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Functional limit theorems for generalized variations of the fractional Brownian sheet

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We prove functional central and non-central limit theorems for generalized variations of the anisotropic d-parameter fractional Brownian sheet (fBs) for any natural number d. Whether the central or the non-central limit theorem applies depends on the Hermite rank of the variation functional and on the smallest component of the Hurst parameter vector of the fBs. The limiting process in the former result is another fBs, independent of the original fBs, whereas the limit given by the latter result is an Hermite sheet, which is driven by the same white noise as the original fBs. As an application, we derive functional limit theorems for power variations of the fBs and discuss what is a proper way to interpolate them to ensure functional convergence.

Keywords: central limit theorem; fractional Brownian sheet; Hermite sheet; Malliavin calculus; non-central limit theorem; power variation

1. Introduction

Since the seminal works by Breuer and Major [7], Dobrushin and Major [9], Giraitis and Surgailis [10], Rosenblatt [27] and Taqqu [28–31], much attention has been given to the study of the asymptotic behaviour of normalized functionals of Gaussian fields, as these quantities arise naturally in applications, for example, where models exhibiting *long-range dependence* are needed. The aforementioned papers focus on nonlinear functionals of a stationary Gaussian field, for which one can derive a *central limit theorem* (in a finite-dimensional sense or in a functional sense) if the correlation function of the field decays sufficiently fast to zero; see [7] for a precise formulation. However, if the correlation function decays too slowly to zero, then only a *non-central limit theorem* can be established, meaning that the limiting distribution fails to be Gaussian; see, for example, [27].

In particular, these results apply to functionals of the *fractional Brownian motion* (fBm). Let $B_H := \{B_H(t)\colon t \in \mathbb{R}\}$ be a fBm with Hurst parameter $H \in (0, 1)$, which is the unique (in law) H-self similar Gaussian process with stationary increments; see (3.2) and (3.3) below for the definitions of these key properties. The behaviour of the so-called *Hermite variations* of B_H, depending on the value of H, can be described as follows. Let $k \in \{1, 2, \dots\}$ and let P_k denote the kth Hermite polynomial, the definition of which we recall in (2.5) below. Applying results

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from [7,9,10,31], one obtains that

(a) If $H \in (0, 1 - \frac{1}{2k})$, then

$$n^{-1/2} \sum_{j=1}^{n} P_k\left(n^H\left(B_H\left(\frac{j}{n}\right) - B_H\left(\frac{j-1}{n}\right)\right)\right) \xrightarrow[n\to\infty]{\mathcal{L}} N\big(0, \sigma_1^2(H,k)\big).$$

(b) If $H = 1 - \frac{1}{2k}$, then

$$\big(n\log(n)\big)^{-1/2} \sum_{j=1}^{n} P_k\left(n^H\left(B_H\left(\frac{j}{n}\right) - B_H\left(\frac{j-1}{n}\right)\right)\right) \xrightarrow[n\to\infty]{\mathcal{L}} N\left(0, \sigma_1^2\left(1 - \frac{1}{2k}, k\right)\right).$$

(c) If $H \in (1 - \frac{1}{2k}, 1)$, then

$$n^{1-2H} \sum_{j=1}^{n} P_k\left(n^H\left(B_H\left(\frac{j}{n}\right) - B_H\left(\frac{j-1}{n}\right)\right)\right) \xrightarrow[n\to\infty]{L^2(\Omega)} \mathrm{Hermite}_{1,k}\big(1 - k(1-H)\big).$$

Above, $\xrightarrow{\mathcal{L}}$ denotes convergence in law, $N(0, \sigma_1^2(H,k))$ denotes the centered Gaussian law with variance $\sigma_1^2(H,k) > 0$, whereas $\mathrm{Hermite}_{1,k}(1-k(1-H))$ stands for a so-called Hermite random variable given by the value of an *Hermite process*, of order k with Hurst parameter $1 - k(1 - H) \in (\frac{1}{2}, 1)$, at time 1. Such an Hermite process can be represented as a k-fold multiple Wiener integral with respect to Brownian motion, as proven by Taqqu [30,31]. Moreover, the process is non-Gaussian if $k \geq 2$. (More details on the Hermite process are provided in Section 2.4.) The key observation here is that there are two regimes: *Gaussian*, subsuming cases (a) and (b), and *Hermite*, case (c), depending on the Hurst parameter H and on the order k.

The convergences in all cases (a), (b), and (c) can be extended to more general functionals, which we call *generalized variations* in this paper, obtained by replacing the Hermite polynomial P_k with a function

$$f(u) := \sum_{k=\underline{k}}^{\infty} a_k P_k(u), \qquad u \in \mathbb{R}, \tag{1.1}$$

where $\underline{k}$ is the so-called *Hermite rank* of f. (Naturally, conditions on the summability of the coefficients $a_{\underline{k}}, a_{\underline{k}+1}, \ldots$ have to be added.) In this setting, the prevailing regime (Gaussian or Hermite) will depend on the Hurst parameter H and on the Hermite rank $\underline{k}$ analogously to the simpler setting discussed above. In addition, functional versions of these asymptotic results (under additional assumptions on the coefficients $a_{\underline{k}}, a_{\underline{k}+1}, \ldots$) can be proven in the Skorohod space $D([0,1])$; see [28,31].

In connection to applications that involve spatial or spatio-temporal modeling, processes of multiple parameters are also of interest. Recently, there has been interest in understanding the asymptotic behaviour of realized quadratic variations and power variations of *ambit fields* [5,21]. An ambit field is an anisotropic multiparameter process driven by white noise, or more generally, by an infinitely-divisible random measure. The problem of finding distributional limits

(central or non-central limit theorems) for such power variations is, however, intricate because the dependence structure of an ambit field can be very general; only a "partial" central limit theorem is obtained in [21]. As a first approximation, it is thus useful to study this problem with simpler processes that incorporate some of the salient features of ambit fields, such as the non-semimartingality of one-parameter "marginal processes" (see [21], Section 2.2) and strong dependence. A tractable process that incorporates some key features of ambit fields is the *fractional Brownian sheet* (fBs), defined by Ayache *et al.* [1], which is a multi-parameter extension of the fBm. In particular, it is a Gaussian process with stationary rectangular increments.

For concreteness, let $Z := \{Z(t)\colon\ t \in [0,1]^2\}$ be a two-parameter anisotropic fBs with Hurst parameter $(H_1, H_2) \in (0,1)^2$; see Section 2.2 for a precise definition. In view of the asymptotic behaviour in cases (a), (b), and (c) involving the fBm, it is natural to ask what is the asymptotic behaviour of Hermite variations of Z with different values of H_1 and H_2. Consider, for example, the "mixed" case where $H_1 < 1 - \frac{1}{2k}$ and $H_2 > 1 - \frac{1}{2k}$, which has no counterpart in the one-parameter setting. Because of the structure of the fBs, it is tempting to conjecture that in this case the limiting law is a mixture of a Gaussian law and a marginal law of an Hermite process. However, as shown in [25], this is not the case and once again only two limiting laws can be obtained:

(a′) If $(H_1, H_2) \in (0,1)^2 \setminus (1 - \frac{1}{2k}, 1)^2$, then

$$\varphi(n, H_1, H_2) \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} P_k\biggl(n^{H_1+H_2} Z\biggl(\biggl[\frac{j_1-1}{n}, \frac{j_1}{n}\biggr) \times \biggl[\frac{j_2-1}{n}, \frac{j_2}{n}\biggr)\biggr)\biggr)$$
$$\xrightarrow[n\to\infty]{\mathcal{L}} N\bigl(0, \sigma_2^2(H_1, H_2, \underline{k})\bigr).$$

(b′) If $(H_1, H_2) \in (1 - \frac{1}{2k}, 1)^2$, then

$$\varphi(n, H_1, H_2) \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} P_k\biggl(n^{H_1+H_2} Z\biggl(\biggl[\frac{j_1-1}{n}, \frac{j_1}{n}\biggr) \times \biggl[\frac{j_2-1}{n}, \frac{j_2}{n}\biggr)\biggr)\biggr)$$
$$\xrightarrow[n\to\infty]{L^2(\Omega)} \mathrm{Hermite}_{2,k}\bigl(1 - k(1-H_1), 1 - k(1-H_2)\bigr).$$

Above, $Z([\frac{j_1-1}{n}, \frac{j_1}{n}) \times [\frac{j_2-1}{n}, \frac{j_2}{n}))$ stands for the increment of Z over the rectangle $[\frac{j_1-1}{n}, \frac{j_1}{n}) \times [\frac{j_2-1}{n}, \frac{j_2}{n})$, defined in Section 2.3 below, and $\varphi(n, H_1, H_2)$ is a suitable scaling factor; see [25], pages 9–10, for its definition. The limit in the case (b′) is the value of a two-parameter *Hermite sheet* (see Section 2.4), of order k with Hurst parameter $(1-k(1-H_1), 1-k(1-H_2)) \in (\frac{1}{2}, 1)^2$, at point $(1,1)$. Contrary to the one-parameter case, the results obtained in [25] are proved only for one-dimensional laws; neither finite-dimensional (except in the particular setting of [24]) nor functional convergence (i.e., tightness in a function space) of Hermite variations has been established so far. (In particular in the d-parameter realm with $d \geq 2$, tightness is a non-trivial issue, which has not been addressed in [25] or in the related paper [24].)

The first main result of this paper addresses the question about functional convergence in the general, d-parameter case for any $d \in \mathbb{N}$. We prove a functional central limit theorem,

Theorem 2.4, for generalized variations of a d-parameter anisotropic fBs Z. (As mentioned above, generalized variations extend Hermite variations by replacing P_k with a function f of the form (1.1).) This result applies if at least one of the components of the Hurst parameter vector $H = (H_1, \dots, H_d) \in (0, 1)^d$ of Z is less than or equal to $1 - \frac{1}{2\underline{k}}$, where $\underline{k}$ is the Hermite rank of f. A novel feature of this result is that the limiting process is a new fBs, independent of Z, with Hurst parameter vector $\widetilde{H} = (\widetilde{H}_1, \dots, \widetilde{H}_d)$ given by

$$\widetilde{H}_\nu := \begin{cases} \dfrac{1}{2}, & H_\nu \leq 1 - \dfrac{1}{2\underline{k}}, \\ 1 - \underline{k}(1 - H_\nu), & H_\nu > 1 - \dfrac{1}{2\underline{k}}, \end{cases}$$

for $\nu \in \{1, \dots, d\}$. Note, in particular, that if $H \in (0, 1 - \frac{1}{2\underline{k}}]^d$, then the limit reduces to an ordinary Brownian sheet. The proof of Theorem 2.4 is based on the limit theory for multiple Wiener integrals, due to Nualart and Peccati [20], and its multivariate extension by Peccati and Tudor [22]. To prove the functional convergence asserted in Theorem 2.4, we use the tightness criterion of Bickel and Wichura [6] in the space $D([0, 1]^d)$, which is d-parameter generalization of $D([0, 1])$, and a moment bound for nonlinear functionals of a stationary Gaussian process on $\mathbb{Z}^d$ (Lemma 4.1).

The second main result of this paper is a functional non-central limit theorem, Theorem 2.7, for generalized variations of Z in the remaining case where each of the components of H is greater than $1 - \frac{1}{2\underline{k}}$. In this case, the limit is a d-parameter Hermite sheet and the convergence holds in probability and also pointwise in $L^2(\Omega)$. Assuming that Z is defined by a moving-average representation with respect to a white noise $\mathcal{W}$ on $\mathbb{R}^d$, we can give a novel and explicit description of the limit; it is defined using the representation introduced by Clarke De la Cerda and Tudor [8] with respect to the same white noise $\mathcal{W}$. This makes the relation between Z and the Hermite sheet precise and constitutes a step further compared to the existing literature (see [15,25]), where the limiting Hermite process/sheet is simply obtained as an abstract limit of a Cauchy sequence, from which the properties of the limiting object are deduced.

As an application of Theorems 2.4 and 2.7, we study the asymptotic behaviour of power variations of the fBs Z. As a straightforward consequence of our main results, we obtain a law of large numbers for these power variations. We then study the more delicate question regarding the asymptotic behaviour of rescaled fluctuations of power variations around the limit given by the law of large numbers. In the case of *odd* power variations, the rescaled fluctuations have a limit, either Gaussian or Hermite, but with *even* power variations, the fluctuations might not converge in a functional way if $d \geq 2$. We show that this convergence issue does not arise at all if one considers instead continuous, multilinear interpolations of power variations.

The paper is organized as follows. In Section 2, we introduce the setting of the paper, some key definitions and the statements of Theorems 2.4 and 2.7. The proofs of these two main results are presented in Sections 3 and 4, the former section collecting the finite-dimensional and the latter the functional arguments. Finally, the application to power variations is given in Section 5.

2. Preliminaries and main results

2.1. Notation

We use the convention that $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{R}_+ := [0, \infty)$. The notation $|A|$ stands for the cardinality of a finite set A. For any $y \in \mathbb{R}$, we write $\lfloor y \rfloor := \max\{n \in \mathbb{Z}\colon n \le v\}$, $\{y\} := y - \lfloor y \rfloor$, and $(y)_+ := \max(y, 0)$. The symbol γ denotes the standard Gaussian measure on $\mathbb{R}$, that is, $\gamma(\mathrm{d}y) := (2\pi)^{-1/2}\exp(-y^2/2)\,\mathrm{d}y$. From now on we fix d in $\mathbb{N}$.

For any vectors $s = (s_1, \dots, s_d) \in \mathbb{R}^d$ and $t = (t_1, \dots, t_d) \in \mathbb{R}^d$, the relation $s \le t$ (resp., $s < t$) signifies that $s_\nu \le t_\nu$ (resp., $s_\nu < t_\nu$) for all $\nu \in \{1, \dots, d\}$. We also use the notation

$$st := (s_1 t_1, \dots, s_d t_d) \in \mathbb{R}^d, \qquad \frac{s}{t} := \left(\frac{s_1}{t_1}, \dots, \frac{s_d}{t_d}\right) \in \mathbb{R}^d,$$

$$\lfloor s \rfloor := \big(\lfloor s_1 \rfloor, \dots, \lfloor s_d \rfloor\big) \in \mathbb{Z}^d, \qquad \langle s \rangle := s_1 \cdots s_d \in \mathbb{R},$$

$$|s| := \big(|s_1|, \dots, |s_d|\big) \in \mathbb{R}_+^d, \qquad \{s\} := \big(\{s_1\}, \dots, \{s_d\}\big) \in [0, 1)^d.$$

Further, when $s \in \mathbb{R}_+^d$, we write $s^t := (s_1^{t_1}, \dots, s_d^{t_d}) \in \mathbb{R}_+^d$, and when $s \le t$, we write $[s, t) := [s_1, t_1) \times \cdots \times [s_d, t_d) \subset \mathbb{R}^d$. Occasionally, we use the norm $\|s\|_\infty := \max(|s_1|, \dots, |s_d|)$ for $s \in \mathbb{R}^d$.

For the sake of clarity, we will consistently use the following convention: $i, i^{(1)}, i^{(2)}, \dots$ are multi-indices (vectors) in $\mathbb{Z}^d$ and $j, j_1, j_2, \dots$ are indices (scalars) in $\mathbb{Z}$.

2.2. Anisotropic fractional Brownian sheet

We consider an anisotropic, d-parameter fractional Brownian sheet (fBs) $Z := \{Z(t)\colon t \in \mathbb{R}^d\}$ with Hurst parameter $H \in (0, 1)^d$, which is a centered Gaussian process with covariance

$$R_H^{(d)}(s, t) := \mathbf{E}\big[Z(s)Z(t)\big] = \prod_{\nu=1}^{d} R_{H_\nu}^{(1)}(s_\nu, t_\nu), \qquad s, t \in \mathbb{R}^d, \tag{2.1}$$

where

$$R_{H_\nu}^{(1)}(s_\nu, t_\nu) := \tfrac{1}{2}\big(|s_\nu|^{2H_\nu} + |t_\nu|^{2H_\nu} - |s_\nu - t_\nu|^{2H_\nu}\big), \qquad s_\nu, t_\nu \in \mathbb{R},$$

is the covariance of a fractional Brownian motion with Hurst parameter H_ν.

In what follows, it will be convenient to assume that the fBs Z has a particular representation. To this end, let us denote by $\mathcal{B}_0(\mathbb{R}^d)$ the family of Borel sets of $\mathbb{R}^d$ with finite Lebesgue measure. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space that supports a *white noise* $\mathcal{W} := \{\mathcal{W}(A)\colon A \in \mathcal{B}_0(\mathbb{R}^d)\}$, which is a centered Gaussian process with covariance

$$\mathbf{E}\big[\mathcal{W}(A)\mathcal{W}(B)\big] = \mathrm{Leb}_d(A \cap B), \qquad A, B \in \mathcal{B}_0\big(\mathbb{R}^d\big),$$

where $\mathrm{Leb}_d(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}^d$. The process Z can be defined as a *Wiener integral* with respect to $\mathcal{W}$ (see, e.g., [19] for the definition), namely

$$Z(t) := \int G_H^{(d)}(t,u)\mathcal{W}(\mathrm{d}u), \qquad t \in \mathbb{R}^d, \tag{2.2}$$

where the kernel

$$G_H^{(d)}(t,u) := \prod_{\nu=1}^{d} G_{H_\nu}^{(1)}(t_\nu, u_\nu), \qquad t, u \in \mathbb{R}^d, \tag{2.3}$$

is defined using the one-dimensional Mandelbrot–Van Ness [13] kernel

$$G_{H_\nu}^{(1)}(t_\nu, u_\nu) := \frac{1}{\chi(H_\nu)}\big((t_\nu - u_\nu)_+^{H_\nu - 1/2} - (-u_\nu)_+^{H_\nu - 1/2}\big), \qquad t_\nu, u_\nu \in \mathbb{R}, \tag{2.4}$$

with the normalizing constant

$$\chi(H_\nu) := \left(\frac{1}{2H_\nu} + \int_0^\infty \big((1+y)^{H_\nu - 1/2} - y^{H_\nu - 1/2}\big)\,\mathrm{d}y\right)^{1/2}.$$

We refer to [1] for a proof that the process Z defined via (2.2) does indeed have the covariance structure (2.1). The fBs admits a continuous modification (see [3], page 1040), so we may assume from now on that Z is continuous.

2.3. Increments and generalized variations

Given a function (or a realization of a stochastic process) $h : \mathbb{R}^d \to \mathbb{R}$, we define the increment of h over the half-open hyperrectangle $[s,t) \subset \mathbb{R}^d$ for any $s \le t$ by

$$h\big([s,t)\big) := \sum_{i \in \{0,1\}^d} (-1)^{d - \sum_{\nu=1}^d i_\nu} h\big((1-i)s + it\big).$$

(Note that i_ν above stands for the νth component of the multi-index i.) This definition can be recovered by differencing iteratively with respect to each of the arguments of the function h. Thus, the increment can be seen as a discrete analogue of the partial derivative $\frac{\partial^d}{\partial t_1 \cdots \partial t_d}$.

Remark 2.1. It is useful to note that if there exists functions $h_\nu : \mathbb{R} \to \mathbb{R}$, $\nu \in \{1, \dots, d\}$, such that $h(t) = h_1(t_1) \cdots h_d(t_d)$ for any $t \in \mathbb{R}^d$, then

$$h\big([s,t)\big) = \prod_{\nu=1}^{d} \big(h_\nu(t_\nu) - h_\nu(s_\nu)\big),$$

which is easily verified by induction with respect to d using iterative differencing.

Let us fix a sequence $(m(n))_{n\in\mathbb{N}} \subset \mathbb{N}^d$ of multi-indices with the property

$$\underline{m}(n) := \min\bigl(m_1(n), \dots, m_d(n)\bigr) \underset{n\to\infty}{\longrightarrow} \infty$$

and a function $f \in L^2(\mathbb{R}, \gamma)$ such that $\int_{\mathbb{R}} f(u)\gamma(\mathrm{d}u) = 0$. Our aim is to study the asymptotic behaviour of a family $\{U_f^{(n)}\colon\ n \in \mathbb{N}\}$ of d-parameter processes, *generalized variations* of Z, defined by

$$U_f^{(n)}(t) := \sum_{1\le i\le \lfloor m(n)t\rfloor} f\left(\langle m(n)^H\rangle Z\left(\left[\frac{i-1}{m(n)}, \frac{i}{m(n)}\right)\right)\right), \qquad t \in [0,1]^d, n \in \mathbb{N}.$$

In this definition, $\langle m(n)^H\rangle = m_1(n)^{H_1}\cdots m_d(n)^{H_d}$ according to the notation and conventions set forth in Section 2.1. The realizations of $U_f^{(n)}$ belong to the space $D([0,1]^d)$, which for $d \ge 2$ is a generalization of the space $D([0,1])$ of càdlàg functions on $[0,1]$. We refer to [6], page 1662, for the definition of the space $D([0,1]^d)$. In particular, $C([0,1]^d) \subset D([0,1]^d)$. We endow $D([0,1]^d)$ with the Skorohod topology described in [6], page 1662. Convergence to a continuous function in this topology is, however, equivalent to uniform convergence (see, e.g., [21], Lemma B.2, for a proof in the case $d = 2$).

2.4. Functional limit theorems for generalized variations

We will now formulate two functional limit theorems for the family $\{U_f^{(n)}\colon n \in \mathbb{N}\}$ of generalized variations, defined above. The class of admissible functions f needs to be restricted somewhat, however, and the choice of f and the Hurst parameter H of Z will determine which of the limit theorems applies. Also, we need to rescale $U_f^{(n)}$ in suitable way that, likewise, depends on both f and H.

To this end, recall that the *Hermite polynomials*,

$$P_0(u) := 1, \qquad P_k(u) := (-1)^k \mathrm{e}^{u^2/2}\frac{\mathrm{d}^k}{\mathrm{d}u^k}\mathrm{e}^{-u^2/2}, \qquad u \in \mathbb{R}, k \in \mathbb{N}, \tag{2.5}$$

form a complete orthogonal system in $L^2(\mathbb{R}, \gamma)$. Thus, we may expand f in $L^2(\mathbb{R}, \gamma)$ as

$$f(u) = \sum_{k=\underline{k}}^{\infty} a_k P_k(u), \tag{2.6}$$

where the *Hermite coefficients* $a_{\underline{k}}, a_{\underline{k}+1}, \dots \in \mathbb{R}$ are such that $a_{\underline{k}} \ne 0$ and

$$\sum_{k=\underline{k}}^{\infty} k! a_k^2 < \infty. \tag{2.7}$$

The index $\underline{k}$ is called the *Hermite rank* of f, and the proviso $\int_{\mathbb{R}} f(u)\gamma(\mathrm{d}u) = 0$ ensures that $\underline{k} \ge 1$. We will assume that the Hermite coefficients decay somewhat faster than what (2.7) entails.

Assumption 2.2. *The Hermite coefficients $a_{\underline{k}}, a_{\underline{k}+1}, \dots$ of the function f satisfy*

$$\sum_{k=\underline{k}}^{\infty} 3^{k/2}\sqrt{k!}|a_k| < \infty.$$

Let us define a sequence $(c(n))_{n\in\mathbb{N}} \subset \mathbb{R}_+^d$ of rescaling factors by setting for any $\nu \in \{1, \dots, d\}$ and $n \in \mathbb{N}$,

$$c_\nu(n) := \begin{cases} m_\nu(n)^{2-2\underline{k}(1-H_\nu)}, & H_\nu \in \left(1 - \dfrac{1}{2\underline{k}}, 1\right), \\ m_\nu(n)\log\big(m_\nu(n)\big), & H_\nu = 1 - \dfrac{1}{2\underline{k}}, \\ m_\nu(n), & H_\nu \in \left(0, 1 - \dfrac{1}{2\underline{k}}\right). \end{cases}$$

Remark 2.3. Note that $\limsup_{n\to\infty} \frac{m_\nu(n)}{c_\nu(n)} < \infty$ and that, in fact, $\lim_{n\to\infty} \frac{m_\nu(n)}{c_\nu(n)} = 0$ if $H_\nu \in [1 - \frac{1}{2\underline{k}}, 1)$.

Now we can define a family $\{\overline{U}_f^{(n)} \colon n \in \mathbb{N}\}$ of rescaled generalized variations as

$$\overline{U}_f^{(n)}(t) := \frac{U_f^{(n)}(t)}{\langle c(n)\rangle^{1/2}}, \qquad t \in [0,1]^d, n \in \mathbb{N}.$$

Our first result is the following functional central limit theorem (FCLT) for generalized variations. Its proof is carried out in Section 3.2 and Section 4.2.

Theorem 2.4 (FCLT). *Let f be as above such that Assumption 2.2 holds and suppose that $H \in (0,1)^d \setminus (1 - \frac{1}{2\underline{k}}, 1)^d$. Then*

$$\big(Z, \overline{U}_f^{(n)}\big) \xrightarrow[n\to\infty]{\mathcal{L}} \big(Z, \Lambda_{H,f}^{1/2}\widetilde{Z}\big) \qquad \text{in } D\big([0,1]^d\big)^2, \tag{2.8}$$

where $\widetilde{Z}$ is a d-parameter fBs with Hurst parameter $\widetilde{H} \in [\frac{1}{2}, 1)^d$, independent of Z (defined, possibly, on an extension of $(\Omega, \mathcal{F}, \mathbf{P})$), and

$$\Lambda_{H,f} := \sum_{k=\max(\underline{k},2)}^{\infty} k! a_k^2 \big\langle b^{(k)}\big\rangle \in \mathbb{R}. \tag{2.9}$$

The vectors $\widetilde{H} \in [\frac{1}{2}, 1)^d$ *and* $b^{(k)} \in \mathbb{R}_+^d$, $k \geq \max(\underline{k}, 2)$, *that appear above are defined by setting for any* $\nu \in \{1, \dots, d\}$,

$$\widetilde{H}_\nu := \begin{cases} \dfrac{1}{2}, & H_\nu \in \left(0, 1 - \dfrac{1}{2\underline{k}}\right], \\ 1 - \underline{k}(1 - H_\nu), & H_\nu \in \left(1 - \dfrac{1}{2\underline{k}}, 1\right), \end{cases} \tag{2.10}$$

and

$$b_\nu^{(k)} := \begin{cases} \displaystyle\sum_{j \in \mathbb{Z}} \left(\frac{|j+1|^{2H_\nu} - 2|j|^{2H_\nu} + |j-1|^{2H_\nu}}{2} \right)^k, & H_\nu \in \left(0, 1 - \dfrac{1}{2\underline{k}}\right), \\ 2\left(\dfrac{(2\underline{k}-1)(\underline{k}-1)}{2\underline{k}^2} \right)^{\underline{k}} =: \iota(\underline{k}), & H_\nu = 1 - \dfrac{1}{2\underline{k}}, k = \underline{k}, \\ \dfrac{H_\nu^{\underline{k}}(2H_\nu - 1)^{\underline{k}}}{(1 - \underline{k}(1 - H_\nu))(1 - 2\underline{k}(1 - H_\nu))} =: \kappa(H_\nu, \underline{k}), & H_\nu \in \left(1 - \dfrac{1}{2\underline{k}}, 1\right), k = \underline{k}, \\ 0, & H_\nu \in [1 - \dfrac{1}{2\underline{k}}, 1), k > \underline{k}. \end{cases} \tag{2.11}$$

Remark 2.5. (1) The counterpart of the convergence (2.8) for finite-dimensional laws holds without Assumption 2.2; see Proposition 3.3 below.

(2) We may use $\max(\underline{k}, 2)$, instead of $\underline{k}$, as the lower bound for the summation index k in (2.9) since $\iota(1) = 0$ and

$$\begin{aligned} &\sum_{j \in \mathbb{Z}} \frac{|j+1|^{2\check{H}} - 2|j|^{2\check{H}} + |j-1|^{2\check{H}}}{2} \\ &\quad = \sum_{j \in \mathbb{Z}} \frac{|j|^{2\check{H}} - |j-1|^{2\check{H}}}{2} - \sum_{j \in \mathbb{Z}} \frac{|j|^{2\check{H}} - |j-1|^{2\check{H}}}{2} = 0 \end{aligned}$$

for any $\check{H} \in (0, \frac{1}{2})$. (Then, $\sum_{j \in \mathbb{Z}} |\frac{|j|^{2\check{H}} - |j-1|^{2\check{H}}}{2}| < \infty$ by the mean value theorem.)

(3) The convergence (2.8) can be understood in the framework of *stable convergence* in law, introduced by Rényi [23]. Equivalently to (2.8), $\overline{U}_f^{(n)}$ converges to $\Lambda_{H,f}^{1/2} \widetilde{Z}$ as $n \to \infty$ stably in law with respect to the σ-algebra generated by $\{Z(t) \colon t \in [0, 1]^d\}$.

Theorem 2.4 excludes the case $H \in (1 - \frac{1}{2\underline{k}}, 1)^d$. Then, the generalized variations do have a limit, but the limit is non-Gaussian, unless $\underline{k} = 1$. To describe the limit, we need the following definition, due to Clarke De la Cerda and Tudor [8].

Definition 2.6. *An anisotropic, d-parameter* Hermite sheet $\widehat{Z} := \{\widehat{Z}(t) \colon t \in \mathbb{R}_+^d\}$ *of order* $k \geq 2$ *with Hurst parameter* $\widetilde{H} \in (\frac{1}{2}, 1)^d$ *is defined as a k-fold multiple Wiener integral* (*see Section* 3.2)

with respect to the white noise $\mathcal{W}$,

$$\widehat{Z}(t) := \int \cdots \int \widehat{G}^{(k)}_{\widetilde{H}}\big(t, u^{(1)}, \dots, u^{(k)}\big)\mathcal{W}\big(\mathrm{d}u^{(1)}\big)\cdots \mathcal{W}\big(\mathrm{d}u^{(k)}\big) := I^{\mathcal{W}}_k\big(\widehat{G}^{(k)}_{\widetilde{H}}(t,\cdot)\big) \tag{2.12}$$

for any $t \in \mathbb{R}^d_+$. *In* (2.12), *the kernel* $\widehat{G}^{(d,k)}_{\widetilde{H}}(t,\cdot) \in L^2(\mathbb{R}^{kd})$ *is given by*

$$\begin{aligned}&\widehat{G}^{(k)}_{\widetilde{H}}\big(t, u^{(1)}, \dots, u^{(k)}\big)\\ &\quad := \frac{1}{\widehat{\chi}(\widetilde{H}, k)} \int_{[0,t)} \prod_{\kappa=1}^{k}\prod_{\nu=1}^{d}\big(y_\nu - u^{(\kappa)}_\nu\big)^{-1/2-(1-\widetilde{H}_\nu)/k}_+ \,\mathrm{d}y, \qquad u^{(1)}, \dots, u^{(k)} \in \mathbb{R}^d,\end{aligned}$$

using the normalizing constant

$$\widehat{\chi}(\widetilde{H}, k) := \prod_{\nu=1}^{d}\left(\frac{\mathrm{B}(1/2 - (1-\widetilde{H}_\nu)/k, (2(1-\widetilde{H}_\nu))/k)}{\widetilde{H}_\nu(2\widetilde{H}_\nu - 1)}\right)^{1/2},$$

where B *stands for the beta function*.

The Hermite sheet $\widehat{Z}$ is self-similar and has the same correlation structure as a fBs with Hurst parameter $\widetilde{H}$. In the case $k = 1$, the process $\widehat{Z}$ is Gaussian (in fact, it coincides with a fractional Brownian sheet with Hurst parameter $\widetilde{H}$) but for $k \geq 2$ it is non-Gaussian. In the case $k = 2$, the name *Rosenblatt sheet* (and *Rosenblatt process*, when $d = 1$; see [32]) is often used, in honor of Murray Rosenblatt's seminal paper [26]. See also the recent papers [12,33] for more details on the Rosenblatt distribution, including proofs that this distribution is *infinitely divisible*.

As our second main result, we obtain the following functional non-central limit theorem (FNCLT) for generalized variations. The proof of this result is carried out in Section 3.3 and Section 4.2.

Theorem 2.7 (FNCLT). *Let* f *be as above such that Assumption* 2.2 *holds and suppose that* $H \in (1 - \frac{1}{2\underline{k}}, 1)^d$. *Then*

$$\overline{U}^{(n)}_f \xrightarrow[n\to\infty]{\mathbf{P}} \Lambda^{1/2}_{H,f}\widehat{Z} \qquad \text{in } D\big([0,1]^d\big), \tag{2.13}$$

where $\widehat{Z}$ *is a* d*-parameter Hermite sheet of order* $\underline{k}$ *with Hurst parameter* $\widetilde{H}$, *given by* (2.10), *and* $\Lambda_{H,f}$ *is given by* (2.9).

Remark 2.8. (1) The convergence (2.13) holds pointwise in $L^2(\Omega, \mathcal{F}, \mathbf{P})$, even when Assumption 2.2 does not hold; see Proposition 3.5 below.

(2) Unlike in Theorem 2.4, the non-central limit $\widehat{Z}$ is defined on the original probability space $(\Omega, \mathcal{F}, \mathbf{P})$. In particular, $\widehat{Z}$ is driven by the same white noise $\mathcal{W}$ as Z.

(3) In the special case $\underline{k} = 1$, the limit in (2.13) is Gaussian. In fact, then $\Lambda_{H,f} = a_1^2$ and $\widehat{Z} = Z$.

Remark 2.9. Our method of proving the convergence of finite-dimensional distributions of $\overline{U}_f^{(n)}$, using chaotic expansions, is particularly suitable for providing estimates on the *speed of convergence* (e.g., in the *Wasserstein distance*) as is done in [18] following the original idea presented in [16], which combines the *Malliavin calculus* and *Stein's method*. In addition, the study of *weighted* variations of the fBs is still partially incomplete, especially with regards to functional convergence (see [24]). To keep the length of this paper within limits – and since proving functional convergence of weighted variations requires slightly different methods – we have decided to treat these two questions in a separate paper.

3. Finite-dimensional convergence

In this section, we begin the proofs of Theorems 2.4 and 2.7. To be more precise, we prove the finite-dimensional statements corresponding to (2.8) and (2.13); see Propositions 3.3 and 3.5, respectively. As a preparation, we study the correlation structure of the increments of the fBs Z and recall the chaotic expansion of functionals of Z.

3.1. Correlation structure of increments

In what follows, it will be convenient to use the shorthand

$$Z_i^{(n)} := \langle m(n)^H \rangle Z\left(\left[\frac{i-1}{m(n)}, \frac{i}{m(n)}\right)\right), \qquad 1 \le i \le m(n), n \in \mathbb{N}. \tag{3.1}$$

For any $n \in \mathbb{N}$, the family $\{Z_i^{(n)}\colon 1 \le i \le m(n)\}$ is clearly centered and Gaussian. We will next derive its correlation structure.

To describe the correlation structure of the rescaled increments (3.1), let $\{B_{\check{H}}(t)\colon t \in \mathbb{R}\}$ be an auxiliary fractional Brownian motion with Hurst parameter $\check{H} \in (0,1)$. Using the kernel (2.4), we may represent it as

$$B_{\check{H}}(t) := \int_{\mathbb{R}} G_{\check{H}}^{(1)}(t,u)\,\mathrm{d}B(u), \qquad t \in \mathbb{R},$$

where $\{B(t)\colon t \in \mathbb{R}\}$ is a standard Brownian motion. Recall that $B_{\check{H}}$ is $\check{H}$*-self similar*, that is,

$$\left\{B_{\check{H}}(at)\colon t \in \mathbb{R}\right\} \stackrel{\mathcal{L}}{=} \left\{a^{\check{H}} B_{\check{H}}(t)\colon t \in \mathbb{R}\right\} \qquad \text{for any } a > 0, \tag{3.2}$$

and has *stationary increments*, that is,

$$\left\{B_{\check{H}}\big([s, s+t)\big)\colon t \in \mathbb{R}\right\} \stackrel{\mathcal{L}}{=} \left\{B_{\check{H}}\big([0,t)\big)\colon t \in \mathbb{R}\right\} \qquad \text{for any } s \in \mathbb{R}. \tag{3.3}$$

The discrete parameter process

$$B_{\check{H}}\big([j, j+1)\big), \qquad j \in \mathbb{Z},$$

which is stationary by (3.3), is called a *fractional Gaussian noise*. Its correlation function can be expressed as

$$\begin{aligned} r_{\check{H}}(j) &:= \mathbf{E}\big[B_{\check{H}}\big([j,j+1)\big)B_{\check{H}}\big([0,1)\big)\big] \\ &= \frac{|j+1|^{2\check{H}} - 2|j|^{2\check{H}} + |j-1|^{2\check{H}}}{2}, \qquad j \in \mathbb{Z}. \end{aligned}$$

One can show, for example, using the mean value theorem, that there exists a constant $C(\check{H}) > 0$ such that

$$\big|r_{\check{H}}(j)\big| \le C(\check{H})|j|^{-2(1-\check{H})}, \qquad j \in \mathbb{Z}. \tag{3.4}$$

Thus, if $k > \frac{1}{2}$ and $\check{H} \in (0, 1 - \frac{1}{2k})$, then

$$\sum_{j\in\mathbb{Z}} \big|r_{\check{H}}(j)\big|^k < \infty. \tag{3.5}$$

If $\check{H} \in [1 - \frac{1}{2k}, 1)$, then the series (3.5) diverges. In this case, it is still useful to have estimates for the partial sums corresponding to (3.5). Using (3.4), one can prove that there exists a constant $C'(\check{H}, k) > 0$ such that

$$\sum_{|j|<l} \big|r_{\check{H}}(j)\big|^k \le \begin{cases} C'(\check{H},k)\log l, & \check{H} = 1 - \dfrac{1}{2k}, \\ C'(\check{H},k) l^{1-2k(1-\check{H})}, & \check{H} \in \left(1 - \dfrac{1}{2k}, 1\right). \end{cases} \tag{3.6}$$

We can now describe the correlations of the rescaled increments (3.1) using the correlation function of the fractional Gaussian noise as follows.

Lemma 3.1 (Correlation structure). *For any $n \in \mathbb{N}$, and $1 \le i^{(1)}, i^{(2)} \le m(n)$,*

$$\mathbf{E}\big[Z^{(n)}_{i^{(1)}} Z^{(n)}_{i^{(2)}}\big] = \prod_{\nu=1}^{d} r_{H_\nu}\big(i^{(1)}_\nu - i^{(2)}_\nu\big).$$

Proof. Using first the linearity of Wiener integrals and then the product structure (2.3) of the kernel $G^{(d)}_H$ and Remark 2.1, we obtain for any $s, t \in [0,1]^d$ such that $s \le t$,

$$\begin{aligned} Z\big([s,t)\big) &= \int G^{(d)}_H\big([s,t),u\big)\mathcal{W}(\mathrm{d}u) \\ &= \int \prod_{\nu=1}^{d} G^{(1)}_{H_\nu}\big([s_\nu,t_\nu),u_\nu\big)\mathcal{W}(\mathrm{d}u). \end{aligned} \tag{3.7}$$

Thus, by Fubini's theorem,

$$\begin{aligned}
&\mathbf{E}\Big[Z\Big(\Big[\frac{i^{(1)}-1}{m(n)},\frac{i^{(1)}}{m(n)}\Big)\Big)Z\Big(\Big[\frac{i^{(2)}-1}{m(n)},\frac{i^{(2)}}{m(n)}\Big)\Big)\Big]\\
&\quad=\prod_{\nu=1}^{d}\int G^{(1)}_{H_\nu}\Big(\Big[\frac{i_\nu^{(1)}-1}{m_\nu(n)},\frac{i_\nu^{(1)}}{m_\nu(n)}\Big),v\Big)G^{(1)}_{H_\nu}\Big(\Big[\frac{i_\nu^{(2)}-1}{m_\nu(n)},\frac{i_\nu^{(2)}}{m_\nu(n)}\Big),v\Big)\,\mathrm{d}v\\
&\quad=\prod_{\nu=1}^{d}\mathbf{E}\Big[B_{H_\nu}\Big(\Big[\frac{i_\nu^{(1)}-1}{m_\nu(n)},\frac{i_\nu^{(1)}}{m_\nu(n)}\Big)\Big)B_{H_\nu}\Big(\Big[\frac{i_\nu^{(2)}-1}{m_\nu(n)},\frac{i_\nu^{(2)}}{m_\nu(n)}\Big)\Big)\Big].
\end{aligned}$$

For any $\nu\in\{1,\dots,d\}$, the fractional Brownian motion B_{H_ν} is H_ν-self similar and has stationary increments, cf. (3.2) and (3.3), so we obtain

$$\mathbf{E}\Big[B_{H_\nu}\Big(\Big[\frac{i_\nu^{(1)}-1}{m_\nu(n)},\frac{i_\nu^{(1)}}{m_\nu(n)}\Big)\Big)B_{H_\nu}\Big(\Big[\frac{i_\nu^{(2)}-1}{m_\nu(n)},\frac{i_\nu^{(2)}}{m_\nu(n)}\Big)\Big)\Big]=\frac{r_{H_\nu}(i_\nu^{(1)}-i_\nu^{(2)})}{m_\nu(n)^{2H_\nu}},$$

from which the assertion follows. □

3.2. Multiple Wiener integrals and central limit theorem

The proofs of Theorems 2.4 and 2.7 rely on particular representations of generalized variations in terms of *multiple Wiener integrals* with respect to the underlying white noise $\mathcal{W}$. We will now briefly review the theory of multiple Wiener integrals and how these integrals can be used to prove central limit theorems. As an application, we take the first step in the proof of Theorem 2.4 by establishing the convergence of finite-dimensional laws.

In what follows, we write $\mathcal{H}:=L^2(\mathbb{R}^d)$. Recall that $\mathcal{H}$ is a separable Hilbert space when we endow it with the usual inner product. For any $k\in\mathbb{N}$, we denote by $\mathcal{H}^{\otimes k}$ the k-fold tensor power of $\mathcal{H}$ and by $\mathcal{H}^{\odot k}\subset\mathcal{H}^{\otimes k}$ the symmetrization of $\mathcal{H}^{\otimes k}$. Note that we can make the identification $\mathcal{H}^{\otimes k}\cong L^2(\mathbb{R}^{kd})$. For any $h\in\mathcal{H}^{\odot k}$, we may define the k-fold multiple Wiener integral $I_k^{\mathcal{W}}(h)$ of h with respect to $\mathcal{W}$. This is done, using Hermite polynomials, by setting for any $\kappa\in\mathbb{N}$, any orthonormal $h_1,\dots,h_\kappa\in\mathcal{H}$, and for any $k_1,\dots,k_\kappa\in\mathbb{N}$ such that $k_1+\cdots+k_\kappa=k$,

$$I_k^{\mathcal{W}}\Big(\bigodot_{j=1}^{\kappa}h_j^{\otimes k_j}\Big):=k!\prod_{j=1}^{\kappa}P_{k_j}\Big(\int h_j(u)\mathcal{W}(\mathrm{d}u)\Big),\tag{3.8}$$

where $\odot$ denotes symmetrization of the tensor product, and extended to general integrands $h\in\mathcal{H}^{\odot k}$ using a density argument. It is worth stressing that the multiple Wiener integral is linear with respect to the integrand and has zero expectation. Moreover, by (3.8), for $h\in\mathcal{H}$ one has

$$I_1^{\mathcal{W}}(h)=\int h(u)\mathcal{W}(\mathrm{d}u),\tag{3.9}$$

and if $\|h\|_{\mathcal{H}} = 1$, then for any $k \in \mathbb{N}$ it holds that $h^{\otimes k} \in \mathcal{H}^{\odot k}$ and

$$P_k\big(I_1^{\mathcal{W}}(h)\big) = I_k^{\mathcal{W}}\big(h^{\otimes k}\big). \tag{3.10}$$

Multiple Wiener integrals have the following isometry and orthogonality properties: for any k_1, $k_2 \in \mathbb{N}$, $h_1 \in \mathcal{H}^{\odot k_1}$, and $h_2 \in \mathcal{H}^{\odot k_2}$,

$$\mathbf{E}\big[I_{k_1}^{\mathcal{W}}(h_1) I_{k_2}^{\mathcal{W}}(h_2)\big] = \begin{cases} k_1!\langle h_1, h_2\rangle_{\mathcal{H}^{\otimes k_1}}, & k_1 = k_2, \\ 0, & k_1 \neq k_2. \end{cases} \tag{3.11}$$

Recall that any random variable $Y \in L^2(\Omega, \mathcal{F}, \mathbf{P})$ has a *chaotic expansion* in terms of kernels $F_k^Y \in \mathcal{H}^{\odot k}$, $k \in \mathbb{N}$, as

$$Y = \mathbf{E}[Y] + \sum_{k=1}^{\infty} I_k^{\mathcal{W}}\big(F_k^Y\big), \tag{3.12}$$

where the series converges in $L^2(\Omega, \mathcal{F}, \mathbf{P})$ (see, e.g., [11], Theorem 13.26). Since the apperance of the seminal paper of Nualart and Peccati [20], the convergence of random variables admitting expansions of the form (3.12) to a Gaussian law has been well understood, based on convenient characterizations using the properties of the kernels. To describe the key result, recall that for any k_1, k_2, $r \in \mathbb{N}$ such the $r < \min\{k_1, k_2\}$, the rth *contraction* of $h_1 \in \mathcal{H}^{\otimes k_1}$ and $h_2 \in \mathcal{H}^{\otimes k_2}$ is defined as

$$\begin{aligned} &(h_1 \otimes_r h_2)\big(t^{(1)}, \dots, t^{(k_1+k_2-2r)}\big) \\ &\quad := \big\langle h_1\big(t^{(1)}, \dots, t^{(k_1-r)}, \cdot\big), h_2\big(\cdot, t^{(k_1-r+1)}, \dots, t^{(k_1+k_2-2r)}\big)\big\rangle_{\mathcal{H}^{\otimes r}} \end{aligned}$$

for any $t^{(1)}, \dots, t^{(k_1+k_2-2r)} \in \mathbb{R}^d$. The following multivariate central limit theorem for chaotic expansions appears in [4], Theorem 5, where it is proven using the results in [22].

Lemma 3.2 (CLT for chaotic expansions). *Let $\kappa \in \mathbb{N}$ and suppose that for any $n \in \mathbb{N}$, we are given random variables $Y_1^{(n)}, \dots, Y_\kappa^{(n)} \in L^2(\Omega, \mathcal{F}, \mathbf{P})$ such that for any $j \in \{1, \dots, \kappa\}$,*

$$Y_j^{(n)} = \sum_{k=1}^{\infty} I_k^{\mathcal{W}}\big(F_k^{(n)}(j, \cdot)\big),$$

where $F_k^{(n)}(j, \cdot) \in \mathcal{H}^{\odot k}$, $k \in \mathbb{N}$. Let us assume that the following conditions hold:

(a) *For any $j \in \{1, \dots, \kappa\}$,*

$$\limsup_{n\to\infty} \sum_{k=K}^{\infty} k!\big\|F_k^{(n)}(j, \cdot)\big\|_{\mathcal{H}^{\otimes k}}^2 \underset{K\to\infty}{\longrightarrow} 0.$$

(b) *There exists a sequence $\Sigma, \Sigma_1, \Sigma_2, \dots$ of positive semidefinite $d \times d$-matrices such that for any $(j_1, j_2) \in \{1, \dots, \kappa\}^2$ and $k \in \mathbb{N}$,*

$$k!\big\langle F_k^{(n)}(j_1, \cdot), F_k^{(n)}(j_2, \cdot)\big\rangle_{\mathcal{H}^{\otimes k}} \underset{n\to\infty}{\longrightarrow} \Sigma_k(j_1, j_2),$$

and that $\sum_{k=1}^{\infty} \Sigma_k = \Sigma$.

(c) *For any* $j \in \{1, \dots, \kappa\}$, $k \geq 2$, *and* $r \in \{1, \dots, k-1\}$,

$$\left\| F_k^{(n)}(j, \cdot) \otimes_r F_k^{(n)}(j, \cdot) \right\|^2_{\mathcal{H}^{\otimes 2(k-r)}} \xrightarrow[n\to\infty]{} 0.$$

Then we have

$$\left(Y_1^{(n)}, \dots, Y_\kappa^{(n)}\right) \xrightarrow[n\to\infty]{\mathcal{L}} N_\kappa(0, \Sigma),$$

where $N_\kappa(0, \Sigma)$ *stands for the* κ*-dimensional Gaussian law with mean* 0 *and covariance matrix* Σ.

We apply now Lemma 3.2 to establish the following finite-dimensional version of Theorem 2.4.

Proposition 3.3 (CLT for finite-dimensional laws). *Suppose that* $H \in (0, 1)^d \setminus (1 - \frac{1}{2\underline{k}}, 1)^d$. *Let* $\kappa \in \mathbb{N}$ *and* $(t^{(1)}, \dots, t^{(\kappa)}) \in ([0, 1]^d)^\kappa$. *Then*

$$\left(Z\left(t^{(1)}\right), \dots, Z\left(t^{(\kappa)}\right), \overline{U}_f^{(n)}\left(t^{(1)}\right), \dots, \overline{U}_f^{(n)}\left(t^{(\kappa)}\right)\right) \xrightarrow[n\to\infty]{\mathcal{L}} N_{2\kappa}\left(0, \begin{bmatrix} \Xi & 0 \\ 0 & \Sigma \end{bmatrix}\right), \tag{3.13}$$

where Ξ *is the covariance matrix of the random vector* $(Z(t^{(1)}), \dots, Z(t^{(\kappa)}))$ *and*

$$\Sigma(j_1, j_2) := \Lambda_{H,f} R_{\widetilde{H}}^{(d)}\left(t^{(j_1)}, t^{(j_2)}\right), \qquad (j_1, j_2) \in \{1, \dots, \kappa\}^2.$$

Remark 3.4. In the case $H \in (0, 1 - \frac{1}{2\underline{k}}]^d$, the convergence

$$\left(\overline{U}_f^{(n)}\left(t^{(1)}\right), \dots, \overline{U}_f^{(n)}\left(t^{(\kappa)}\right)\right) \xrightarrow[n\to\infty]{\mathcal{L}} N_\kappa(\Sigma)$$

follows from the classical results of Breuer and Major [7].

Proof of Proposition 3.3. By (3.9), we have $Z(t) = I_1^{\mathcal{W}}(G_H^{(d)}(t, \cdot))$ for any $t \in [0, 1]^d$. In particular, by (3.7) and linearity, we find that for any $n \in \mathbb{N}$ and $1 \leq i \leq m(n)$,

$$Z_i^{(n)} = I_1^{\mathcal{W}}\left(h_i^{(n)}\right),$$

where

$$h_i^{(n)} := \left(m(n)^H\right) g_i^{(n)}, \qquad g_i^{(n)} := G_H^{(d)}\left(\left[\frac{i-1}{m(n)}, \frac{i}{m(n)}\right), \cdot\right), \tag{3.14}$$

satisfying $\|h_i^{(n)}\|_{\mathcal{H}} = 1$, due to the relation (3.11) and Lemma 3.1. The expansion (2.6) and the connection of Hermite polynomials and multiple Wiener integrals (3.10) allows then us to write

$$\overline{U}_f^{(n)}(t) = \sum_{k=\underline{k}}^{\infty} I_k^{\mathcal{W}}\left(F_k^{(n)}(t, \cdot)\right), \qquad t \in [0, 1]^d, n \in \mathbb{N}, \tag{3.15}$$

where

$$F_k^{(n)}(t,\cdot) := \frac{a_k}{\langle c(n)\rangle^{1/2}} \sum_{1\le i\le \lfloor m(n)t\rfloor} \big(h_i^{(n)}\big)^{\otimes k}, \qquad k \ge \underline{k}. \tag{3.16}$$

For the remainder of the proof, let $s, t \in \{t^{(1)}, \dots, t^{(\kappa)}\}$. Let us first look into condition (a) of Lemma 3.2. By Lemma 3.1 and the relation (3.11), we obtain for any $n \in \mathbb{N}$ and $k \ge \underline{k}$,

$$\begin{aligned}
\big\langle F_k^{(n)}(s,\cdot), F_k^{(n)}(t,\cdot)\big\rangle_{\mathcal{H}^{\otimes k}} &= \frac{a_k^2}{\langle c(n)\rangle} \sum_{1\le i^{(1)}\le \lfloor m(n)s\rfloor} \sum_{1\le i^{(2)}\le \lfloor m(n)t\rfloor} \big\langle \big(h_{i^{(1)}}^{(n)}\big)^{\otimes k}, \big(h_{i^{(2)}}^{(n)}\big)^{\otimes k}\big\rangle_{\mathcal{H}^{\otimes k}} \\
&= \frac{a_k^2}{\langle c(n)\rangle} \sum_{1\le i^{(1)}\le \lfloor m(n)s\rfloor} \sum_{1\le i^{(2)}\le \lfloor m(n)t\rfloor} \big\langle h_{i^{(1)}}^{(n)}, h_{i^{(2)}}^{(n)}\big\rangle_{\mathcal{H}}^k \\
&= a_k^2 \prod_{\nu=1}^d \frac{1}{c_\nu(n)} \sum_{j_1=1}^{\lfloor m_\nu(n)s_\nu\rfloor} \sum_{j_2=1}^{\lfloor m_\nu(n)t_\nu\rfloor} r_{H_\nu}(j_1-j_2)^k.
\end{aligned} \tag{3.17}$$

Let $k_0 \in \mathbb{N}$ be large enough so that $H_\nu \in (0, 1-\frac{1}{2k_0})$ for any $\nu \in \{1,\dots,d\}$. Then we have for any $k \ge k_0$,

$$\begin{aligned}
0 &\le \prod_{\nu=1}^d \frac{1}{c_\nu(n)} \sum_{j_1=1}^{\lfloor m_\nu(n)s_\nu\rfloor} \sum_{j_2=1}^{\lfloor m_\nu(n)s_\nu\rfloor} r_{H_\nu}(j_1-j_2)^k \\
&\le \prod_{\nu=1}^d \frac{1}{c_\nu(n)} \sum_{j_1=1}^{m_\nu(n)} \sum_{j_2=1}^{m_\nu(n)} \big|r_{H_\nu}(j_1-j_2)\big|^{k_0} \\
&\le \prod_{\nu=1}^d \Big(\sup_{n\in\mathbb{N}} \frac{m_\nu(n)}{c_\nu(n)}\Big) \sum_{j\in\mathbb{Z}} \big|r_{H_\nu}(j)\big|^{k_0} < \infty,
\end{aligned}$$

which follows from Remark 2.3 and the elementary estimate

$$\sup_{1\le j_1\le \ell} \sum_{j_2=1}^{\ell} \big|r_{H_\nu}(j_1-j_2)\big|^q \le \sum_{|j|<\ell} \big|r_{H_\nu}(j)\big|^q, \qquad \ell \in \mathbb{N}, q \in \mathbb{R}_+. \tag{3.18}$$

Thus, by (2.7), we have for $K \ge k_0$,

$$0 \le \limsup_{n\to\infty} \sum_{k=K}^\infty k! \big\| F_k^{(n)}(s)\big\|_{\mathcal{H}^{\otimes k}}^2 \le \sum_{k=K}^\infty k! a_k^2 \prod_{\nu=1}^d \Big(\sup_{n\in\mathbb{N}} \frac{m_\nu(n)}{c_\nu(n)}\Big) \sum_{j\in\mathbb{Z}} \big|r_{H_\nu}(j)\big|^{k_0} \underset{K\to\infty}{\longrightarrow} 0,$$

and the condition (a) is verified.

To check condition (b) of Lemma 3.2, note that we can write for any $\nu \in \{1, \dots, d\}$, assuming without loss of generality that $t_\nu \geq s_\nu$,

$$\begin{aligned}
&\frac{1}{c_\nu(n)} \sum_{j_1=1}^{\lfloor m_\nu(n)s_\nu \rfloor} \sum_{j_2=1}^{\lfloor m_\nu(n)t_\nu \rfloor} r_{H_\nu}(j_1 - j_2)^k \\
&= \frac{1}{2}\Bigg(\frac{1}{c_\nu(n)} \sum_{j_1=1}^{\lfloor m_\nu(n)s_\nu \rfloor} \sum_{j_2=1}^{\lfloor m_\nu(n)s_\nu \rfloor} r_{H_\nu}(j_1 - j_2)^k + \frac{1}{c_\nu(n)} \sum_{j_1=1}^{\lfloor m_\nu(n)t_\nu \rfloor} \sum_{j_2=1}^{\lfloor m_\nu(n)t_\nu \rfloor} r_{H_\nu}(j_1 - j_2)^k \\
&\qquad - \frac{1}{c_\nu(n)} \sum_{j_1=1}^{\lfloor m_\nu(n)t_\nu \rfloor - \lfloor m_\nu(n)s_\nu \rfloor} \sum_{j_2=1}^{\lfloor m_\nu(n)t_\nu \rfloor - \lfloor m_\nu(n)s_\nu \rfloor} r_{H_\nu}(j_1 - j_2)^k \Bigg).
\end{aligned} \tag{3.19}$$

We will now compute the limit of (3.19) separately in the following three possible cases:

(i) $H_\nu \in (1 - \frac{1}{2\underline{k}}, 1)$,
(ii) $H_\nu = 1 - \frac{1}{2\underline{k}}$,
(iii) $H_\nu \in (0, 1 - \frac{1}{2\underline{k}})$.

In the case (i), we obtain, by Lemma A.1 of [25],

$$\begin{aligned}
&\frac{1}{c_\nu(n)} \sum_{j_1=1}^{\lfloor m_\nu(n)s_\nu \rfloor} \sum_{j_2=1}^{\lfloor m_\nu(n)s_\nu \rfloor} r_{H_\nu}(j_1 - j_2)^{\underline{k}} \\
&= \left(\frac{\lfloor m_\nu(n)s_\nu \rfloor}{m_\nu(n)} \right)^{2-2\underline{k}(1-H_\nu)} \lfloor m_\nu(n)s_\nu \rfloor^{-2+2\underline{k}(1-H_\nu)} \sum_{j_1=1}^{\lfloor m_\nu(n)s_\nu \rfloor} \sum_{j_2=1}^{\lfloor m_\nu(n)s_\nu \rfloor} r_{H_\nu}(j_1 - j_2)^{\underline{k}} \\
&\underset{n\to\infty}{\longrightarrow} \kappa(H_\nu, \underline{k}) s_\nu^{2-2\underline{k}(1-H_\nu)} = \kappa(H_\nu, \underline{k}) s_\nu^{\widetilde{H}_\nu},
\end{aligned}$$

where $\kappa(H_\nu, \underline{k})$ is given by (2.11). (In fact, Lemma A.1 of [25] requires that $\underline{k} \geq 2$, but it is straightforward to check that the limits stated therein are valid also when $\underline{k} = 1$.) With $k > \underline{k}$ we may choose $\varepsilon > 0$ so that $\underline{k} + \varepsilon < \min(\frac{1}{2(1-H_\nu)}, k)$, whence

$$\begin{aligned}
\left| \frac{1}{c_\nu(n)} \sum_{j_1=1}^{\lfloor m_\nu(n)s_\nu \rfloor} \sum_{j_2=1}^{\lfloor m_\nu(n)s_\nu \rfloor} r_{H_\nu}(j_1 - j_2)^k \right| &\leq \frac{1}{c_\nu(n)} \sum_{j_1=1}^{m_\nu(n)} \sum_{j_2=1}^{m_\nu(n)} \left| r_{H_\nu}(j_1 - j_2) \right|^{\underline{k}+\varepsilon} \\
&\leq \frac{1}{m_\nu(n)^{1-2\underline{k}(1-H_\nu)}} \sum_{|j|<m_\nu(n)} \left| r_{H_\nu}(j) \right|^{\underline{k}+\varepsilon} \\
&\underset{n\to\infty}{\longrightarrow} 0
\end{aligned} \tag{3.20}$$

by the estimate (3.6). Treating the other summands on the right-hand side of (3.19) similarly, we arrive at

$$\lim_{n\to\infty}\frac{1}{c_\nu(n)}\sum_{j_1=1}^{\lfloor m_\nu(n)s_\nu\rfloor}\sum_{j_2=1}^{\lfloor m_\nu(n)t_\nu\rfloor} r_{H_\nu}(j_1-j_2)^k$$
$$=\begin{cases}\dfrac{\kappa(H_\nu,\underline{k})}{2}\big(s_\nu^{\widetilde{H}_\nu}+t_\nu^{\widetilde{H}_\nu}-(t_\nu-s_\nu)^{\widetilde{H}_\nu}\big)=\kappa(H_\nu,\underline{k})R^{(1)}_{\widetilde{H}_\nu}(s_\nu,t_\nu), & k=\underline{k},\\ 0, & k>\underline{k}.\end{cases}$$

In the case (ii), rearranging and applying Lemma A.1 of [25] yields

$$\frac{1}{c_\nu(n)}\sum_{j_1=1}^{\lfloor m_\nu(n)s_\nu\rfloor}\sum_{j_2=1}^{\lfloor m_\nu(n)s_\nu\rfloor} r_{H_\nu}(j_1-j_2)^{\underline{k}}$$
$$=\left(1+\frac{\log(\lfloor m_\nu(n)s_\nu\rfloor/m_\nu(n))}{\log(m_\nu(n))}\right)\frac{\lfloor m_\nu(n)s_\nu\rfloor/m_\nu(n)}{\lfloor m_\nu(n)s_\nu\rfloor\log(\lfloor m_\nu(n)s_\nu\rfloor)}\sum_{j_1=1}^{\lfloor m_\nu(n)s_\nu\rfloor}\sum_{j_2=1}^{\lfloor m_\nu(n)s_\nu\rfloor} r_{H_\nu}(j_1-j_2)^{\underline{k}}$$
$$\xrightarrow[n\to\infty]{}\iota(\underline{k})s_\nu,$$

where $\iota(\underline{k})$ is given by (2.11). When $k>\underline{k}$, we have $H_\nu\in(0,1-\frac{1}{2k})$ and, consequently,

$$\left|\frac{1}{c_\nu(n)}\sum_{j_1=1}^{\lfloor m_\nu(n)s_\nu\rfloor}\sum_{j_2=1}^{\lfloor m_\nu(n)s_\nu\rfloor} r_{H_\nu}(j_1-j_2)^k\right|\le\frac{1}{\log(m_\nu(n))}\sum_{j\in\mathbb{Z}}\big|r_{H_\nu}(j)\big|^k\xrightarrow[n\to\infty]{}0.$$

Again, a similar treatment of the other summands on right-hand side of (3.19) establishes that

$$\lim_{n\to\infty}\frac{1}{c_\nu(n)}\sum_{j_1=1}^{\lfloor m_\nu(n)s_\nu\rfloor}\sum_{j_2=1}^{\lfloor m_\nu(n)t_\nu\rfloor} r_{H_\nu}(j_1-j_2)^k$$
$$=\begin{cases}\dfrac{\iota(\underline{k})}{2}\big(s_\nu+t_\nu-(t_\nu-s_\nu)\big)=\iota(\underline{k})R^{(1)}_{\widetilde{H}_\nu}(s_\nu,t_\nu), & k=\underline{k},\\ 0, & k>\underline{k}.\end{cases}$$

Finally, in the case (iii), we deduce in a straightforward manner that for any $k\ge\underline{k}$,

$$\lim_{n\to\infty}\frac{1}{c_\nu(n)}\sum_{j_1=1}^{\lfloor m_\nu(n)s_\nu\rfloor}\sum_{j_2=1}^{\lfloor m_\nu(n)t_\nu\rfloor} r_{H_\nu}(j_1-j_2)^k=\frac{1}{2}\sum_{j\in\mathbb{Z}}r_{H_\nu}(j)^k\big(s_\nu+t_\nu-(t_\nu-s_\nu)\big)$$
$$=\sum_{j\in\mathbb{Z}}r_{H_\nu}(j)^kR^{(1)}_{\widetilde{H}_\nu}(s_\nu,t_\nu)$$

using Lemma A.1 of [25].

Returning to the expression (3.17), we have shown that for any $k \geq \underline{k}$,

$$k!\big\langle F_k^{(n)}(s), F_k^{(n)}(t)\big\rangle_{\mathcal{H}^{\otimes k}} \underset{n\to\infty}{\longrightarrow} k! a_k^2 \big\langle b^{(k)}\big\rangle R_{\widetilde{H}}^{(d)}(s,t). \tag{3.21}$$

When $\underline{k} = 1$, we need to check, additionally, that the covariance matrix appearing in the limit (3.13) is block-diagonal. To this end, note that it follows from the assumption $H \in (0,1)^d \setminus (\frac{1}{2},1)^d$, that $b_\nu^{(1)} = 0$ for some $\nu \in \{1,\dots,d\}$, which in turn implies that

$$\big\| F_1^{(n)}(s,\cdot)\big\|_{\mathcal{H}}^2 \underset{n\to\infty}{\longrightarrow} 0.$$

By the Cauchy–Schwarz inequality, we have then

$$\big\langle F_1^{(n)}(s,\cdot), G_H^{(d)}(t,\cdot)\big\rangle_{\mathcal{H}} \underset{n\to\infty}{\longrightarrow} 0,$$

which ensures block diagonality, and concludes the verification of condition (b).

In order to check condition (c) of Lemma 3.2, let $k \geq \max(\underline{k}, 2)$ and $r \in \{1,\dots,k-1\}$. Using the bilinearity of contractions and inner products, we obtain

$$\begin{aligned}
&\big\| F_k^{(n)}(t,\cdot) \otimes_r F_k^{(n)}(t,\cdot)\big\|_{\mathcal{H}^{\otimes 2(k-r)}}^2 \\
&\quad = \frac{a_k^4}{\langle c(n)\rangle^2} \sum_{\substack{1\leq i^{(j)}\leq \lfloor m(n)t\rfloor \\ j\in\{1,2,3,4\}}} \big\langle \big(h_{i^{(1)}}^{(n)}\big)^{\otimes k} \otimes_r \big(h_{i^{(2)}}^{(n)}\big)^{\otimes k}, \big(h_{i^{(3)}}^{(n)}\big)^{\otimes k} \otimes_r \big(h_{i^{(4)}}^{(n)}\big)^{\otimes k}\big\rangle_{\mathcal{H}^{\otimes 2(k-r)}} \\
&\quad = \frac{a_k^4}{\langle c(n)\rangle^2} \sum_{\substack{1\leq i^{(j)}\leq \lfloor m(n)t\rfloor \\ j\in\{1,2,3,4\}}} \big\langle h_{i^{(1)}}^{(n)}, h_{i^{(2)}}^{(n)}\big\rangle_{\mathcal{H}}^r \big\langle h_{i^{(3)}}^{(n)}, h_{i^{(4)}}^{(n)}\big\rangle_{\mathcal{H}}^r \big\langle h_{i^{(1)}}^{(n)}, h_{i^{(3)}}^{(n)}\big\rangle_{\mathcal{H}}^{k-r} \big\langle h_{i^{(2)}}^{(n)}, h_{i^{(4)}}^{(n)}\big\rangle_{\mathcal{H}}^{k-r} \\
&\quad = a_k^4 \prod_{\nu=1}^d \frac{1}{c_\nu(n)^2} \sum_{j_1,j_2,j_3,j_4=1}^{\lfloor m_\nu(n)t_\nu\rfloor} r_{H_\nu}(j_1-j_2)^r r_{H_\nu}(j_3-j_4)^r r_{H_\nu}(j_1-j_3)^{k-r} r_{H_\nu}(j_2-j_4)^{k-r}.
\end{aligned}$$

Following the proof of Lemma 4.1 of [17], we apply the bound

$$\big|r_{H_\nu}(j_1)\big|^r \big|r_{H_\nu}(j_2)\big|^{k-r} \leq \big|r_{H_\nu}(j_1)\big|^k + \big|r_{H_\nu}(j_2)\big|^k, \qquad j_1, j_2 \in \mathbb{Z},$$

which is a consequence of Young's inequality, and use repeatedly (3.18) to deduce that

$$\big\| F_k^{(n)}(t,\cdot) \otimes_r F_k^{(n)}(t,\cdot)\big\|_{\mathcal{H}^{\otimes 2(k-r)}}^2 \leq 16^d a_k^4 \prod_{\nu=1}^d \frac{m_\nu(n)\phi_\nu(n)}{c_\nu(n)^2}, \tag{3.22}$$

where

$$\phi_\nu(n) := \sum_{|j_1|<m_\nu(n)} \big|r_{H_\nu}(j_1)\big|^r \sum_{|j_2|<m_\nu(n)} \big|r_{H_\nu}(j_2)\big|^{k-r} \sum_{|j_3|<m_\nu(n)} \big|r_{H_\nu}(j_3)\big|^k.$$

We need to analyze the asymptotic behaviour of $\phi_\nu(n)$ as $n \to \infty$. This can be accomplished by considering separately the three possible cases:

(i′) $H_\nu \in (1 - \frac{1}{2k}, 1)$,
(ii′) $H_\nu = 1 - \frac{1}{2k}$,
(iii′) $H_\nu \in (0, 1 - \frac{1}{2k})$.

In the case (i′) we have clearly $H_\nu \in (1 - \frac{1}{2(k-r)}, 1) \cap (1 - \frac{1}{2r}, 1)$, and by the estimate (3.6), it follows that

$$\phi_\nu(n) \le C''(H_\nu, k, r) m_\nu(n)^{3-4k(1-H_\nu)},$$

where $C''(H_\nu, k, r) := C'(H_\nu, r)C'(H_\nu, k-r)C'(H_\nu, k)$. Since $H_\nu \in (1 - \frac{1}{2k}, 1) \subset (1 - \frac{1}{2\underline{k}}, 1)$, we obtain

$$\limsup_{n\to\infty} \frac{m_\nu(n)\phi_\nu(n)}{c_\nu(n)^2} \le \limsup_{n\to\infty} \frac{C''(H_\nu, k, r)}{m_\nu(n)^{4(k-\underline{k})(1-H_\nu)}} < \infty.$$

Let us then consider to the case (ii′). We have still $H_\nu \in (1 - \frac{1}{2(k-r)}, 1) \cap (1 - \frac{1}{2r}, 1)$, so by (3.6) we find that

$$\phi_\nu(n) \le C''(H_\nu, k, r) m_\nu(n)^{2-2k(1-H_\nu)} \log\bigl(m_\nu(n)\bigr) = C''(H_\nu, k, r) m_\nu(n) \log\bigl(m_\nu(n)\bigr).$$

Necessarily $H_\nu \in [1 - \frac{1}{2\underline{k}}, 1)$, whence there is an index $n_0 \in \mathbb{N}$ such that $c_\nu(n) \ge m_\nu(n) \times \log(m_\nu(n))$ for all $n \ge n_0$. We deduce then that

$$\lim_{n\to\infty} \frac{m_\nu(n)\phi_\nu(n)}{c_\nu(n)^2} \le \lim_{n\to\infty} \frac{C''(H_\nu, k, r)}{\log(m_\nu(n))} = 0.$$

In the remaining case (iii′) we have $\sum_{j\in\mathbb{Z}} |r_{H_\nu}(j)|^k < \infty$. Since there is $n_0 \in \mathbb{N}$ such that $c_\nu(n) \ge m_\nu(n)$ for all $n \ge n_0$, we find that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{m_\nu(n)\phi_\nu(n)}{c_\nu(n)^2} \\
&\quad \le \lim_{n\to\infty} \frac{1}{m_\nu(n)} \sum_{|j_1|<m_\nu(n)} \bigl|r_{H_\nu}(j_1)\bigr|^r \sum_{|j_2|<m_\nu(n)} \bigl|r_{H_\nu}(j_2)\bigr|^{k-r} \sum_{j_3\in\mathbb{Z}} \bigl|r_{H_\nu}(j_3)\bigr|^k = 0
\end{aligned}$$

by Lemma 2.2 of [17].

Finally, let us return to the upper bound (3.22). The crucial observation is that the assumption $H \in (0,1)^d \setminus (1 - \frac{1}{2\underline{k}}, 1)^d$ implies that there is at least one coordinate $\nu \in \{1, \dots, d\}$ that falls within case (ii′) or (iii′). Thus,

$$\bigl\| F_k^{(n)}(t,\cdot) \otimes_r F_k^{(n)}(t,\cdot) \bigr\|^2_{\mathcal{H}^{\otimes 2(k-r)}} \xrightarrow[n\to\infty]{} 0,$$

concluding the verification of the condition (c), and the convergence (3.13) follows. □

3.3. Convergence to the Hermite sheet

We prove next a pointwise version of Theorem 2.7 in $L^2(\Omega)$. The argument is based mainly on the chaotic expansion (3.15) and the isometry property (3.11) of multiple Wiener integrals. However, compared to the proof of Proposition 3.3, we need to analyze the asymptotic behaviour of the associated kernels more carefully.

Proposition 3.5 (Pointwise NCLT). *Suppose that $H \in (1-\frac{1}{2\underline{k}}, 1)^d$. Then, for any $t \in [0,1]^d$,*

$$\overline{U}_f^{(n)}(t) \xrightarrow[n\to\infty]{L^2(\Omega)} \Lambda_{H,f}^{1/2}\widehat{Z}(t), \tag{3.23}$$

where $\widehat{Z}$ is the Hermite sheet appearing in Theorem 2.7.

Proof. Fix $t \in [0,1]^d$. By the chaotic expansion (3.15), we have for any $n \in \mathbb{N}$,

$$\overline{U}_f^{(n)}(t) = I_{\underline{k}}^{\mathcal{W}}\big(F_{\underline{k}}^{(n)}(t,\cdot)\big) + \sum_{k=\underline{k}+1}^{\infty} I_k^{\mathcal{W}}\big(F_k^{(n)}(t,\cdot)\big).$$

Using the property (3.11) and Parseval's identity, we find that

$$\mathbf{E}\left[\left(\sum_{k=\underline{k}+1}^{\infty} I_k^{\mathcal{W}}\big(F_k^{(n)}(t,\cdot)\big)\right)^2\right] = \sum_{k=\underline{k}+1}^{\infty} k!\big\|F_k^{(n)}(t,\cdot)\big\|_{\mathcal{H}^{\otimes k}}^2.$$

Since $H \in (1-\frac{1}{2\underline{k}}, 1)^d$, we may choose $\varepsilon \in (0,1]$ so that $H \in (1-\frac{1}{2(\underline{k}+\varepsilon)}, 1)^d$. Combining (3.17) and (3.20), we find that

$$\sum_{k=\underline{k}+1}^{\infty} k!\big\|F_k^{(n)}(t,\cdot)\big\|_{\mathcal{H}^{\otimes k}}^2 \le \sum_{k=\underline{k}+1}^{\infty} k! a_k^2 \prod_{\nu=1}^{d} \frac{1}{m_\nu(n)^{1-2\underline{k}(1-H_\nu)}} \sum_{|j|<m_\nu(n)} \big|r_{H_\nu}(j)\big|^{\underline{k}+\varepsilon} \xrightarrow[n\to\infty]{} 0,$$

where convergence to zero is a consequence of the bound (3.6). Thus, it remains to show that

$$I_{\underline{k}}^{\mathcal{W}}\big(F_{\underline{k}}^{(n)}(t,\cdot)\big) \xrightarrow[n\to\infty]{L^2(\Omega)} I_{\underline{k}}^{\mathcal{W}}\big(\Lambda_{H,f}^{1/2}\widehat{G}_{\widetilde{H}}^{(\underline{k})}(t,\cdot)\big) = \Lambda_{H,f}^{1/2}\widehat{Z}(t),$$

which follows by (3.11), if we can show that

$$F_{\underline{k}}^{(n)}(t,\cdot) \xrightarrow[n\to\infty]{\mathcal{H}^{\otimes\underline{k}}} \Lambda_{H,f}^{1/2}\widehat{G}_{\widetilde{H}}^{(\underline{k})}(t,\cdot). \tag{3.24}$$

In the special case $\underline{k}=1$, the convergence (3.23) follows already. Namely,

$$I_1^{\mathcal{W}}\big(F_1^{(n)}(t,\cdot)\big) = a_1 Z\left(\frac{\lfloor m(n)t\rfloor}{m(n)}\right) \xrightarrow[n\to\infty]{L^2(\Omega)} a_1 Z(t) = \Lambda_{H,f}^{1/2} Z(t) = \Lambda_{H,f}^{1/2}\widehat{Z}(t)$$

by the L^2-continuity of Z. Thus, we can assume that $\underline{k} \ge 2$ from now on.

We will prove the convergence (3.24) in two steps. First, we show that $(F_{\underline{k}}^{(n)}(t,\cdot))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}^{\otimes\underline{k}}$. Later, we characterize the limit. Let $n_1, n_2 \in \mathbb{N}$ and consider

$$\begin{aligned}&\|F_{\underline{k}}^{(n_1)}(t,\cdot) - F_{\underline{k}}^{(n_2)}(t,\cdot)\|_{\mathcal{H}^{\otimes\underline{k}}}^2\\&\quad= \|F_{\underline{k}}^{(n_1)}(t,\cdot)\|_{\mathcal{H}^{\otimes\underline{k}}}^2 + \|F_{\underline{k}}^{(n_2)}(t,\cdot)\|_{\mathcal{H}^{\otimes\underline{k}}}^2 - 2\langle F_{\underline{k}}^{(n_1)}(t,\cdot), F_{\underline{k}}^{(n_2)}(t,\cdot)\rangle_{\mathcal{H}^{\otimes\underline{k}}}.\end{aligned} \tag{3.25}$$

By Definition (3.16), we have

$$\begin{aligned}&\langle F_{\underline{k}}^{(n_1)}(t,\cdot), F_{\underline{k}}^{(n_2)}(t,\cdot)\rangle_{\mathcal{H}^{\otimes\underline{k}}}\\&\quad= a_{\underline{k}}^2 \langle m(n_1)\rangle^{\underline{k}-1}\langle m(n_2)\rangle^{\underline{k}-1} \sum_{1\le i^{(1)}\le\lfloor m(n_1)t\rfloor}\ \sum_{1\le i^{(2)}\le\lfloor m(n_2)t\rfloor} \langle g_{i^{(1)}}^{(n_1)}, g_{i^{(2)}}^{(n_2)}\rangle_{\mathcal{H}^{\otimes\underline{k}}}^{\underline{k}},\end{aligned}$$

where $g_i^{(n)}$ is given by (3.14). Mimicking the proof of Lemma 3.1, we obtain

$$\begin{aligned}&\langle g_{i^{(1)}}^{(n_1)}, g_{i^{(2)}}^{(n_2)}\rangle_{\mathcal{H}^{\otimes\underline{k}}}\\&\quad= \prod_{\nu=1}^d \int G_{H_\nu}^{(1)}\left(\left[\frac{i_\nu^{(1)}-1}{m_\nu(n_1)}, \frac{i_\nu^{(1)}}{m_\nu(n_1)}\right), v\right) G_{H_\nu}^{(1)}\left(\left[\frac{i_\nu^{(2)}-1}{m_\nu(n_2)}, \frac{i_\nu^{(2)}}{m_\nu(n_2)}\right), v\right) \mathrm{d}v\\&\quad= \prod_{\nu=1}^d \mathbf{E}\left[B_{H_\nu}\left(\left[\frac{i_\nu^{(1)}-1}{m_\nu(n_1)}, \frac{i_\nu^{(1)}}{m_\nu(n_1)}\right)\right) B_{H_\nu}\left(\left[\frac{i_\nu^{(2)}-1}{m_\nu(n_2)}, \frac{i_\nu^{(2)}}{m_\nu(n_2)}\right)\right)\right]\\&\quad= \prod_{\nu=1}^d H_\nu(2H_\nu-1)\int_{(i_\nu^{(1)}-1)/m_\nu(n_1)}^{i_\nu^{(1)}/m_\nu(n_1)} \int_{(i_\nu^{(2)}-1)/m_\nu(n_2)}^{i_\nu^{(2)}/m_\nu(n_2)} |v_1-v_2|^{-2(1-H_\nu)}\,\mathrm{d}v_1\,\mathrm{d}v_2,\end{aligned}$$

where the final equality follows (see, e.g., [14], page 574) since $H_\nu > 1 - \frac{1}{2\underline{k}} > \frac{1}{2}$ for any $\nu \in \{1,\dots,d\}$. Adapting the argument used in [15], pages 1064–1065, we deduce that

$$\begin{aligned}&\lim_{n_1,n_2\to\infty}\langle F_{\underline{k}}^{(n_1)}(t,\cdot), F_{\underline{k}}^{(n_2)}(t,\cdot)\rangle_{\mathcal{H}^{\otimes\underline{k}}}\\&\quad= a_{\underline{k}}^2 \prod_{\nu=1}^d H_\nu^{\underline{k}}(2H_\nu-1)^{\underline{k}} \int_0^t\int_0^t |v_1-v_2|^{-2\underline{k}(1-H_\nu)}\,\mathrm{d}v_1\,\mathrm{d}v_2\\&\quad= a_{\underline{k}}^2 \prod_{\nu=1}^d t_\nu^{2\widetilde{H}_\nu} H_\nu^{\underline{k}}(2H_\nu-1)^{\underline{k}} \int_0^1\int_0^1 |v_1-v_2|^{-2\underline{k}(1-H_\nu)}\,\mathrm{d}v_1\,\mathrm{d}v_2\\&\quad= a_{\underline{k}}^2 \prod_{\nu=1}^d t_\nu^{2\widetilde{H}_\nu}\kappa(H_\nu,\underline{k}) = a_{\underline{k}}^2\langle b^{(\underline{k})}\rangle R_{\widetilde{H}}^{(d)}(t,t).\end{aligned} \tag{3.26}$$

Thus, by (3.21) and (3.25),

$$\lim_{n_1,n_2\to\infty}\big\|F_{\underline{k}}^{(n_1)}(t,\cdot)-F_{\underline{k}}^{(n_2)}(t,\cdot)\big\|^2_{\mathcal{H}^{\otimes\underline{k}}}=0,$$

whence $(F_{\underline{k}}^{(n)}(t,\cdot))_{n\in\mathbb{N}}$ is a Cauchy sequence.

To characterize the limit of $(F_{\underline{k}}^{(n)}(t,\cdot))_{n\in\mathbb{N}}$, let us consider for any $s^{(1)},\dots,s^{(\underline{k})}\in\mathbb{R}^d$,

$$\begin{aligned}
&F_{\underline{k}}^{(n)}\big(t,s^{(1)},\dots,s^{(\underline{k})}\big)\\
&\quad=a_{\underline{k}}\langle m(n)\rangle^{\underline{k}-1}\sum_{1\le i\le\lfloor m(n)t\rfloor}\prod_{\kappa=1}^{\underline{k}}G_H^{(d)}\left(\left[\frac{i-1}{m(n)},\frac{i}{m(n)}\right),s^{(\kappa)}\right)\\
&\quad=a_{\underline{k}}\langle m(n)\rangle^{\underline{k}-1}\sum_{1\le i\le\lfloor m(n)t\rfloor}\prod_{\kappa=1}^{\underline{k}}\prod_{\nu=1}^{d}G_{H_\nu}^{(1)}\left(\left[\frac{i_\nu-1}{m_\nu(n)},\frac{i_\nu}{m_\nu(n)}\right),s_\nu^{(\kappa)}\right)\\
&\quad=a_{\underline{k}}\prod_{\nu=1}^{d}\frac{1}{m_\nu(n)}\sum_{j=1}^{\lfloor m_\nu(n)t_\nu\rfloor}\prod_{\kappa=1}^{\underline{k}}m_\nu(n)G_{H_\nu}^{(1)}\left(\left[\frac{j-1}{m_\nu(n)},\frac{j}{m_\nu(n)}\right),s_\nu^{(\kappa)}\right),
\end{aligned}$$

where the second equality is a consequence of Remark 2.1. Since

$$\begin{aligned}
&G_{H_\nu}^{(1)}\left(\left[\frac{j-1}{m_\nu(n)},\frac{j}{m_\nu(n)}\right),s_\nu^{(\kappa)}\right)\\
&\quad=\frac{1}{\chi(H_\nu)}\left(\left(\frac{j}{m_\nu(n)}-s_\nu^{(\kappa)}\right)_+^{H_\nu-1/2}-\left(\frac{j-1}{m_\nu(n)}-s_\nu^{(\kappa)}\right)_+^{H_\nu-1/2}\right),
\end{aligned}$$

it follows from Lemma 3.6, below, that

$$F_{\underline{k}}^{(n)}(t,\cdot)\underset{n\to\infty}{\longrightarrow}C'''(a_{\underline{k}},H,\underline{k})\widehat{G}_{\widetilde{H}}^{(\underline{k})}(t,\cdot)\qquad\text{a.e. on }\mathbb{R}^{\underline{k}d}\tag{3.27}$$

for some constant $C'''(a_{\underline{k}},H,\underline{k})>0$. By the Cauchy property of $(F_{\underline{k}}^{(n)}(t,\cdot))_{n\in\mathbb{N}}$, the convergence (3.27) holds also in $\mathcal{H}^{\otimes\underline{k}}$. Clarke De la Cerda and Tudor [8], pages 4–6, have shown that $\mathbf{E}[\widehat{Z}(t)^2]=\underline{k}!\|\widehat{G}_{\widetilde{H}}^{(\underline{k})}(t,\cdot)\|^2_{\mathcal{H}^{\otimes\underline{k}}}=R_{\widetilde{H}}^{(d)}(t,t)$. In view of (3.26), we find that

$$C'''(a_{\underline{k}},H,\underline{k})^2=\underline{k}!a_{\underline{k}}^2\langle b^{(\underline{k})}\rangle=\Lambda_{H,f},$$

whence (3.24) follows. □

The following technical lemma was essential in the proof of Proposition 3.5.

Lemma 3.6. *Suppose that* $k \geq 2$, $\check{H} \in (\frac{1}{2}, 1)$, *and* $v > 0$. *Then*

$$\begin{aligned}&\frac{1}{n}\sum_{j=1}^{\lfloor nv\rfloor}\prod_{\kappa=1}^{k} n\left(\left(\frac{j}{n}-s_\kappa\right)_+^{\check{H}-1/2}-\left(\frac{j-1}{n}-s_\kappa\right)_+^{\check{H}-1/2}\right)\\&\underset{n\to\infty}{\longrightarrow}\left(\check{H}-\frac{1}{2}\right)^k\int_0^v\prod_{\kappa=1}^{k}(u-s_\kappa)_+^{\check{H}-3/2}\,\mathrm{d}u\end{aligned}\tag{3.28}$$

for almost any $s = (s_1, \ldots, s_k) \in \mathbb{R}^k$.

Proof. We may assume that $\bar{s} := \max(s_1, \ldots, s_k) < v$, as otherwise (3.28) is trivially true. In fact,

$$\int_0^v \prod_{\kappa=1}^{k}(y-s_\kappa)_+^{\check{H}-3/2}\,\mathrm{d}y = \int_{\bar{s}}^v \prod_{\kappa=1}^{k}(y-s_\kappa)^{\check{H}-3/2}\,\mathrm{d}y.$$

We split the sum on the left-hand side of (3.28) for any $n \in \mathbb{N}$, such that $\lfloor nv\rfloor > \lfloor n\bar{s}\rfloor + 3$, as

$$\begin{aligned}&\frac{1}{n}\sum_{j=1}^{\lfloor nv\rfloor}\prod_{\kappa=1}^{k} n\left(\left(\frac{j}{n}-s_\kappa\right)_+^{\check{H}-1/2}-\left(\frac{j-1}{n}-s_\kappa\right)_+^{\check{H}-1/2}\right)\\&=\frac{1}{n}\sum_{j=\lfloor n\bar{s}\rfloor+1}^{\lfloor n\bar{s}\rfloor+2}\prod_{\kappa=1}^{k} n\left(\left(\frac{j}{n}-s_\kappa\right)^{\check{H}-1/2}-\left(\frac{j-1}{n}-s_\kappa\right)_+^{\check{H}-1/2}\right)\\&\quad+\frac{1}{n}\sum_{j=\lfloor n\bar{s}\rfloor+3}^{\lfloor nv\rfloor}\prod_{\kappa=1}^{k} n\left(\left(\frac{j}{n}-s_\kappa\right)^{\check{H}-1/2}-\left(\frac{j-1}{n}-s_\kappa\right)^{\check{H}-1/2}\right)\\&=: S_n^{(1)}+S_n^{(2)}.\end{aligned}$$

Using the mean value theorem, we obtain for any $y \in \mathbb{R}$ and $n, j \in \mathbb{N}$, such that $\frac{j-1}{n} > y$, the bounds

$$n\left(\left(\frac{j}{n}-y\right)^{\check{H}-1/2}-\left(\frac{j-1}{n}-y\right)^{\check{H}-1/2}\right)\leq\left(\check{H}-\frac{1}{2}\right)\left(\frac{j-1}{n}-y\right)^{\check{H}-3/2},\tag{3.29}$$

$$n\left(\left(\frac{j}{n}-y\right)^{\check{H}-1/2}-\left(\frac{j-1}{n}-y\right)^{\check{H}-1/2}\right)\geq\left(\check{H}-\frac{1}{2}\right)\left(\frac{j}{n}-y\right)^{\check{H}-3/2}.\tag{3.30}$$

Since we are aiming to prove (3.28) for *almost* any $s \in \mathbb{R}^k$, we may assume (by symmetry) that $\overline{s} = s_1 > s_\kappa$ for any $\kappa \in \{2, \dots, k\}$. Then we have for $j \in \{1, 2\}$,

$$\limsup_{n\to\infty} \prod_{\kappa=2}^{k} n\left(\left(\frac{\lfloor n\overline{s}\rfloor + j}{n} - s_\kappa\right)^{\check{H}-1/2} - \left(\frac{\lfloor n\overline{s}\rfloor + j - 1}{n} - s_\kappa\right)_+^{\check{H}-1/2}\right) < \infty$$

by (3.29), and

$$0 \le \left(\frac{\lfloor n\overline{s}\rfloor + j}{n} - s_1\right)^{\check{H}-1/2} - \left(\frac{\lfloor n\overline{s}\rfloor + j - 1}{n} - s_1\right)_+^{\check{H}-1/2} \le \left(\frac{\lfloor n\overline{s}\rfloor + 2}{n} - s_1\right)^{\check{H}-1/2} \underset{n\to\infty}{\longrightarrow} 0.$$

Hence, we find that $S_n^{(1)} \to 0$ as $n \to \infty$.

Finally, invoking (3.29), we obtain

$$\begin{aligned}
S_n^{(2)} &\le \left(\check{H} - \frac{1}{2}\right)^k \frac{1}{n} \sum_{j=\lfloor n\overline{s}\rfloor+3}^{\lfloor nv\rfloor} \prod_{\kappa=1}^{k} \left(\frac{j-1}{n} - s_\kappa\right)^{\check{H}-3/2} \\
&= \left(\check{H} - \frac{1}{2}\right)^k \int_{(\lfloor n\overline{s}\rfloor+2)/n}^{\lfloor nv\rfloor/n} \prod_{\kappa=1}^{k} \left(\frac{\lfloor ny\rfloor + 1}{n} - \frac{1}{n} - s_\kappa\right)^{\check{H}-3/2} \mathrm{d}y \\
&\le \left(\check{H} - \frac{1}{2}\right)^k \int_{(\lfloor n\overline{s}\rfloor+1)/n}^{(\lfloor nv\rfloor-1)/n} \prod_{\kappa=1}^{k} (y - s_\kappa)^{\check{H}-3/2} \mathrm{d}y \underset{n\to\infty}{\longrightarrow} \left(\widehat{H} - \frac{1}{2}\right)^k \int_{\overline{s}}^{v} \prod_{\kappa=1}^{k} (y - s_\kappa)^{\check{H}-3/2} \mathrm{d}y
\end{aligned}$$

and similarly by (3.30),

$$\begin{aligned}
S_n^{(2)} &\ge \left(\check{H} - \frac{1}{2}\right)^k \frac{1}{n} \sum_{j=\lfloor n\overline{s}\rfloor+3}^{\lfloor nv\rfloor} \prod_{\kappa=1}^{k} \left(\frac{j}{n} - s_\kappa\right)^{\check{H}-3/2} \\
&= \left(\check{H} - \frac{1}{2}\right)^k \int_{(\lfloor n\overline{s}\rfloor+2)/n}^{\lfloor nv\rfloor/n} \prod_{\kappa=1}^{k} \left(\frac{\lfloor ny\rfloor + 1}{n} - s_\kappa\right)^{\check{H}-3/2} \mathrm{d}y \\
&\ge \left(\check{H} - \frac{1}{2}\right)^k \int_{(\lfloor n\overline{s}\rfloor+3)/n}^{(\lfloor nv\rfloor+1)/n} \prod_{\kappa=1}^{k} (y - s_\kappa)^{\check{H}-3/2} \mathrm{d}y \underset{n\to\infty}{\longrightarrow} \left(\check{H} - \frac{1}{2}\right)^k \int_{\overline{s}}^{v} \prod_{\kappa=1}^{k} (y - s_\kappa)^{\check{H}-3/2} \mathrm{d}y.
\end{aligned}$$

(The convergence of the bounding integrals above, as $n \to \infty$, is ensured by Lebesgue's dominated convergence theorem.) Thus, the convergence (3.28) follows from the sandwich lemma. □

4. Functional convergence

To show that Theorems 2.4 and 2.7 indeed hold in the functional sense, we need to establish tightness of the relevant families of processes in the space $D([0,1]^d)$. To this end, we use the

tightness criterion due to Bickel and Wichura [6], Theorem 3. To apply this criterion, we need to bound the fourth moments of the increments of $\overline{U}_f^{(n)}$ uniformly over $n \in \mathbb{N}$.

4.1. Moment bound and diagrams

As a preparation for the proof of tightness, we establish a moment bound for nonlinear functionals of stationary Gaussian processes indexed by $\mathbb{N}^d$. The bound is a multi-parameter extension of Proposition 4.2 of [29], albeit under more restrictive assumptions.

Lemma 4.1 (Moment bound). *Let f be as in Section 2 and $\{Y_i\colon i \in \mathbb{N}^d\}$ a Gaussian process such that $\mathbf{E}[Y_i] = 0$ and $\mathbf{E}[Y_i^2] = 1$ for any $i \in \mathbb{N}^d$. Moreover, suppose that there exists a function $\rho\colon \mathbb{Z}^d \to [-1, 1]$ such that $\mathbf{E}[Y_{i^{(1)}} Y_{i^{(2)}}] = \rho(i^{(1)} - i^{(2)})$ for any $i^{(1)}, i^{(2)} \in \mathbb{N}^d$. If $p \in \{2, 3, \dots\}$ and the Hermite coefficients $a_{\underline{k}}, a_{\underline{k}+1}, \dots$ of the function f satisfy*

$$C''''(f, p) := \sum_{k=\underline{k}}^{\infty} (p-1)^{k/2} \sqrt{k!} |a_k| < \infty,$$

then for any $l \in \mathbb{N}^d$,

$$\left| \mathbf{E}\left[\left(\langle l \rangle^{-1/2} \sum_{1 \le i \le l} f(Y_i) \right)^p \right] \right| \le \left(2^d C''''(f, p)^2 \sum_{|i| < l} \left| \rho(i) \right|^{\underline{k}} \right)^{p/2}.$$

The proof of Proposition 4.2 of [29] is based on a graph theoretic argument that involves *multigraphs*. We prove Lemma 4.1 using slightly different (but essentially analogous) formalism based on *diagrams*, defined below. Breuer and Major [7] used diagrams to prove their central limit theorem for nonlinear functionals of Gaussian random fields via the *method of moments*. In fact in the proof of Lemma 4.1, we adapt some of the arguments used in [7].

Definition 4.2. *Let $p \in \{2, 3, \dots\}$ and $(k_1, \dots, k_p) \in \mathbb{N}^p$ be such that $k_1 + \cdots + k_p$ is an even number. A* diagram *of order $(k_1, \dots, k_p)$ is a graph $G = (V_G, E_G)$ with the following three properties*:

(1) *We have $V_G = \bigcup_{j=1}^p \{(j, 1), \dots, (j, k_j)\}$.*
(2) *The degree of any vertex $v \in V_G$ is one.*
(3) *Any edge $((j, k), (j', k')) \in E_G$ has the property that $j \neq j'$.*

We denote the class of diagrams of order $(k_1, \dots, k_p)$ by $\mathcal{G}(k_1, \dots, k_p)$. For the sake of completeness we set $\mathcal{G}(k_1, \dots, k_p) := \varnothing$ when $k_1 + \cdots + k_p$ is an odd number (no diagrams can then exist by the handshaking lemma *of graph theory). Let us also define two functions λ_1 and λ_2 of an edge $e = ((j, k), (j', k')) \in E_G$, where $j < j'$, by setting $\lambda_1(e) := j$ and $\lambda_2(e) := j'$.*

Diagrams are connected to Hermite polynomials and Gaussian random variables via the so-called *diagram formula*, which is originally due to Taqqu [29], Lemma 3.2. Below, we state a version of the formula that appears in [7], page 431.

Lemma 4.3 (Diagram formula). *Let $p \in \{2, 3, \dots\}$ and let $Y_1, \dots, Y_p$ be jointly Gaussian random variables with $\mathbf{E}[Y_i] = 0$ and $\mathbf{E}[Y_i^2] = 1$ for any $i \in \{1, \dots, p\}$. For any $(k_1, \dots, k_p) \in \mathbb{N}^p$, we have*

$$\mathbf{E}\left[\prod_{j=1}^{p} P_{k_j}(Y_j)\right] = \sum_{G \in \mathcal{G}(k_1,\dots,k_p)} \prod_{e \in E_G} \mathbf{E}[Y_{\lambda_1(e)} Y_{\lambda_2(e)}],$$

where a sum over an empty index set is interpreted as zero.

Remark 4.4. The diagram formula can be used to estimate the cardinalities of classes of diagrams. As pointed out by Bardet and Surgailis [2], page 461, using Lemma 4.3 and Lemma 3.1 of [29] in the special case $Y := Y_1 = \cdots = Y_p$, we obtain

$$\left|\mathcal{G}(k_1, \dots, k_p)\right| = \mathbf{E}\left[\prod_{j=1}^{p} P_{k_j}(Y)\right] \le (p-1)^{(k_1+\cdots+k_p)/2} \sqrt{k_1! \cdots k_p!}. \tag{4.1}$$

Proof of Lemma 4.1. Fix $l \in \mathbb{N}^d$. Let us define for any $K \ge \underline{k}$, a polynomial function

$$f_K(x) = \sum_{k=\underline{k}}^{K} a_k P_k(x), \qquad x \in \mathbb{R}.$$

By Fatou's lemma, Lemma 4.3 and inequality (4.1), it follows that

$$\begin{aligned}\mathbf{E}\left[\left|f(Y_i) - f_K(Y_i)\right|^p\right] &\le \sum_{k_1,\dots,k_p=K+1}^{\infty} |a_{k_1} \cdots a_{k_p}| \left|\mathcal{G}(k_1, \dots, k_p)\right| \\ &\le \left(\sum_{k=K+1}^{\infty} (p-1)^{k/2} \sqrt{k!} |a_k|\right)^p \underset{K\to\infty}{\longrightarrow} 0\end{aligned}$$

for any $i \in \mathbb{N}^d$. Thus, if $\varepsilon > 0$, then there exists $K(l) \in \mathbb{N}$ such that

$$\left|\mathbf{E}\left[\left(\langle l\rangle^{-1/2} \sum_{1\le i\le l} f(Y_i)\right)^p\right] - \mathbf{E}\left[\left(\langle l\rangle^{-1/2} \sum_{1\le i\le l} f_{K(l)}(Y_i)\right)^p\right]\right| \le \varepsilon, \tag{4.2}$$

by Minkowski's inequality and the fact that $X_n \overset{L^p(\Omega)}{\longrightarrow} X$ implies $\mathbf{E}[X_n^p] \to \mathbf{E}[X^p]$ when $p \in \{2, 3, \dots\}$. Lemma 4.3 yields now the expansion

$$\begin{aligned}&\mathbf{E}\left[\left(\langle l\rangle^{-1/2} \sum_{1\le i\le l} f_{K(l)}(Y_i)\right)^p\right] \\ &= \langle l\rangle^{-p/2} \sum_{\substack{1\le i^{(j)}\le l \\ j\in\{1,\dots,p\}}} \sum_{k_1,\dots,k_p=\underline{k}}^{K(l)} a_{k_1} \cdots a_{k_p} \sum_{G\in\mathcal{G}(k_1,\dots,k_p)} \prod_{e\in E_G} \mathbf{E}[Y_{i^{(\lambda_1(e))}} Y_{i^{(\lambda_2(e))}}] \\ &= \sum_{k_1,\dots,k_p=\underline{k}}^{K(l)} a_{k_1} \cdots a_{k_p} \sum_{G\in\mathcal{G}(k_1,\dots,k_p)} I_G(l),\end{aligned}$$

where

$$I_G(l) := \langle l\rangle^{-p/2} \sum_{\substack{1\le i^{(j)}\le l\\ j\in\{1,\dots,p\}}} \prod_{e\in E_G} \rho\big(i^{(\lambda_1(e))} - i^{(\lambda_2(e))}\big), \qquad G\in\mathcal{G}(k_1,\dots,k_p). \tag{4.3}$$

By Lemma 4.5 below and inequality (4.1), we obtain the bound

$$\begin{aligned}&\left|\sum_{k_1,\dots,k_p=\underline{k}}^{K(l)} a_{k_1}\cdots a_{k_p} \sum_{G\in\mathcal{G}(k_1,\dots,k_p)} I_G(l)\right|\\ &\quad\le \left(\sum_{k=\underline{k}}^{K(l)} (p-1)^{k/2}\sqrt{k!}|a_k|\right)^p \left(2^d \sum_{|i|<l} \big|\rho(i)\big|^{\underline{k}}\right)^{p/2} \le \left(2^d C''''(f,p)^2 \sum_{|i|<l} \big|\rho(i)\big|^{\underline{k}}\right)^{p/2}.\end{aligned}$$

In view of (4.2),

$$\left|\mathbf{E}\left[\left(\langle l\rangle^{-1/2} \sum_{1\le i\le l} f(Y_i)\right)^p\right]\right| \le \left(2^d C''''(f,p)^2 \sum_{|i|<l} \big|\rho(i)\big|^{\underline{k}}\right)^{p/2} + \varepsilon,$$

and letting $\varepsilon\to 0$ concludes the proof. □

The key ingredient in the proof of Lemma 4.1 was the following uniform bound for the absolute value of the quantity $I_G(l)$. We will derive this bound by adapting the asymptotic analysis of the moments of a nonlinear functional of a Gaussian random field, carried out in [7], pages 435–436.

Lemma 4.5. *For any $k_1,\dots,k_p\ge\underline{k}$, $G\in\mathcal{G}(k_1,\dots,k_p)$, and $l\in\mathbb{N}^d$,*

$$\big|I_G(l)\big| \le \left(2^d \sum_{|i|<l} \big|\rho(i)\big|^{\underline{k}}\right)^{p/2},$$

where $I_G(l)$ is defined by (4.3).

Proof. As pointed out by Breuer and Major [7], page 435, the quantity $I_G(l)$ is invariant under permutation of the levels of the diagram G. More precisely, if σ is a permutation of the set $\{1,\dots,p\}$, then we define a new diagram $\widetilde{G}\in\mathcal{G}(k_{\sigma(1)},\dots,k_{\sigma(p)})$ such that $((j,k),(j',k'))\in E_{\widetilde{G}}$ if and only if $((\sigma^{-1}(j),k),(\sigma^{-1}(j'),k'))\in E_G$. For such a diagram $\widetilde{G}$ it holds that $I_G(l) = I_{\widetilde{G}}(l)$. Relying on this invariance property we assume, without loss of generality, that

$$k_1\le k_2\le\cdots\le k_{p-1}\le k_p. \tag{4.4}$$

Let us introduce the notation $k_G(j) := |\{e \in E_G\colon\ \lambda_1(e) = j\}| \in \{0, 1, \dots, k_j\}$ for any $j \in \{1, \dots, p\}$. Since $\lambda_1(e) < \lambda_2(e)$ for any $e \in E_G$, we have

$$\begin{aligned}\big|I_G(l)\big| &\le \langle l\rangle^{-p/2} \sum_{\substack{1\le i^{(\kappa)}\le l\\ \kappa\in\{1,\dots,p\}}} \prod_{j=1}^{p} \prod_{\substack{e\in E_G\\ \lambda_1(e)=j}} \big|\rho\big(i^{(j)} - i^{(\lambda_2(e))}\big)\big| \\ &= \langle l\rangle^{-p/2} \sum_{\substack{1\le i^{(\kappa)}\le l\\ \kappa\in\{2,\dots,p\}}} \prod_{j=2}^{p} \prod_{\substack{e\in E_G\\ \lambda_1(e)=j}} \big|\rho\big(i^{(j)} - i^{(\lambda_2(e))}\big)\big| \sum_{1\le i^{(1)}\le l} \prod_{\substack{e\in E_G\\ \lambda_1(e)=1}} \big|\rho\big(i^{(1)} - i^{(\lambda_2(e))}\big)\big|.\end{aligned} \tag{4.5}$$

Using Young's inequality (see [7], page 435) and the trivial estimate

$$\sup_{1\le i\le l} \sum_{1\le i^{(1)}\le l} \big|\rho\big(i^{(1)} - i\big)\big|^q \le \sum_{|i|<l} \big|\rho(i)\big|^q, \qquad q \ge 0,$$

one can show that

$$\sup_{\substack{1\le i^{(\kappa)}\le l\\ \kappa\in\{2,\dots,p\}}} \sum_{1\le i^{(1)}\le l} \prod_{\substack{e\in E_G\\ \lambda_1(e)=1}} \big|\rho\big(i^{(1)} - i^{(\lambda_2(e))}\big)\big| \le \sum_{|i|<l} \big|\rho(i)\big|^{k_G(1)}.$$

Applying this procedure, mutatis mutandis, to (4.5) repeatedly we arrive at

$$\big|I_G(l)\big| \le \langle l\rangle^{-p/2} \prod_{j=1}^{p} \sum_{|i|<l} \big|\rho(i)\big|^{k_G(j)}. \tag{4.6}$$

By Hölder's inequality, we have for any $j \in \{1, \dots, p\}$,

$$\begin{aligned}\sum_{|i|<l} \big|\rho(i)\big|^{k_G(j)} &\le \langle 2l\rangle^{1-k_G(j)/k_j} \Bigg(\sum_{|i|<l} \big|\rho(i)\big|^{k_j}\Bigg)^{k_G(j)/k_j} \\ &\le \langle 2l\rangle^{1-k_G(j)/k_j} \Bigg(\sum_{|j|<l} \big|\rho(i)\big|^{\underline{k}}\Bigg)^{k_G(j)/k_j},\end{aligned}$$

where we use the proviso $k_j \ge \underline{k}$ to deduce the second inequality. Returning to (4.6), we have thus established that

$$\big|I_G(l)\big| \le \big(2^d\big)^{p/2} \langle 2l\rangle^{p/2-\sum_{j=1}^{p} k_G(j)/k_j} \Bigg(\sum_{|i|<l} \big|\rho(i)\big|^{\underline{k}}\Bigg)^{\sum_{j=1}^{p} k_G(j)/k_j}. \tag{4.7}$$

Breuer and Major [7], page 436, have shown that whenever (4.4) holds, we have

$$\sum_{j=1}^{p} \frac{k_G(j)}{k_j} - \frac{p}{2} \ge 0 \tag{4.8}$$

(see also Remark 4.6, below). By (4.8), we may use the rough estimate $\sum_{|i|<l} |\rho(i)|^{\underline{k}} \le \langle 2l \rangle$ to deduce that

$$\begin{aligned}\Big(\sum_{|i|<l} \big|\rho(i)\big|^{\underline{k}}\Big)^{\sum_{j=1}^{p} k_G(j)/k_j} &= \Big(\sum_{|i|<l} \big|\rho(i)\big|^{\underline{k}}\Big)^{\sum_{j=1}^{p} k_G(j)/k_j - p/2} \Big(\sum_{|i|<l} \big|\rho(i)\big|^{\underline{k}}\Big)^{p/2} \\ &\le \langle 2l \rangle^{\sum_{j=1}^{p} k_G(j)/k_j - p/2} \Big(\sum_{|i|<l} \big|\rho(i)\big|^{\underline{k}}\Big)^{p/2}.\end{aligned} \tag{4.9}$$

The assertion follows now by applying (4.9) to (4.7). □

Remark 4.6. Strictly speaking, the inequality (4.8) is shown in [7] as a part of a more extensive argument that uses the assumption that the diagram G is not *regular* (see [7], page 432, for the definition of regularity). However, the assumption of non-regularity of G is completely immaterial concerning the validity of (4.8) and, in fact, not used in the proof in [7], page 436.

4.2. Tightness

Furnished with the moment bound of Lemma 4.1, we prove the following lemma that enables us to complete the proofs of Theorems 2.4 and 2.7.

Lemma 4.7 (Tightness). *Suppose that $H \in (0,1)^d$ and that Assumption 2.2 holds. Then, the family $\{\overline{U}_f^{(n)} \colon n \in \mathbb{N}\}$ is tight in $D([0,1]^d)$.*

Proof. The assertion follows from Theorem 3 of [6], provided that

$$\sup_{n \in \mathbb{N}^d} \sup_{\substack{s,t \in [0,1]^d \\ s<t}} \frac{\mathbf{E}[\overline{U}_f^{(n)}([s,t))^4]}{\langle t-s \rangle^2} < \infty. \tag{4.10}$$

But since for any $n \in \mathbb{N}$, the realization of $\overline{U}_f^{(n)}$ is constant on any set of the form

$$\left[\frac{i-1}{m(n)}, \frac{i}{m(n)}\right), \qquad 1 \le i \le m(n),$$

it suffices to show (see [6], page 1665) that

$$\sup_{n \in \mathbb{N}} \sup_{\substack{s,t \in \mathcal{E}_n \\ s<t}} \frac{\mathbf{E}[\overline{U}_f^{(n)}([s,t))^4]}{\langle t-s \rangle^2} < \infty, \tag{4.11}$$

where $\mathcal{E}_n := \{i/m(n) \colon 0 \le i \le m(n)\}$, instead of (4.10).

Using Lemmas 3.1 and 4.1, we arrive at

$$\sup_{n\in\mathbb{N}}\sup_{\substack{s,t\in\mathcal{E}_n\\ s<t}}\frac{\mathbf{E}[\overline{U}_f^{(n)}([s,t))^4]}{\langle t-s\rangle^2}=\sup_{n\in\mathbb{N}}\left\langle\frac{m(n)}{c(n)}\right\rangle^2\sup_{1\le l\le m(n)}\mathbf{E}\left[\left(\langle l\rangle^{-1/2}\sum_{1\le i\le l}f\big(X_i^{(n)}\big)\right)^4\right]$$

$$\le\sup_{n\in\mathbb{N}}\left(2^dC''''(f,4)\prod_{\nu=1}^d\psi_\nu(n)\right)^2,$$

where

$$\psi_\nu(n):=\begin{cases}\dfrac{1}{m_\nu(n)^{1-2\underline{k}(1-H_\nu)}}\displaystyle\sum_{|j|<m_\nu(n)}\big|r_{H_\nu}(j)\big|^{\underline{k}}\le C'(H_\nu,\underline{k}), & H_\nu\in\left(1-\dfrac{1}{2\underline{k}},1\right),\\[2ex] \dfrac{1}{\log(m_\nu(n))}\displaystyle\sum_{|j|<m_\nu(n)}\big|r_{H_\nu}(j)\big|^{\underline{k}}\le C'(H_\nu,\underline{k}), & H_\nu=1-\dfrac{1}{2\underline{k}},\\[2ex] \displaystyle\sum_{|j|<m_\nu(n)}\big|r_{H_\nu}(j)\big|^{\underline{k}}\le\sum_{j\in\mathbb{Z}}\big|r_{H_\nu}(j)\big|^{\underline{k}}<\infty, & H_\nu\in\left(0,1-\dfrac{1}{2\underline{k}}\right).\end{cases}$$

(The first two inequalities above follow from the estimate (3.6).) We have, thus, verified the tightness condition (4.11). □

Proof of Theorem 2.4. Recall that, for a family of pairs of random elements, tightness of marginals implies joint tightness. Thus, it follows from Lemma 4.7 that the family $\{(Z,\overline{U}_f^{(n)}):\ n\in\mathbb{N}\}$ is tight in $D([0,1]^d)^2$. The assertion follows then from Proposition 3.3 and Theorem 2 of [6]. □

Proof of Theorem 2.7. Analogously to the proof of Theorem 2.4, above, we deduce from Lemma 4.7 that $\{(\Lambda_{H,f}^{1/2}\widehat{Z},\overline{U}_f^{(n)}):\ n\in\mathbb{N}\}$ is tight in $D([0,1]^d)^2$. Moreover, Proposition 3.5 implies that

$$\overline{U}_f^{(n)}(t)\xrightarrow[n\to\infty]{\mathbf{P}}\Lambda_{H,f}^{1/2}\widehat{Z}(t),\qquad t\in[0,1]^d,$$

which, in turn, implies the corresponding convergence of finite-dimensional laws. Thus, by Theorem 2 of [6], we have

$$\big(\Lambda_{H,f}^{1/2}\widehat{Z},\overline{U}_f^{(n)}\big)\xrightarrow[n\to\infty]{\mathcal{L}}\big(\Lambda_{H,f}^{1/2}\widehat{Z},\Lambda_{H,f}^{1/2}\widehat{Z}\big)\qquad\text{in } D\big([0,1]^d\big)^2.\tag{4.12}$$

Since the limit in (4.12) belongs to $C([0,1]^d)^2$ and since substraction is a continuous operation on $C([0,1]^d)^2$ (with respect to the Skorohod topology), the continuous mapping theorem implies that

$$\overline{U}_f^{(n)}-\Lambda_{H,f}^{1/2}\widehat{Z}\xrightarrow[n\to\infty]{\mathcal{L}}0\qquad\text{in } D\big([0,1]^d\big).\tag{4.13}$$

It remains to note that the convergence (4.13) holds also in probability as the limit is deterministic. □

5. Application to power variations

5.1. Convergence of power variations and their fluctuations

As an application of Theorems 2.4 and 2.7, we study the asymptotic behaviour of *signed power variations* of the fBs Z. Let $p \in \mathbb{N}$ be fixed throughout this section. We consider a family $\{V_p^{(n)}\colon n \in \mathbb{N}\}$ of d-parameter processes, given by

$$V_p^{(n)}(t) := \langle m(n)^{pH-1} \rangle \sum_{1 \le i \le \lfloor m(n)t \rfloor} Z\left(\left[\frac{i-1}{m(n)}, \frac{i}{m(n)}\right]\right)^p, \qquad t \in [0,1]^d, n \in \mathbb{N}.$$

The realizations of $V_p^{(n)}$ belong to the space $D([0,1]^d)$, as was the case with generalized variations. To describe the asymptotic behaviour of $V_p^{(n)}$, we introduce

$$v_p(t) := \gamma_p \langle t \rangle, \qquad t \in [0,1]^d,$$
$$\rho_p(y) := y^p - \gamma_p, \qquad y \in \mathbb{R},$$

where γ_p is the pth moment of the standard Gaussian law, that is,

$$\gamma_p := \int_{\mathbb{R}} y^p \gamma(\mathrm{d}y) = \begin{cases} 0, & p \text{ is odd}, \\ \prod_{j=1}^{p/2} (2j-1), & p \text{ is even}. \end{cases}$$

Since the function ρ_p is a polynomial, it belongs to $L^2(\mathbb{R}, \gamma)$ and is a linear combination of finitely many Hermite polynomials. Moreover, it is easy to check that the Hermite rank of ρ_p is given by

$$\underline{k} = \underline{k}_p = \begin{cases} 1, & p \text{ is odd}, \\ 2, & p \text{ is even}. \end{cases}$$

Thus, the Hermite coefficients of ρ_p satisfy Assumption 2.2. In what follows, we denote by Λ_{H,ρ_p} the constant given by (2.9), substituting f with ρ_p therein.

As a straightforward application of Theorems 2.4 and 2.7, we can prove a functional law of large numbers (FLLN) for $V_p^{(n)}$ as $n \to \infty$, namely,

$$V_p^{(n)} \xrightarrow[n \to \infty]{\mathbf{P}} v_p \qquad \text{in } D\big([0,1]^d\big).$$

It would then be natural to expect that the rescaled fluctuation process

$$\frac{\langle m(n)\rangle}{\langle c(n)\rangle^{1/2}}\big(V_p^{(n)}(t) - v_p(t)\big), \qquad t \in [0,1]^d, \tag{5.1}$$

has a non-trivial limit as $n \to \infty$. In fact, we can write for any $t \in [0,1]^d$ and $n \in \mathbb{N}$,

$$\frac{\langle m(n)\rangle}{\langle c(n)\rangle^{1/2}}\big(V_p^{(n)}(t) - v_p(t)\big) = \overline{U}_{\rho_p}^{(n)}(t) - \beta_p^{(n)}(t), \tag{5.2}$$

where

$$\beta_p^{(n)}(t) := \frac{\langle m(n)\rangle}{\langle c(n)\rangle^{1/2}}\left(v_p(t) - v_p\left(\frac{\lfloor m(n)t\rfloor}{m(n)}\right)\right) \geq 0.$$

If the remainder $\beta_p^{(n)}$ were asymptotically negligible in $D([0,1]^d)$, the limit of the fluctuation process (5.1) when $n \to \infty$ would be easy to deduce from Theorems 2.4 and 2.7. If p is odd, then indeed $\beta_p^{(n)} = 0 = v_p$ for any $n \in \mathbb{N}$. However, when p is even, the situation is more delicate. In the special case $d = 1$, it is not difficult to see that $\beta_p^{(n)}(t) < c(n)^{-1/2} \to 0$ when $n \to \infty$ for any $t \in [0,1]$. But when $d \geq 2$, the fluctuations of $\beta_p^{(n)}$ may be non-negligible or even explosive when $n \to \infty$, as the following example shows.

Example 5.1. Consider the case where p is even, $d \geq 2$, $m(n) := (n, \dots, n)$ for any $n \in \mathbb{N}$, and $H \in (0, \frac{3}{4})^d$. Then we have by the mean value theorem,

$$\beta_p^{(n)}(t) = n^{d/2-1}\sum_{\nu=1}^{d}\left(\prod_{\kappa\neq\nu}\xi_\kappa^{(n)}(t)\right)\{nt_\nu\}, \qquad t \in [0,1]^d, n \in \mathbb{N},$$

where $\xi^{(n)}(t)$ is some convex combination of $n^{-1}\lfloor nt\rfloor$ and t. We will now show that $\beta_p^{(n)}$ cannot converge to a continuous function in $D([0,1]^d)$ as $n \to \infty$ (similar, but slightly longer, argument shows that a discontinuous limit in $D([0,1]^d)$ is also impossible).

To this end, suppose that $\beta_p^{(n)} \to \beta$ in $D([0,1]^d)$, where $\beta \in C([0,1]^d)$. Then it follows that $\beta_p^{(n)} \to \beta$ uniformly. By the continuity of β, there exists an open set $E \subset [\frac{2}{3}, 1]^d$ such that

$$\sup_{s,t\in E}\big|\beta(s) - \beta(t)\big| \leq \frac{1}{2^d}. \tag{5.3}$$

Note that there exists $n_0 \in \mathbb{N}$ such that $E \cap \mathcal{E}_n \neq \varnothing$ for any $n \geq n_0$, where $\mathcal{E}_n = \{i/m(n)\colon 0 \leq i \leq m(n)\}$. Moreover, we can find $n_1 \geq n_0$ such that

$$\inf_{t\in E}\prod_{\kappa\neq\nu}\xi_\kappa^{(n)}(t) \geq \frac{1}{2^{d-1}} \qquad \text{for any } n \geq n_1.$$

Thus, we find that for any $n \geq n_1$,

$$\sup_{t \in E} \beta_p^{(n)}(t) \geq \frac{n^{d/2-1}}{2^{d-1}}, \tag{5.4}$$

while

$$\inf_{t \in E} \beta_p^{(n)}(t) = 0. \tag{5.5}$$

But when $\beta_p^{(n)} \to \beta$ uniformly, the estimate (5.3) is not compatible with (5.4) and (5.5), which is a contradiction. (This also shows that $\beta_p^{(n)}$ cannot converge to β along a subsequence.)

5.2. Multilinear interpolations

We have just seen that the rescaled fluctuations (5.1) of the power variations $V_p^{(n)}$, $n \in \mathbb{N}$, around their FLLN limit v_p do not necessarily satisfy a functional limit theorem in $D([0,1]^d)$ when $d \geq 2$ and p is even. Note that it is implicit in the definition of $V_p^{(n)}$ that the corresponding partial sums are interpolated in a *piecewise constant* manner. Such an interpolation can have very poor precision in higher dimensions. In fact, interpolating $V_p^{(n)}$ using a more appropriate, multilinear method enables functional convergence in the general case.

Definition 5.2. *For any $n \in \mathbb{N}$, we define a (piecewise)* multilinear interpolation operator $L_n : \mathbb{R}^{[0,1]^d} \to C([0,1]^d)$ *acting on a function $g : [0,1]^d \to \mathbb{R}$, sampled on the lattice $\mathcal{E}_n$, by*

$$(L_n g)(t) := \sum_{i \in \{0,1\}^d} g\left(\frac{\lfloor m(n)t \rfloor + i}{m(n)}\right) \alpha_i^{(n)}(t), \qquad t \in [0,1]^d, \tag{5.6}$$

where the weights

$$\alpha_i^{(n)}(t) := \left\langle \left\{m(n)t\right\}^i \left(1 - \left\{m(n)t\right\}\right)^{1-i} \right\rangle, \qquad i \in \{0,1\}^d,$$

belong to $[0,1]$ *and satisfy*

$$\sum_{i \in \{0,1\}^d} \alpha_i^{(n)}(t) = 1. \tag{5.7}$$

Remark 5.3. (1) In the cases $d = 1$ and $d = 2$, the definition (5.6) reduces to the well-known (piecewise) linear and bilinear interpolation formulae, respectively.

(2) The definition (5.6) involves slight abuse of notation. Namely,

$$\frac{\lfloor m(n)t \rfloor + i}{m(n)} \notin [0,1]^d \tag{5.8}$$

when $t_\nu = 1$ and $i_\nu = 1$ for some $\nu \in \{1, \ldots, d\}$. But then $\alpha_i^{(n)}(t) = 0$, whence (5.8) is of no concern.

The fluctuation process, analogous to (5.1), obtained by substituting the power variation $V_p^{(n)}$ with its multilinear interpolation $\widetilde{V}_p^{(n)} := L_n V_p^{(n)}$ satisfies the following functional limit theorem. In particular, it applies with any $d \in \mathbb{N}$ and $p \in \mathbb{N}$.

Theorem 5.4 (Interpolated power variations). (1) *If* $H \in (0,1)^d \setminus (1 - \frac{1}{2\underline{k}_p}, 1)^d$, *then*

$$\left(Z, \frac{\langle m(n)\rangle}{\langle c(n)\rangle^{1/2}}\big(\widetilde{V}_p^{(n)} - v_p\big)\right) \xrightarrow[n\to\infty]{\mathcal{L}} (Z, \Lambda_{H,\rho_p}\widetilde{Z}) \qquad \text{in } C\big([0,1]^d\big)^2,$$

where $\widetilde{Z}$ *is the fBs of Theorem* 2.4.

(2) *If* $H \in (1 - \frac{1}{2\underline{k}_p}, 1)^d$, *then*

$$\frac{\langle m(n)\rangle}{\langle c(n)\rangle^{1/2}}\big(\widetilde{V}_p^{(n)} - v_p\big) \xrightarrow[n\to\infty]{\mathbf{P}} \Lambda_{H,\rho_p}\widehat{Z} \qquad \text{in } C\big([0,1]^d\big),$$

where $\widehat{Z}$ *is the Hermite sheet of Theorem* 2.7.

Remark 5.5. As mentioned above, the remainder term $\beta_p^{(n)}$ in the decomposition (5.2) is asymptotically negligible in $D([0,1]^d)$ if $d = 1$ or p is odd. In these special cases, multilinear interpolations can be dispensed with, to wit the convergences of Theorem 5.4 hold also with the original power variation $V_p^{(n)}$ in place of $\widetilde{V}_p^{(n)}$, in the spaces $D([0,1]^d)^2$ and $D([0,1]^d)$, respectively.

The proof of Theorem 5.4 is based on the following two simple lemmas concerning the multilinear interpolation operators. First, we show that the function v_p is a fixed point of the operator L_n for any $n \in \mathbb{N}$.

Lemma 5.6 (Fixed point). *We have* $L_n v_p = v_p$ *for any* $n \in \mathbb{N}$.

Proof. Let $t \in [0,1]^d$ and $n \in \mathbb{N}$. By rearranging, we obtain that

$$\begin{aligned}(L_n v_p)(t) &= \sum_{i\in\{0,1\}^d} \gamma_p \left\langle \frac{\lfloor m(n)t\rfloor + i}{m(n)} \big\{m(n)t\big\}^i \big(1 - \{m(n)t\}\big)^{1-i} \right\rangle \\ &= \gamma_p \prod_{\nu=1}^d \sum_{j\in\{0,1\}} \frac{\lfloor m_\nu(n)t_\nu\rfloor + j}{m_\nu(n)} \big\{m_\nu(n)t_\nu\big\}^j \big(1 - \{m_\nu(n)t_\nu\}\big)^{1-j}.\end{aligned}$$

It remains to observe that for any $\nu \in \{1,\dots,d\}$,

$$\sum_{j\in\{0,1\}} \frac{\lfloor m_\nu(n)t_\nu\rfloor + j}{m_\nu(n)} \big\{m_\nu(n)t_\nu\big\}^j \big(1 - \{m_\nu(n)t_\nu\}\big)^{1-j} = \frac{\lfloor m_\nu(n)t_\nu\rfloor + \{m_\nu(n)t_\nu\}}{m_\nu(n)} = t_\nu,$$

and the assertion follows. □

Second, we show that convergence in probability in the space $D([0,1]^d)$ can be converted to convergence in probability in $C([0,1]^d)$ via interpolations.

Lemma 5.7 (Convergence and interpolation). *Let $X_1, X_2, \dots$ be random elements in $D([0,1]^d)$ and X a random element in $C([0,1]^d)$, all defined on a common probability space. If $X_n \xrightarrow{\mathbf{P}} X$ in $D([0,1]^d)$ as $n \to \infty$, then*

$$L_n X_n \xrightarrow[n\to\infty]{\mathbf{P}} X \qquad \text{in } C\big([0,1]^d\big).$$

Proof. By (5.7), we can write for any $t \in [0,1]^d$ and $n \in \mathbb{N}$,

$$\begin{aligned}(L_n X_n)(t) - X(t) = &\sum_{i\in\{0,1\}^d} \left(X_n\left(\frac{\lfloor m(n)t\rfloor + i}{m(n)}\right) - X\left(\frac{\lfloor m(n)t\rfloor + i}{m(n)}\right)\right)\alpha_i^{(n)}(t) \\ &+ \sum_{i\in\{0,1\}^d} \left(X\left(\frac{\lfloor m(n)t\rfloor + i}{m(n)}\right) - X(t)\right)\alpha_i^{(n)}(t).\end{aligned}$$

Thus, invoking (5.7) again, we obtain the bound

$$\sup_{t\in[0,1]^d} \big|(L_n X_n)(t) - X(t)\big| \le \sup_{t\in[0,1]^d} \big|X_n(t) - X(t)\big| + w_X\big(\underline{m}(n)^{-1}\big),$$

where

$$w_X(u) := \sup\big\{\big|X(s) - X(t)\big|\colon s,t \in [0,1]^d, \|s-t\|_\infty \le u\big\}, \qquad u > 0,$$

is the modulus of continuity of X, which satisfies $\lim_{u\to 0} w_X(u) = 0$ a.s. since the realizations of X are uniformly continuous. Thus, $\lim_{n\to\infty} w_X(\underline{m}(n)^{-1}) = 0$ a.s. Finally, since convergence to a continuous function in $D([0,1]^d)$ is equivalent to uniform convergence, it follows that $\sup_{t\in[0,1]^d} |X_n(t) - X(t)| \xrightarrow{\mathbf{P}} 0$ as $n \to \infty$. □

Proof of Theorem 5.4. We have for any $n \in \mathbb{N}$, by Lemma 5.6, decomposition (5.2), and the linearity of the operator L_n,

$$\frac{\langle m(n)\rangle}{\langle c(n)\rangle^{1/2}}\big(\widetilde{V}_p^{(n)} - v_p\big) = L_n\left(\frac{\langle m(n)\rangle}{\langle c(n)\rangle^{1/2}}\big(V_p^{(n)} - v_p\big)\right) = L_n \overline{U}_{\rho_p}^{(n)} + L_n \beta_p^{(n)}.$$

Note that the function

$$t \mapsto v_p\left(\frac{\lfloor m(n)t\rfloor}{m(n)}\right)$$

coincides with v_p on $\mathcal{E}_n$. Since $L_n g$ depends on the function g only through the values of g on $\mathcal{E}_n$, we find that

$$L_n v_p = L_n v_p\left(\frac{\lfloor m(n)\cdot\rfloor}{m(n)}\right),$$

whence

$$L_n \beta_p^{(n)} = \frac{\langle m(n) \rangle}{\langle c(n) \rangle^{1/2}} \left(L_n v_p - L_n v_p \left(\frac{\lfloor m(n) \cdot \rfloor}{m(n)} \right) \right) = 0.$$

The assertion in the case (2) follows now from Theorem 2.7 and Lemma 5.7. In the case (1), one can apply Theorem 2.4, Lemma 5.7 and Skorohod's representation theorem [11], Theorem 4.30. □

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