Equilibrium of the interface of the grass-bushes-trees process

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We consider the grass-bushes-trees process, which is a two-type contact process in which one of the types is dominant. Individuals of the dominant type can give birth on empty sites and sites occupied by non-dominant individuals, whereas non-dominant individuals can only give birth at empty sites. We study the shifted version of this process so that it is 'seen from the rightmost dominant individual' (which is well defined if the process occurs in an appropriate subset of the configuration space); we call this shifted process the grass-bushes-trees interface (GBTI) process. The set of stationary distributions of the GBTI process is fully characterized, and precise conditions for convergence to these distributions are given.

Keywords: contact process; interacting particle systems

1. Introduction

The grass-bushes-trees (GBT) process is a continuous-time Markov process $(\xi_t)_{t \geq 0}$ on $\{0, 1, 2\}^{\mathbb{Z}}$ defined as follows. We endow $\{0, 1, 2\}^{\mathbb{Z}}$ with the product topology and endow the vector space of continuous real-valued functions on $\{0, 1, 2\}^{\mathbb{Z}}$ with the supremum norm, making it a Banach space. We then consider the operator, defined on a suitable subspace of this Banach space, given by

$$
\mathcal{L}f(\xi) = \sum_{x:\xi(x)=1} \left(\delta_1 \cdot \left(f\left(\xi^{0 \to x}\right) - f(\xi) \right) + \lambda_1 \cdot \sum_{\substack{y:0<|y-x|\leq K_1: \\ \xi(y)=0 \text{ or } 2}} \left(f\left(\xi^{1 \to y}\right) - f(\xi) \right) \right)
$$
$$
+ \sum_{x:\xi(x)=2} \left(\delta_2 \cdot \left(f\left(\xi^{0 \to x}\right) - f(\xi) \right) + \lambda_2 \cdot \sum_{\substack{y:0<|y-x|\leq K_2: \\ \xi(y)=0}} \left(f\left(\xi^{2 \to y}\right) - f(\xi) \right) \right),
$$

where $\delta_1, \delta_2, \lambda_1, \lambda_2 \geq 0$, $K_1, K_2 \in \mathbb{N}$, and

$$
\xi^{i \to x}(z) = \begin{cases} i & \text{if } z = x, \\ \xi(z) & \text{otherwise.} \end{cases}
$$

The domain of $\mathcal{L}$ can be taken as the set of functions f satisfying

$$\sum_{x\in\mathbb{Z}}\sup\{|f(\xi)-f(\xi')|:\xi,\xi'\in\{0,1,2\}^{\mathbb{Z}},\xi(y)=\xi'(y)\text{ for all }y\neq x\}<\infty.$$

By Theorems 2.9 and 3.9 of Chapter 1 of [10], the closure of $\mathcal{L}$ is a Markov generator, which uniquely determines the Markov process $(\xi_t)_{t\geq 0}$ on the space of trajectories on $\{0,1,2\}^{\mathbb{Z}}$ which are right continuous and have left limits.

Given disjoint sets $A,B\subset\mathbb{Z}$, we will write $(\xi_t^{A,B})_{t\geq 0}$ to denote the GBT process with initial configuration $\xi_0^{A,B}=\mathbb{1}_A+2\cdot\mathbb{1}_B$ (though we will omit the superscripts when the initial configuration is clear from the context or unimportant). We refer the reader to [5] and [6], where the grass-bushes-trees process was first considered.

This process can be seen as a model for biological competition between two species, denoted 1 and 2: a vertex in state 0 is empty, whereas a vertex in state 1 or 2 contains an individual of the corresponding species. The above infinitesimal generator gives the following rules for the dynamics (with $i=1$ or 2):

- an individual of species i dies with rate δ_i;
- an individual of species i gives birth at sites within range K_i with rate λ_i, *but*
- an individual of type 2 cannot be born at a site containing an individual of type 1.

The name of the process is due to the interpretation in which a vertex in state 0, 1 or 2 is respectively said to contain grass, a tree or a bush (so that trees can produce offspring over grass and bushes, whereas bushes can only produce offspring over grass).

Here we will be interested in the following choice of parameters:

$$\delta_1=\delta_2=1,\qquad \lambda_1=\lambda_2=\lambda>0,\qquad K_1=K_2=K\in\mathbb{N}. \tag{1.1}$$

The important feature of this choice of parameters is that (using the common abuse of notation in which a set $A\subset\mathbb{Z}$ is identified with its indicator function $\mathbb{1}_A$) both processes

$$\left(\{x:\xi_t(x)\neq 0\}\right)_{t\geq 0}\quad\text{and}\quad\left(\{x:\xi_t(x)=1\}\right)_{t\geq 0}$$

are contact processes with rate λ and range K (see [10] and [11] for expositions on the contact process). Thus, in the grass-bushes-trees dynamics, 1's evolve as a contact process, whereas 2's evolve as a contact process in a dynamic random environment: they can only occupy vertices that are not taken by 1's (this idea is made precise in the proof of Lemma 2.1 below).

The contact process on $\mathbb{Z}^d$ exhibits a phase transition delimited by $\lambda_c(\mathbb{Z}^d,K)\in(0,\infty)$. If $\lambda\leq\lambda_c$, then the process is ergodic and the only stationary distribution is $\delta_\varnothing$, which gives full mass to the configuration in which all vertices are zero. If $\lambda>\lambda_c$, the process is not ergodic and apart from $\delta_\varnothing$, there is one more extremal stationary distribution, obtained as the distributional limit, as time is taken to infinity, of the process started from full occupancy. Throughout this paper, we fix $d=1$, $K\in\mathbb{N}$ and also fix λ in the corresponding supercritical region, that is,

$$\lambda>\lambda_c(\mathbb{Z},K). \tag{1.2}$$

In [2], motivated by a conjecture in [4], the authors considered the grass-bushes-trees process with parameters given by (1.1) and (1.2) and the initial configuration in which all vertices $x \leq 0$ are in state 1 and all vertices $x > 0$ are in state 2. For this process, defining $R_t = \sup\{x : \xi_t(x) = 1\}$ and $L_t = \inf\{x : \xi_t(x) = 2\}$, the interval delimited by R_t and L_t is called the interface and $|R_t - L_t|$ is the interface size (note that $R_t - L_t$ is necessarily negative when the range $K = 1$, whereas it can be positive or negative if $K > 1$). It was then shown that the interface size is stochastically tight (in (2.10) below we reproduce the exact statement). This leads to the natural conjecture that the process "seen from the interface" converges in distribution, and in the present paper we address this point (moreover, as we will explain shortly, we allow for more general initial configurations).

Let us give some definitions in order to state our results. We define the set of configurations

$$\mathcal{Y} = \left\{ \xi \in \{0, 1, 2\}^{\mathbb{Z}} : \inf\{x : \xi(x) = 1\} = -\infty, \sup\{x : \xi(x) = 1\} < \infty \right\}. \tag{1.3}$$

We remark that the GBT process $(\xi_t)_{t \geq 0}$ started from a configuration in $\mathcal{Y}$ almost surely never leaves $\mathcal{Y}$. Then, defining as above $R_t = \sup\{x : \xi_t(x) = 1\}$, we have $-\infty < R_t < \infty$ for all t and we can introduce the shifted version of the process,

$$\tilde{\xi}_t(x) = \xi(x + R_t), \qquad x \in \mathbb{Z}, t \geq 0.$$

$(\tilde{\xi}_t)_{t \geq 0}$ is itself a Markov process in the set of configurations

$$\mathcal{Y}_0 = \left\{ \tilde{\xi} \in \{0, 1, 2\}^{\mathbb{Z}} : \inf\{x : \tilde{\xi}(x) = 1\} = -\infty, \sup\{x : \tilde{\xi}(x) = 1\} = 0 \right\}. \tag{1.4}$$

We call $(\tilde{\xi}_t)$ the grass-bushes-trees interface (GBTI) process. We fully describe the set of extremal stationary distributions for the GBTI process and give sharp conditions for convergence to these distributions.

Theorem 1.1. *For the GBTI process with rates given by* (1.1) *and* (1.2)*, the set of stationary and extremal distributions consists of two measures $\underline{v}$ and $\bar{v}$. These measures are mutually singular: $\underline{v}$ is supported on configurations where 2's are absent, and $\bar{v}$ is supported on the set of configurations*

$$\mathcal{Y}_0' = \left\{ \tilde{\xi} \in \{0, 1, 2\}^{\mathbb{Z}} : \begin{array}{l} \inf\{x : \tilde{\xi}(x) = 1\} = -\infty, \sup\{x : \tilde{\xi}(x) = 1\} = 0, \\ \inf\{x : \tilde{\xi}(x) = 2\} > -\infty, \sup\{x : \tilde{\xi}(x) = 2\} = \infty \end{array} \right\}. \tag{1.5}$$

Theorem 1.2. *Let $(\tilde{\xi}_t)_{t \geq 0}$ be the GBTI process with parameters given by* (1.1) *and* (1.2) *and started from a (deterministic) initial configuration $\tilde{\xi}_0 \in \mathcal{Y}_0$. Then,*

(a) $(\tilde{\xi}_t)_{t \geq 0}$ *converges to $\bar{v}$ if and only if*

$$\text{for all } M > 0, \qquad \lim_{n \to \infty} \#\left\{ x \in [M\sqrt{n}, 2n - M\sqrt{n}] : \tilde{\xi}_0(x) = 2 \right\} = \infty. \tag{$\star$}$$

(b) $(\tilde{\xi}_t)_{t \geq 0}$ *converges to $\underline{v}$ if and only if*

$$\sup\{x : \tilde{\xi}_0(x) = 2\} < \infty. \tag{$*$}$$

Condition ($\star$) fails for initial configurations in which the vertices in state 2 appear either in finite number or quite sparsely. For example, if $\tilde{\xi}_0$ is such that $\tilde{\xi}_0(x) = 2$ if and only if $x = 2^{3^n}$ for some $n \in \mathbb{N}$, then ($\star$) fails for any M.

A byproduct of our proofs of the above results is of independent interest. Namely, we establish the impossibility of coexistence of 1's and 2's in the GBT process.

Theorem 1.3. *Let* $(\xi_t)_{t \geq 0}$ *be the GBT process with parameters given by* (1.1) *and* (1.2) *and started from a configuration with finitely many* 2's. *Then,*

$$\mathbb{P}\big(\forall t \; \exists x, y : \xi_t(x) = 1, \xi_t(y) = 2\big) = 0. \tag{1.6}$$

In particular, if the initial configuration has infinitely many 1's and finitely many 2's, then the 2's eventually disappear, and if the initial configuration has finitely many 1's and 2's, then the 2's can only survive if the 1's disappear.

It is worth contrasting this result with the case of a related competition model, Neuhauser's multitype contact process (MCP) introduced in [14]. The MCP differs from the GBT in that in the MCP, both 1's and 2's are forbidden from giving birth at occupied vertices, so that the model is symmetric (as long as one takes birth and death rates to be the same for the two types). In [1] and [15], it was shown that for the (symmetric) MCP with $\lambda > \lambda_c(\mathbb{Z})$, coexistence of the two types is in fact possible: for example, if the process is started from finitely many 1's and 2's, then with positive probability neither type ever disappears. It would be very interesting to determine whether or not the corresponding fact holds for the multidimensional versions of the GBT and MCP.

While on this topic, let us also mention that it would be interesting to investigate the stationary distributions of the interface process obtained from the MCP. As of now, what is known is that, if the process is started from all 1's to the left of the origin and all 2's to the right of the origin, then the size of the interface is tight ([15]) and its position moves diffusively ([12]).

To conclude this Introduction, let us detail how the rest of the paper is organized. In Section 2, we introduce notation and give a few results about the original (one-type) contact process and the grass-bushes-trees process, including a useful stochastic domination result (Lemma 2.1). In Section 3, we prove Theorem 1.3. Sections 4 and 5 are dedicated to the definitions of the measures $\underline{\nu}$ and $\overline{\nu}$, respectively. Finally, Section 6 is dedicated to the proof of Theorem 1.2 and Section 7 to the proof of Theorem 1.1.

2. Notation and preliminary results

Sets and configurations

We denote $\mathbb{Z}_+ = \{1, 2, \ldots\}$, $\mathbb{Z}_- = -\mathbb{Z}_+$, $\mathbb{Z}_+^* = (\mathbb{Z}_-)^c$ and $\mathbb{Z}_-^* = (\mathbb{Z}_+)^c$. We will often abuse notation in our treatment of intervals, for example writing an interval as (a, b) when we mean the integer interval $\{x \in \mathbb{Z} : a < x < b\}$. We adopt the usual convention that $\inf \varnothing = \infty$ and $\sup \varnothing = -\infty$. The cardinality of a set A will be denoted by $\#A$.

We will often refer to the sets $\mathcal{Y}$ and $\mathcal{Y}_0$ introduced in (1.3) and (1.4), and also define

$$\mathcal{X} = \left\{ \zeta \in \{0, 1\}^{\mathbb{Z}} : \inf\{x : \zeta(x) = 1\} = -\infty, \sup\{x : \zeta(x) = 1\} < \infty \right\}, \qquad (2.1)$$

$$\mathcal{X}_0 = \left\{ \zeta \in \{0, 1\}^{\mathbb{Z}} : \inf\{x : \zeta(x) = 1\} = -\infty, \sup\{x : \zeta(x) = 1\} = 0 \right\}. \qquad (2.2)$$

We will often associate a set A to its indicator function $\mathbb{1}_A$, which will allow us to write things like $A \in \mathcal{X}$ or $A \in \mathcal{X}_0$. Similarly, if $A, B \subset \mathbb{Z}$ are disjoint, we will identify the pair (A, B) with the configuration $\mathbb{1}_A + 2 \cdot \mathbb{1}_B$, so we will for example, write $(A, B) \in \mathcal{Y}$ or $(A, B) \in \mathcal{Y}_0$. Throughout this paper, spaces as $\{0, 1\}^{\mathbb{Z}}$ and $\{0, 1, 2\}^{\mathbb{Z}}$ are endowed with the product topology and any of their subspaces with the corresponding subspace topology.

Graphical construction

As mentioned in the Introduction, throughout the paper we fix $K \in \mathbb{N}$ and λ larger than $\lambda_c(\mathbb{Z}, K)$, the critical parameter of the one-dimensional contact process with range K. We will construct all the processes we are interested in using a single graphical construction, that is, a family of Poisson processes commanding the transitions in the dynamics; although this construction is quite well known, let us present it in order to fix notation. A Harris system is a collection H of independent Poisson point processes on $[0, \infty)$,

$$\left\{ D^x : x \in \mathbb{Z} \right\}, \qquad \text{each with rate } 1,$$

$$\left\{ D^{(x,y)} : x, y \in \mathbb{Z}, |x - y| \le K \right\}, \qquad \text{each with rate } \lambda.$$

Given a realization of all these processes and $(x, t_1), (y, t_2) \in \mathbb{Z} \times [0, \infty)$ with $t_1 < t_2$, an infection path from (x, t_1) to (y, t_2) is a càdlàg function $\gamma : [t_1, t_2] \to \mathbb{Z}$ such that: (1) $\gamma(t_1) = x$, (2) $\gamma(t_2) = y$, (3) $t \notin D^{\gamma(t)}$ for all t and (4) $t \in D^{(\gamma(t-),\gamma(t))}$ whenever $\gamma(t) \ne \gamma(t-)$. In case there is an infection path from (x, t_1) to (y, t_2), we say that the two space–time points are connected by an infection path, and write $(x, t_1) \leftrightarrow (y, t_2)$ in H (though dependence on H will in general be omitted). Given sets $A, B \subset \mathbb{Z}$, we write $A \times \{t_1\} \leftrightarrow B \times \{t_2\}$ if $(x, t_1) \leftrightarrow (y, t_2)$ for some $x \in A$ and $y \in B$. We write $(x, t_1) \leftrightarrow B \times \{t_2\}$ instead of $\{x\} \times \{t_1\} \leftrightarrow B \times \{t_2\}$ and write $A \times \{t_1\} \leftrightarrow (y, t_2)$ instead of $A \times \{t_1\} \leftrightarrow \{y\} \times \{t_2\}$.

For $A \subset \mathbb{Z}$, $x \in \mathbb{Z}$ and $t \ge 0$, we let

$$\zeta_t^A(x) = \mathbb{1}\left\{ A \times \{0\} \leftrightarrow (x, t) \right\}.$$

Then, $(\zeta_t^A)_{t \ge 0}$ is a contact process with initial occupancy on A. In case $\sup A < \infty$, we almost surely have $\sup \zeta_t^A < \infty$ for all t, so we can define

$$R_t^A = \sup \zeta_t^A, \qquad \tilde{\zeta}_t^A(x) = \begin{cases} \zeta_t^A(x + R_t^A), & \text{if } \zeta_t^A \ne \varnothing, \\ \triangle & \text{otherwise,} \end{cases}$$

where $\triangle$ denotes a cemetery state. Then, $(\tilde{\zeta}_t^A)_{t \ge 0}$ is the contact process with initial occupancy on A seen from the rightmost site in state 1.

For disjoint sets $A, B \subset \mathbb{Z}$, $x \in \mathbb{Z}$ and $t \geq 0$, we let

$$
\xi_t^{A,B}(x) = \begin{cases} 1 & \text{if } A \times \{0\} \leftrightarrow (x,t), \\ 2 & \text{if } A \times \{0\} \nleftrightarrow (x,t), B \times \{0\} \leftrightarrow (x,t), \\ 0 & \text{otherwise.} \end{cases}
$$

Then, $(\xi_t^{A,B})_{t \geq 0}$ is a grass-bushes-trees process started with 1's on A and 2's on B. In case $\sup A < \infty$, also let

$$
\tilde{\xi}_t^{A,B}(x) = \begin{cases} \xi_t^{A,B}(x + R_t^A) & \text{if } \zeta_t^A \neq \varnothing, \\ \triangle & \text{otherwise.} \end{cases}
$$

Then, $(\tilde{\xi}_t^{A,B})_{t \geq 0}$ is the process defined above seen from the rightmost 1.

Let us remark that, in case $\#A = \infty$, we almost surely have $\zeta_t^A \neq \varnothing$, hence $\tilde{\zeta}_t^A \neq \triangle$ and $\tilde{\xi}_t^{A,B} \neq \triangle$ for all t.

Unless we explicitly state otherwise, we will always assume that the processes we consider are all defined in the same probability space using a single graphical construction. We will then be able to take advantage of useful properties of this coupling, such as

$$
\{x : \xi_t^{A,B}(x) = 1\} = \{x : \zeta_t^A(x) = 1\}, \qquad \{x : \xi_t^{A,B}(x) \neq 0\} = \{x : \zeta_t^{A \cup B}(x) \neq 0\}
$$

(we could of course not have introduced the notations ζ_t^A and $\tilde{\zeta}_t^A$ and instead write $\xi_t^{A,\varnothing}$ and $\tilde{\xi}_t^{A,\varnothing}$, respectively, but we find it convenient to be able to refer to the one-type processes exclusively). We will also omit the superscripts of ζ and ξ when the initial configuration is clear from the context or unimportant.

Behavior of the right edge

One of the elementary facts about the supercritical contact process is that the right edge moves with positive speed, that is,

$$
\exists \alpha = \alpha(\lambda, K) > 0 : \qquad \frac{R_t^{\mathbb{Z}_-^*}}{t} \xrightarrow{t \to \infty} \alpha \text{ almost surely and in } L^1; \tag{2.3}
$$

the proof, which is based on the subadditive ergodic theorem, is carried out in [10] for $K = 1$, but works equally well for any $K \in \mathbb{N}$. A Central Limit theorem is also known to hold:

$$
\exists \sigma = \sigma(\lambda, K) : \qquad \frac{R_t^{\mathbb{Z}_-^*} - \alpha \cdot t}{\sqrt{t}} \xrightarrow[\text{(d)}]{t \to \infty} \mathcal{N}(0, \sigma^2). \tag{2.4}
$$

This was proved in [7] for $K = 1$ and in [13] for $K \in \mathbb{N}$. The constants α and σ of (2.3) and (2.4) will be fixed throughout the paper.

Partial order on configurations

We define a partial order $\preceq$ on $\{0, 1, 2\}^{\mathbb{Z}}$ by setting $\xi \preceq \xi'$ if and only if

$$\{x : \xi(x) = 1\} \subseteq \{x : \xi'(x) = 1\} \quad \text{and} \quad \{x : \xi(x) = 2\} \supseteq \{x : \xi'(x) = 2\}. \qquad (2.5)$$

This induces a relation of stochastic domination, also denoted by $\preceq$, on pairs of random configurations (or pairs of probability measures) on $\{0, 1, 2\}^{\mathbb{Z}}$. However, whenever we write $\xi \preceq \xi'$ for a pair of random configurations ξ and ξ', it should be understood that the configurations are defined in the same probability space and the inequality holds in the almost sure sense.

The joint graphical construction given above reveals that:

Claim 2.1. *If ξ and ξ' are (deterministic or random) configurations such that $\xi \preceq \xi'$ and $(\xi_t)_{t \geq 0}$ and $(\xi'_t)_{t \geq 0}$ are grass-bushes-trees processes started from ξ and ξ', respectively, then $\xi_t \preceq \xi'_t$ for all $t \geq 0$.*

Still regarding the partial order $\preceq$, the following will be a useful tool.

Lemma 2.1. *Assume $A, B \subset \mathbb{Z}$ are disjoint, and let $(\zeta_t^A)_{t \geq 0}$, $(\hat{\zeta}_t^B)_{t \geq 0}$ be two independent contact processes started from occupancy in A and B, respectively. Then, there exists a version $(\xi_t^{A,B})_{t \geq 0}$ of the grass-bushes-trees process started from $\mathbb{1}_A + 2 \cdot \mathbb{1}_B$ such that*

$$\zeta_t^A + 2(1 - \zeta_t^A)\hat{\zeta}_t^B \preceq \xi_t^{A,B} \qquad \text{for all } t \geq 0. \qquad (2.6)$$

Proof. Let H and $\hat{H}$ be two independent Harris systems with rate λ and range K. We construct $(\zeta_t^A)_{t \geq 0}$ using H and $(\hat{\zeta}_t^B)_{t \geq 0}$ using $\hat{H}$. Then, for each $t \geq 0$, we let ξ_t be defined as follows. In case $\zeta_t^A(x) = 1$, we set $\xi_t(x) = 1$. In case there exists an infection path γ in $\hat{H}$ from $B \times \{0\}$ to (x, t) such that $\zeta_s^A(\gamma(s)) = 0$ for each $s \in [0, t]$, we set $\xi_t(x) = 2$.

Inspecting the rates at which the transitions occur in the process $(\xi_t)_{t \geq 0}$ reveals that it is a version of the grass-bushes-trees process started from $\mathbb{1}_A + 2 \cdot \mathbb{1}_B$. Moreover, we have

$$\{x : \zeta_t^A(x) = 1\} = \{x : \xi_t^{A,B}(x) = 1\}, \qquad \{x : \zeta_t^A(x) = 0, \hat{\zeta}_t^B(x) = 1\} \supseteq \{x : \xi_t^{A,B}(x) = 2\}.$$

$\square$

Insulating points

Given $\beta > 0$ and a Harris system H, we say that a point $x \in \mathbb{Z}$ is β-insulating if the following hold:

- for all t, $\zeta_t^x \neq \varnothing$, $\sup \zeta_t^x \geq x$ and $\inf \zeta_t^x \leq x$;
- if $(y, 0) \leftrightarrow (z, t)$ for some $y \leq x$ and $z \geq x - \beta t$, then $(x, 0) \leftrightarrow (z, t)$;
- if $(y, 0) \leftrightarrow (z, t)$ for some $y \geq x$ and $z \leq x + \beta t$, then $(x, 0) \leftrightarrow (z, t)$.

In case x is β-insulating, the cone $\{y \in \mathbb{Z} : x - \beta t \leq y \leq x + \beta t\}$ is called a descendancy barrier. In [13] and [2], it was shown by the following proposition.

Proposition 2.1. *For any* $K \in \mathbb{N}$ *and* $\lambda > \lambda_c(\mathbb{Z}, K)$, *there exist* $\bar{\beta} > 0$, $\bar{\delta} > 0$ *such that*

$$\mathbb{P}(0 \text{ is } \bar{\beta}\text{-insulating}) > \bar{\delta} \quad \text{and} \tag{2.7}$$

$$\forall \varepsilon > 0 \; \exists n : \forall A \subset \mathbb{Z} \text{ with } \#A \geq n, \qquad \mathbb{P}(\text{no point of } A \text{ is } \bar{\beta}\text{-insulating}) < \varepsilon. \tag{2.8}$$

(In Lemma 2.6(i) of [2] it is shown that a vertex x satisfies certain properties in the Harris system with positive probability, and then Proposition 2.7(i) and (iii) of [2] imply that a vertex satisfying the mentioned list of properties is $\bar{\beta}$-insulating in the sense that we give here. Property (2.8) above can be obtained from Lemma 2.6(ii) of [2] through a routine argument that we will omit). The constants $\bar{\beta}$ and $\bar{\delta}$ of the above proposition will be fixed throughout the paper.

One immediate consequence of the definition of $\bar{\beta}$-insulating points is:

$$(0, 0) \text{ is } \bar{\beta}\text{-insulating}$$
$$\implies \quad \text{for all } t, \qquad R_t^{\mathbb{Z}^*_-} = R_t^0 \geq -\bar{\beta}t \quad \text{and} \quad \zeta_t^{\mathbb{Z}^*_-} \equiv \zeta_t^0 \qquad \text{on } [-\bar{\beta}t, \infty). \tag{2.9}$$

Interface tightness. In [2], the following has been proved:

$$\forall \varepsilon > 0 \; \exists L > 0 :$$
$$\mathbb{P}\big(\tilde{\xi}_t^{\mathbb{Z}^*_-, \mathbb{Z}_+}(x) \neq 2 \text{ for all } x \leq -L, \tilde{\xi}_t^{\mathbb{Z}^*_-, \mathbb{Z}_+}(x) = 2 \text{ for some } x \leq L\big) > 1 - \varepsilon \qquad \forall t \geq 0. \tag{2.10}$$

Using the coupled construction of the one-type process and the GBT process using a single Harris system, we see that the above is the same as

$$\forall \varepsilon > 0 \; \exists L > 0 :$$
$$\mathbb{P}\left(\begin{array}{c} \zeta_t^{\mathbb{Z}^*_-} \text{ and } \zeta_t^{\mathbb{Z}} \text{ coincide on } \big(-\infty, R_t^{\mathbb{Z}^*_-} - L\big] \\ \text{but not on } \big(R_t^{\mathbb{Z}^*_-} - L, R_t^{\mathbb{Z}^*_-} + L\big] \end{array} \right) > 1 - \varepsilon \qquad \forall t \geq 0. \tag{2.11}$$

3. Extinction of bushes and a consequence

Lemma 3.1. *For any* $L > 0$ *there exists* $t^* > 0$ *such that*

$$\mathbb{P}\big(R_{t^*}^{\mathbb{Z}^*_-} = R_{t^*}^{(-\infty, L]}\big) > \frac{1}{4}.$$

Proof. For any $t > 0$ we have

$$\mathbb{P}\big(R_t^{\mathbb{Z}^*_-} = R_t^{(-\infty, L]}\big) = \mathbb{P}\big(\sup\{x : \xi_t^{\mathbb{Z}^*_-, [1, L]}(x) = 1\} > \sup\{x : \xi_t^{\mathbb{Z}^*_-, [1, L]}(x) = 2\}\big)$$
$$\geq \mathbb{P}\big(\sup \zeta_t^{\mathbb{Z}^*_-} > \sup \hat{\zeta}_t^{(-\infty, L]}\big), \tag{3.1}$$

where $(\zeta_t^{\mathbb{Z}^*_-})_{t\geq 0}$ and $(\hat{\zeta}_t^{(-\infty,L]})_{t\geq 0}$ are two independent contact processes (see Lemma 2.1). Now, by (2.4), the probability in (3.1) converges to $\frac{1}{2}$ as $t \to \infty$. $\qquad\square$

Lemma 3.2. *For any disjoint sets A, B satisfying* $\inf A = -\infty$ *and* $\#B < \infty$, *there exists t^* such that*

$$\mathbb{P}\big(\sup\{x : \xi_{t^*}^{A,B}(x) = 1\} > \sup\{x : \xi_{t^*}^{A,B}(x) = 2\}\big) > \frac{1}{8}. \qquad (3.2)$$

Proof. Using (2.8), we choose $N > 0$ such that, for any $D \subset \mathbb{Z}$ with $\#D \geq N$, the probability that some point of D is $\bar{\beta}$-insulating is larger than $\frac{7}{8}$. We then take $a, b, c \in \mathbb{Z}$, $a < b < c$ so that $\#(A \cap [a,b]) \geq N$ and $B \subset [b,c]$. Next, we choose t^* corresponding to $L = c - a$ in the previous lemma. Then, with probability larger than $\frac{1}{8}$ both the following events occur:

$$E_1 = \big\{\exists x^* \in A \cap [a,b] : x^* \text{ is } \bar{\beta}\text{-insulating}\big\},$$
$$E_2 = \big\{R_{t^*}^{(-\infty,a]} = R_{t^*}^{(-\infty,c]}\big\}.$$

On $E_1 \cap E_2$ we have

$$R_{t^*}^A \geq R_{t^*}^{x^*} \overset{(2.9)}{=} R_{t^*}^{(-\infty,x^*]} \geq R_{t^*}^{(-\infty,a]} = R_{t^*}^{(-\infty,c]} \geq R_{t^*}^B;$$

the inequality $R_{t^*}^A \geq R_{t^*}^B$ is the same as the event in (3.2). $\qquad\square$

Lemma 3.3. *For any disjoint sets A, B satisfying* $\inf A = -\infty$, $\#B < \infty$ *and* $B \subset [\inf A, \sup A]$, *there exists t^* such that*

$$\mathbb{P}\big(\xi_{t^*}^{A,B}(x) \neq 2 \,\forall x\big) > \frac{\bar{\delta}}{2},$$

where $\bar{\delta}$ is as in Proposition 2.1

Proof. Using (2.8), we can find $a, b \in \mathbb{Z}$, $a < b < \inf B$,

$$\mathbb{P}\big(\exists x^* \in A \cap [a,b] : x^* \text{ is } \bar{\beta}\text{-insulating}\big) > 1 - \frac{\bar{\delta}}{2}.$$

Hence, with probability larger than $\frac{\bar{\delta}}{2}$, the event in the above probability occurs and moreover, $y^* = \sup A$ is $\bar{\beta}$-insulating. In that case, at time $t^* = (y^* - a)/(2\bar{\beta})$, the descendancy barrier growing from some point x^* in $A \cap [a,b]$ and the one growing from y^* intersect, and it then follows from the definition of descendancy barriers that $\{x : \xi_{t^*}^{A,B}(x) = 2\} = \varnothing$. $\qquad\square$

Lemma 3.4. *If $A, B \subset \mathbb{Z}$ are disjoint sets with* $\#A = \infty$ *and* $\#B < \infty$, *then*

$$\mathbb{P}\big(\exists t : \xi_t^{A,B}(x) \neq 2 \,\forall x\big) = 1.$$

Proof. By symmetry, it suffices to treat the case in which inf $A = -\infty$. The result is an immediate consequence of Lemmas 3.2 and 3.3 and the Markov property. $\qquad\square$

Proof of Theorem 1.3. Since on $\{\zeta_t^A \neq \varnothing \; \forall t\}$ we have $\#\zeta_t^A \xrightarrow{t\to\infty} \infty$ almost surely, the theorem will follow from proving

$$\forall \varepsilon > 0 \; \exists N : \#A \geq N, \#B < \infty \quad \Longrightarrow \quad \mathbb{P}\big(\exists t : \xi_t^{A,B}(x) \neq 2 \; \forall x\big) > 1 - \varepsilon. \tag{3.3}$$

Fix $\varepsilon > 0$. Using (2.8), we choose N such that any subset of $\mathbb{Z}$ with at least N points has at least one $\bar{\beta}$-insulating point with probability larger than $1 - \varepsilon$. Then assume $A, B \subset \mathbb{Z}$ are disjoint sets with $\#A \geq N$ and $\#B < \infty$. We define, for all $x \in \mathbb{Z}$,

$$\tau(x, B) = \inf\big\{t : \zeta_t^{(-\infty,x]} = \zeta_t^{(-\infty,x]\cup B}, \zeta_t^{[x,\infty)} = \zeta_t^{[x,\infty)\cup B}\big\}.$$

By Lemma 3.4, we have $\tau(x, B) < \infty$ almost surely for all x, hence the event

$$\bigcup_{x \in A} \big\{\tau(x, B) < \infty, x \text{ is } \bar{\beta}\text{-insulating}\big\}$$

has probability larger than $1 - \varepsilon$. We now claim that if this event occurs, there exists $t > 0$ such that $\xi_t(x) \neq 2$ for all x. Indeed, assume that x^* is a point of A which is $\bar{\beta}$-insulating and so that $\tau(x^*, B) < \infty$. Assume $(z, 0) \leftrightarrow (y, \tau(x^*, B))$ for some $z \in B$ and $y \in \mathbb{Z}$. If $z \geq x^*$, it follows from $\zeta_{\tau(x^*,B)}^{(-\infty,x^*]} = \zeta_{\tau(x^*,B)}^{(-\infty,x^*]\cup B}$ that $(x^*, 0) \leftrightarrow (y, \tau(x^*, B))$, so $\xi_{\tau(x^*,B)}^{A,B}(y) = 1$. The case $z < x^*$ is treated similarly, so the proof is complete. $\qquad\square$

Corollary 3.1. *For any $\varepsilon > 0$ and any $A \subset \mathbb{Z}$ infinite and bounded from above, there exists $t_0 = t_0(\varepsilon, A)$ such that, for all $t \geq t_0$,*

$$\mathbb{P}\big(R_t^A = R_t^{\mathbb{Z}^*_-}, \zeta_t^A \equiv \zeta_t^{\mathbb{Z}^*_-} \text{ on } \big[R_t^A - \bar{\beta}t, R_t^A\big]\big) > 1 - \varepsilon. \tag{3.4}$$

Proof. Fix $\varepsilon > 0$. Using (2.8), we choose $a \in \mathbb{Z}^*_-$ such that

$$\mathbb{P}\big(\exists x \in A \cap [a, 0] : x \text{ is } \bar{\beta}\text{-insulating}\big) > 1 - \varepsilon/3 \tag{3.5}$$

and then, using (2.4) and Theorem 1.3, we choose t_0 such that, for all $t \geq t_0$,

$$\mathbb{P}\big(R_t^{(-\infty,a]} \geq 0\big) > 1 - \varepsilon/3 \quad \text{and} \tag{3.6}$$

$$\mathbb{P}\big(\zeta_t^{(-\infty,a]} = \zeta_t^{\mathbb{Z}^*_- \cup A}\big) > 1 - \varepsilon/3. \tag{3.7}$$

Now let $t \geq t_0$ and assume the events that appear in (3.5), (3.6) and (3.7) all occur. Fix a $\bar{\beta}$-insulating point $x^* \in A \cap [a, 0]$. We will prove that

$$R_t^{x^*} > 0, \qquad \zeta_t^{\mathbb{Z}^*_- \cup A} \equiv \zeta_t^{x^*} \qquad \text{on } \big[x^* - \bar{\beta}t, \infty\big). \tag{3.8}$$

It is not hard to see that the event described in (3.8) is contained in the event inside the probability in (3.4).

In order to prove (3.8), we start with

$$R_t^{x^*} \overset{(2.9)}{=} R_t^{(-\infty, x^*]} \geq R_t^{(-\infty, a]} > 0.$$

Next, fix $y \geq x^* - \bar{\beta} t$ such that $\zeta_t^{\mathbb{Z}_-^* \cup A}(y) = 1$. Since we assume that the event in (3.7) occurs, there exists $z \leq a$ such that $(z, 0) \leftrightarrow (y, t)$. Since $(z, 0)$ and (y, t) are on opposite sides of the line $\{(x^* - \bar{\beta} s, s) : s \geq 0\}$, we must also have $(x^*, 0) \leftrightarrow (y, t)$. This shows that $\zeta_t^{\mathbb{Z}_-^* \cup A} \cap [x^* - \bar{\beta} t, \infty) \subset \zeta_t^{x^*} \cap [x^* - \bar{\beta} t, \infty)$; the reverse inequality is trivial, so we are done. $\qquad\square$

4. One-type process seen from the right edge: The measure $\underline{\nu}$

In this section, we focus on the (supercritical, one-type) contact process seen from the right edge, $(\tilde{\zeta}_t)_{t \geq 0}$. We will prove the following.

Proposition 4.1. *The supercritical contact process seen from the right edge has a unique stationary distribution $\underline{\nu}$ on $\mathcal{X}_0$. For any $A \in \mathcal{X}_0$, $\tilde{\zeta}_t^A$ converges in distribution to $\underline{\nu}$ as $t \to \infty$.*

Remark 4.1. The measure $\underline{\nu}$ of Proposition 4.1 is obviously also stationary for the GBTI process $(\tilde{\xi}_t)_{t \geq 0}$, and is indeed one of the two measures mentioned in Theorem 1.1.

Remark 4.2. For the case when the process has range $K = 1$, the statement of Proposition 4.1 has already been proved in [7]. Since here we allow for range $K \geq 1$, we give a full proof.

In order to prove Proposition 4.1, we will first need to prove the following lemma.

Lemma 4.1. *Let μ_t denote the distribution of $\tilde{\zeta}_t^{\mathbb{Z}_-^*}$. Then, for any $\varepsilon > 0$ and $k > 0$ there exists $L > 0$ such that*

$$\limsup_{T \to \infty} \frac{1}{T} \int_0^T \mu_t \left(\left\{ \tilde{\zeta} \in \mathcal{X}_0 : \sum_{i=1}^L \tilde{\zeta}(-i) \leq k \right\} \right) dt < \varepsilon. \tag{4.1}$$

Proof. First, note that there exists a constant $c_k > 0$ such that

$$\mu_{t+1} \left(\left\{ \tilde{\zeta} \in \mathcal{X}_0 : \sum_{i=1}^L \tilde{\zeta}(-i) = 0 \right\} \right) \geq c_k \cdot \mu_t \left(\left\{ \tilde{\zeta} \in \mathcal{X}_0 : \sum_{i=1}^L \tilde{\zeta}(-i) \leq k \right\} \right). \tag{4.2}$$

Hence,

$$\limsup_{T \to \infty} \frac{1}{T} \int_0^T \mu_t \left(\left\{ \tilde{\zeta} : \sum_{i=1}^L \tilde{\zeta}(-i) \leq k \right\} \right) dt \leq \frac{1}{c_k} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \mu_t \left(\left\{ \tilde{\zeta} : \sum_{i=1}^L \tilde{\zeta}(-i) = 0 \right\} \right) dt.$$

Noting that

$$\frac{d\mathbb{E}(R_t^{\mathbb{Z}_-^*})}{dt} \leq -L \cdot \mu_t\left(\left\{\tilde{\xi} : \sum_{i=1}^{L} \tilde{\xi}(-i) = 0\right\}\right) + C,$$

where C is constant depending only on the rate λ and range R, we obtain

$$\frac{\mathbb{E}(R_T^{\mathbb{Z}_-^*})}{T} \leq -\frac{L}{T}\int_0^T \mu_t\left(\left\{\tilde{\xi} : \sum_{i=1}^{L} \tilde{\xi}(-i) = 0\right\}\right)dt + C.$$

By (2.3), $\lim_{T\to\infty} \mathbb{E}(R_T^{\mathbb{Z}_-^*})/T = \alpha > 0$, so

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \mu_t\left(\left\{\tilde{\xi} : \sum_{i=1}^{L} \tilde{\xi}(-i) = 0\right\}\right)dt \leq \frac{C}{L}.$$

Therefore the conclusion of the lemma holds for any $L > \frac{C}{c_k \varepsilon}$. □

Proof of Proposition 4.1. For each $n \in \mathbb{Z}_+$, let

$$\mu^n = \frac{1}{n}\int_0^n \mu_t\, dt. \tag{4.3}$$

For each $\varepsilon > 0$, using Lemma 4.1, we can obtain an increasing sequence $(L_k)_{k\in\mathbb{Z}_+}$ such that, setting

$$K = \left\{\tilde{\xi} \in \mathcal{X}_0 : \sum_{i=1}^{L_k} \tilde{\xi}(-i) \geq k \text{ for all } k\right\} \tag{4.4}$$

we have

$$\mu^n(K) > 1 - \varepsilon \qquad \text{for all } n.$$

Noting that K is a compact subset of $\mathcal{X}_0$, this shows that the family $\{\mu^n : n \in \mathbb{Z}_+\}$ is tight. Hence, by Prohorov's theorem (see Section 5 of [3]), there exists an increasing sequence $(n_i)_{i\in\mathbb{Z}_+}$ and a measure $\underline{\nu}$ on $\mathcal{X}_0$ such that $\mu^{(n_i)}$ converges weakly to $\underline{\nu}$ on $\mathcal{X}_0$. Any measure on $\mathcal{X}_0$ obtained as a limit of the measures (4.3) is stationary for $(\tilde{\xi}_t)$; for a proof of this, see Proposition 1.8(e) in [10]. Hence, $\underline{\nu}$ is stationary.

Now, Corollary 3.1 implies that

$$\forall L > 0, \varepsilon > 0, \qquad \lim_{t\to\infty} \underline{\nu}\left(\left\{A \in \mathcal{X}_0 : \mathbb{P}(\tilde{\xi}_t^A \equiv \tilde{\xi}_t^{\mathbb{Z}_-^*} \text{ on } [-L, 0]) > 1 - \varepsilon\right\}\right) = 1. \tag{4.5}$$

This shows that $\tilde{\xi}_t^{\mathbb{Z}_-^*}$ converges in distribution to $\underline{\nu}$. Now, for any $A \in \mathcal{X}_0$, another application of Corollary 3.1 shows that

$$\forall L > 0, \qquad \lim_{t\to\infty} \mathbb{P}(\tilde{\xi}_t^A \equiv \tilde{\xi}_t^{\mathbb{Z}_-^*} \text{ on } [-L, 0]) = 1,$$

so that $\tilde{\xi}_t^A$ converges to $\underline{\nu}$ as well. The uniqueness of $\underline{\nu}$ then readily follows. □

5. Two-type process seen from the interface: The measure $\overline{\nu}$

We now define the second stationary measure mentioned in Theorem 1.1 as the limit of the GBTI process in which, in the initial configuration, the set of 1's is given by $\underline{\nu}$ and every vertex not occupied by a 1 is occupied by a 2.

Proposition 5.1. *Let $(\tilde{\xi}_t^*)_{t\geq 0}$ be the GBTI process with initial distribution*

$$\tilde{\xi}_0^* = \mathbb{1}_A + 2 \cdot \mathbb{1}_{A^c}, \qquad \text{with } A \sim \underline{\nu}.$$

Then, as $t \to \infty$, $\tilde{\xi}_t^*$ converges in distribution to a measure $\overline{\nu}$ on $\mathcal{Y}_0$, which is stationary for the GBTI.

Proof. For each $t \geq 0$, let ν_t denote the distribution of $\tilde{\xi}_t^*$. Since $\{x : \tilde{\xi}_t(x) = 1\} \sim \underline{\nu}$ for every t, it can be shown using sets K similar to the one in (4.4) that $\{\nu_t : t \geq 0\}$ is a tight family of probabilities on $\mathcal{Y}_0$.

Define

$$E(L, A_0, B_0) = \big\{(A, B) \in \mathcal{Y}_0 : A \cap [-L, 0] = A_0, \, B \cap [-L, L] \supset B_0\big\},$$

$$\text{for } L \in \mathbb{Z}_+, \, A_0 \subset [-L, 0], \, B_0 \subset [-L, L], \, A_0 \cap B_0 = \varnothing.$$

We claim that, for all L, A_0, B_0,

$$t \mapsto \nu_t\big(E(L, A_0, B_0)\big) \text{ is decreasing.} \tag{5.1}$$

To see this, fix $r, s \geq 0$. Let $(\tilde{\xi}_t^{**})_{t\geq 0}$ be the GBTI process with initial distribution ν_r. Now, construct $(\tilde{\xi}_t^*)$ and $(\tilde{\xi}_t^{**})$ with the same graphical representation and with initial conditions verifying $\{x : \tilde{\xi}_0^*(x) = 1\} = \{x : \tilde{\xi}_0^{**}(x) = 1\}$. Then,

$$\big\{x : \tilde{\xi}_s^*(x) = 1\big\} = \big\{x : \tilde{\xi}_s^{**}(x) = 1\big\}, \quad \text{and} \quad \big\{x : \tilde{\xi}_s^*(x) = 2\big\} \supseteq \big\{x : \tilde{\xi}_s^{**}(x) = 2\big\}.$$

Since $\tilde{\xi}_s^{**} \stackrel{(d)}{=} \tilde{\xi}_{r+s}^*$, this proves (5.1).

Now, the statement that ν_t converges weakly as $t \to \infty$ is equivalent to the statement that any sequence $(\nu_{t_i})_{i\in\mathbb{Z}_+}$ with (t_i) increasing and $t_i \to \infty$ has a weakly convergent subsequence, and the limiting measure does not depend on the choice of (t_i). With this in mind, fix (t_i). By tightness and Prohorov's theorem, there exists a subsequence $(t_{i_j})_{j\in\mathbb{Z}_+}$ and a probability $\overline{\nu}$ on $\mathcal{Y}_0$ such that $\nu_{t_{i_j}} \to \overline{\nu}$. Additionally, for all L, A_0 and B_0,

$$\overline{\nu}\big(E(L, A_0, B_0)\big) = \lim_{j\to\infty} \nu_{t_{i_j}}\big(E(L, A_0, B_0)\big) \stackrel{(5.1)}{=} \lim_{t\to\infty} \nu_t\big(E(L, A_0, B_0)\big).$$

This and the inclusion-exclusion formula imply that, defining

$$E'(L, A_0, B_0) = \big\{(A, B) \in \mathcal{Y}_0 : A \cap [-L, 0] = A_0, \, B \cap [-L, L] = B_0\big\},$$

$$\text{for } L \in \mathbb{Z}_+, \, A_0 \subset [-L, 0], \, B_0 \subset [-L, L], \, A_0 \cap B_0 = \varnothing,$$

we also have

$$\overline{\nu}\big(E'(L, A_0, B_0)\big) = \lim_{t \to \infty} \nu_t\big(E'(L, A_0, B_0)\big) \qquad \forall L, A_0, B_0.$$

This shows that $\overline{\nu}$ is uniquely determined.

Finally, the fact that $\overline{\nu}$ is stationary follows from Proposition 1.8(d) in [10]. □

6. Convergence to $\underline{\nu}$ and $\overline{\nu}$

6.1. Condition ($\star$) implies convergence to $\overline{\nu}$

Lemma 6.1. *We have*

$$\lim_{M \to \infty} \liminf_{n \to \infty} \mathbb{P}\big(\zeta_{n/\alpha}^{\mathbb{Z}_-^*} \equiv \zeta_{n/\alpha}^{\mathbb{Z}} \text{ on } (-\infty, n - M\sqrt{n}]\big) = 1, \tag{6.1}$$

where α is as in (2.3).

Proof. Using (2.4) and (2.11), given $\varepsilon > 0$ we can choose L and M such that, for n large enough,

$$\mathbb{P}\bigg(R_{n/\alpha}^{\mathbb{Z}_-^*} \geq n - \frac{M}{2}\sqrt{n}\bigg) > 1 - \varepsilon/2,$$

$$\mathbb{P}\big(\zeta_{n/\alpha}^{\mathbb{Z}_-^*} \equiv \zeta_{n/\alpha}^{\mathbb{Z}} \text{ on } \big(-\infty, R_{n/\alpha}^{\mathbb{Z}_-^*} - L\big]\big) > 1 - \varepsilon/2.$$

Additionally assuming that n is large enough that $\frac{M}{2}\sqrt{n} \geq L$, the probability in (6.1) is larger than $1 - \varepsilon$, so we are done. □

Lemma 6.2. *Letting*

$$\mathcal{B}(M, n) = \big\{B \subset \mathbb{Z} : \#\big(B \cap [M\sqrt{n}, 2n - M\sqrt{n}]\big) \geq M\big\}, \tag{6.2}$$

we have

$$\lim_{M \to \infty} \liminf_{n \to \infty} \inf_{B \in \mathcal{B}(M,n)} \mathbb{P}\bigg(\zeta_{n/\alpha}^B \equiv \zeta_{n/\alpha}^{\mathbb{Z}} \text{ on } \bigg[n - \frac{M}{2}\sqrt{n}, n + \frac{M}{2}\sqrt{n}\bigg]\bigg) = 1. \tag{6.3}$$

Proof. Fix $\varepsilon > 0$. By the previous lemma we can choose $M_0 > 0$ and $n_0 > 0$ such that, if $n \geq n_0$, $\mathbb{P}(\zeta_{n/\alpha}^{\mathbb{Z}_-^*} \equiv \zeta_{n/\alpha}^{\mathbb{Z}} \text{ on } (-\infty, n - \frac{M_0}{2}\sqrt{n}]) > 1 - \varepsilon/2$. By (2.8), we can choose $M_1 \geq M_0$ such that any subset of $\mathbb{Z}$ with at least $M_1/2$ points has at least one $\bar{\beta}$-insulating point with probability at least $1 - \varepsilon/2$.

Now let $n \geq n_0$ so that $n \gg M_1\sqrt{n}$ and assume $B \subset \mathbb{Z}$ satisfies $\#(B \cap [M_1\sqrt{n}, 2n - M_1\sqrt{n}]) \geq M_1$. We either have $\#(B \cap [M_1\sqrt{n}, n]) \geq M_1/2$ or $\#(B \cap [n, 2n - M_1\sqrt{n}]) \geq M_1/2$; we can assume that the first case holds, as the other can then be treated by symmetry.

Define the events

$$E_1 = \{\exists x \in B \cap [M_1\sqrt{n}, n] : x \text{ is } \bar{\beta}\text{-insulating}\},$$

$$E_2 = \{\zeta_{n/\alpha}^{(-\infty, M_1\sqrt{n}]} \equiv \zeta_{n/\alpha}^{\mathbb{Z}} \text{ on } (-\infty, n + M_1\sqrt{n}/2]\},$$

so that $\mathbb{P}(E_1 \cap E_2) > 1 - \varepsilon$. Let us prove that $E_1 \cap E_2$ is contained in the event that appears in (6.3). Let $x^* \in B \cap [M_1\sqrt{n}, n]$ be a $\bar{\beta}$-insulating point. Fix $y \in [n - \frac{M_1}{2}\sqrt{n}, n + \frac{M_1}{2}\sqrt{n}]$ with $\zeta_{n/\alpha}^{\mathbb{Z}}(y) = 1$. By the definition of E_2, there exists $z \le M_1\sqrt{n}$ such that $(z, 0) \leftrightarrow (y, n/\alpha)$. Since $(z, 0)$ and $(y, n/\alpha)$ are on opposite sides of the line $\{(x^* - \bar{\beta}s, s) : s \ge 0\}$, we must also have $\zeta_{n/\alpha}^{x^*}(y) = 1$, so $\zeta_{n/\alpha}^{B}(y) = 1$. $\qquad\square$

Lemma 6.3. *For all $\varepsilon > 0$ there exists $M_0 > 0$ such that the following holds. For any $A \subset \mathbb{Z}$ with*

$$\inf A = -\infty, \qquad \sup A = 0,$$

there exists $n_0 = n_0(\varepsilon, A)$ such that, if $n \ge n_0$ and $B \subset \mathbb{Z}$ satisfies

$$B \cap A = \varnothing, \qquad \#\big(B \cap [M_0\sqrt{n}, 2n - M_0\sqrt{n}]\big) \ge M_0,$$

then

$$\mathbb{P}\left(R_{n/\alpha}^{A} = R_{n/\alpha}^{\mathbb{Z}_-^*}, \xi_{n/\alpha}^{A,B} \equiv \xi_{n/\alpha}^{\mathbb{Z}_-^*, \mathbb{Z}_+} \text{ on } \left[R_{n/\alpha}^{A} - \frac{M_0}{4}\sqrt{n}, R_{n/\alpha}^{A} + \frac{M_0}{4}\sqrt{n}\right]\right) > 1 - \varepsilon. \qquad (6.4)$$

Proof. For $M > 0$, $n \in \mathbb{N}$, $A, B \subset \mathbb{Z}$, define the events

$$E_1(M, n) = \left\{R_{n/\alpha}^{\mathbb{Z}_-^*} \in \left[n - \frac{M}{4}\sqrt{n}, n + \frac{M}{4}\sqrt{n}\right]\right\},$$

$$E_2(A, n) = \left\{R_{n/\alpha}^{A} = R_{n/\alpha}^{\mathbb{Z}_-^*}, \zeta_{n/\alpha}^{\mathbb{Z}_-^*} \equiv \zeta_{n/\alpha}^{A} \text{ on } \left(-\bar{\beta} \cdot \frac{n}{\alpha}, \infty\right)\right\},$$

$$E_3(B, M, n) = \left\{\zeta_{n/\alpha}^{\mathbb{Z}} \equiv \zeta_{n/\alpha}^{B} \text{ on } \left[n - \frac{M}{2}\sqrt{n}, n + \frac{M}{2}\sqrt{n}\right]\right\}.$$

Recall the definition of $\mathcal{B}(M, n)$ in (6.2). Given $\varepsilon > 0$, using (2.4) and Lemma 6.2, we choose M_0 such that, for n large enough and any $B \in \mathcal{B}(M_0, n)$ we have

$$\mathbb{P}\big(E_1(M_0, n) \cap E_3(B, M_0, n)\big) > 1 - \varepsilon/2. \qquad (6.5)$$

Now fix $A \subset \mathbb{Z}$ with $\inf A = -\infty$ and $\sup A = 0$. Choose n_0 so that for $n \ge n_0$ the following conditions hold:

$$n > M_0\sqrt{n}, \qquad (6.5) \text{ holds}, \quad \text{and} \quad \mathbb{P}\big(E_2(A, n)\big) > 1 - \varepsilon/2$$

(the third condition is satisfied for n large enough due to Corollary 3.1). Now let us show that, if E_1, E_2 and E_3 all occur, we have

$$\xi_{n/\alpha}^{\mathbb{Z}_-^*,\mathbb{Z}_+} \equiv \xi_{n/\alpha}^{A,B} \qquad \text{on } \left[n - \frac{M_0}{2}\sqrt{n}, n + \frac{M_0}{2}\sqrt{n}\right]; \qquad (6.6)$$

together with $n - \frac{M_0}{4}\sqrt{n} \le R_{n/\alpha}^A = R_{n/\alpha}^{\mathbb{Z}_-^*} \le n + \frac{M_0}{4}\sqrt{n}$, this guarantees that the event inside the probability in (6.4) occurs. Fix $x \in [n - \frac{M_0}{2}\sqrt{n}, n + \frac{M_0}{2}\sqrt{n}]$; there are three possibilities:

1. if $\xi_{n/\alpha}^{\mathbb{Z}_-^*,\mathbb{Z}_+}(x) = 0$, then $\zeta_{n/\alpha}^{\mathbb{Z}}(x) = 0$, so $\xi_{n/\alpha}^{A,B}(x) = 0$;
2. if $\xi_{n/\alpha}^{\mathbb{Z}_-^*,\mathbb{Z}_+}(x) = 1$, then $\zeta_{n/\alpha}^{\mathbb{Z}_-^*}(x) = 1$, so (using the definitions of E_1 and E_2) $\zeta_{n/\alpha}^A(x) = 1$, so $\xi_{n/\alpha}^{A,B}(x) = 1$;
3. if $\xi_{n/\alpha}^{\mathbb{Z}_-^*,\mathbb{Z}_+}(x) = 2$, then $\zeta_{n/\alpha}^{\mathbb{Z}_-^*}(x) = 0$, so (by the definition E_2) $\zeta_{n/\alpha}^A(x) = 0$. $\xi_{n/\alpha}^{\mathbb{Z}_-^*,\mathbb{Z}_+}(x) = 2$ also implies that $\zeta_{n/\alpha}^{\mathbb{Z}}(x) = 1$, so (by the definition of E_3) $\zeta_{n/\alpha}^B(x) = 1$. Therefore, $\xi_{n/\alpha}^{A,B}(x) = 2$. $\qquad\square$

Corollary 6.1. *The process* $(\tilde{\xi}_t^{\mathbb{Z}_-^*,\mathbb{Z}_+})_{t \ge 0}$ *converges in distribution to* $\bar{\nu}$.

Proof. By Lemma 6.3, for any infinite set $A \subset \mathbb{Z}$ with $\sup A = 0$ and any $K > 0$ we have

$$\mathbb{P}\big(\tilde{\xi}_t^{A,A^c} \equiv \tilde{\xi}_t^{\mathbb{Z}_-^*,\mathbb{Z}_+} \text{ on } [-K, K]\big) \xrightarrow{t\to\infty} 1.$$

The statement of the corollary then follows from recalling that

$$\bar{\nu} = \lim_{\substack{t\to\infty \\ (d)}} \tilde{\xi}_t^{A,A^c}, \qquad \text{where } A \sim \underline{\nu}. \qquad\square$$

Proof of Theorem 1.2(a), sufficiency of $(\star)$**.** Assume $(A, B) \in \mathcal{Y}_0$ satisfies condition $(\star)$. Then, for all M we have $\#(B \cap [M\sqrt{n}, 2n - M\sqrt{n}]) \ge M$ if n is large enough. By Lemma 6.3, for any $K > 0$ we have

$$\mathbb{P}\big(\tilde{\xi}_{n/\alpha}^{A,B} \equiv \tilde{\xi}_{n/\alpha}^{\mathbb{Z}_-^*,\mathbb{Z}_+} \text{ on } [-K, K]\big) \xrightarrow{n\to\infty} 1.$$

From this, it is straightforward to show that

$$\mathbb{P}\big(\tilde{\xi}_t^{A,B} \equiv \tilde{\xi}_t^{\mathbb{Z}_-^*,\mathbb{Z}_+} \text{ on } [-K, K]\big) \xrightarrow{t\to\infty} 1.$$

Then, by Corollary 6.1, $\tilde{\xi}_t^{A,B}$ converges to $\bar{\nu}$ in distribution as $t \to \infty$. $\qquad\square$

Corollary 6.2. *The measure* $\overline{\nu}$ *is supported on the set* $\mathcal{Y}_0'$ *defined in* (1.5).

Proof. By Corollary 6.1, for any $L > 0$,

$$\overline{\nu}\big(\{(A, B) : B \cap (-\infty, -L] = \varnothing, B \cap [-L, L] \neq \varnothing\}\big)$$

$$= \lim_{t \to \infty} \mathbb{P}\big(\tilde{\xi}_t^{\mathbb{Z}_-^*, \mathbb{Z}_+}(x) \neq 2 \text{ for all } x \leq -L, \tilde{\xi}_t^{\mathbb{Z}_-^*, \mathbb{Z}_+}(x) = 2 \text{ for some } x \in [-L, L]\big).$$

By (2.10), we can make the right-hand side arbitrarily close to 1 by choosing L large.   □

6.2. Convergence to $\overline{\nu}$ implies condition ($\star$)

Recall the definition of $\mathcal{B}(M, n)$ from (6.2).

Lemma 6.4. *For all $M > 0$,*

$$\liminf_{n \to \infty} \inf_{B \in \mathcal{B}(M,n)^c} \mathbb{P}\big(\zeta_{n/\alpha}^B \equiv 0 \text{ on } [n - 2M\sqrt{n}, n + 2M\sqrt{n}]\big) > 0. \tag{6.7}$$

Proof. It is easy to verify that

$$\liminf_{n \to \infty} \inf_{B \in \mathcal{B}(M,n)^c} \mathbb{P}\big(\zeta_1^B \equiv 0 \text{ on } [M\sqrt{n}, 2n - M\sqrt{n}]\big) > 0.$$

Hence, it is enough to prove (6.7) with $\mathcal{B}(M, N)^c$ replaced by

$$\mathcal{C}(M, n) = \big\{B \subset \mathbb{Z} : B \cap [M\sqrt{n}, 2n - M\sqrt{n}] = \varnothing\big\}.$$

Fix M and n and let $B \in \mathcal{C}(M, n)$. Let $B_1 = (-\infty, M\sqrt{n}] \cap \mathbb{Z}$ and $B_2 = [2n - M\sqrt{n}, \infty) \cap \mathbb{Z}$, so that $B \subset B_1 \cup B_2$. We then have

$$\mathbb{P}\big(\zeta_{n/\alpha}^B \equiv 0 \text{ on } [n - 2M\sqrt{n}, n + 2M\sqrt{n}]\big)$$

$$\geq \mathbb{P}\big(\zeta_{n/\alpha}^{B_1 \cup B_2} \cap [n - 2M\sqrt{n}, n + 2M\sqrt{n}] = \varnothing\big)$$

$$\geq \mathbb{P}\big(R_{n/\alpha}^{B_1} < n - 2M\sqrt{n}\big) \cdot \mathbb{P}\big(L_{n/\alpha}^{B_2} > n + 2M\sqrt{n}\big)$$

$$= \mathbb{P}\big(R_{n/\alpha}^{\mathbb{Z}_-^*} < n - 3M\sqrt{n}\big)^2.$$

We now observe that, by (2.4), for fixed M, the right-hand side is bounded away from zero as $n \to \infty$.   □

Lemma 6.5. *For $M > 0$ sufficiently large there exists $\delta > 0$ such that the following holds for n sufficiently large. If $A, B \subset \mathbb{Z}$ are disjoint sets with*

$$\sup A = 0, \qquad B \in \mathcal{B}(M, n)^c,$$

then

$$\mathbb{P}\big(\xi_{n/\alpha}^{A,B}(x) \neq 2 \ \forall x \in \big[R_{n/\alpha}^A - M\sqrt{n}, R_{n/\alpha}^A + M\sqrt{n}\big]\big) > \delta. \tag{6.8}$$

Proof. Using (2.4), we choose $M > 0$ such that, for n large enough,

$$\mathbb{P}\big(\big|R_{n/\alpha}^{\mathbb{Z}_-^*} - n\big| \leq M\sqrt{n}\big) > 1 - \bar{\delta}/2, \tag{6.9}$$

where $\bar{\delta}$ is as in (2.7). Then choose $\delta > 0$ smaller than the square of the left-hand side of (6.7) and so that

$$\bar{\delta}/2 > \delta^{1/2}. \tag{6.10}$$

Now, fix $n \in \mathbb{N}$ and A, B as in the statement of the claim. Let $(\zeta_t^A)_{t \geq 0}$ and $(\hat{\zeta}_t^B)_{t \geq 0}$ be two independent contact processes started from occupancy in A and B, respectively. By Lemma 2.1, the desired bound (6.8) follows from

$$\mathbb{P}\big(\sup \zeta_{n/\alpha}^A \in [n - M\sqrt{n}, n + M\sqrt{n}]\big) > \delta^{1/2} \quad \text{and} \tag{6.11}$$

$$\mathbb{P}\big(\hat{\zeta}_{n/\alpha}^B \cap [n - 2M\sqrt{n}, n + 2M\sqrt{n}] = \varnothing\big) > \delta^{1/2}. \tag{6.12}$$

Now, (6.12) holds for n large by the choice of δ and (6.7). By (2.9), the left-hand side of (6.11) is larger than

$$\mathbb{P}\big(0 \text{ is } \bar{\beta}\text{-insulating and } \big|R_{n/\alpha}^{\mathbb{Z}_-^*} - n\big| \leq M\sqrt{n}\big)$$

$$\geq \mathbb{P}(0 \text{ is } \bar{\beta}\text{-insulating}) - \mathbb{P}\big(\big|R_{n/\alpha}^{\mathbb{Z}_-^*} - n\big| > M\sqrt{n}\big) \overset{(6.9)}{\geq} \frac{\bar{\delta}}{2} \overset{(6.10)}{\geq} \delta^{1/2}$$

if n is large. $\qquad\square$

Proof of Theorem 1.2(a), necessity of $(\star)$. If $(\star)$ fails, then there exists $M > 0$ and an increasing sequence $(n_k)_{k \geq 0}$ such that, for all k, $\#\{x \in [M\sqrt{n}, 2n - M\sqrt{n}] : \tilde{\xi}_0(x) = 2\} \leq M$. Lemma 6.5 then implies that

$$\liminf_{k \to \infty} \mathbb{P}\big(\tilde{\xi}_{n_k/\alpha}(x) \neq 2 \; \forall x \in [-M\sqrt{n_k}, M\sqrt{n_k}]\big) > 0.$$

Together with Corollary 6.2, this shows that $(\tilde{\xi}_t)$ cannot converge to $\bar{\nu}$. $\qquad\square$

6.3. Condition $(\ast)$ implies convergence to $\underline{\nu}$

Proof of Theorem 1.2(b), sufficiency of $(\ast)$. Fix disjoint sets $A, B \subset \mathbb{Z}$ with

$$\inf A = -\infty, \qquad \sup A = 0, \qquad \sup B < \infty.$$

By Proposition 4.1,

$$\big(\{x : \tilde{\xi}_t^{A,B}(x) = 1\}\big)_{t \geq 0} \xrightarrow[(d)]{t \to \infty} \underline{\nu},$$

so the fact that $\tilde{\xi}_t^{A,B}$ also converges in distribution to $\underline{v}$ will follow from

$$\forall L > 0, \qquad \lim_{t \to \infty} \mathbb{P}\big(\xi_t^{A,B}(x) \neq 2 \; \forall x \in \big[R_t^A - L, R_t^A + L\big]\big) = 1. \tag{6.13}$$

Letting $D = (-\infty, \sup B] \cap \mathbb{Z}$, it is straightforward to check that

$$\big\{\xi_t^{A,B}(x) \neq 2 \; \forall x \in \big[R_t^A - L, R_t^A + L\big]\big\} \supseteq \big\{R_t^A = R_t^D, \zeta_t^A \equiv \zeta_t^D \text{ on } \big[R_t^A - L, R_t^A\big]\big\},$$

so (6.13) follows from Corollary 3.1 and translation invariance. $\qquad\square$

6.4. Convergence to $\underline{v}$ implies condition ($*$)

Proof of Theorem 1.2(b), necessity of ($*$). Fix disjoint sets A, B satisfying

$$\inf A = -\infty, \qquad \sup A = 0, \qquad \sup B = \infty;$$

let us show that $\tilde{\xi}_t^{A,B}$ does not converge to $\underline{v}$ in distribution as $t \to \infty$.

Fix an increasing sequence $(n_k)_{k \geq 0}$ such that $n_k \in B$ for each k. Using (2.7), (2.4) and Corollary 3.1, we can first choose $M > 0$ and then $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$,

$$\mathbb{P}\big(R_{n_k/\alpha}^{\mathbb{Z}_-^*} \in [n_k - M\sqrt{n_k}, n_k + M\sqrt{n_k}]\big) > 1 - \bar{\delta}/4,$$

$$\mathbb{P}\big(\zeta_{n_k/\alpha}^{n_k} \equiv \zeta_{n_k/\alpha}^{\mathbb{Z}} \text{ on } [n_k - \bar{\beta}n_k/\alpha, n_k + \bar{\beta}n_k/\alpha]\big) > \bar{\delta},$$

$$\mathbb{P}\big(R_{n_k/\alpha}^{A} = R_{n_k/\alpha}^{\mathbb{Z}_-^*}, \zeta_{n_k/\alpha}^{A} \equiv \zeta_{n_k/\alpha}^{\mathbb{Z}_-^*} \text{ on } \big[-\bar{\beta}n_k/\alpha, R_{n_k/\alpha}^{A}\big]\big) > 1 - \bar{\delta}/4,$$

so the probability that the three above events all occur is at least $\bar{\delta}/2$. Similarly to the proof of Lemma 6.3, we can show that if these three events occur, we have

$$\tilde{\xi}_{n_k/\alpha}^{A,B} \equiv \tilde{\xi}_{n_k/\alpha}^{\mathbb{Z}_-^*, \mathbb{Z}_+} \qquad \text{on } [-M\sqrt{n_k}, M\sqrt{n_k}].$$

We then have, for any $L > 0$,

$$\liminf_{k \to \infty} \mathbb{P}\big(\exists x \in [-L, L] : \tilde{\xi}_{n_k/\alpha}^{A,B}(x) = 2\big) \geq \frac{\bar{\delta}}{2} - \limsup_{k \to \infty} \mathbb{P}\big(\tilde{\xi}_{n_k/\alpha}^{\mathbb{Z}_-^*, \mathbb{Z}_+}(x) \neq 2 \; \forall x \in [-L, L]\big);$$

by (2.10), the second term on the right-hand side can be made arbitrarily close to zero if L is taken large enough. $\qquad\square$

7. $\underline{v}$ and $\bar{v}$ are the only extremal stationary measures of GBTI

In this section, we will prove Theorem 1.1. The fact that $\underline{v}$ is supported on $\mathcal{X}_0$ is obvious and the fact that $\bar{v}$ is supported on the set given in (1.5) was proved in Corollary 6.2. It will follow from Lemma 7.1 below that, if v is a stationary and extremal measure for the GBTI process

$(\tilde{\xi}_t)_{t\geq 0}$, then either $\nu = \underline{\nu}$ or $\nu = \overline{\nu}$. It will be useful to consider measures ν in $\mathcal{Y}_0$ which satisfy the following property:

$$\forall M > 0, \qquad \lim_{n\to\infty} \nu\big\{(A, B) : \#\big(B \cap [M\sqrt{n}, 2n - M\sqrt{n}]\big) \geq M\big\} = 1. \qquad (\maltese)$$

Lemma 7.1. *Let ν be a stationary measure for $(\tilde{\xi}_t)_{t\geq 0}$.*

1. *If ν is extremal, then $\nu(\mathcal{X}_0) \in \{0, 1\}$.*
2. *If $\nu(\mathcal{X}_0) = 1$, then $\nu = \underline{\nu}$.*
3. *If ν does not satisfy $(\maltese)$, then $\nu(\mathcal{X}_0) > 0$.*
4. *If ν satisfies $(\maltese)$, then $\nu = \overline{\nu}$.*

Proof.

1. Assume ν is stationary and $\nu(\mathcal{X}_0) \in (0, 1)$; let us show that ν is not extremal. We can write

$$\nu = \nu(\mathcal{X}_0) \cdot \hat{\nu} + \big(1 - \nu(\mathcal{X}_0)\big) \cdot \hat{\hat{\nu}}, \qquad (7.1)$$

where $\hat{\nu}(\cdot) = \nu(\cdot | \mathcal{X}_0)$ and $\hat{\hat{\nu}}(\cdot) = \nu(\cdot | \mathcal{X}_0^c)$. For $t \geq 0$, let $\hat{\nu}_t$ and $\hat{\hat{\nu}}_t$ denote the distribution of $\tilde{\xi}_t$ when $\tilde{\xi}_0$ is distributed as $\hat{\nu}$ and $\hat{\hat{\nu}}$, respectively. Since ν is stationary, we have

$$\nu = \nu(\mathcal{X}_0) \cdot \hat{\nu}_t + \big(1 - \nu(\mathcal{X}_0)\big) \cdot \hat{\hat{\nu}}_t. \qquad (7.2)$$

We evidently have $\hat{\nu}(\mathcal{X}_0) = \hat{\nu}_t(\mathcal{X}_0) = 1$ and $\hat{\hat{\nu}}(\mathcal{X}_0) = 0$; using these facts and also $\nu(\mathcal{X}_0) \in (0, 1)$ in equations (7.1) and (7.2), we obtain $\hat{\hat{\nu}}_t(\mathcal{X}_0) = 0$. This implies that, if E is any measurable subset of $\mathcal{X}_0$, we have $\hat{\hat{\nu}}(E) = \hat{\hat{\nu}}_t(E) = 0$, so (7.1) and (7.2) yield $\hat{\nu}(E) = \hat{\nu}_t(E)$. This shows that $\hat{\nu} = \hat{\nu}_t$, that is, $\hat{\nu}$ is stationary, from which it follows that $\hat{\hat{\nu}}$ is also stationary. Hence, ν is not extremal.
2. This is an immediate consequence of Proposition 4.1.
3. If ν does not satisfy $(\maltese)$, then there exist $M > 0$, $\kappa > 0$ and a sequence $(n_k)_{k\geq 0}$ such that, for all k,

$$\nu\big\{(A, B) : \#\big(B \cap [M\sqrt{n_k}, 2n_k - M\sqrt{n_k}]\big) < M\big\} > \kappa.$$

We now choose $\delta = \delta(M)$ and $n_0 = n_0(M)$ as in Lemma 6.5. Then, taking $n_k \geq n$ shows that

$$\nu\big\{(A, B) : \mathbb{P}\big(\xi_{n_k/\alpha}^{A,B}(x) \neq 2 \; \forall x \in \big[R_{n_k/\alpha}^A - M\sqrt{n_k}, R_{n_k/\alpha}^A + M\sqrt{n_k}\big]\big) > \delta\big\} > \kappa$$

or equivalently, if $(\xi_t)_{t\geq 0}$ is a grass-bushes-trees process with initial distribution ν, then with probability larger than $\delta \cdot \kappa$, at time n_k there is no 2 within distance $M\sqrt{n_k}$ of the rightmost 1. Since ν is stationary, this shows that for any $L > 0$ we have

$$\nu\big\{(A, B) : B \cap [-L, L] = \varnothing\big\} > \delta \cdot \kappa,$$

so $\nu(\mathcal{X}_0) \geq \delta \cdot \kappa > 0$.

4. Assume (❖) holds. Given $\varepsilon > 0$ and $K > 0$, using Lemma 6.3, we can find $M > 0$ and then $n \in \mathbb{N}$ so that

$$\nu\big(\big\{(A, B) : \mathbb{P}\big(\tilde{\xi}_{n/\alpha}^{A,B} \equiv \tilde{\xi}_{n/\alpha}^{\mathbb{Z}^*_-, \mathbb{Z}_+} \text{ on } [-K, K]\big) > 1 - \varepsilon\big\}\big) > 1 - \varepsilon.$$

The result now follows from Corollary 6.1.

$\square$

Acknowledgements

References

[1] Andjel, E., Miller, J. and Pardoux, E. (2010). Survival of a single mutant in one dimension. *Electron. J. Probab.* **15** 386–408.

[2] Andjel, E., Mountford, T., Pimentel, L.P. and Valesin, D. (2010). Tightness for the interface of the one-dimensional contact process. *Bernoulli* **16** 909–925.

[3] Billingsley, P. (2013). *Convergence of Probability Measures*. New York: Wiley.

[4] Cox, J.T. and Durrett, R. (1995). Hybrid zones and voter model interfaces. *Bernoulli* **1** 343–370. MR1369166

[5] Durrett, R. and Møller, A.M. (1991). Complete convergence theorem for a competition model. *Probab. Theory Related Fields* **88** 121–136. MR1094080

[6] Durrett, R. and Swindle, G. (1991). Are there bushes in a forest? *Stochastic Process. Appl.* **37** 19–31. MR1091691

[7] Galves, A. and Presutti, E. (1987). Edge fluctuations for the one dimensional supercritical contact process. *Ann. Probab.* 1131–1145.

[8] Harris, T.E. (1974). Contact interactions on a lattice. *Ann. Probab.* 969–988.

[9] Kuczek, T. (1989). The central limit theorem for the right edge of supercritical oriented percolation. *Ann. Probab.* **17** 1322–1332. MR1048929

[10] Liggett, T.M. (2005). *Interacting Particle Systems*. *Classics in Mathematics*. Berlin: Springer. Reprint of the 1985 original. MR2108619

[11] Liggett, T.M. (2013). *Stochastic Interacting Systems*: *Contact*, *Voter and Exclusion Processes* **324**. Berlin: Springer.

[12] Mountford, T. and Valesin, D. (2016). Functional central limit theorem for the interface of the symmetric multitype contact process. *ALEA Lat. Am. J. Probab. Math. Stat.* **13** 481–519. MR3519255

[13] Mountford, T.S. and Sweet, T.D. (2000). An extension of Kuczek's argument to nonnearest neighbor contact processes. *J. Theoret. Probab.* **13** 1061–1081. MR1820502

[14] Neuhauser, C. (1992). Ergodic theorems for the multitype contact process. *Probab. Theory Related Fields* **91** 467–506.

[15] Valesin, D. (2010). Multitype contact process on $\mathbb{Z}$: Extinction and interface. *Electron. J. Probab.* **15** 2220–2260. MR2748404