately becomes non-colliding and there are $p - 1$ particles different from X_1 . Once again, by Strong Markov property, we get that X_1 becomes negative with positive probability. Finally, knowing that $X_1(t) = \dots = X_{p-\alpha}(t) = 0$ for every t , the equations for the remaining $X_{p-\alpha+1}, \dots, X_p$ are

$$dX_i = 2\sqrt{|X_i|}dB_i + \left(p + \sum_{\substack{j=p-\alpha+1, \dots, p \\ j \neq i}} \frac{|X_i| + |X_j|}{X_i - X_j} \mathbf{1}_{\{X_i \neq X_j\}} \right) dt,$$

$$i = p - \alpha + 1, \dots, p.$$

Note that this is just the system (1.1) of SDEs describing $\tilde{p} = \alpha$ particles with drift parameter $\tilde{\alpha} = p$. Since $\tilde{\alpha} > \tilde{p} + 1$, by Theorem 1 there exists unique non-negative solution, which ends the proof. \square

Proof of Theorem 5. Let (X_1, \dots, X_p) be a non-colliding solution to (1.1) with given Brownian motions (B_1, \dots, B_p) . Bru in [2] showed that for $\alpha \in (p - 1, p + 1)$, the first particle hits zero almost surely ($T_0^{(1)} < \infty$), but it remains nonnegative ($T_-^{(1)} = \infty$).

For $\alpha \leq p - 1$, we define \tilde{X}_1 as a solution to the following SDE

$$d\tilde{X}_1 = 2\sqrt{|\tilde{X}_1|}dB_1 + (\alpha - p + 1)dt$$

starting from x_1 . This process is $BESQ^{(\alpha-p+1)}(x_1)$ driven by the same Brownian motion as X_1 . Following the proof of the comparison theorem (see Theorem 3.7, p. 394 in [11]), we notice that the local time at zero $L^0(\tilde{X}_1 - X_1)$ vanishes and consequently, using the Tanaka's formula, we can write

$$\mathbf{E}(X_1 - \tilde{X}_1)^+ = \mathbf{E} \int_0^t \mathbf{1}_{\{X_1(s) > \tilde{X}_1(s)\}} \left(p - 1 + \sum_{i=2}^p \frac{|X_1(s)| + |X_i(s)|}{X_1(s) - X_i(s)} \right) ds \leq 0.$$

The last inequality follows from the inequality $(|x| + |y|)/(x - y) \leq -1$ for $y > x$. Thus, $X_1(t) \leq \tilde{X}_1(t)$ for every $t \geq 0$ a.s. This implies that X_1 hits zero. Moreover, for $\alpha < p - 1$ the process $X_1(t)$ becomes strictly negative ($T_0^{(1)} = T_-^{(1)} < \infty$) and remains nonpositive for $t > T_0^{(1)}$, because the same holds for the squared Bessel process \tilde{X}_1 with negative dimension $\alpha - p + 1$. For $\alpha = p - 1$ the process X is non-negative (by Theorems 2 and 3, that is, the unique non-colliding solution is nonnegative), that is, $T_-^{(1)} = \infty$. Consequently, in the case $\alpha = p - 1$, we have $X_1(t) = 0$ for $t \geq T_0^{(1)}$.

To examine the behaviour of the system after the time $T_-^{(1)}$ (for $\alpha < p - 1$), we define $X_i^*(t) = X_i(T_-^{(1)} + t)$ and $B_i^*(s) = B_i(T_-^{(1)} + s) - B_i(T_-^{(1)})$ for $i = 1, \dots, p$. Note that, by Strong Markov property, the process (B_1^*, \dots, B_p^*) is again a p -dimensional Brownian motion and in particular

B_i^* are independent. Moreover, we have $X_1^*(0) = 0$ and for $t < T_-^{(1)} - T_-^{(2)}$ we have

$$\begin{aligned} X_1^*(t) &= \int_{T_-^{(1)}}^{T_-^{(1)}+t} 2\sqrt{|X_1(s)|} dB_1(s) + t\alpha + \int_{T_-^{(1)}}^{T_-^{(1)}+t} \sum_{k=2}^p \frac{|X_1(s)| + |X_k(s)|}{X_1(s) - X_k(s)} ds \\ &= \int_{T_-^{(1)}}^{T_-^{(1)}+t} 2\sqrt{|X_1(s)|} dB_1(s) + (\alpha - p + 1)t = \int_0^t 2\sqrt{|X_1^*(s)|} dB_1^*(s) + (\alpha - p + 1)t, \end{aligned}$$

where we used the fact that $(|x| + |y|)/(x - y) = -1$ whenever $x \leq 0 \leq y$. Similarly, for $i = 2, \dots, p$ we get

$$\begin{aligned} X_i^*(t) - X_i^*(0) &= \int_{T_-^{(1)}}^{T_-^{(1)}+t} 2\sqrt{|X_i(s)|} dB_i(s) + t\alpha + \int_{T_-^{(1)}}^{T_-^{(1)}+t} \sum_{k \neq i}^p \frac{|X_i(s)| + |X_k(s)|}{X_i(s) - X_k(s)} ds \\ &= \int_{T_-^{(1)}}^{T_-^{(1)}+t} 2\sqrt{|X_i(s)|} dB_i(s) + t(\alpha + 1) \\ &\quad + \int_{T_-^{(1)}}^{T_-^{(1)}+t} \sum_{k > 1, k \neq i} \frac{|X_i(s)| + |X_k(s)|}{X_i(s) - X_k(s)} ds \\ &= \int_0^t 2\sqrt{|X_i^*(s)|} dB_i^*(s) + t(\alpha + 1) + \int_0^t \sum_{k > 1, k \neq i} \frac{X_i^*(s) + X_k^*(s)}{X_i^*(s) - X_k^*(s)} ds. \end{aligned}$$

Note that the interactions between particles X_1^* and X_2^*, \dots, X_p^* disappeared from the corresponding drift parts and, consequently, the processes $\mathbf{Y}_1 = X_1$ and $\mathbf{Z}_1 = (X_2, \dots, X_p)$ on $[T_-^{(1)}, T_-^{(2)})$ are conditionally independent, given the starting point $\mathbf{Z}_1(T_-^{(1)})$. Moreover, \mathbf{Y}_1 is $-BESQ^{(p-1-\alpha)}(0)$ and \mathbf{Z}_1 evolves as a non-colliding squared Bessel system of $p - 1$ particles with drift parameter $\alpha + 1$.

By Strong Markov property, we can apply the above-given argument to the squared Bessel system of $p^* = p - 1$ particles (X_2^*, \dots, X_p^*) with drift parameter $\alpha^* = \alpha + 1$ and show that if $\alpha < p - 3$ (which is equivalent to $\alpha^* < p^* - 1$) then $T_-^{(2)} < \infty$. Moreover, after going into $(-\infty, 0]$ the second particle becomes invisible (independent) for the nonnegative particles, but starts to interact with the first one. Indeed, we have

$$\bar{X}_i(t) - \bar{X}_i(0) = \int_0^t 2\sqrt{|\bar{X}_i(s)|} d\bar{B}_i(s) + t(\alpha + 2) + \int_0^t \sum_{k > 2, k \neq i} \frac{\bar{X}_i(s) + \bar{X}_k(s)}{\bar{X}_i(s) - \bar{X}_k(s)} ds$$

for $i = 3, 4, \dots, p$ and

$$\bar{X}_j(t) = \int_0^t 2\sqrt{|\bar{X}_j(s)|} d\bar{B}_j(s) + (\alpha - p + 2)t, \quad j = 1, 2,$$

where $\bar{X}(t) = X(T_-^{(2)} + t)$ and $\bar{B}(t) = B(T_-^{(2)} + t) - B(T_-^{(2)})$.

We complete the proof by iterating this procedure. When α is small enough the consecutive particles become negative and then the nonnegative and nonpositive particle subsystems evolve independently as squared Bessel particle systems with appropriate drift parameters. \square

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