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Quantile spectral processes: Asymptotic analysis and inference

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Quantile- and copula-related spectral concepts recently have been considered by various authors. Those spectra, in their most general form, provide a full characterization of the copulas associated with the pairs (X_t, X_{t-k}) in a process $(X_t)_{t\in\mathbb{Z}}$, and account for important dynamic features, such as changes in the conditional shape (skewness, kurtosis), time-irreversibility, or dependence in the extremes that their traditional counterparts cannot capture. Despite various proposals for estimation strategies, only quite incomplete asymptotic distributional results are available so far for the proposed estimators, which constitutes an important obstacle for their practical application. In this paper, we provide a detailed asymptotic analysis of a class of smoothed rank-based cross-periodograms associated with the *copula spectral density kernels* introduced in Dette *et al.* [*Bernoulli* **21** (2015) 781–831]. We show that, for a very general class of (possibly nonlinear) processes, properly scaled and centered smoothed versions of those cross-periodograms, indexed by couples of quantile levels, converge weakly, as stochastic processes, to Gaussian processes. A first application of those results is the construction of asymptotic confidence intervals for *copula spectral density kernels*. The same convergence results also provide asymptotic distributions (under serially dependent observations) for a new class of rank-based spectral methods involving the Fourier transforms of rank-based serial statistics such as the *Spearman, Blomqvist* or *Gini autocovariance coefficients*.

Keywords: Blomqvist; copulas; Gini spectra; periodogram; quantiles; ranks; Spearman; spectral analysis; time series

1. Introduction

Spectral analysis and frequency domain methods play a central role in the nonparametric analysis of time series data. The classical frequency domain representation is based on the *spectral density* – call it the L^2-*spectral density* in order to distinguish it from other *spectral densities* to be defined in the sequel – which is traditionally defined as the Fourier transform of the autocovariance function of the process under study. Fundamental tools for the estimation of spectral densities are the *periodogram* and its *smoothed* versions. The classical periodogram – similarly call it the L^2-*periodogram* – can be defined either as the discrete Fourier transform of empirical autocovariances, or through L^2-projections of the observed series on a harmonic basis. The success of periodograms in time series analysis is rooted in their fast and simple computation (through the *fast Fourier transform* algorithm) and their interpretation in terms of cyclic behavior, both of a stochastic and of deterministic nature, which in specific applications are more illuminating than time-domain representations. L^2-periodograms are particularly attractive in the analysis of Gaussian time series, since the distribution of a Gaussian process is completely characterized by its spectral density. Classical references are Priestley [46], Brillinger [4] or Chapters 4 and 10 of Brockwell and Davis [5].

Being intrinsically connected to means and covariances, the L^2-spectral density and L^2-periodogram inherit the nice features (such as optimality properties in the analysis of Gaussian series) of L^2-methods, but also their weaknesses: they are lacking robustness against outliers and heavy tails, and are unable to capture important dynamic features such as changes in the conditional shape (skewness, kurtosis), time-irreversibility, or dependence in the extremes. This was realized by many researchers, and various extensions and modifications of the L^2-periodogram have been proposed to remedy those drawbacks.

Robust nonparametric approaches to frequency domain estimation have been considered first; see Kleiner, Martin and Thomson [30] for an early contribution, and Chapter 8 of Maronna, Martin and Yohai [43] for an overview. More recently, Klüppelberg and Mikosch [34] proposed a weighted ("self-normalized") version of the periodogram; see also Mikosch [44]. Hill and McCloskey [25] used a robust version of autocovariances and a robustified periodogram with the goal to obtain L^2-spectrum-based parameter estimates that are robust to heavy-tailed data. In the context of signal detection, Katkovnik [28] introduced a periodogram based on robust loss functions. The objective of all those attempts is a robustification of classical tools: they essentially aim at protecting existing L^2-spectral methods against the impact of possible outliers or violations of distributional assumptions.

Other attempts, more recent and somewhat less developed, are introducing alternative spectral concepts and tools, mostly related with quantiles or copulas, and accounting for more general dynamic features. A first step in that direction was taken by Hong [26], who proposes a generalized spectral density with covariances replaced by joint characteristic functions. In the specific problem of testing pairwise independence, Hong [27] introduces a test statistic based on the Fourier transforms of (empirical) joint distribution functions and copulas at different lags. Recently, there has been a renewed surge of interest in that type of concept, with the introduction, under the names of *Laplace-, quantile-* and *copula* spectral density and spectral density *kernels*, of various quantile-related spectral concepts, along with the corresponding sample-based periodograms and smoothed periodograms. That strand of literature includes Li [38–40], Hagemann [19], Dette *et al.* [12] and Lee and Rao [37]. A Fourier analysis of extreme events, which is related in spirit but quite different in many other respects, was considered by Davis, Mikosch and Zhao [11]. Finally, in the time domain, Linton and Whang [41], Davis and Mikosch [10] and Han *et al.* [24] introduced the related concepts of *quantilograms* and *extremograms*. A more detailed account of some of those contributions is given in Section 2.

A deep understanding of the distributional properties of any statistical tool is crucial for its successful application. The construction of confidence intervals, testing procedures and efficient estimators all rest on results concerning finite-sample or asymptotic properties of related statistics – here the appropriate (smoothed) periodograms associated with the quantile-related spectral density under study. Obtaining such asymptotic results, unfortunately, is not trivial, and to the best of our knowledge, no results on the asymptotic distribution of the aforementioned (smoothed) quantile and copula periodograms are available so far.

In the case of i.i.d. observations, Hong [27] derived the asymptotic distribution of an empirical version of the integrated version of his quantile spectral density, while Lee and Rao [37] investigated the distributions of Cramér–von Mises-type statistics based on empirical joint distributions. No results on the asymptotic distribution of the periodogram itself are given, though. Li [38,39] does not consider asymptotics for smoothed versions of his quantile periodograms, while the asymptotic results in Hagemann [19] and Dette *et al.* [12] are quite incomplete. This is perhaps not so surprising: the asymptotic distribution of classical L^2-spectral density estimators for general nonlinear processes also has remained an active domain of research for several decades; see Brillinger [4] for early results, Shao and Wu [50], Liu and Wu [42] or Giraitis and Koul [18] for more recent references.

The present paper has two major objectives. First, it aims at providing a rigorous analysis of the asymptotic properties of a general class of smoothed rank-based copula cross-periodograms generalizing the quantile periodograms introduced by Hagemann [19] and, in an integrated version, by Hong [27]. In Section 3, we show that, for general nonlinear time series, properly centered and smoothed versions of those cross-periodograms, indexed by couples of quantile levels, converge in distribution to centered Gaussian processes. A first application of those results is the construction of asymptotic confidence intervals which we discuss in detail in Section 5.

The second objective of this paper is to introduce a new class of rank-based frequency domain methods that can be described as a non-standard rank-based Fourier analysis of the serial features of time series. Examples of such methods are discussed in detail in Section 4, where we study a class of spectral densities, such as the *Spearman*, *Blomqvist and Gini spectra*, and the corresponding periodograms, associated with rank-based autocovariance concepts. Denoting by F the marginal distribution function of X_t, the *Spearman spectral density*, for instance, is defined as $\sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\omega k}\rho_k^{\mathrm{Sp}}$, where $\rho_k^{\mathrm{Sp}} := \mathrm{Corr}(F(X_t), F(X_{t-k}))$ denotes the lag-k *Spearman autocorrelation*. We show that estimators of those spectral densities can be obtained as functionals of the rank-based copula periodograms investigated in this paper. This connection, and our process-level convergence results on the rank-based copula periodograms, allow us to establish the asymptotic normality of the smoothed versions of the newly defined rank-based periodograms. Those results can be considered as frequency domain versions of Hájek's celebrated asymptotic representation and normality results for (non-serial) linear rank statistics under non-i.i.d. observations (Hájek [20]).

The paper is organized as follows. Section 2 provides precise definitions of the spectral concepts to be considered throughout, and motivates the use of our quantile-related methods by a graphical comparison of the copula spectra of white noise, QAR(1) and ARCH(1) processes, respectively – all of which share the same helplessly flat L^2-spectral density. Section 3 is devoted to the asymptotics of rank-based copula (cross-)periodograms and their smoothed versions, presenting the main results of this paper: the convergence, for fixed frequencies ω, of the smoothed copula rank-based periodogram indexed by couples of (τ_1, τ_2) of quantile orders to a Gaussian process (Theorem 3.5). That theorem is based on an equally interesting asymptotic representation result (Theorem 3.6). Section 4 deals with the relation with Spearman, Gini, and Blomqvist autocorrelation coefficients and the related spectra. Based on a short Monte-Carlo study, Section 5 discusses the practical performances of the methods proposed, and Section 6 provides some conclusions and directions for future research. Proofs are concentrated in an Appendix and an the online supplement [33].

2. Copula spectral density kernels and rank-based periodograms

In this section, we provide more precise definitions of the various quantile- and copula-related spectra mentioned in the Introduction, along with the corresponding periodograms.

Throughout, let $(X_t)_{t\in\mathbb{Z}}$ denote a strictly stationary process, of which we observe a finite stretch $X_0, \ldots, X_{n-1}$, say. Denote by F the marginal distribution function of X_t, and by $q_\tau := \inf\{x \in \mathbb{R}\colon \tau \le F(x)\}$, $\tau \in [0, 1]$ the corresponding quantile function, where we use the convention $\inf\varnothing = \infty$. Note that if $\tau \in \{0, 1\}$ then $-\infty$ and ∞ are possible values for q_τ. Our main object of interest is the *copula spectral density kernel*

$$\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) := \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}\omega k}\gamma_k^U(\tau_1, \tau_2), \qquad \omega \in \mathbb{R}, (\tau_1, \tau_2) \in [0, 1]^2, \tag{2.1}$$

based on the *copula cross-covariances*

$$\gamma_k^U(\tau_1, \tau_2) := \operatorname{Cov}\bigl(I\{U_t \le \tau_1\}, I\{U_{t-k} \le \tau_2\}\bigr), \qquad k \in \mathbb{Z},$$

where $U_t := F(X_t)$. Those copula spectral density kernels were introduced in Dette *et al.* [12], and generalize the τ*th quantile spectral densities* of Hagemann [19], with which they coincide for $\tau_1 = \tau_2 = \tau$; an integrated version actually was first considered by Hong [27]. The same copula spectral density kernel also takes the form

$$\begin{aligned}&\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega)\\ &\quad := \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}\omega k}\bigl(\mathbb{P}(X_k \le q_{\tau_1}, X_0 \le q_{\tau_2}) - \tau_1\tau_2\bigr), \qquad \omega \in \mathbb{R}, (\tau_1, \tau_2) \in [0, 1]^2,\end{aligned} \tag{2.2}$$

where $\mathbb{P}(X_k \le q_{\tau_1}, X_0 \le q_{\tau_2})$ is the joint distribution function of the pair (X_k, X_0) taken at (q_{τ_1}, q_{τ_2}). This is, by definition, the copula of the pair (X_k, X_0) evaluated at (τ_1, τ_2), while $\tau_1\tau_2$ is the independence copula evaluated at the same (τ_1, τ_2). The copula spectral density kernel thus can be interpreted as the Fourier transform of the difference between pairwise copulas at lag k and the independence copula, which justifies the notation and the terminology.

As shown by Dette *et al.* [12], the copula spectral densities provide a complete description of the pairwise copulas of a time series. Similar to the regression setting, where joint distributions and quantiles provide more information than covariances and means, the copula spectral density kernel, by accounting for much more than the covariance structure of a series, extends and supplements the classical L^2-spectral density.

As an illustration, the L^2-spectra and copula spectral densities are shown in Figures 1 and 2, respectively, for three different processes: (a) a Gaussian white noise process, (b) a QAR(1) process (Koenker and Xiao [36]) and (c) an ARCH(1) process [the same processes are also considered in the simulations of Section 5]. All processes were standardized so that the marginal distributions have unit variance. Although their dynamics obviously are quite different, those three processes are uncorrelated, and thus all exhibit the same flat L^2-spectrum. This very clearly appears in Figure 1. In Figure 2, the copula spectral densities associated with various values

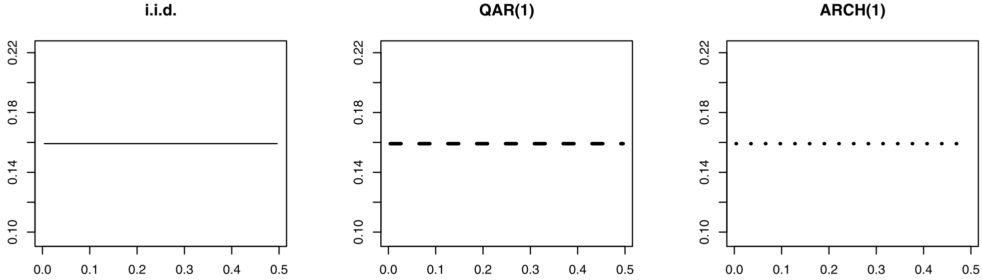

Figure 1. Traditional L^2-spectra $(2\pi)^{-1}\sum_{k\in\mathbb{Z}}\operatorname{Cov}(Y_{t+k}, Y_t)\mathrm{e}^{-\mathrm{i}\omega k}$. The process (Y_t) in the left-hand picture is independent standard normal white noise; in the middle picture, $Y_t = X_t/\operatorname{Var}(X_t)^{1/2}$ where (X_t) is QAR(1) as defined in (5.1); in the right-hand picture, $Y_t = X_t/\operatorname{Var}(X_t)^{1/2}$ where (X_t) is the ARCH(1) process defined in (5.3). All curves are plotted against $\omega/(2\pi)$.

of τ_1 and τ_2 are shown for the same processes. Obviously, the three copula spectral densities differ considerably from each other and, therefore, provide a much richer information about the dynamics of those three processes.

For an interpretation of Figure 2, recall (2.1) and (2.2), and note that

$$\begin{aligned}
-\gamma_k^U(\tau_1,\tau_2) &= -\operatorname{Cov}\bigl(I\{U_t \le \tau_1\}, I\{U_{t-k} \le \tau_2\}\bigr)\\
&= \operatorname{Cov}\bigl(I\{U_t \le \tau_1\}, I\{U_{t-k} > \tau_2\}\bigr)\\
&= \mathbb{P}(X_t \le q_{\tau_1}, X_{t-k} > q_{\tau_2}) - \tau_1(1-\tau_2)\\
&= \mathbb{P}(X_t > q_{\tau_1}, X_{t-k} \le q_{\tau_2}) - (1-\tau_1)\tau_2.
\end{aligned}$$

Hence, $\gamma_k^U(\tau_1,\tau_2)$ is the probability for $\{X_t\}$ to switch from the upper τ_2 tail to the lower τ_1 tail in k steps, minus the corresponding probability for white noise, which is also the probability for $\{X_t\}$ to switch from the lower τ_2 tail to the upper τ_1 tail in k steps, minus the corresponding probability for white noise.

Copula spectral density kernels, as represented in Figure 2, thus provide information on those quantile-crossing, or tail-switching probabilities. In particular, a non-vanishing imaginary part for $\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega)$ indicates that $\mathbb{P}(X_t \le q_{\tau_1}, X_{t-k} \le q_{\tau_2})$, for some values of k, differs from $\mathbb{P}(X_t \le q_{\tau_1}, X_{t+k} \le q_{\tau_2})$, which implies that $\{X_t\}$ is not time-revertible. Figure 2, where imaginary parts are depicted above the diagonal, clearly indicates that the QAR(1) process is not time-revertible.

Note that, in order to distinguish between the ARCH(1) and the i.i.d. process, it is common practice to compare the L_2-spectral densities of the squared processes. This approach can also be used for the QAR(1) process, but is bound to miss important features. For example, the asymmetric nature of QAR(1) dynamics, revealed, for example, by the difference between its $(0.1, 0.1)$ (top left panel) and $(0.9, 0.9)$ (bottom right) spectra cannot be detected in the L^2-spectrum of a squared QAR(1) process.

For a more detailed discussion of the advantages of the copula spectrum compared to the classical one, see Hong [27], Dette *et al.* [12], Hagemann [19] and Lee and Rao [37].

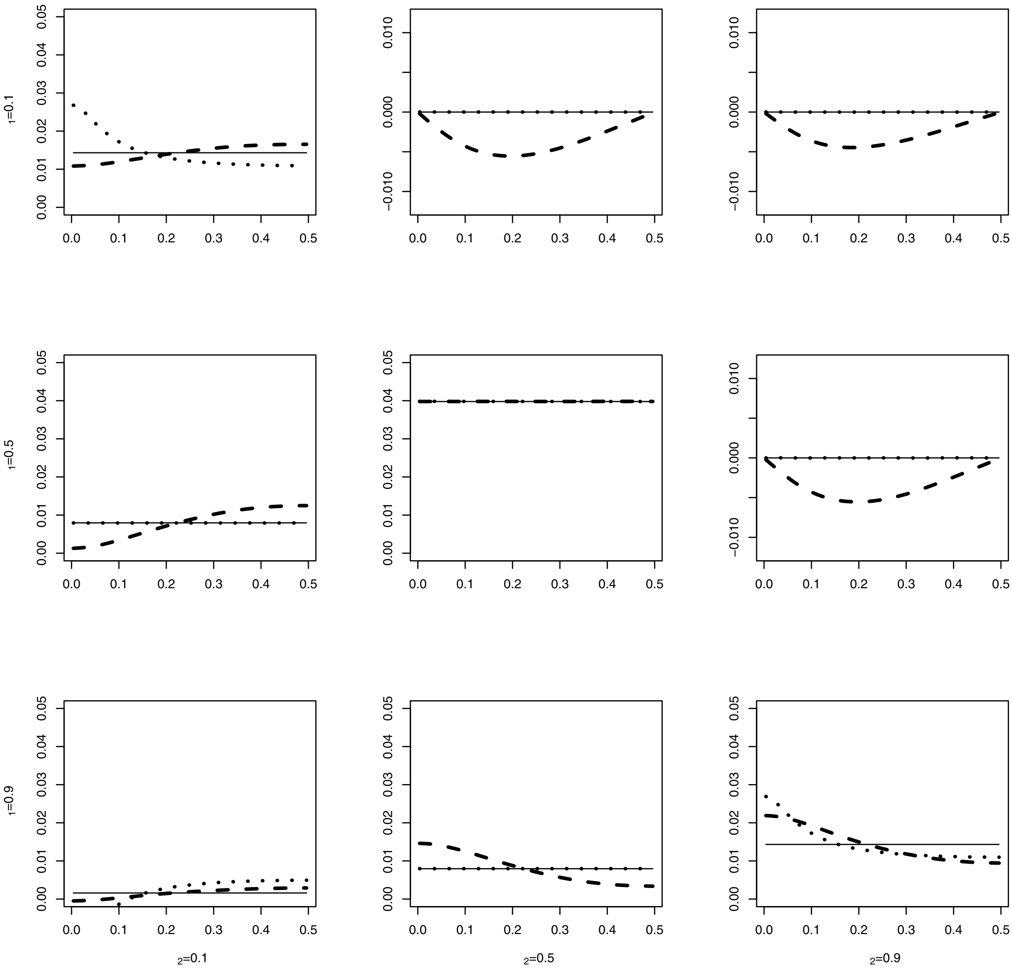

Figure 2. Copula spectra $(2\pi)^{-1}\sum_{k\in\mathbb{Z}}\mathrm{Cov}(I\{F(Y_{t+k})\le\tau_1\}, I\{F(Y_t)\le\tau_2\})\mathrm{e}^{-\mathrm{i}\omega k}$ for $\tau_1,\tau_2 = 0.1, 0.5,$ and 0.9. Real parts (imaginary parts) are shown in sub-figures with $\tau_2\le\tau_1$ ($\tau_2>\tau_1$). Solid, dashed, and dotted lines correspond to the white noise, QAR(1) and ARCH(1) processes in Figure 1. All curves are plotted against $\omega/(2\pi)$.

Consistent estimation of $\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega)$ was independently considered in Hagemann [19] for the special case $\tau_1=\tau_2\in(0,1)$, and by Dette *et al.* [12] for general couples $(\tau_1,\tau_2)\in(0,1)^2$ of quantile levels, under different assumptions such as $m(n)$-decomposa bility and β-mixing.

Hagemann's estimator, called the *τth quantile periodogram*, is a traditional L^2-periodogram where observations are replaced with the indicators

$$I\{\hat{F}_n(X_t)\le\tau\} = I\{R_{n;t}\le n\tau\},$$

where $\hat{F}_n(x) := n^{-1}\sum_{t=0}^{n-1} I\{X_t \le x\}$ denotes the empirical marginal distribution function and $R_{n;t}$ the rank of X_t among $X_0, \dots, X_{n-1}$. Dette *et al.* [12] introduce their Laplace rank-based periodograms by substituting an L^1-approach for the L^2 one, and considering the cross-periodograms associated with arbitrary couples (τ_1, τ_2) of quantile levels. See Remark 2.1 for details.

In this paper, we stick to the L^2-approach, but extend Hagemann's concept by considering, as in Dette *et al.* [12], the cross-periodograms associated with arbitrary couples (τ_1, τ_2). More precisely, we define the *rank-based copula periodogram* $I_{n,R}$, shortly, the *CR-periodogram* as the collection

$$I_{n,R}^{\tau_1,\tau_2}(\omega) := \frac{1}{2\pi n} d_{n,R}^{\tau_1}(\omega) d_{n,R}^{\tau_2}(-\omega), \qquad \omega \in \mathbb{R}, (\tau_1, \tau_2) \in [0,1]^2, \tag{2.3}$$

with

$$d_{n,R}^{\tau}(\omega) := \sum_{t=0}^{n-1} I\{\hat{F}_n(X_t) \le \tau\} \mathrm{e}^{-\mathrm{i}\omega t} = \sum_{t=0}^{n-1} I\{R_{n;t} \le n\tau\} \mathrm{e}^{-\mathrm{i}\omega t}.$$

Those cross-periodograms, as well as Hagemann's τth quantile periodograms, are measurable functions of the marginal ranks $R_{n;t}$, whence the terminology and the notation.

Classical periodograms and rank-based Laplace periodograms converge, as $n \to \infty$, to random variables whose expectations are the corresponding spectral densities; but they fail estimating those spectral densities in a consistent way. Similarly, the CR-periodogram $I_{n,R}^{\tau_1,\tau_2}(\omega)$ fails to estimate $\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega)$ consistently. More precisely, let $\rightsquigarrow$ denote the *Hoffman–Jørgensen convergence*, namely, the weak convergence in the space of bounded functions $[0,1]^2 \to \mathbb{C}$, which we denote by $\ell_{\mathbb{C}}^{\infty}([0,1]^2)$. Note that results in empirical process theory are typically stated for spaces of real-valued, bounded functions; see Chapter 1 of van der Vaart and Wellner [54]. By identifying $\ell_{\mathbb{C}}^{\infty}([0,1]^2)$ with the product space $\ell^{\infty}([0,1]^2) \times \ell^{\infty}([0,1]^2)$ these results transfer immediately. We show (see Proposition 3.4 for details) that, under suitable assumptions, for any fixed frequencies $\omega \ne 0 \mod 2\pi$,

$$\big(I_{n,R}^{\tau_1,\tau_2}(\omega)\big)_{(\tau_1,\tau_2)\in[0,1]^2} \rightsquigarrow \big(\mathbb{I}(\tau_1,\tau_2;\omega)\big)_{(\tau_1,\tau_2)\in[0,1]^2} \qquad \text{as } n \to \infty,$$

where the limiting process $\mathbb{I}$ is such that

$$\mathbb{E}\big[\mathbb{I}(\tau_1,\tau_2;\omega)\big] = \mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) \qquad \text{for all } (\tau_1,\tau_2) \in [0,1]^2 \text{ and } \omega \ne 0 \mod 2\pi$$

and $\mathbb{I}(\tau_1, \tau_2; \omega_1)$ and $\mathbb{I}(\tau_3, \tau_4; \omega_2)$ are independent for any $\tau_1, \dots, \tau_4$ as soon as $\omega_1 \ne \omega_2$.

In view of this asymptotic independence at different frequencies, it seems natural to consider smoothed versions of $I_{n,R}^{\tau_1,\tau_2}(\omega)$, namely, for $(\tau_1, \tau_2) \in [0,1]^2$ and $\omega \in \mathbb{R}$, averages of the form

$$\hat{G}_{n,R}(\tau_1,\tau_2;\omega) := \frac{2\pi}{n} \sum_{s=1}^{n-1} W_n(\omega - 2\pi s/n) I_{n,R}^{\tau_1,\tau_2}(2\pi s/n), \tag{2.4}$$

where W_n denotes a sequence of weighting functions. For the special case $\tau_1 = \tau_2$, the consistency of a closely related estimator is established by Hagemann [19]. However, even for $\tau_1 = \tau_2$, obtaining the asymptotic distribution of *smoothed CR-periodograms* of the form (2.4) is not trivial, and so far has remained an open problem. Similarly, Dette *et al.* [12] do not provide any results on the asymptotic distributions of their (smoothed) *Laplace rank-based periodograms*. Note that even consistency results in Hagemann [19], as well as in Dette *et al.* [12] are only pointwise in τ_1, τ_2.

In the present paper, we fill that gap. Theorem 3.5 below does not only provide pointwise asymptotic distributions for smoothed CR-periodograms, but also describes the asymptotic behavior of a properly centered and rescaled version of the full collection $\{\hat{G}_{n,R}(\tau_1, \tau_2; \omega), (\tau_1, \tau_2) \in [0,1]^2\}$ as a sequence of *stochastic processes*. Such convergence results (process convergence rather than pointwise) are of particular importance, as they can be used to obtain the asymptotic distribution of functionals of smoothed CR-periodograms as estimators of functionals of the corresponding copula spectral density kernel. As an example, we derive, in Section 4, the asymptotic distributions of periodograms computed from various rank-based autocorrelation concepts (Spearman, Gini, Blomqvist, etc.).

In the process of analyzing the asymptotic behavior of $\{\hat{G}_{n,R}(\tau_1, \tau_2; \omega)\}$, we establish several intermediate results of independent interest. For instance, we prove an asymptotic representation theorem (Theorem 3.6(i)), where we show that, uniformly in $\tau_1, \tau_2 \in [0,1]^2, \omega \in \mathbb{R}$, the smoothed periodogram $\hat{G}_{n,R}(\tau_1, \tau_2; \omega)$ can be approximated by

$$\hat{G}_{n,U}(\tau_1, \tau_2; \omega) := \frac{2\pi}{n} \sum_{s=1}^{n-1} W_n(\omega - 2\pi s/n) I_{n,U}^{\tau_1,\tau_2}(2\pi s/n), \tag{2.5}$$

where

$$I_{n,U}^{\tau_1,\tau_2}(\omega) := \frac{1}{2\pi} \frac{1}{n} d_{n,U}^{\tau_1}(\omega) d_{n,U}^{\tau_2}(-\omega), \tag{2.6}$$

and

$$d_{n,U}^{\tau}(\omega) := \sum_{t=0}^{n-1} I\{U_t \le \tau\} \mathrm{e}^{-\mathrm{i}\omega t} \qquad \text{with } U_t := F(X_t).$$

We conclude this section with two remarks clarifying the relation between the approach considered here that of Dette *et al.* [12], and some other copula-based approaches in the analysis of time series.

Remark 2.1. The classical L^2-periodogram of a real-valued time series can be represented in two distinct ways, providing two distinct interpretations. First, it can be defined as the Fourier transform of the empirical autocovariance function. More precisely, considering the empirical autocovariance

$$\hat{\gamma}_k := \frac{1}{n-k} \sum_{t=1}^{n-k} (X_{t+k} - \bar{X})(X_t - \bar{X}), \qquad k \ge 0, \hat{\gamma}_k := \hat{\gamma}_{-k}, k < 0,$$

the classical L^2-periodogram can be defined as

$$I_n(\omega) := \frac{1}{2\pi} \sum_{|k|<n} \frac{n-k}{n} \hat{\gamma}_k \mathrm{e}^{-\mathrm{i}k\omega}. \tag{2.7}$$

However, an alternative definition is

$$I_n(\omega) := \frac{1}{2\pi}\frac{1}{n}\left|\sum_{t=0}^{n-1} X_t \mathrm{e}^{-\mathrm{i}t\omega}\right|^2 = \frac{n}{4}\big(\hat{b}_1^2 + \hat{b}_2^2\big), \tag{2.8}$$

where b_1, b_2 are the coefficients of the L^2-projection of the observations $X_0, \dots, X_{n-1}$ on the basis $(1, \sin(\omega t), \cos(\omega t))$, that is,

$$(\hat{a}, \hat{b}_1, \hat{b}_2) = \operatorname*{Argmin}_{(a,b_1,b_2)' \in \mathbb{R}^3} \sum_{t=0}^{n-1}\big(X_t - a - b_1\cos(\omega t) - b_2 \sin(\omega t)\big)^2. \tag{2.9}$$

This suggests two different starting points for generalization. We either can replace autocovariances in (2.7) by alternative measures of dependence such as (empirical) joint distributions or copulas, or consider alternative loss functions in the minimization step (2.9). Replacing the autocovariance function by the pairwise copula with $\tau_1 = \tau_2 = \tau$ yields the τ-quantile periodogram of Hagemann [19], which we also consider here, under the name of CR-periodogram, albeit for general $(\tau_1, \tau_2) \in [0, 1]^2$. Replacing the quadratic loss in (2.9) was, in a time series context, first considered by Li [38,39] and Dette *et al.* [12], who observed that substituting the check function $\rho_\tau(x) = x(\tau - I\{x < 0\})$ of Koenker and Bassett [35] for the standard L^2-loss leads to an estimator for the quantity

$$\tilde{\mathfrak{f}}_{\tau,\tau}(\omega) := \frac{1}{2\pi f^2(q_\tau)} \sum_{k\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}\omega k}\big(\mathbb{P}(X_0 \le q_\tau, X_{-k} \le q_\tau) - \tau^2\big).$$

This latter expression is a weighted version of the copula spectral density kernel at $\tau_1 = \tau_2 = \tau$ introduced in (2.2). This weighting, which involves $f(q_\tau)$, is undesirable, since it involves the unknown marginal distribution of X_t, which is unrelated with its dynamics. Dette *et al.* [12] demonstrate that, by considering ranks instead of the original data, that weighting can be removed. The same authors also proposed a generalization to cross-periodograms associated with distinct quantile levels. See Li [39], Dette *et al.* [12] and Hagemann [19] for details and discussion.

Remark 2.2. The benefits of considering joint distributions and copulas as measures of serial dependence in a nonparametric time-domain analysis of time series has been realized by many authors. Skaug and Tjøstheim [51] and Hong [26] used joint distribution functions to test for serial independence at given lag. Subsequently, related approaches were taken by many authors, and an overview of related results can be found in Tjøstheim [53] and Hong [26]. Copula-based tests of serial independence were considered by Genest and Rémillard [17], among others. Linton and Whang [41] introduced the so-called *quantilogram*, defined as the autocorrelation of the series of indicators $I\{X_t \le \hat{q}_\tau\}$, $t = 0, \dots, n-1$, where $\hat{q}_\tau$ denotes the empirical τ-quantile; they discuss the application of this quantilogram (closely related to Hagemann's τ-quantile periodogram) to measuring directional predictability of time series. They do not, however, enter into any spectral considerations. An extension of those concepts to the dependence between several time series was recently considered in Han *et al.* [24]. Finally, Davis and Mikosch [10] also considered a related quantity which is based on autocorrelations of indicators of extreme events.

3. Asymptotic properties of rank-based copula periodograms

The derivation of the asymptotic properties of CR-periodograms requires some assumptions on the underlying process and the weighting functions W_n.

Recall that the *rth order joint cumulant* $\mathrm{cum}(\zeta_1, \ldots, \zeta_r)$ of the random vector $(\zeta_1, \ldots, \zeta_r)$ is defined as

$$\mathrm{cum}(\zeta_1, \ldots, \zeta_r) := \sum_{\{\nu_1,\ldots,\nu_p\}} (-1)^{p-1}(p-1)! \Bigl(\mathbb{E} \prod_{j\in\nu_1} \zeta_j\Bigr) \cdots \Bigl(\mathbb{E} \prod_{j\in\nu_p} \zeta_j\Bigr),$$

with summation extending over all partitions $\{\nu_1, \ldots, \nu_p\}$, $p = 1, \ldots, r$ of $\{1, \ldots, r\}$ (cf. Brillinger [4], page 19).

The assumption we make on the dependence structure of the process $(X_t)_{t\in\mathbb{Z}}$ is as follows. Its relation to more classical assumptions of weak dependence is discussed in Propositions 3.1 and 3.2 below, and in Lemma 3.3.

(C) *There exist constants* $\rho \in (0, 1)$ *and* $K < \infty$ *such that*, *for arbitrary intervals* $A_1, \ldots, A_p \subset \mathbb{R}$ *and arbitrary* $t_1, \ldots, t_p \in \mathbb{Z}$,

$$\bigl|\mathrm{cum}\bigl(I\{X_{t_1} \in A_1\}, \ldots, I\{X_{t_p} \in A_p\}\bigr)\bigr| \le K\rho^{\max_{i,j} |t_i - t_j|}. \tag{3.1}$$

The crucial point here is that we replace an assumption on the cumulants of the original observations by an assumption on the cumulants of certain indicators. Thus, in contrast to classical assumptions, condition (C) does not require the existence of any moments. Additionally, note that the sets A_j in (3.1) only need to be intervals, not general Borel sets as in classical mixing assumptions.

Proposition 3.1. *Assume that the process* $(X_t)_{t\in\mathbb{Z}}$ *is strictly stationary and exponentially* α*-mixing*, *that is*,

$$\alpha(n) := \sup_{\substack{A\in\sigma(X_0,X_{-1},\ldots)\\ B\in\sigma(X_n,X_{n+1},\ldots)}} \bigl|\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)\bigr| \le K\kappa^n, \qquad n \in \mathbb{N} \tag{3.2}$$

for some $K < \infty$ *and* $\kappa \in (0, 1)$. *Then assumption* (C) *holds.*

While mixing assumptions are very general and intuitively interpretable, which makes them quite attractive from a probabilistic point of view, verifying conditions such as (3.1) or (3.2) can be difficult in specific applications. An alternative description of dependence that is often easier to check for was recently proposed by Wu and Shao [56]. More precisely, these authors assume that the process $(X_t)_{t\in\mathbb{Z}}$ can be represented as

$$X_t = g(\ldots, \varepsilon_{t-2}, \varepsilon_{t-1}, \varepsilon_t), \qquad t \in \mathbb{N}, \tag{3.3}$$

where g denotes some measurable function and $(\varepsilon_t)_{t\in\mathbb{Z}}$ is a collection of i.i.d. random variables. Note that the function g is not assumed to be linear, which makes this kind of process very general. To quantify the long-run dependence between $(\ldots, X_{-1}, X_0)$, and $(X_t, X_{t+1}, \ldots)$, denote by $(\varepsilon_t^*)_{t\leq 0}$ an independent copy of $(\varepsilon_t)_{t\leq 0}$ and define

$$X_t^* := g\big(\ldots,\varepsilon_{-1}^*,\varepsilon_0^*,\varepsilon_1,\ldots,\varepsilon_t\big), \qquad t\in\mathbb{N}.$$

The process $(X_t)_{t\in\mathbb{Z}}$ satisfies a *geometric moment contraction of order a* property (shortly, GMC(a) throughout this paper) if, for some $K<\infty$ and $\sigma\in(0,1)$,

$$\mathbb{E}\big|X_n - X_n^*\big|^a \leq K\sigma^n, \qquad n\in\mathbb{N}; \tag{3.4}$$

see Wu and Shao [56]. Examples of processes that satisfy this condition include, (possibly, under mild additional conditions on the parameters) ARMA, ARCH, GARCH, asymmetric GARCH, random coefficient autoregressive, quantile autoregressive and Markov models, to name just a few. Proofs and additional examples can be found in Shao and Wu [50] and Shao [49]. The definition in (3.4) still requires the existence of moments, which is quite undesirable in our setting. However, the following result shows that a modified version of (3.4) is sufficient for our purposes.

Proposition 3.2. *Assume that the strictly stationary process $(X_t)_{t\in\mathbb{Z}}$ can be represented as in* (3.3), *and that X_0 has distribution function F. Let the process $(F(X_t))_{t\in\mathbb{Z}}$ satisfy GMC(a) for some $a>0$, that is, assume that there exist $K<\infty$ and $\sigma\in(0,1)$ such that*

$$\mathbb{E}\big|F(X_n) - F\big(X_n^*\big)\big|^a \leq K\sigma^n, \qquad n\in\mathbb{N}. \tag{3.5}$$

Then assumption (C) *holds.*

The important difference between assumptions (3.4) and (3.5) lies in the fact that, in condition (3.5), only the random variables $F(X_t)=U_t$, which possess moments of arbitrary order, appear. This implies that a GMC(a) condition on X_t with arbitrarily small values of a, together with a very mild regularity condition on F, are sufficient to imply assumption (C). More precisely, we have the following result.

Lemma 3.3. *Assume that $(X_t)_{t\in\mathbb{Z}}$ is strictly stationary. Let $(X_t)_{t\in\mathbb{Z}}$ satisfy the GMC(b) condition for some $b>0$, and assume that the distribution function F of X_0 is Hölder-continuous of order $\gamma>0$. Then* (3.5) *holds for any $a>0$.*

For a proof of Lemma 3.3, note that

$$\mathbb{E}\big|F(X_t) - F\big(X_t^*\big)\big|^a \leq 2^{a-b/\gamma}\mathbb{E}\big|F(X_t) - F\big(X_t^*\big)\big|^{b/\gamma} \leq C\mathbb{E}\big|X_t - X_t^*\big|^b \leq CK\sigma^t,$$

where $\sigma\in(0,1)$ and $K>0$ are the constants from the GMC(b) condition.

Remark 3.1. Although not very deep at first sight, the above result has some remarkable implications. In particular, we show in the Appendix that, under a very mild regularity condition on F, the copula spectra of a GMC(a) process are analytical functions of the frequency ω. This is in sharp contrast with classical spectral density analysis, where higher-order moments are required to obtain smoothness of the spectral density.

We now are ready to state a first result on the asymptotic properties of the CR-periodogram $I_{n,R}^{\tau_1,\tau_2}$ defined in (2.3).

Proposition 3.4. *Assume that F is continuous and that $(X_t)_{t\in\mathbb{Z}}$ is strictly stationary and satisfies assumption* (C). *Then, for every fixed $\omega \neq 0 \mod 2\pi$,*

$$\big(I_{n,R}^{\tau_1,\tau_2}(\omega)\big)_{(\tau_1,\tau_2)\in[0,1]^2} \rightsquigarrow \big(\mathbb{I}(\tau_1,\tau_2;\omega)\big)_{(\tau_1,\tau_2)\in[0,1]^2} \qquad \text{in } \ell_{\mathbb{C}}^{\infty}\big([0,1]^2\big).$$

The (*complex-valued*) *limiting processes $\mathbb{I}$, indexed by $(\tau_1,\tau_2)\in[0,1]^2$, are of the form*

$$\mathbb{I}(\tau_1,\tau_2;\omega) = \frac{1}{2\pi}\mathbb{D}(\tau_1;\omega)\overline{\mathbb{D}(\tau_2;\omega)}$$

with $\mathbb{D}(\tau;\omega) = \mathbb{C}(\tau;\omega) + \mathrm{i}\mathbb{S}(\tau;\omega)$ where $\mathbb{C}$ and $\mathbb{S}$ denote two centered jointly Gaussian processes. For $\omega\in\mathbb{R}$, their covariance structure takes the form

$$\mathbb{E}\big[\big(\mathbb{C}(\tau_1;\omega),\mathbb{S}(\tau_1;\omega)\big)'\big(\mathbb{C}(\tau_2;\omega),\mathbb{S}(\tau_2;\omega)\big)\big] = \pi\begin{pmatrix}\Re\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) & -\Im\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega)\\ \Im\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) & \Re\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega)\end{pmatrix}.$$

Moreover, $\mathbb{D}(\tau;\omega) = \mathbb{D}(\tau;\omega+2\pi) = \overline{\mathbb{D}(\tau;-\omega)}$, and the family $\{\mathbb{D}(\cdot;\omega) : \omega\in[0,\pi]\}$ is a collection of independent processes.

Note that, for $\omega = 0 \mod 2\pi$ we have $d_{n,R}^{\tau}(0) = n\tau + o_P(n^{1/2})$, where the exact form of the remainder term depends on the number of ties in the observations. Therefore, under the assumptions of Proposition 3.4, $I_{n,R}^{\tau_1,\tau_2}(0) = n(2\pi)^{-1}\tau_1\tau_2 + o_P(1)$ for $\omega = 0 \mod 2\pi$.

Remark 3.2. Proposition 3.4 implies that the CR-periodograms corresponding to different frequencies are asymptotically independent. Therefore, it can be used to obtain the asymptotic distribution of the smoothed periodogram for any $b_n = m/n$, with $m\in\mathbb{N}$ *not depending on n*. In this case, m CR-periodograms are used for smoothing and the asymptotic distributions of the smoothed CR-periodograms follow from Proposition 3.4. However, in this case, the smoothed periodogram is not necessarily a consistent estimator (its variance does not tend to zero) of the spectral density kernel $\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega)$. As we shall see, in order to be consistent, a smoothed CR-periodogram requires $nb_n = m(n)\to\infty$ as $n\to\infty$.

In order to establish the convergence of the smoothed CR-periodogram process (2.4), we require the weights W_n in (2.4) to satisfy the following assumption, which is quite standard in classical time series analysis (see, e.g., page 147 of Brillinger [4]).

(W) The weight function W is real-valued and even, with support $[-\pi,\pi]$; moreover, it has bounded variation, and satisfies $\int_{-\pi}^{\pi} W(u)\,\mathrm{d}u = 1$.

Denoting by $b_n > 0$, $n = 1,2,\ldots,$ a sequence of scaling parameters such that $b_n\to 0$ and $nb_n\to\infty$ as $n\to\infty$, define

$$W_n(u) := \sum_{j=-\infty}^{\infty} b_n^{-1} W\big(b_n^{-1}[u+2\pi j]\big).$$

We now are ready to state our main result.

Theorem 3.5. *Let assumptions* (C) *and* (W) *hold. Assume that* X_0 *has a continuous distribution function* F *and that there exist constants* $\kappa > 0$ *and* $k \in \mathbb{N}$, *such that*

$$b_n = \mathrm{o}\bigl(n^{-1/(2k+1)}\bigr) \quad \textit{and} \quad b_n n^{1-\kappa} \to \infty.$$

Then, for any fixed $\omega \in \mathbb{R}$, *the process*

$$\mathbb{G}_n(\cdot,\cdot;\omega) := \sqrt{nb_n}\bigl(\hat{G}_{n,R}(\tau_1,\tau_2;\omega) - \mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) - B_n^{(k)}(\tau_1,\tau_2;\omega)\bigr)_{\tau_1,\tau_2\in[0,1]}$$

satisfies

$$\mathbb{G}_n(\cdot,\cdot;\omega) \rightsquigarrow H(\cdot,\cdot;\omega) \tag{3.6}$$

in $\ell^\infty_{\mathbb{C}}([0,1]^2)$, *where the bias* $B_n^{(k)}$ *is given by*

$$B_n^{(k)}(\tau_1,\tau_2;\omega) := \sum_{j=2}^{k} \frac{b_n^j}{j!}\int_{-\pi}^{\pi} v^j W(v)\,\mathrm{d}v \frac{\mathrm{d}^j}{\mathrm{d}\omega^j}\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega), \qquad \omega \in \mathbb{R}, \tag{3.7}$$

and $\mathfrak{f}_{q_{\tau_1},q_{\tau_2}}$ *is defined in* (2.2). *The process* $H(\cdot,\cdot;\omega)$ *in* (3.6) *is a centered Gaussian process characterized by*

$$\begin{aligned}
&\operatorname{Cov}\bigl(H(u_1,v_1;\omega), H(u_2,v_2;\omega)\bigr)\\
&\quad = 2\pi\left(\int_{-\pi}^{\pi} W^2(w)\,\mathrm{d}w\right)\bigl(\mathfrak{f}_{q_{u_1},q_{u_2}}(\omega)\mathfrak{f}_{q_{v_2},q_{v_1}}(\omega) + \mathfrak{f}_{q_{u_1},q_{v_2}}(\omega)\mathfrak{f}_{q_{v_1},q_{u_2}}(\omega)I\{\omega = 0 \mod \pi\}\bigr).
\end{aligned}$$

Moreover, $H(\omega) = H(\omega + 2\pi) = \overline{H(-\omega)}$, *and the family* $\{H(\omega), \omega \in [0,\pi]\}$ *is a collection of independent processes. In particular, the weak convergence* (3.6) *holds jointly for any finite fixed collection of frequencies* ω.

Remark 3.3. Assume that W is a kernel of order d, that is, $\int_{-\pi}^{\pi} v^j W(v)\,\mathrm{d}v = 0$, for $j < d$ and $0 < \int_{-\pi}^{\pi} v^d W(v)\,\mathrm{d}v < \infty$. The Epanechnikov kernel, for example, is of order 2. Then, for $\omega \neq 0 \mod 2\pi$, the bias is of order b_n^d. Since the variance is of order $(nb_n)^{-1}$, the mean squared error will be minimized when b_n decays at rate $n^{-1/(2d+1)}$. Therefore, for kernels of finite order, the optimal bandwidth fulfils the assumptions of Theorem 3.5.

Remark 3.4. Theorem 3.5 can be used to conduct asymptotic inference in various ways. An important example is the construction of asymptotic confidence intervals. For illustration purposes, consider the case $\tau_1 = \tau_2 = \tau$. Assume that W is a kernel of order d, and that the bandwidth b_n is chosen such that $b_n = \mathrm{o}(n^{-1/(2d+1)})$. In this case, the bias is of order b_n^d, and thus is asymptotically negligible compared to the variance. An asymptotic confidence interval thus can be constructed by using Theorem 3.5 to obtain the approximation

$$\sqrt{nb_n}\bigl(\hat{G}_{n,R}(\tau,\tau;\omega) - \mathfrak{f}_{q_\tau,q_\tau}(\omega)\bigr) \approx \mathcal{N}\bigl(0,\sigma^2\bigr) \quad \text{for } \sigma^2 = 2\pi\int W^2(u)\,\mathrm{d}u\,\mathfrak{f}^2_{q_\tau,q_\tau}(\omega)$$

and estimating σ^2 by plugging in $\hat{G}_{n,R}(\tau, \tau; \omega)$ as an estimator for $\mathfrak{f}_{q_\tau,q_\tau}(\omega)$. A more detailed discussion of confidence interval construction that also includes the case where $\tau_1 \neq \tau_2$ is deferred to Section 5.

Remark 3.5. Process convergence with respect to the frequencies ω cannot hold since the limiting processes are independent for different values of ω. This implies that there exists no tight random element in $\ell^\infty_{\mathbb{C}}([0, 1]^2 \times [0, \pi])$ with the right finite-dimensional distributions, as would be required for process convergence in $\ell^\infty_{\mathbb{C}}([0, 1]^2 \times [0, \pi])$ to take place. Note that a similar situation occurs for the classical L^2-spectral density which does not converge as a process when indexed by frequencies.

For fixed quantile levels τ_1, τ_2, the asymptotic distribution of $\mathbb{G}_n(\tau_1, \tau_2; \omega)$ is the same as the distribution of the smoothed L^2-cross-periodogram (see Chapter 7 of Brillinger [4]) corresponding to the (unobservable) bivariate time series

$$\big(I\big\{F(X_t) \le \tau_1\big\}, I\big\{F(X_t) \le \tau_2\big\}\big)_{0\le t\le n-1}.$$

In particular, the estimation of the marginal quantiles has no impact on the asymptotic distribution of $\mathbb{G}_n$. Intuitively, this can be explained by the fact that $(\hat{q}_{\tau_1}, \hat{q}_{\tau_2})$ converges at $n^{-1/2}$ rate while the normalization $\sqrt{nb_n}$ appearing in $\mathbb{G}_n$ is strictly slower.

One aspect of Theorem 3.5 that does not appear in the context of classical spectral density estimation is the convergence of $\mathbb{G}_n$ as a process. Establishing this result is challenging, and it requires the development of new tools. On the other hand, once convergence has been established at process level, it can be applied to derive the asymptotic distributions of various related statistics; see Section 4.

Remark 3.6. In the derivation of Theorem 3.5, it would be natural to show that $d^\tau_{n,R}(\omega)$ and $d^\tau_{n,U}(\omega)$ are sufficiently close to each other uniformly with respect to τ and ω, as $n \to \infty$. Indeed, using modifications of standard arguments from empirical process theory, it is possible to establish that

$$n^{-1/2} \sup_{\substack{\omega\in\mathbb{R}\\ \tau\in[0,1]}} \big|d^\tau_{n,R}(\omega) - d^\tau_{n,U}(\omega)\big| = \mathrm{o}_P(r_n) \tag{3.8}$$

for some rate $r_n \to 0$ depending on the underlying dependence structure. Unfortunately, the best rate r_n that can theoretically be obtained *must* be slower than $\mathrm{o}(n^{-1/4})$, and this makes the approximation (3.8) useless in establishing Theorem 3.5 for practically relevant choices of the bandwidth parameter.

Remark 3.7. Another type of process convergence is frequently discussed in the literature on classical L^2-based spectral analysis, which is dealing with *empirical spectral processes* of the form

$$\left(\int_{-\pi}^{\pi} g(\omega) I_n(\omega)\, \mathrm{d}\omega\right)_{g\in\mathcal{G}}$$

with $\mathcal{G}$ denoting a suitable class of functions. For more details, see Dahlhaus [8], Dahlhaus and Polonik [9], and the references therein. Those processes are completely different from the processes considered above, and the mathematical tools that need to be developed for their analysis also differ substantially. It would be very interesting to extend our results to classes of integrated periodograms that are indexed by classes of functions. Such an extension, however, is beyond of the scope of the present paper.

Remark 3.8. At first glance, it seems surprising that the asymptotic theory developed here does not require the marginal distribution function F to have a continuous Lebesgue density, although the CR-periodograms in (2.3) are based on marginal quantiles. The reason is that the estimators which are constructed from $X_0, \dots, X_{n-1}$ are almost surely equal to estimators based on the (unobserved) transformed variables $F(X_0), \dots, F(X_{n-1})$. A similar phenomenon can be observed in the estimation of copulas; see, for example, Fermanian, Radulović and Wegkamp [14].

In order to establish Theorem 3.5, we prove (an *asymptotic representation* result) that the estimator $\hat{G}_{n,R}$ can be approximated by $\hat{G}_{n,U}$ in a suitable uniform sense. Theorem 3.5 then follows from the asymptotic properties of $\hat{G}_{n,U}$, which we state now.

Theorem 3.6. *Let assumptions* (C) *and* (W) *hold, and assume that the distribution function F of X_0 is continuous. Let b_n satisfy the assumptions of Theorem* 3.5. *Then*,

(i) *for any fixed $\omega \in \mathbb{R}$, as $n \to \infty$,*

$$\sqrt{nb_n}\big(\hat{G}_{n,U}(\tau_1, \tau_2; \omega) - \mathbb{E}\hat{G}_{n,U}(\tau_1, \tau_2; \omega)\big)_{\tau_1,\tau_2\in[0,1]} \rightsquigarrow H(\cdot, \cdot; \omega)$$

in $\ell^{\infty}_{\mathbb{C}}([0,1]^2)$, where the process $H(\cdot,\cdot;\omega)$ is defined in Theorem 3.5;

(ii) *still as $n \to \infty$,*

$$\sup_{\substack{\tau_1,\tau_2\in[0,1]\\ \omega\in\mathbb{R}}} \big|\mathbb{E}\hat{G}_{n,U}(\tau_1, \tau_2; \omega) - \mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) - B_n^{(k)}(\tau_1, \tau_2, \omega)\big| = \mathrm{O}\big((nb_n)^{-1}\big) + \mathrm{o}\big(b_n^k\big),$$

where $B_n^{(k)}$ is defined in (3.7);

(iii) *for any fixed $\omega \in \mathbb{R}$,*

$$\sup_{\tau_1,\tau_2\in[0,1]} \big|\hat{G}_{n,R}(\tau_1, \tau_2; \omega) - \hat{G}_{n,U}(\tau_1, \tau_2; \omega)\big| = o_P\big((nb_n)^{-1/2} + b_n^k\big);$$

if moreover the kernel W is uniformly Lipschitz-continuous, this bound is uniform with respect to $\omega \in \mathbb{R}$.

4. Spearman, Blomqvist and Gini spectra

In the past decades, considerable effort has been put into replacing empirical autocovariances by alternative (scalar) measures of dependence; see, for example, Kendall [29], Blomqvist [3], Cifarelli, Conti and Regazzini [7], Ferguson, Genest and Hallin [13] and Schmid *et al.* [48] for a recent survey. Such measures of association provide a good compromise between the limited information contained in autocovariances on one hand, and the fully nonparametric nature of joint distributions and copulas on the other.

A particularly appealing class of such dependence measures is given by general rank-based autocorrelations (see Hallin and Puri [22,23] or Hallin [21] for a survey). The idea of using ranks in a time-series context is not new. In fact, it is possible to trace back rank-based measures of serial dependence to the early developments of rank-based inference: early examples include run statistics or the serial version of Spearman's rho (see Wald and Wolfowitz [55]). The asymptotics of rank-based autocorrelations are well studied under the assumption of white noise or, at least, *exchangeability*, and under contiguous alternatives of serial dependence. An alternative approach to deriving the asymptotic distribution of rank-based autocorrelations, which is applicable under general kinds of dependence, is based on their representation as functionals of (weighted) empirical copula processes and was considered, for instance, in Fermanian, Radulović and Wegkamp [14], Berghaus, Bücher and Volgushev [2].

Despite the great success of the L^2-periodogram in time series analysis, the only attempt to consider Fourier transforms of rank-based autocorrelations (or any other rank-based scalar measures of dependence), to the best of our knowledge, is that of Ahdesmäki *et al.* [1]. The latter paper is of a more empirical nature, and no theoretical foundation is provided. The aim of the present section is to introduce a general class of frequency domain methods, and discuss their connection to rank-based extensions of autocovariances.

4.1. The Spearman periodogram

To illustrate our purpose, first consider in detail the classical example of Spearman's rank autocorrelation coefficients (more precisely, a version of it – see Remark 4.1); at lag k, that coefficient can be defined as

$$\hat{\rho}_n^k := \frac{12}{n^3} \sum_{t=0}^{n-|k|-1} \left(R_{n;t} - \frac{n+1}{2}\right)\left(R_{n;t+|k|} - \frac{n+1}{2}\right).$$

Letting $\mathcal{F}_n := \{2\pi j/n | j = 1, \dots, \lfloor \frac{n-1}{2} \rfloor - 1, \lfloor \frac{n-1}{2} \rfloor\}$, define the *Spearman* and *smoothed Spearman periodograms* as

$$I_{n,\rho}(\omega) := \frac{1}{2\pi} \sum_{|k|<n} \mathrm{e}^{-\mathrm{i}\omega k} \hat{\rho}_n^k, \qquad \omega \in \mathcal{F}_n$$

and

$$\hat{G}_{n,\rho}(\omega) := \frac{2\pi}{n} \sum_{s=1}^{n-1} W_n(\omega - 2\pi s/n) I_{n,\rho}(2\pi s/n), \qquad \omega \in \mathbb{R},$$

respectively. Intuition suggests that the (smoothed) rank-based periodogram $\hat{G}_{n,\rho}$ should be an estimator for the Fourier transform

$$\mathfrak{f}_\rho(\omega) := \frac{1}{2\pi} \frac{1}{12} \sum_{k \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}\omega k} \rho_k$$

of the population counterpart

$$\rho_k = \rho(C_k) = 12 \int_{[0,1]^2} \big(C_k(u,v) - uv\big)\,\mathrm{d}u\,\mathrm{d}v, \tag{4.1}$$

of $\hat{\rho}_n^k$, where C_k is the copula associated with (X_t, X_{t+k}) (see, e.g., Schmid *et al.* [48]). Due to the presence of ranks, the investigation of the asymptotic properties of the Spearman periodogram under non-exchangeable observations seems highly non-trivial. However, as we shall demonstrate now, those properties can be obtained via Theorem 3.5 by establishing a connection between the Spearman periodogram and the CR-periodogram.

Proposition 4.1. *For any $\omega \in \mathcal{F}_n$,*

$$I_{n,\rho}(\omega) = 12 \int_{[0,1]^2} I_{n,R}^{u,v}(\omega)\,\mathrm{d}u\,\mathrm{d}v, \tag{4.2}$$

where $I_{n,R}^{u,v}$ is defined in (2.3) Moreover, for any fixed $\omega \in \mathbb{R}$,

$$\hat{G}_{n,\rho}(\omega) = 12 \int_{[0,1]^2} \hat{G}_{n,R}(u,v;\omega)\,\mathrm{d}u\,\mathrm{d}v,$$

where $\hat{G}_{n,R}$ is defined in (2.4).

Proof of Proposition 4.1. Simple algebra yields

$$I_{n,\rho}(\omega) = \frac{12}{2\pi}\frac{1}{n} d_{n,\rho}(\omega) d_{n,\rho}(-\omega) \qquad \text{with } d_{n,\rho}(\omega) := \frac{1}{n}\sum_{t=0}^{n-1} R_{n;t}\mathrm{e}^{-\mathrm{i}\omega t}.$$

Observe that

$$I_{n,\rho}(\omega) = \frac{12}{2\pi}\frac{1}{n^3}\sum_{s,t=0}^{n-1} R_{n;t}R_{n;s}\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}\omega s}.$$

On the other hand,

$$\begin{aligned}
\int_{[0,1]^2} I_{n,R}^{u,v}(\omega)\,\mathrm{d}u\,\mathrm{d}v &= \frac{12}{2\pi}\frac{1}{n}\sum_{s,t=0}^{n-1}\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}\omega s}\int_{[0,1]^2} I\{R_{n;t} \le nu, R_{n;s} \le nv\}\,\mathrm{d}u\,\mathrm{d}v \\
&= \frac{12}{2\pi}\frac{1}{n}\sum_{s,t=0}^{n-1}\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}\omega s}\big(1 - n^{-1}R_{n;t}\big)\big(1 - n^{-1}R_{n;s}\big) \qquad (4.3)\\
&= I_{n,\rho}(\omega) + \frac{12}{2\pi}\frac{1}{n^2}\sum_{s,t=0}^{n-1}\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}\omega s}(n - R_{n;t} - R_{n;s}).
\end{aligned}$$

For $\omega \in \mathcal{F}_n$, $\sum_{t=0}^{n-1}\mathrm{e}^{\mathrm{i}\omega t} = 0$, so that the second term in (4.3) vanishes. The claim follows. □

This result is useful in several ways. On one hand, it allows to easily derive the asymptotic distribution of the smoothed Spearman periodogram by applying the continuous mapping theorem in combination with Theorem 3.5; see Proposition 4.2 below. On the other hand, it motivates the definition of a general class of rank-based spectra to be discussed in the next section.

Proposition 4.2. *Under the assumptions of Theorem 3.5, for any fixed frequency $\omega \neq 0 \mod 2\pi$,*

$$I_{n,\rho}(\omega) \rightsquigarrow 12 \int_0^1 \int_0^1 \mathbb{I}(\tau_1, \tau_2; \omega) \, \mathrm{d}\tau_1 \, \mathrm{d}\tau_2$$

and, for every fixed $\omega \in \mathbb{R}$,

$$\sqrt{nb_n}\big(\hat{G}_{n,\rho}(\omega) - \mathfrak{f}_\rho(\omega) - B_{n,\rho}^{(k)}(\omega)\big) \xrightarrow{\mathcal{D}} Z_\rho(\omega),$$

where $Z_\rho(\omega) \sim \mathcal{N}(0, 2\pi \mathfrak{f}_\rho^2(\omega)(1 + I\{\omega = 0 \mod \pi\}) \int W^2(w) \, \mathrm{d}w)$ and

$$B_{n,\rho}^{(k)}(\omega) := \sum_{j=2}^{k} \frac{b_n^j}{j!} \int_{-\pi}^{\pi} v^j W(v) \, \mathrm{d}v \frac{\mathrm{d}^j}{\mathrm{d}\omega^j} \mathfrak{f}_\mu(\omega), \qquad \omega \in \mathbb{R}.$$

Moreover, $Z_\rho(\omega) = Z_\rho(-\omega)$, $Z_\rho(\omega) = Z_\rho(2\pi + \omega)$ and $Z_\rho(\omega)$, $\omega \in [0, \pi]$ are mutually independent random variables. The weak convergence above holds jointly for any finite, fixed collection of frequencies ω.

This result is a direct consequence of the more general Proposition 4.3, which we establish in the next section. Note that, following the method described in Remark 3.4, Proposition 4.2 can be used to construct pointwise asymptotic confidence bands for $\mathfrak{f}_\rho(\omega)$.

Remark 4.1. A closely related version of the Spearman periodogram was recently considered by Ahdesmäki *et al.* [1]. The main difference with our approach is that these authors use a slightly different version of the lag-k Spearman coefficient, namely

$$\tilde{\rho}_k := \frac{1}{n} \frac{12}{(n-k)^2 - 1} \sum_{t=0}^{n-k-1} \left(R_{n;t}^k - \frac{n-k+1}{2}\right)\left(\bar{R}_{n;t+k}^k - \frac{n-k+1}{2}\right),$$

where $R_{n;t}^k$ denotes the rank of X_t among $X_0, \dots, X_{n-k-1}$ and $\bar{R}_{n;t}^k$ the rank of X_t among $X_{k-1}, \dots, X_{n-1}$, respectively. Letting $\tilde{\rho}_k := \tilde{\rho}_{-k}$ for $k < 0$, Ahdesmäki *et al.* [1] then consider a statistic of the form $\sum_{|k|<n} \mathrm{e}^{\mathrm{i}k\omega} \tilde{\rho}_k$. Note that these authors investigate their method by means of a simulation study and do not provide any asymptotic theory.

4.2. A general class of rank-based spectra

The findings in the previous section suggest considering a general class of rank-based periodograms which are defined in terms of the CR-periodogram as

$$I_{n,\mu}(\omega) := \int_{[0,1]^2} I_{n,R}^{u,v}(\omega)\,\mathrm{d}\mu(u,v), \qquad \omega \in \mathcal{F}_n, \tag{4.4}$$

where μ denotes an arbitrary finite measure on $[0,1]^2$. A smoothed version of $I_{n,\mu}$ is defined through

$$\hat{G}_{n,\mu}(\omega) := \frac{2\pi}{n}\sum_{s=1}^{n-1} W_n(\omega - 2\pi s/n) I_{n,\mu}(2\pi s/n), \qquad \omega \in \mathbb{R}.$$

As discussed in the previous section, taking μ as 12 times the uniform distribution on $[0,1]^2$ yields the Fourier transform of Spearman autocorrelations.

The general results in Theorem 3.5 combined with the continuous mapping theorem imply that the smoothed periodogram $\hat{G}_{n,\mu}$ is a consistent and asymptotically normal estimator of a spectrum of the form

$$\mathfrak{f}_\mu(\omega) := \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}\omega k}\int_{[0,1]^2}\big(C_k(u,v) - uv\big)\,\mathrm{d}\mu(u,v),$$

where C_k denotes the copula of the pair (X_0, X_k).

Proposition 4.3. *Under the assumptions of Theorem 3.5, for any fixed frequency $\omega \in \mathbb{R}$,*

$$\sqrt{nb_n}\big(\hat{G}_{n,\mu}(\omega) - \mathfrak{f}_\mu(\omega) - B_{n,\mu}^{(k)}(\omega)\big) \xrightarrow{\mathcal{D}} Z_\mu(\omega) \sim \mathcal{N}\big(0, \sigma_\mu^2\big),$$

where the variance σ_μ^2 takes the form

$$\begin{aligned}\sigma_\mu^2 = 2\pi \int_{-\pi}^{\pi} W^2(w)\,\mathrm{d}w \\ &\times \int_{[0,1]^2}\int_{[0,1]^2}\big(\mathfrak{f}_{q_u,q_{u'}}(\omega)\mathfrak{f}_{q_v,q_{v'}}(\omega) + \mathfrak{f}_{q_u,q_{v'}}(\omega)\mathfrak{f}_{q_v,q_{u'}}(\omega) I\{\omega = 0 \mod \pi\}\big) \\ &\qquad \times \mathrm{d}\mu(u,v)\,\mathrm{d}\mu\big(u',v'\big)\end{aligned}$$

and the bias is given by

$$B_{n,\mu}^{(k)}(\omega) := \sum_{j=2}^{k}\frac{b_n^j}{j!}\int_{-\pi}^{\pi} v^j W(v)\,\mathrm{d}v \frac{\mathrm{d}^j}{\mathrm{d}\omega^j}\mathfrak{f}_\mu(\omega), \qquad \omega \in \mathbb{R}.$$

Moreover, $Z_\mu(\omega) = Z_\mu(-\omega)$, $Z_\mu(\omega) = Z_\mu(2\pi + \omega)$, and $Z_\mu(\omega)$, $\omega \in [0,\pi]$ are mutually independent random variables. The weak convergence above holds jointly for any finite, fixed collection of frequencies ω.

Proof. Assumption (C) entails

$$\mathfrak{f}_\mu(\omega) - B_{n,\mu}^{(k)}(\omega) = \int_{[0,1]^2} \mathfrak{f}_{q_u,q_v}(\omega) - B_n^{(k)}(u,v;\omega)\,\mathrm{d}\mu(u,v).$$

This yields

$$\hat{G}_{n,\mu}(\omega) - \mathfrak{f}_\mu(\omega) + B_{n,\mu}^{(k)}(\omega) = \int_{[0,1]^2} \mathbb{G}_n(u,v;\omega)\,\mathrm{d}\mu(u,v),$$

where $\mathbb{G}_n$ was defined in Theorem 3.5. An application of the continuous mapping theorem implies

$$\sqrt{nb_n}\big(\hat{G}_{n,\mu}(\omega) - \mathfrak{f}_\mu(\omega) - B_{n,\mu}^{(k)}(\omega)\big) \xrightarrow{\mathcal{D}} \int_{[0,1]^2} H(u,v;\omega)\,\mathrm{d}\mu(u,v).$$

Since $H(\cdot,\cdot;\omega)$ is a centered Gaussian process, the integral $\int_{[0,1]^2} H(u,v;\omega)\,\mathrm{d}u\,\mathrm{d}v$ follows a normal distribution with mean zero and variance:

$$\begin{aligned}
&\int_{[0,1]^2}\int_{[0,1]^2} \mathrm{Cov}\big(H(u,v;\omega), H\big(u',v';\omega\big)\big)\,\mathrm{d}\mu(u,v)\,\mathrm{d}\mu\big(u',v'\big)\\
&\quad = 2\pi\int W^2(w)\,\mathrm{d}w \int_{[0,1]^2}\int_{[0,1]^2}\big(\mathfrak{f}_{q_u,q_{u'}}(\omega)\mathfrak{f}_{q_v,q_{v'}}(\omega) + \mathfrak{f}_{q_u,q_{v'}}(\omega)\mathfrak{f}_{q_v,q_{u'}}(\omega)I\{\omega = 0 \mod \pi\}\big)\\
&\qquad\qquad \times \mathrm{d}\mu(u,v)\,\mathrm{d}\mu\big(u',v'\big).
\end{aligned}$$

This completes the proof. □

4.3. The Blomqvist and Gini periodograms

In this section, we identify two measures μ that correspond to two classical measures of serial dependence, Blomqvist's beta (see Blomqvist [3], Schmid *et al.* [48], Genest and Carabarín-Aguirre [16]) and Gini's gamma (see Schechtman and Yitzhaki [47], Nelsen [45], Carcea and Serfling [6]) coefficients, which lead to the definition of the *Blomqvist* and *Gini spectra*, respectively.

Let C_k denote the copula of the pair (X_0, X_k) and assume that it is continuous. The corresponding *Blomqvist beta coefficient at lag k* is

$$\beta_k := 4C_k(1/2, 1/2) - 1. \tag{4.5}$$

Similarly, Gini's gamma, also known as *Gini's lag k rank association coefficient* is the copula-based quantity

$$\begin{aligned}
\Gamma_k &:= 2\int_{[0,1]^2}\big(|u+v-1| - |v-u|\big)\,\mathrm{d}C_k(u,v)\\
&= 4\bigg(\int_{[0,1]} C_k(u,u) - u^2\,\mathrm{d}u + \int_{[0,1]} C_k(u,1-u) - u(1-u)\,\mathrm{d}u\bigg).
\end{aligned} \tag{4.6}$$

This motivates the definition of the *Blomqvist spectrum*

$$\mathfrak{f}_\beta(\omega) := \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}\omega k}\beta_k$$

and the *Gini spectrum*

$$\mathfrak{f}_\Gamma(\omega) := \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}\omega k}\Gamma_k.$$

Sample versions of the Blomqvist and Gini coefficients are

$$\hat{\beta}_n^k := \frac{1}{n-|k|}\sum_{t=1}^{n-|k|-1}\big(4I\{R_{n;t} \le 1/2, R_{n;t+|k|} \le 1/2\} - 1\big),$$

and

$$\hat{\Gamma}_n^k := \frac{2}{n(n-|k|)}\sum_{t=0}^{n-|k|-1}\big(|R_{n;t} + R_{n;t+|k|} - n| - |R_{n;t} - R_{n;t+|k|}|\big),$$

respectively. To establish the connection with the general periodogram defined in the previous section, consider the measures μ_β which puts mass 4 in the point $(1/2, 1/2)$ and μ_Γ which puts mass 4 on the sets $\{(u,u)\colon u \in [0,1]\}$ and $\{(u, 1-u)\colon u \in [0,1]\}$, respectively.

Proposition 4.4. *For any $\omega \in \mathcal{F}_n$,*

$$I_{n,\beta}(\omega) := \int_{[0,1]^2} I_{n,R}^{u,v}(\omega)\,\mathrm{d}\mu_\beta(u,v) = \frac{1}{2\pi}\sum_{|k|<n}\frac{n-k}{n}\mathrm{e}^{\mathrm{i}\omega k}\hat{\beta}_n^k$$

and

$$I_{n,\Gamma}(\omega) := \int_{[0,1]^2} I_{n,R}^{u,v}(\omega)\,\mathrm{d}\mu_\Gamma(u,v) = \frac{1}{2\pi}\sum_{|k|<n}\frac{n-k}{n}\mathrm{e}^{\mathrm{i}\omega k}\hat{\Gamma}_n^k.$$

Proof. Observing that

$$|n - R_{n;t} - R_{n;t+k}| = 2\max(n - R_{n;t} - R_{n;t+k}, 0) - (n - R_{n;t} - R_{n;t+k})$$

and

$$|R_{n;t} - R_{n;t+k}| = 2\max(R_{n;t}, R_{n;t+k}) - (R_{n;t} + R_{n;t+k})$$

yields

$$\begin{aligned}&|R_{n;t} + R_{n;t+k} - n| - |R_{n;t} - R_{n;t+k}| \\ &\quad = 2\max(n - R_{n;t} - R_{n;t+k}, 0) - 2\max(R_{n;t}, R_{n;t+k}) + 2(R_{n;t} + R_{n;t+k}) - n.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_0^1 I_{n,R}^{u,u}(\omega)\,\mathrm{d}u &= \frac{1}{2\pi}\frac{1}{n}\sum_{s,t=0}^{n-1} \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}\omega s}\int_0^1 I\{R_{n;t} \le nu, R_{n;s} \le nu\}\,\mathrm{d}u \\ &= \frac{1}{2\pi}\frac{1}{n}\sum_{s,t=0}^{n-1} \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}\omega s}\bigl(1 - n^{-1}\max(R_{n;t}, R_{n;s})\bigr) \\ &= -\frac{1}{2\pi}\frac{1}{n^2}\sum_{s,t=0}^{n-1} \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}\omega s}\max(R_{n;t}, R_{n;s})\end{aligned}$$

and

$$\begin{aligned}\int_0^1 I_{n,R}^{u,1-u}(\omega)\,\mathrm{d}u &= \frac{1}{2\pi}\frac{1}{n}\sum_{s,t=0}^{n-1} \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}\omega s}\int_0^1 I\bigl\{R_{n;t} \le nu, R_{n;s} \le n(1-u)\bigr\}\,\mathrm{d}u \\ &= \frac{1}{2\pi}\frac{1}{n}\sum_{s,t=0}^{n-1} \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}\omega s}\max\bigl(1 - n^{-1}R_{n;t} - n^{-1}R_{n;s}, 0\bigr).\end{aligned}$$

Elementary algebra yields, for arbitrary functions a from $\mathbb{Z}^2$ to $\mathbb{Z}$ such that $a(j,k) = a(k,j)$ for all j,k,

$$\sum_{|k|<n}\sum_{t=0}^{n-1-|k|} \mathrm{e}^{\mathrm{i}\omega k} a\bigl(t, t+|k|\bigr) = \sum_{s=0}^{n-1}\sum_{t=0}^{n-1} \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}\omega s} a(t,s).$$

This implies (recall that $\omega \in \mathcal{F}_n$)

$$\begin{aligned}I_{n,\Gamma}(\omega) &= \frac{1}{2\pi}\frac{2}{n}\sum_{|k|<n}\sum_{t=0}^{n-1-|k|} \mathrm{e}^{\mathrm{i}\omega k}\bigl(|R_{n;t} + R_{n;t+k} - n| - |R_{n;t} - R_{n;t+k}|\bigr) \\ &= \frac{1}{2\pi}\frac{4}{n^2}\sum_{s=0}^{n-1}\sum_{t=0}^{n-1} \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}\omega s}\bigl(\max(n - R_{n;t} - R_{n;s}, 0) - \max(R_{n;t}, R_{n;s})\bigr) \\ &\quad + \frac{1}{2\pi}\frac{2}{n^2}\sum_{s=0}^{n-1}\sum_{t=0}^{n-1} \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{\mathrm{i}\omega s}\bigl(2(R_{n;t} + R_{n;s}) - n\bigr) \\ &= 4\biggl(\int_{[0,1]} I_{n,R}^{u,u}(\omega)\,\mathrm{d}u + \int_{[0,1]} I_{n,R}^{u,1-u}(\omega)\,\mathrm{d}u\biggr).\end{aligned}$$

The representation for $I_{n,\beta}$ can be derived similarly; details are omitted for the sake of brevity. □

Smoothed versions of the Blomqvist and Gini periodograms can be defined accordingly, and their asymptotic distributions follow from Proposition 4.3. In particular, this yields consistent estimators of the Blomqvist and Gini spectra defined above.

We conclude this section with some general remarks. First, note that the approach above can be applied to any scalar dependence measure that can be represented as a continuous linear functional of the copula. For instance, Cifarelli, Conti and Regazzini [7] consider a general measure of monotone dependence of the form

$$\int_{[0,1]^2} g\big(|u+v-1|\big) - g\big(|u-v|\big)\, \mathrm{d}C(u,v), \tag{4.7}$$

where $g:[0,1] \to \mathbb{R}$ is strictly increasing and convex. Choosing $g(x) = x$ and $g(x) = x^2$ yields (up to constants) the Gini and Spearman rank correlations, respectively. Under suitable assumptions on g, the monotone dependence measure in (4.7) can be written (by applying integration-by-parts) in the form of equation (4.4), and the results from Section 4.2 apply.

Other measures of serial dependence such as Kendall's τ (see Ferguson, Genest and Hallin [13]) only can be represented as nonlinear functionals of the copula. More general rank-based autocorrelation coefficients also have been introduced in the context of inference for ARMA models (see Hallin and Puri [22,23] or Hallin [21]); they involve score functions, typically are not time-revertible, and lead to possibly unbounded measures μ. We expect that the general results presented here can be extended to the periodograms associated with such coefficients, but leave this question to future research.

5. Simulation study

In this section, we show how Theorem 3.5 can be used to construct asymptotic confidence intervals for the copula spectra. An analysis of the finite sample performance was conducted using the R package **quantspec** (Kley [31,32]). We consider three different models:

(a) the QAR(1) process

$$Y_t = 0.1\Phi^{-1}(U_t) + 1.9(U_t - 0.5)Y_{t-1} \tag{5.1}$$

(cf. Koenker and Xiao [36]), where (U_t) is a sequence of i.i.d. standard uniform random variables, and Φ denotes the distribution function of the standard normal distribution;

(b) the AR(2) process

$$Y_t = -0.36Y_{t-2} + \varepsilon_t, \tag{5.2}$$

where (ε_t) is standard normal white noise (cf. Li [39]);

(c) the ARCH(1) process

$$Y_t = \big(1/1.9 + 0.9Y_{t-1}^2\big)^{1/2}\varepsilon_t, \tag{5.3}$$

where (ε_t) is standard normal white noise (cf. Lee and Rao [37]).

For each model, 10 000 independent copies of length $n \in \{2^8, 2^9, 2^{10}, 2^{11}\}$ were generated. For each of them, the smoothed CR-periodograms

$$\tilde{G}_{n,R}(\tau_1, \tau_2; \omega_{jn}) := \hat{G}_{n,R}(\tau_1, \tau_2; \omega_{jn})/W_n^j, \qquad W_n^j := \frac{2\pi}{n} \sum_{0=s\neq j}^{n-1} W_n(\omega_{jn} - \omega_{sn}), \tag{5.4}$$

were computed for $\omega_{jn} := 2\pi j/n$, $j = 1, \dots, n/2 - 1$ and $\tau_1, \tau_2 \in \{0.1, 0.5, 0.9\}$, where we used the kernel of order 4

$$W(u) := \frac{15}{32}\frac{1}{\pi}\bigl(7(u/\pi)^4 - 10(u/\pi)^2 + 3\bigr)I\bigl\{|u| \leq \pi\bigr\}$$

minimizing the asymptotic IMSE (see Gasser, Müller and Mammitzsch [15]). The bandwidth was chosen as $b_n = 0.4n^{-1/4}$ which is of lower order than the IMSE-optimal bandwidth $n^{-1/9}$ to reduce bias and the factor $(W_n^j)^{-1}$ ensures that the weights in (5.4) sum up to one for every n.

Based on Theorem 3.6, we then computed pointwise asymptotic $(1 - \alpha)$-level confidence bands for the real and imaginary parts of the spectrum, namely,

$$IC_{1,n}(\tau_1, \tau_2; \omega_{jn}) := \Re\tilde{G}_{n,R}(\tau_1, \tau_2; \omega_{jn}) \pm \Re\sigma(\tau_1, \tau_2; \omega_{jn})\Phi^{-1}(1 - \alpha/2), \tag{5.5}$$

for the real part, and

$$IC_{2,n}(\tau_1, \tau_2; \omega_{jn}) := \Im\tilde{G}_{n,R}(\tau_1, \tau_2; \omega_{jn}) \pm \Im\sigma(\tau_1, \tau_2; \omega_{jn})\Phi^{-1}(1 - \alpha/2), \tag{5.6}$$

for the imaginary part of the copula spectrum. As usual, Φ stands for the standard normal distribution function, and

$$\bigl(\Re\sigma(\tau_1, \tau_2; \omega_{jn})\bigr)^2 := 0 \vee \begin{cases} c(\tau_1, \tau_2; \omega_{jn}, \omega_{jn}), & \text{if } \tau_1 = \tau_2, \\ \frac{1}{2}\bigl(c(\tau_1, \tau_2; \omega_{jn}, \omega_{jn}) + c(\tau_1, \tau_2; \omega_{jn}, -\omega_{jn})\bigr), & \text{if } \tau_1 \neq \tau_2, \end{cases}$$

and

$$\bigl(\Im\sigma(\tau_1, \tau_2; \omega_{jn})\bigr)^2 := 0 \vee \begin{cases} 0, & \text{if } \tau_1 = \tau_2, \\ \frac{1}{2}\bigl(c(\tau_1, \tau_2; \omega_{jn}, \omega_{jn}) - c(\tau_1, \tau_2; \omega_{jn}, -\omega_{jn})\bigr), & \text{if } \tau_1 \neq \tau_2 \end{cases}$$

are estimators for $\operatorname{Var}(\Re\tilde{G}_{n,R}(\tau_1, \tau_2; \omega_{jn}))$ and $\operatorname{Var}(\Im\tilde{G}_{n,R}(\tau_1, \tau_2; \omega_{jn}))$, respectively. Here,

$$\begin{aligned} &c\bigl(\tau_1, \tau_2; \omega_{jn}, \omega_{j'n}\bigr) \\ &\quad := \left(\frac{2\pi}{n}\Big/W_n^j\right)^2 \\ &\qquad \times \Biggl[\sum_{s=1}^{n-1} W_n(\omega_{jn} - 2\pi s/n)W_n\bigl(\omega_{j'n} - 2\pi s/n\bigr)\tilde{G}_{n,R}(\tau_1, \tau_1; 2\pi s/n)\tilde{G}_{n,R}(\tau_2, \tau_2; 2\pi s/n) \\ &\qquad\qquad + \sum_{s=1}^{n-1} W_n(\omega_{jn} - 2\pi s/n)W_n\bigl(\omega_{j'n} + 2\pi s/n\bigr)\bigl|\tilde{G}_{n,R}(\tau_1, \tau_2; 2\pi s/n)\bigr|^2\Biggr] \end{aligned}$$

is an estimator for the covariance of $\tilde{G}_{n,R}(\tau_1, \tau_2; \omega_{jn})$ and $\tilde{G}_{n,R}(\tau_1, \tau_2; \omega_{j'n})$; this follows from the representation in Theorem 3.6(iii) and Theorem 7.4.3 in Brillinger [4]. To motivate this approach, recall that, for any complex-valued random variable Z with complex conjugate $\bar{Z}$,

$$\operatorname{Var}(\Re Z) = \frac{1}{2}\big(\operatorname{Var}(Z) + \Re \operatorname{Cov}(Z, \bar{Z})\big); \qquad \operatorname{Var}(\Im Z) = \frac{1}{2}\big(\operatorname{Var}(Z) - \Re \operatorname{Cov}(Z, \bar{Z})\big).$$

For $n \to \infty$, the estimated variances above converge to the asymptotic variance in Theorem 3.5. However, in small samples the more elaborate version considered here typically leads to better coverage probabilities.

In Tables 1–4, we report the simulated coverage frequencies associated with

$$\mathbb{P}\big(\Re \mathfrak{f}_{q_{\tau_1}, q_{\tau_2}}(\omega) \in IC_{1,n}(\tau_1, \tau_2, \omega)\big) \quad \text{and} \quad \mathbb{P}\big(\Im \mathfrak{f}_{q_{\tau_1}, q_{\tau_2}}(\omega) \in IC_{2,n}(\tau_1, \tau_2, \omega)\big).$$

Inspection of Tables 1–4 reveals that, as n gets larger, the coverage frequencies converge to the confidence level $1 - \alpha$. For models (5.1)–(5.2), those frequencies are quite close to $1 - \alpha$ even for moderately large values of n. Due to boundary effects, the coverage frequencies for ω close to multiples of π are too low in all three models, but, as noted earlier, they improve as n increases. Finally, in models (5.1) and (5.3) for smaller values of n, the confidence intervals involving extreme quantiles tend to cover less frequently, as can be expected. Again, the accuracy improves with increasing sample size.

Table 1. Coverage frequencies for the confidence intervals $IC_n(\tau_1, \tau_2, \omega)$, $n = 2^8$, $b_n = 0.4n^{-1/4}$, $1 - \alpha = 0.95$

Model	(τ_1, τ_2) ω/π	$(0.1, 0.1)$ $(\Re)$	$(0.1, 0.9)$ $(\Im)$	$(0.5, 0.5)$ $(\Re)$	$(0.1, 0.9)$ $(\Re)$	$(0.9, 0.9)$ $(\Re)$
(a) QAR(1) (5.1)	1/8	0.911	0.921	0.906	0.987	0.899
	1/4	0.934	0.917	0.920	0.979	0.910
	1/2	0.947	0.919	0.932	0.976	0.915
	3/4	0.946	0.918	0.927	0.979	0.916
	7/8	0.941	0.915	0.931	0.979	0.921
(b) AR(2) (5.2)	1/8	0.913	0.926	0.900	0.975	0.916
	1/4	0.935	0.925	0.917	0.967	0.940
	1/2	0.940	0.927	0.929	0.966	0.949
	3/4	0.939	0.924	0.928	0.969	0.947
	7/8	0.937	0.920	0.928	0.972	0.945
(c) ARCH(1) (5.3)	1/8	0.860	0.910	0.906	0.902	0.878
	1/4	0.872	0.905	0.922	0.909	0.887
	1/2	0.902	0.897	0.937	0.946	0.914
	3/4	0.906	0.894	0.934	0.959	0.924
	7/8	0.906	0.891	0.935	0.962	0.920

Table 2. Coverage frequencies for the confidence intervals $IC_n(\tau_1, \tau_2, \omega)$, $n = 2^9$, $b_n = 0.4n^{-1/4}$, $1 - \alpha = 0.95$

Model	(τ_1, τ_2) ω/π	$(0.1, 0.1)$ $(\Re)$	$(0.1, 0.9)$ $(\Im)$	$(0.5, 0.5)$ $(\Re)$	$(0.1, 0.9)$ $(\Re)$	$(0.9, 0.9)$ $(\Re)$
(a) QAR(1) (5.1)	1/8	0.934	0.932	0.915	0.974	0.916
	1/4	0.953	0.933	0.931	0.968	0.925
	1/2	0.954	0.932	0.940	0.968	0.934
	3/4	0.952	0.926	0.939	0.973	0.932
	7/8	0.953	0.923	0.941	0.975	0.934
(b) AR(2) (5.2)	1/8	0.930	0.934	0.913	0.962	0.932
	1/4	0.950	0.932	0.928	0.956	0.951
	1/2	0.948	0.935	0.933	0.957	0.949
	3/4	0.951	0.932	0.936	0.964	0.952
	7/8	0.949	0.931	0.937	0.965	0.955
(c) ARCH(1) (5.3)	1/8	0.890	0.932	0.918	0.913	0.892
	1/4	0.900	0.924	0.938	0.917	0.903
	1/2	0.922	0.912	0.939	0.948	0.928
	3/4	0.926	0.913	0.944	0.957	0.934
	7/8	0.928	0.908	0.943	0.958	0.937

Table 3. Coverage frequencies for the confidence intervals $IC_n(\tau_1, \tau_2, \omega)$, $n = 2^{10}$, $b_n = 0.4n^{-1/4}$, $1 - \alpha = 0.95$

Model	(τ_1, τ_2) ω/π	$(0.1, 0.1)$ $(\Re)$	$(0.1, 0.9)$ $(\Im)$	$(0.5, 0.5)$ $(\Re)$	$(0.1, 0.9)$ $(\Re)$	$(0.9, 0.9)$ $(\Re)$
(a) QAR(1) (5.1)	1/8	0.942	0.943	0.933	0.961	0.924
	1/4	0.959	0.938	0.941	0.963	0.929
	1/2	0.953	0.938	0.941	0.962	0.934
	3/4	0.954	0.935	0.941	0.967	0.933
	7/8	0.956	0.935	0.943	0.969	0.936
(b) AR(2) (5.2)	1/8	0.939	0.943	0.931	0.953	0.940
	1/4	0.954	0.939	0.942	0.954	0.952
	1/2	0.954	0.944	0.945	0.953	0.955
	3/4	0.950	0.937	0.942	0.956	0.954
	7/8	0.954	0.937	0.940	0.959	0.952
(c) ARCH(1) (5.3)	1/8	0.900	0.935	0.933	0.911	0.906
	1/4	0.901	0.930	0.945	0.916	0.908
	1/2	0.929	0.928	0.945	0.942	0.928
	3/4	0.941	0.916	0.948	0.954	0.937
	7/8	0.940	0.918	0.948	0.953	0.936

Table 4. Coverage frequencies for the confidence intervals $IC_n(\tau_1, \tau_2, \omega)$, $n = 2^{11}$, $b_n = 0.4n^{-1/4}$, $1 - \alpha = 0.95$

Model	(τ_1, τ_2) ω/π	$(0.1, 0.1)$ $(\Re)$	$(0.1, 0.9)$ $(\Im)$	$(0.5, 0.5)$ $(\Re)$	$(0.1, 0.9)$ $(\Re)$	$(0.9, 0.9)$ $(\Re)$
(a) QAR(1) (5.1)	1/8	0.953	0.945	0.944	0.957	0.933
	1/4	0.957	0.943	0.945	0.961	0.932
	1/2	0.955	0.938	0.949	0.960	0.938
	3/4	0.952	0.938	0.946	0.963	0.939
	7/8	0.954	0.936	0.945	0.964	0.945
(b) AR(2) (5.2)	1/8	0.953	0.944	0.943	0.954	0.947
	1/4	0.954	0.944	0.945	0.953	0.956
	1/2	0.955	0.946	0.945	0.951	0.954
	3/4	0.954	0.947	0.940	0.954	0.957
	7/8	0.952	0.945	0.943	0.956	0.951
(c) ARCH(1) (5.3)	1/8	0.911	0.942	0.944	0.918	0.908
	1/4	0.918	0.937	0.950	0.926	0.917
	1/2	0.934	0.931	0.947	0.946	0.937
	3/4	0.944	0.931	0.949	0.954	0.943
	7/8	0.944	0.928	0.950	0.958	0.945

6. Conclusions

Spectral analysis for the past fifty years has been a major tool in the analysis of time series. Being essentially covariance-based, however, classical L^2-spectral methods have obvious limitations, for instance (see Figures 1 and 2), they cannot discriminate between QAR or ARCH and white noise processes. Quantile-related spectral concepts have been proposed, which palliate those limitations. Only quite incomplete asymptotic distributional results, however, have been available in the literature for the consistent estimation of such concepts, which so far has precluded most practical applications.

In this paper, we provide (Theorem 3.5), in the very strong form of convergence to a Gaussian process, such asymptotic results for the smoothed copula rank-based periodogram process. That rank-based periodogram is the generalization (Dette *et al.* [12]) of the copula rank periodograms proposed by Hagemann [19]. Theorem 3.5 was used to construct confidence intervals. A simulation study was conducted using the R package **quantspec** (Kley [31,32]).

Being copula- or rank-based, our spectral concepts furthermore are invariant under monotone increasing continuous marginal transformations of the data, and are likely to enjoy appealing robustness features their traditional L^2-counterparts are severely lacking. Another application is in the asymptotic behavior of the spectra associated with more classical rank-based autocorrelation coefficients, such as the Spearman, Gini or Blomqvist spectra.

Copula rank-based periodogram methods are improving over the classical ones both from the point of view of efficiency (detection of nonlinear features) and from the point of view of robustness (no finite variance assumption is required). They are likely to be ideal tools for a large variety of problems of practical interest, such as change-point analysis, tail dependence, model diagnostics, or local stationary procedures (cf. Skowronek [52]) – essentially, all problems covered in the traditional spectral context can be extended here, with the huge advantage that nonlinear features that cannot be accounted for by traditional methods can be analyzed via the new ones. This seems to offer most promising perspectives for future research.

Appendix: Proof of Theorem 3.6

The proof of Theorem 3.6 relies on a series of technical lemmas; for the readers' convenience, we begin by giving a general overview of the main steps and the corresponding lemmas.

For all $n \in \mathbb{N}$, consider the stochastic process

$$\hat{H}_{n,U}(\tau_1, \tau_2; \omega) := \sqrt{nb_n}\big(\hat{G}_{n,U}(\tau_1, \tau_2; \omega) - \mathbb{E}\hat{G}_{n,U}(\tau_1, \tau_2; \omega)\big), \tag{A.1}$$

indexed by $(\tau_1, \tau_2) \in [0, 1]^2$ and $\omega \in \mathbb{R}$; for $a = (a_1, a_2) \in [0, 1]^2$, write $\hat{H}_n(a; \omega)$ for $\hat{H}_{n,U}(a_1, a_2; \omega)$.

The key step in the process of establishing parts (i) and (iii) of Theorem 3.6 is a uniform bound on the increments of the process $\hat{H}_{n,U}$. That bound is required, for example, when showing the stochastic equicontinuity of $\hat{H}_n(a; \omega) - \hat{H}_n(b; \omega)$. We derive such a bound by a restricted chaining technique, which is described in Lemma A.1. The application of Lemma A.1 requires two ingredients. First, we need a general bound, uniform in a and b, on the moments of $\hat{H}_n(a; \omega) - \hat{H}_n(b; \omega)$. Such a bound is derived in Lemma A.2. Second, we need a sharper bound on the increments $\hat{H}_n(a; \omega) - \hat{H}_n(b; \omega)$ when a and b are "sufficiently close". We provide this result in Lemma A.7.

Lemma A.2 is a very general result, relying on an abstract condition on the cumulants of discrete Fourier transforms of certain indicator functions; see (A.5). The link between assumption (C) and (A.5) is established in Lemma A.4.

Finally, the proof of part (ii) of Theorem 3.6 follows by a series of uniform generalizations of results from Brillinger [4], the details of which are provided in the online supplement [33] [Lemmas 8.1–8.5].

A.1. Proof of part (i) of Theorem 3.6

In view of Theorems 1.5.4 and 1.5.7 in van der Vaart and Wellner [54], it is sufficient to prove the following two claims:

(i1) convergence of the finite-dimensional distributions of the process (A.1), that is,

$$\big(\hat{H}_n(a_{1j}, a_{2j}; \omega_j)\big)_{j=1,\dots,k} \xrightarrow{d} \big(H(a_{1j}, a_{2j}; \omega_j)\big)_{j=1,\dots,k} \tag{A.2}$$

for any $(a_{1j}, a_{2j}, \omega_j) \in [0, 1]^2 \times \mathbb{R}$, $j = 1, \dots, k$ and $k \in \mathbb{N}$;

(i2) stochastic equicontinuity: for any $x > 0$ and any $\omega \in \mathbb{R}$,

$$\lim_{\delta \downarrow 0} \limsup_{n\to\infty} \mathbb{P}\Big(\sup_{\substack{a,b\in[0,1]^2 \\ \|a-b\|_1 \le \delta}} \big|\hat{H}_n(a;\omega) - \hat{H}_n(b;\omega)\big| > x\Big) = 0. \tag{A.3}$$

Note indeed that (A.3) implies stochastic equicontinuity of both the real part $(\Re \hat{H}_n(a;\omega))_{a\in[0,1]^2}$ and the imaginary part $(\Im \hat{H}_n(a;\omega))_{a\in[0,1]^2}$ of $\hat{H}_n$.

First consider (i1). Observe that $\hat{G}_{n,U}(\tau_1,\tau_2;\omega)$ is the traditional smoothed periodogram estimator (see Chapter 7.1 in Brillinger [4]) of the cross-spectrum of the *clipped processes* $(I\{F(X_t) \le \tau_1\})_{t\in\mathbb{Z}}$ and $(I\{F(X_t) \le \tau_2\})_{t\in\mathbb{Z}}$. Thus, (A.2) is an immediate corollary of Theorem 7.4.4 in Brillinger [4]. The limiting first and second moment structures are given by Theorem 7.4.1 and Corollary 7.4.3 in Brillinger [4]. This implies the desired convergence (A.2) of finite-dimensional distributions. Note that, by condition (C), the summability condition required for the three theorems holds (Assumption 2.6.2(ℓ), for every ℓ; cf. Brillinger [4]).

Turning to (i2), in the notation from van der Vaart and Wellner [54], page 95, put $\Psi(x) := x^6$: the Orlicz norm $\|X\|_\Psi = \inf\{C > 0\colon \mathbb{E}\Psi(|X|/C) \le 1\}$ coincides with the L_6 norm $\|X\|_6 = (\mathbb{E}|X|^6)^{1/6}$. Therefore, by Lemma A.2 and Lemma A.4, we have, for any $\kappa \in (0,1)$ and sufficiently small $\|a-b\|_1$,

$$\big\|\hat{H}_n(a;\omega) - \hat{H}_n(b;\omega)\big\|_\Psi \le K\bigg(\frac{\|a-b\|_1^\kappa}{(nb_n)^2} + \frac{\|a-b\|_1^{2\kappa}}{nb_n} + \|a-b\|_1^{3\kappa}\bigg)^{1/6}.$$

It follows that, for all a, b with $\|a-b\|_1$ sufficiently small and $\|a-b\|_1 \ge (nb_n)^{-1/\gamma}$ and all $\gamma \in (0,1)$ such that $\gamma < \kappa$,

$$\begin{aligned}\big\|\hat{H}_n(a;\omega) - \hat{H}_n(b;\omega)\big\|_\Psi &\le K\big(\|a-b\|_1^{\kappa+2\gamma} + \|a-b\|_1^{2\kappa+\gamma} + \|a-b\|_1^{3\kappa}\big)^{1/6} \\ &\le \bar{K}\|a-b\|_1^{\gamma/2}.\end{aligned}$$

Note that $\|a-b\|_1 \ge (nb_n)^{-1/\gamma}$ iff $d(a,b) := \|a-b\|_1^{\gamma/2} \ge (nb_n)^{-1/2} =: \bar{\eta}_n/2$.

Denoting by $D(\varepsilon, d)$ the *packing number* of $([0,1]^2, d)$ (cf. van der Vaart and Wellner [54], page 98), we have $D(\varepsilon,d) \asymp \varepsilon^{-4/\gamma}$. Therefore, by Lemma A.1, for all $x, \delta > 0$ and $\eta \ge \bar{\eta}_n$,

$$\begin{aligned}&\mathbb{P}\Big(\sup_{\|a-b\|_1 \le \delta^{2/\gamma}} \big|\hat{H}_n(a;\omega) - \hat{H}_n(b;\omega)\big| > x\Big) \\ &\quad = \mathbb{P}\Big(\sup_{d(a,b)\le\delta} \big|\hat{H}_n(a;\omega) - \hat{H}_n(b;\omega)\big| > x\Big) \\ &\quad \le \bigg[\frac{8\tilde{K}}{x}\bigg(\int_{\bar{\eta}_n/2}^{\eta} \epsilon^{-2/(3\gamma)}\,\mathrm{d}\epsilon + (\delta + 2\bar{\eta}_n)\eta^{-4/(3\gamma)}\bigg)\bigg]^6 \\ &\qquad + \mathbb{P}\Big(\sup_{d(a,b)\le\bar{\eta}_n} \big|\hat{H}_n(a;\omega) - \hat{H}_n(b;\omega)\big| > x/4\Big).\end{aligned}$$

Now choose $1 > \gamma > 2/3$. Letting n tend to infinity, the second term tends to zero by Lemma A.7 since, by construction, $1/\gamma > 1$ and

$$d(a,b) \le \bar{\eta}_n \quad \text{iff} \quad \|a-b\|_1 \le 2^{2/\gamma}(nb_n)^{-1/\gamma}.$$

All together, this implies

$$\lim_{\delta \downarrow 0} \limsup_{n\to\infty} \mathbb{P}\Big(\sup_{d(a,b)\le\delta} \big|\hat{H}_n(a;\omega) - \hat{H}_n(b;\omega)\big| > x\Big) \le \left[\frac{8\tilde{K}}{x}\int_0^{\eta} \epsilon^{-2/(3\gamma)}\,\mathrm{d}\epsilon\right]^6,$$

for every $x, \eta > 0$; the claim follows, since the integral in the right-hand side can be made arbitrarily small by choosing η accordingly.

A.2. Proof of part (ii) of Theorem 3.6

Essentially, this part of Theorem 3.6 is a uniform version of Theorems 7.4.1 and 7.4.2 in Brillinger [4] in the present setting of Laplace spectra. The proof is based on a series of uniform versions of results from Brillinger [4]; details are provided in the online supplement [33] (see in particular Lemma 8.5).

A.3. Proof of part (iii) of Theorem 3.6

It follows from the continuity of F that the ranks of the random variables $X_0, \dots, X_{n-1}$ and $F(X_0), \dots, F(X_{n-1})$ coincide almost surely. Thus, without loss of generality, we can assume that the estimator is computed from the unobservable data $F(X_0), \dots, F(X_{n-1})$. In particular, this implies that we can assume the marginals to be uniform.

Denote by $\hat{F}_n^{-1}(\tau) := \inf\{x\colon\ \hat{F}_n(x) \ge \tau\}$ the generalized inverse of $\hat{F}_n$ and let $\inf\varnothing := 0$. Elementary computation shows that, for any $k \in \mathbb{N}$,

$$\sup_{\omega\in\mathbb{R}} \sup_{\tau\in[0,1]} \big|d_{n,R}^{\tau}(\omega) - d_{n,U}^{\hat{F}_n^{-1}(\tau)}(\omega)\big| \le n \sup_{\tau\in[0,1]} \big|\hat{F}_n(\tau) - \hat{F}_n(\tau-)\big| = O_P\big(n^{1/2k}\big), \tag{A.4}$$

where $\hat{F}_n(\tau-) := \lim_{\xi\uparrow 0} \hat{F}_n(\tau-\xi)$ and the O_P-bound in the above equation follows from Lemma 8.6 (online supplement [33]). By the definition of $\hat{G}_{n,R}$ and arguments similar to the ones used in the proof of Lemma A.7, it follows that

$$\sup_{\omega\in\mathbb{R}} \sup_{\tau_1,\tau_2\in[0,1]} \big|\hat{G}_{n,R}(\tau_1,\tau_2;\omega) - \hat{G}_{n,U}\big(\hat{F}_n^{-1}(\tau_1), \hat{F}_n^{-1}(\tau_2);\omega\big)\big| = o_P(1).$$

It therefore suffices to bound the differences

$$\sup_{\tau_1,\tau_2\in[0,1]} \big|\hat{G}_{n,U}(\tau_1,\tau_2;\omega) - \hat{G}_{n,U}\big(\hat{F}_n^{-1}(\tau_1), \hat{F}_n^{-1}(\tau_2);\omega\big)\big|$$

pointwise and uniformly in ω.

In what follows, we give a detailed proof of the statement for fixed $\omega \in \mathbb{R}$ and sketch the arguments needed for the proof of the uniform result.

By (A.1) we have, for any $x > 0$ and δ_n with

$$n^{-1/2} \ll \delta_n = \mathrm{o}\big(n^{-1/2} b_n^{-1/2} (\log n)^{-d}\big),$$

where d is the constant from Lemma A.3 corresponding to $j = k$,

$$\begin{aligned}
&P^n(\omega) \\
&\quad := \mathbb{P}\Big(\sup_{\tau_1,\tau_2\in[0,1]} \big|\hat{G}_{n,U}\big(\hat{F}_n^{-1}(\tau_1), \hat{F}_n^{-1}(\tau_2); \omega\big) - \hat{G}_{n,U}(\tau_1, \tau_2; \omega)\big| > x\big((nb_n)^{-1/2} + b_n^k\big)\Big) \\
&\quad \le \mathbb{P}\Big(\sup_{\tau_1,\tau_2\in[0,1]} \sup_{\substack{\|(u,v)-(\tau_1,\tau_2)\|_\infty \\ \le \sup_{\tau\in[0,1]}|\hat{F}_n^{-1}(\tau)-\tau|}} \big|\hat{G}_{n,U}(u, v; \omega) - \hat{G}_{n,U}(\tau_1, \tau_2; \omega)\big| > x\big((nb_n)^{-1/2} + b_n^k\big)\Big) \\
&\quad \le \mathbb{P}\Big(\sup_{\tau_1,\tau_2\in[0,1]} \sup_{\substack{|u-\tau_1|\le\delta_n \\ |v-\tau_2|\le\delta_n}} \big|\hat{G}_{n,U}(u, v; \omega) - \hat{G}_{n,U}(\tau_1, \tau_2; \omega)\big| > x\big((nb_n)^{-1/2} + b_n^k\big), \\
&\qquad\qquad \sup_{\tau\in[0,1]} \big|\hat{F}_n^{-1}(\tau) - \tau\big| \le \delta_n\Big) + \mathbb{P}\Big(\sup_{\tau\in[0,1]} \big|\hat{F}_n^{-1}(\tau) - \tau\big| > \delta_n\Big) \\
&\quad = P_1^n + P_2^n, \qquad \text{say.}
\end{aligned}$$

It follows from Lemma A.5 that P_2^n is $\mathrm{o}(1)$. As for P_1^n, it is bounded by

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{\tau_1,\tau_2\in[0,1]} \sup_{\substack{|u-\tau_1|\le\delta_n \\ |v-\tau_2|\le\delta_n}} \big|\hat{H}_{n,U}(u, v; \omega) - \hat{H}_{n,U}(\tau_1, \tau_2; \omega)\big| > \big(1 + (nb_n)^{1/2} b_n^k\big)x/2\Big) \\
&\quad + I\Big\{\sup_{\tau_1,\tau_2\in[0,1]} \sup_{\substack{|u-\tau_1|\le\delta_n \\ |v-\tau_2|\le\delta_n}} \big|\mathbb{E}\hat{G}_{n,U}(u, v; \omega) - \mathbb{E}\hat{G}_{n,U}(\tau_1, \tau_2; \omega)\big| > \big((nb_n)^{-1/2} + b_n^k\big)x/2\Big\},
\end{aligned}$$

where the first term tends to zero in view of (A.3). To see that the indicator in the second term also is $\mathrm{o}(1)$, note that

$$\begin{aligned}
&\sup_{\tau_1,\tau_2\in[0,1]} \sup_{\substack{|u-\tau_1|\le\delta_n \\ |v-\tau_2|\le\delta_n}} \big|\mathbb{E}\hat{G}_{n,U}(u, v; \omega) - \mathbb{E}\hat{G}_{n,U}(\tau_1, \tau_2; \omega)\big| \\
&\quad \le \sup_{\tau_1,\tau_2\in[0,1]} \sup_{\substack{|u-\tau_1|\le\delta_n \\ |v-\tau_2|\le\delta_n}} \big|\mathbb{E}\hat{G}_{n,U}(u, v; \omega) - \mathfrak{f}_{q_u,q_v}(\omega) - B_n^{(k)}(u, v, \omega)\big| \\
&\qquad + \sup_{\tau_1,\tau_2\in[0,1]} \sup_{\substack{|u-\tau_1|\le\delta_n \\ |v-\tau_2|\le\delta_n}} \big|B_n^{(k)}(\tau_1, \tau_2, \omega) + \mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) - \mathbb{E}\hat{G}_{n,U}(\tau_1, \tau_2; \omega)\big|
\end{aligned}$$

$$+ \sup_{\tau_1,\tau_2\in[0,1]} \sup_{\substack{|u-\tau_1|\le\delta_n \\ |v-\tau_2|\le\delta_n}} \left| \mathfrak{f}_{q_u,q_v}(\omega) + B_n^{(k)}(u,v,\omega) - \mathfrak{f}_{q_{\tau_1},q_{\tau_2}}(\omega) - B_n^{(k)}(\tau_1,\tau_2,\omega) \right|$$

$$= \mathrm{o}\big(n^{-1/2} b_n^{-1/2} + b_n^k\big) + \mathrm{O}\big(\delta_n \big(1 + |\log \delta_n|\big)^d\big),$$

where d still is the constant from Lemma A.3 corresponding to $j = k$. Here, we have applied part (ii) of Theorem 3.6 to bound the first two terms and Lemma A.3 for the third one. For any fixed ω, thus, $P^n(\omega) = \mathrm{o}(1)$, which establishes the pointwise version of the claim.

We now turn to the uniformity (with respect to ω) issue. For an arbitrary $y_n > 0$, similar arguments as above yield, with the same δ_n,

$$\mathbb{P}\Big(\sup_{\omega\in\mathbb{R}} \sup_{\tau_1,\tau_2\in[0,1]} \big| \hat{G}_{n,R}(\tau_1,\tau_2;\omega) - \hat{G}_{n,U}(\tau_1,\tau_2;\omega) \big| > y_n \Big)$$

$$\le \mathbb{P}\Big(\sup_{\omega\in\mathbb{R}} \sup_{\tau_1,\tau_2\in[0,1]} \sup_{\substack{|u-\tau_1|\le\delta_n \\ |v-\tau_2|\le\delta_n}} \big| \hat{H}_{n,U}(u,v;\omega) - \hat{H}_{n,U}(\tau_1,\tau_2;\omega) \big| > (nb_n)^{1/2} y_n/2 \Big)$$

$$+ I\Big\{ \sup_{\omega\in\mathbb{R}} \sup_{\tau_1,\tau_2\in[0,1]} \sup_{\substack{|u-\tau_1|\le\delta_n \\ |v-\tau_2|\le\delta_n}} \big| \mathbb{E}\hat{G}_{n,U}(u,v;\omega) - \mathbb{E}\hat{G}_{n,U}(\tau_1,\tau_2;\omega) \big| > y_n/2 \Big\} + \mathrm{o}(1).$$

That the indicator in the latter expression is o(1) follows by the same arguments as above [note that Lemma A.3 and the statement of part (ii) both hold uniformly in $\omega \in \mathbb{R}$]. To bound the probability term, observe that by Lemma A.6, $\sup_{\tau_1,\tau_2} \sup_{j=1,\dots,n} |I_{n,U}^{\tau_1,\tau_2}(2\pi j/n)|$ is $\mathrm{O}_P(n^{2/K})$ for any $K > 0$. Moreover, the uniform Lipschitz continuity of W implies that W_n also is uniformly Lipschitz continuous with constant of order $\mathrm{O}(b_n^{-2})$. Combining those facts with Lemma A.3 and the assumptions on b_n, we obtain

$$\sup_{\substack{\omega_1,\omega_2\in\mathbb{R} \\ |\omega_1-\omega_2|\le n^{-3}}} \sup_{\tau_1,\tau_2\in[0,1]} \big| \hat{H}_{n,U}(\tau_1,\tau_2;\omega_1) - \hat{H}_{n,U}(\tau_1,\tau_2;\omega_2) \big| = \mathrm{o}_P(1).$$

By periodicity of $\hat{H}_{n,U}$ in the argument ω, it thus remains to show that

$$\max_{\omega=0,2\pi n^{-3},\dots,2\pi} \sup_{\tau_1,\tau_2\in[0,1]} \sup_{\substack{|u-\tau_1|\le\delta_n \\ |v-\tau_2|\le\delta_n}} \big| \hat{H}_{n,U}(u,v;\omega) - \hat{H}_{n,U}(\tau_1,\tau_2;\omega) \big| = \mathrm{o}_P(1).$$

Lemmas A.1 and A.7 entail the existence of a random variable $S(\omega)$ such that, for any fixed $\omega \in \mathbb{R}$,

$$\sup_{\tau_1,\tau_2\in[0,1]} \sup_{\substack{|u-\tau_1|\le\delta_n \\ |v-\tau_2|\le\delta_n}} \big| \hat{H}_{n,U}(u,v;\omega) - \hat{H}_{n,U}(\tau_1,\tau_2;\omega) \big| \le \big|S(\omega)\big| + R_n(\omega),$$

where $\sup_{\omega\in\mathbb{R}} |R_n(\omega)| = \mathrm{o}_P(1)$ and

$$\max_{\omega=0,2\pi n^{-3},\dots,2\pi} \mathbb{E}\big[\big|S^{2L}(\omega)\big|\big] \le K_L^{2L} \bigg(\int_0^\eta \epsilon^{-4/(2L\gamma)}\,\mathrm{d}\epsilon + \big(\delta_n^{\gamma/2} + 2(nb_n)^{-1/2}\big)\eta^{-8/(2L\gamma)} \bigg)^{2L}$$

for any $0 < \gamma < 1$, $L \in \mathbb{N}$, $0 < \eta < \delta_n$, and a constant K_L depending on L only. For appropriate choice of L and γ, this latter bound is $\mathrm{o}(n^{-3})$; since the maximum is over a set with $\mathrm{O}(n^3)$ elements. This completes the proof of part (iii).

A.4. Details for the proof of parts (i) and (iii) of Theorem 3.6

This section contains the main lemmas used in Sections A.1 and A.3 above. We use the notation introduced at the beginning of the proof of Theorem 3.6. The proofs of the results presented here can be found in the online supplement [33] [Section 1.3].

For the statement of the first result, recall that, for any non-decreasing, convex function $\Psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\Psi(0) = 0$ the *Orlicz norm* of a real-valued random variable Z is defined as (see, e.g., van der Vaart and Wellner [54], Chapter 2.2)

$$\|Z\|_\Psi = \inf\bigl\{C > 0\colon \mathbb{E}\Psi\bigl(|Z|/C\bigr) \le 1\bigr\}.$$

Lemma A.1. *Let $\{\mathbb{G}_t\colon t \in T\}$ be a separable stochastic process with $\|\mathbb{G}_s - \mathbb{G}_t\|_\Psi \le C\,\mathrm{d}(s,t)$ for all s, t with $d(s,t) \ge \bar{\eta}/2 \ge 0$. Denote by $D(\epsilon, d)$ the packing number of the metric space (T, d). Then, for any $\delta > 0$, $\eta \ge \bar{\eta}$, there exists a random variable S_1 and a constant $K < \infty$ such that*

$$\sup_{d(s,t)\le\delta} |\mathbb{G}_s - \mathbb{G}_t| \le S_1 + 2 \sup_{d(s,t)\le\bar{\eta}, t\in\tilde{T}} |\mathbb{G}_s - \mathbb{G}_t|$$

and

$$\|S_1\|_\Psi \le K\biggl[\int_{\bar{\eta}/2}^{\eta} \Psi^{-1}\bigl(D(\epsilon, d)\bigr)\,\mathrm{d}\epsilon + (\delta + 2\bar{\eta})\Psi^{-1}\bigl(D^2(\eta, d)\bigr)\biggr],$$

where the set $\tilde{T}$ contains at most $D(\bar{\eta}, d)$ points. In particular, by Markov's inequality (cf. van der Vaart and Wellner [54], page 96),

$$\mathbb{P}\bigl(|S_1| > x\bigr) \le \biggl(\Psi\biggl(x\biggl[8K\biggl(\int_{\bar{\eta}/2}^{\eta} \Psi^{-1}\bigl(D(\epsilon, d)\bigr)\,\mathrm{d}\epsilon + (\delta + 2\bar{\eta})\Psi^{-1}\bigl(D^2(\eta, d)\bigr)\biggr)\biggr]^{-1}\biggr)\biggr)^{-1}$$

for any $x > 0$.

Lemma A.2. *Let $X_0, \ldots, X_{n-1}$ be the finite realization of a strictly stationary process with $X_0 \sim U[0,1]$, and let* (W) *hold. For $x = (x_1, x_2)$ let $\hat{H}_n(x; \omega) := \sqrt{nb_n}(\hat{G}_n(x_1, x_2; \omega) - \mathbb{E}[\hat{G}_n(x_1, x_2; \omega)])$. For any Borel set A, define*

$$d_n^A(\omega) := \sum_{t=0}^{n-1} I\{X_t \in A\}\mathrm{e}^{-\mathrm{i}t\omega}.$$

Assume that, for $p = 1, \dots, P$, there exist a constant C and a function $g : \mathbb{R}^+ \to \mathbb{R}^+$, both independent of $\omega_1, \dots, \omega_p \in \mathbb{R}$, n and $A_1, \dots, A_p$, such that

$$\left|\operatorname{cum}\left(d_n^{A_1}(\omega_1), \dots, d_n^{A_p}(\omega_p)\right)\right| \le C\left(\left|\Delta_n\left(\sum_{i=1}^p \omega_i\right)\right| + 1\right) g(\varepsilon) \tag{A.5}$$

for any Borel sets $A_1, \dots, A_p$ with $\min_j \mathbb{P}(X_0 \in A_j) \le \varepsilon$. Then there exists a constant K (depending on C, L, g only) such that

$$\sup_{\omega \in \mathbb{R}} \sup_{\|a-b\|_1 \le \varepsilon} \mathbb{E}\left|\hat{H}_n(a; \omega) - \hat{H}_n(b; \omega)\right|^{2L} \le K \sum_{\ell=0}^{L-1} \frac{g^{L-\ell}(\varepsilon)}{(nb_n)^\ell}$$

for all ε with $g(\varepsilon) < 1$ and all $L = 1, \dots, P$.

Lemma A.3. *Under the assumptions of Theorem 3.5, the derivative $(\tau_1, \tau_2) \mapsto \frac{\mathrm{d}^j}{\mathrm{d}\omega^j} \mathfrak{f}_{q_{\tau_1}, q_{\tau_2}}(\omega)$ exists and satisfies, for any $j \in \mathbb{N}_0$ and some constants C, d that are independent of $a = (a_1, a_2)$, $b = (b_1, b_2)$ but may depend on j,*

$$\sup_{\omega \in \mathbb{R}} \left| \frac{\mathrm{d}^j}{\mathrm{d}\omega^j} \mathfrak{f}_{q_{a_1}, q_{a_2}}(\omega) - \frac{\mathrm{d}^j}{\mathrm{d}\omega^j} \mathfrak{f}_{q_{b_1}, q_{b_2}}(\omega) \right| \le C\|a - b\|_1 \left(1 + \left|\log \|a - b\|_1\right|\right)^d.$$

Lemma A.4. *Let the strictly stationary process $(X_t)_{t \in \mathbb{Z}}$ satisfy assumption* (C). *For any Borel set A, define*

$$d_n^A(\omega) := \sum_{t=0}^{n-1} I\{X_t \in A\} \mathrm{e}^{-\mathrm{i}t\omega}.$$

Let $A_1, \dots, A_p \subset [0, 1]$ be intervals, and let $\varepsilon := \min_{j=1,\dots,p} \mathbb{P}(X_0 \in A_j)$. Then, for any p-tuple $\omega_1, \dots, \omega_p \in \mathbb{R}$,

$$\left|\operatorname{cum}\left(d_n^{A_1}(\omega_1), \dots, d_n^{A_p}(\omega_p)\right)\right| \le C\left(\left|\Delta_n\left(\sum_{i=1}^p \omega_i\right)\right| + 1\right) \varepsilon\left(|\log \varepsilon| + 1\right)^d,$$

where $\Delta_n(\lambda) := \sum_{t=0}^{n-1} \mathrm{e}^{\mathrm{i}t\lambda}$ and the constants C, d depend only on K, p, and ρ [with ρ from condition (C)].

Lemma A.5. *Let $X_0, \dots, X_{n-1}$ be the finite realization of a strictly stationary process satisfying* (C) *and such that $X_0 \sim U[0, 1]$. Then*

$$\sup_{\tau \in [0,1]} \left|\hat{F}_n^{-1}(\tau) - \tau\right| = \mathrm{O}_P\left(n^{-1/2}\right).$$

Lemma A.6. *Let the strictly stationary process* $(X_t)_{t\in\mathbb{Z}}$ *satisfy assumption* (C); *assume moreover that* $X_0 \sim U[0,1]$. *For any* $y \in [0,1]$, *define*

$$d_n^y(\omega) := \sum_{t=0}^{n-1} I\{X_t \le y\} \mathrm{e}^{-\mathrm{i}\omega t}.$$

Then, for any $k \in \mathbb{N}$,

$$\sup_{\omega\in\mathcal{F}_n} \sup_{y\in[0,1]} \left|d_n^y(\omega)\right| = \mathrm{O}_P\left(n^{1/2+1/k}\right).$$

Lemma A.7. *Under the assumptions of Theorem* 3.6, *let* δ_n *be a sequence of non-negative real numbers. Assume that there exists* $\gamma \in (0,1)$, *such that* $\delta_n = \mathrm{O}((nb_n)^{-1/\gamma})$. *Then*

$$\sup_{\omega\in\mathbb{R}} \sup_{\substack{u,v\in[0,1]^2 \\ \|u-v\|_1\le\delta_n}} \left|\hat{H}_n(u;\omega) - \hat{H}_n(v;\omega)\right| = \mathrm{o}_P(1).$$

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Supplementary Material

Supplement to "Quantile spectral processes: Asymptotic analysis and inference". (DOI: 10.3150/15-BEJ711SUPP; .pdf). We provide details for the proof of part (ii) of Theorem 3.6, and proofs for Propositions 3.1, 3.2, and 3.4. Further, we prove results from Section A.4, namely Lemmas A.1–A.7.

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