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Non-Gaussian semi-stable laws arising in sampling of finite point processes

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A finite point process is characterized by the distribution of the number of points (the size) of the process. In some applications, for example, in the context of packet flows in modern communication networks, it is of interest to infer this size distribution from the observed sizes of sampled point processes, that is, processes obtained by sampling independently the points of i.i.d. realizations of the original point process. A standard nonparametric estimator of the size distribution has already been suggested in the literature, and has been shown to be asymptotically normal under suitable but restrictive assumptions. When these assumptions are not satisfied, it is shown here that the estimator can be attracted to a semi-stable law. The assumptions are discussed in the case of several concrete examples. A major theoretical contribution of this work are new and quite general sufficient conditions for a sequence of i.i.d. random variables to be attracted to a semi-stable law.

Keywords: domain of attraction; finite point process; sampling; semi-stable law

1. Introduction

We first explain the motivation behind this work, namely, understanding statistical properties of certain estimators arising when sampling finite point process. The issues raised in the motivation require developing new theoretical results on the domain of attraction of the so-called semi-stable laws. We conclude this section by describing this theoretical contribution, along with the structure of this work.

Let $W, W^{(i)}$, $i = 1, 2, \dots, N$, be i.i.d. integer-valued random variables with the probability mass function (p.m.f.) $f_W(w)$, $w \geq 1$. Let also $\mathrm{Bin}(n, q)$ denote a binomial distribution with parameters $n \geq 1$, $q \in (0, 1)$. Consider random variables $W_q, W_q^{(i)}$, $i = 1, 2, \dots, N$, obtained from $W, W^{(i)}$, $i = 1, 2, \dots, N$, through the relationships $W_q = \mathrm{Bin}(W, q)$ and $W_q^{(i)} = \mathrm{Bin}(W^{(i)}, q)$, $i = 1, 2, \dots, N$ (independently across i). Note that W_q takes values in $0, 1, 2, \dots, W$. Let the probability mass function of W_q be $f_{W_q}(s)$, $s \geq 0$. The basic interpretation of W_q is as follows. If an object consists of W points (a finite point process) and each point is sampled with a probability q, then the number of sampled points is $W_q = \mathrm{Bin}(W, q)$.

One application of the above setting arises in modern communication networks. A finite point process (an object) is associated with the so-called packet flow (and a point is associated with a single packet). Sampling is used in order to reduce the amount of data being collected and processed. One basic problem that has attracted much attention recently is the inference of f_W from the observed sampled data $W_q^{(i)}$, $i = 1, 2, \dots, N$ (in principle, $W_q^{(i)} = 0$ is not observed directly, but the inference about the number of times $W_q^{(i)} = 0$ is made through other means). See,

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for example, Duffield, Lund and Thorup [7], Hohn and Veitch [9], Yang and Michailidis [14]. For other, more recent progress on sampling in communication networks, see Antunes and Pipiras [2,3], and references therein.

We are interested here in some statistical properties of a nonparametric estimator of $f_W(w)$, introduced in Hohn and Veitch [9] and also considered in Antunes and Pipiras [1]. We first briefly outline how the estimator is derived. Estimation of $f_W(w)$ is based on a theoretical inversion of the relation

$$\begin{aligned} f_{W_q}(s) &= \sum_{w=s}^{\infty} P(W_q = s | W = w) P(W = w) \\ &= \sum_{w=s}^{\infty} \binom{w}{s} q^s (1-q)^{w-s} f_W(w), \qquad s \geq 0. \end{aligned} \tag{1.1}$$

In terms of the moment generating functions $G_{W_q}(z) = \sum_{s=0}^{\infty} z^s f_{W_q}(s)$ and $G_W(z) = \sum_{w=1}^{\infty} z^w f_W(w)$, the relation (1.1) can be written as $G_{W_q}(z) = G_W(zq + 1 - q)$. By changing the variables $zq + 1 - q = x$, one has $G_W(x) = G_{W_q}(q^{-1}x - q^{-1}(1-q))$ which has the earlier form but with q replaced by q^{-1} (and z replaced by x). This suggests that (1.1) can be inverted as

$$\begin{aligned} f_W(w) &= \sum_{s=w}^{\infty} \binom{s}{w} (q^{-1})^w (1-q^{-1})^{s-w} f_{W_q}(s) \\ &= \sum_{s=w}^{\infty} \binom{s}{w} \frac{(-1)^{s-w}}{q^s} (1-q)^{s-w} f_{W_q}(s), \qquad w \geq 1. \end{aligned} \tag{1.2}$$

Antunes and Pipiras [1], Proposition 4.1, showed that the inversion relation (1.2) holds when

$$\sum_{s=n}^{\infty} \binom{s}{n} \frac{(1-q)^{s-n}}{q^s} f_{W_q}(s) = \sum_{w=n}^{\infty} \binom{w}{n} 2^{w-n} (1-q)^{w-n} f_W(w) < \infty, \qquad n \geq 1. \tag{1.3}$$

Observe that (1.3) always holds when $q \in (0.5, 1)$. But when $q \in (0, 0.5]$, the finiteness of the above expression depends on the behavior of $f_W(w)$ as $w \to \infty$. We shall make the assumption (1.3) throughout this work.

In view of (1.2), a natural nonparametric estimator of f_W is

$$\widehat{f}_W(w) = \sum_{s=w}^{\infty} \binom{s}{w} \frac{(-1)^{s-w}}{q^s} (1-q)^{s-w} \widehat{f}_{W_q}(s), \qquad w \geq 1, \tag{1.4}$$

where

$$\widehat{f}_{W_q}(s) = \frac{1}{N} \sum_{i=1}^{N} 1_{\{W_q^{(i)} = s\}}, \qquad s \geq 0, \tag{1.5}$$

is the empirical p.m.f. of f_{W_q}, and 1_A denotes the indicator function of an event A. Note that, by using (1.4) and (1.2),

$$\sqrt{N}\big(\widehat{f}_W(w) - f_W(w)\big) = \sum_{s=w}^{\infty} \binom{s}{w} \frac{(-1)^{s-w}}{q^s}(1-q)^{s-w}\sqrt{N}\big(\widehat{f}_{W_q}(s) - f_{W_q}(s)\big). \tag{1.6}$$

Since

$$\big\{\sqrt{N}\big(\widehat{f}_{W_q}(s) - f_{W_q}(s)\big)\big\}_{s=0}^{\infty} \xrightarrow{d} \big\{\xi(s)\big\}_{s=0}^{\infty}, \tag{1.7}$$

where $\{\xi(s)\}_{s=0}^{\infty}$ is a Gaussian process with zero mean and covariance structure

$$E\big(\xi(s_1)\xi(s_2)\big) = f_{W_q}(s_1)1_{\{s_1=s_2\}} - f_{W_q}(s_1)f_{W_q}(s_2),$$

one may naturally expect that under suitable assumptions, (1.6) is asymptotically normal in the sense that

$$\big\{\sqrt{N}\big(\widehat{f}_W(w) - f_W(w)\big)\big\}_{w=1}^{\infty} \xrightarrow{d} \big\{S(\xi)_w\big\}_{w=1}^{\infty}, \tag{1.8}$$

where $\{S(\xi)_w\}_{w=1}^{\infty}$ is a Gaussian process. Antunes and Pipiras [1], Theorem 4.1, showed that (1.8) holds indeed if $R_{q,w} < \infty$, $w \geq 1$, where

$$\begin{aligned} R_{q,w} &= \sum_{s=w}^{\infty} \binom{s}{w}^2 \frac{(1-q)^{2(s-w)}}{q^{2s}} f_{W_q}(s) \\ &= \sum_{i=w}^{\infty} f_W(i)(1-q)^{i-2w}\binom{i}{w}\sum_{s=w}^{i}\binom{s}{w}\binom{i-w}{s-w}\big(q^{-1}-1\big)^s. \end{aligned} \tag{1.9}$$

The quantity $R_{q,w}$ is naturally related to the limiting variance of $\sqrt{N}\widehat{f}_W(w)$. Indeed, since $NE(\widehat{f}_{W_q}(s_1) - f_{W_q}(s_1))(\widehat{f}_{W_q}(s_2) - f_{W_q}(s_2)) = f_{W_q}(s_1)1_{\{s_1=s_2\}} - f_{W_q}(s_1)f_{W_q}(s_2)$ and by using (1.6) and (1.2), the asymptotic variance of $\sqrt{N}\widehat{f}_W(w)$ is expected to be $R_{q,w} - (f_W(w))^2$. Requiring $R_{q,w} < \infty$ is then a natural assumption in proving (1.8).

We are interested in $\widehat{f}_W(w)$ when the condition $R_{q,w} < \infty$, $w \geq 1$, is not satisfied. In fact, such a situation is expected with many distributions. For example, we show in Section 4 below that if $f_W(w) = (1-c)c^{w-1}$, $w \geq 1$, is a geometric distribution with parameter $c \in (0,1)$, then the distribution of $f_{W_q}(s)$ is given by

$$f_{W_q}(s) = \begin{cases} \dfrac{(1-q)(1-c)}{1-c(1-q)}, & \text{if } s = 0, \\ \dfrac{1}{c}c_q^s(1-c_q), & \text{if } s \geq 1, \end{cases} \tag{1.10}$$

where $c_q = \frac{cq}{1-c(1-q)}$. Moreover, the condition $R_{q,w} < \infty$ holds if and only if $c < \frac{q}{1-q}$ (see Section 4). Thus, for example, we are interested what happens with $\widehat{f}_W(w)$ when W_q has p.m.f. given by (1.10) with $c \geq \frac{q}{1-q}$.

To understand what happens when $R_{q,w} = \infty$, observe from (1.4) and (1.5) that $\widehat{f}_W(w)$ can also be written as

$$\widehat{f}_W(w) = \frac{1}{N}\sum_{i=1}^{N} X_i, \tag{1.11}$$

where X_i, $i = 1, 2, \dots, N$, are i.i.d. random variables defined as

$$X_i = \binom{W_q^{(i)}}{w} \frac{(-1)^{W_q^{(i)}-w}}{q^{W_q^{(i)}}} (1-q)^{W_q^{(i)}-w} 1_{\{W_q^{(i)} \geq w\}}. \tag{1.12}$$

Focus on the key term $\frac{(1-q)^{W_q^{(i)}}}{q^{W_q^{(i)}}} = (q^{-1}-1)^{W_q^{(i)}}$ entering (1.12). For example, when W is geometric with parameter c, $W_q^{(i)}$ has p.m.f. in (1.10). One then expects that

$$\begin{aligned} P\bigl((q^{-1}-1)^{W_q^{(i)}} > x\bigr) &= P\biggl(W_q^{(i)} > \frac{\log x}{\log(q^{-1}-1)}\biggr) \\ &\approx \frac{1}{c} c_q^{\log x/\log(q^{-1}-1)} = \frac{1}{c} x^{-\alpha}, \end{aligned} \tag{1.13}$$

where $\alpha = \frac{\log c_q^{-1}}{\log(q^{-1}-1)}$. This suggests that the distribution of X_i, $i = 1, 2, \dots, N$, has heavy tail and that the estimator $\widehat{f}_W(w)$ is asymptotically non-Gaussian stable when $\alpha < 2$. In fact, the story turns out to be more complex. Because of the discrete nature of $W_q^{(i)}$, the relation (1.13) does not hold in the asymptotic sense as $x \to \infty$. An appropriate setting in this case involves the so-called semi-stable laws. In the semi-stable context, moreover, the convergence of (1.11) is expected only along subsequences of N.

Semi-stable laws have been studied quite extensively (see Section 2 for references). They are infinitely divisible and extend the stable laws by allowing the power function in the Lévy measure (of the stable law) to be multiplied by a function with a multiplicative period. In particular, necessary and sufficient conditions are known for a distribution to be attracted to a semi-stable law (see Theorem 2.2 below), that is, for the sum of independent copies following the distribution to converge to a semi-stable law (along a subsequence and after suitable normalization and centering). A common example (and, in fact, one of the few concrete examples) of such a distribution is that of a log-geometric random variable

$$X = a^{W_q} \qquad \text{with } P(W_q = s) = (1-c_q)c_q^s, s = 0, 1, \dots, \tag{1.14}$$

where $a > 0$ and $c_q \in (0, 1)$. (Strictly speaking, the log-geometric case is when $a = e$.) Note that in (1.14), we use purposely the notation of (1.10) and (1.12).

In fact, motivated by (1.12) and the desire to consider more general distributions than log-geometric, we will show that the domain of attraction of semi-stable laws also includes the distributions of random variables of the form

$$X = k(W_q)a^{W_q} \qquad \text{with } P(W_q = s) = h(s)c_q^s, s = 0, 1, \dots, \tag{1.15}$$

where k and h are functions satisfying suitable but also flexible conditions. Our approach goes through verifying that the distributions determined by (1.15) satisfy the necessary and sufficient conditions to be attracted to a semi-stable law. Somewhat surprising perhaps, the proof turns out to be highly nontrivial. The difficulty lies in dealing with the general case when both functions k and h in (1.15) are not constant. Much of this work, in fact, concerns this problem.

The rest of this work is structured as follows. Preliminaries on semi-stable laws can be found in Section 2. In Section 3, we state and prove the main general results of this work concerning semi-stable distributions and their domains of attraction. In Section 4, we apply the main results from Section 3 to sampling of finite point processes. Several concrete examples, in particular, are considered. A few auxiliary results are given in the Appendix. Some numerical illustrations can be found in Chaudhuri and Pipiras [5].

2. Preliminaries on semi-stable laws

One way to characterize a semi-stable distribution is through its characteristic function (Maejima [11]).

Definition 2.1. *A probability distribution μ on $\mathbb{R}$ (or a random variable with distribution μ) is called semi-stable if there exist $r, b \in (0,1)$ and $c \in \mathbb{R}$ such that*

$$\widehat{\mu}(\theta)^r = \widehat{\mu}(b\theta)\mathrm{e}^{\mathrm{i}c\theta} \qquad \textit{for all } \theta \in \mathbb{R}, \tag{2.1}$$

and $\widehat{\mu}(\theta) \neq 0$, for all $\theta \in \mathbb{R}$, where $\widehat{\mu}(\theta)$ denotes the characteristic function of μ.

A semi-stable distribution is known to be infinitely divisible (Maejima [11]) with a location parameter $\eta \in \mathbb{R}$, a Gaussian part with variance $\sigma^2 \geq 0$ and a non-Gaussian part with Lévy measure characterized by (distribution) functions

$$L(x) = \frac{M_L(x)}{|x|^\alpha}, \qquad x < 0, \qquad R(x) = -\frac{M_R(x)}{x^\alpha}, \qquad x > 0, \tag{2.2}$$

where $\alpha \in (0,2)$, $M_L(c^{1/\alpha}x) = M_L(x)$ when $x < 0$, and $M_R(c^{1/\alpha}x) = M_R(x)$ when $x > 0$, for some $c > 0$. The functions M_L and M_R are thus periodic with multiplicative period $c^{1/\alpha}$. The functions $L(x)$ and $R(x)$ are left-continuous and non-decreasing on $(-\infty, 0)$ and right-continuous and non-decreasing on $(0,\infty)$, respectively. The characteristic function of a semi-stable distribution with a location parameter η and without a Gaussian part is given by

$$\log \widehat{\mu}(t) = \mathrm{i}\eta t + \int_{-\infty}^0 \left(\mathrm{e}^{\mathrm{i}tx} - 1 - \frac{\mathrm{i}tx}{1+x^2}\right) \mathrm{d}L(x) + \int_0^\infty \left(\mathrm{e}^{\mathrm{i}tx} - 1 - \frac{\mathrm{i}tx}{1+x^2}\right) \mathrm{d}R(x). \tag{2.3}$$

Semi-stable distributions arise as limits of partial sums of i.i.d. random variables. Let $X_1, X_2, \dots$ be a sequence of i.i.d. random variables with a common distribution function F.

Consider the sequence of partial sums

$$S_n^* = \frac{1}{A_{k_n}} \left\{ \sum_{j=1}^{k_n} X_j - B_{k_n} \right\}, \tag{2.4}$$

where $\{A_{k_n}\}$ and $\{B_{k_n}\}$ are normalizing and centering sequences. Semi-stable laws arise as limits of partial sums S_n^*, supposing that $\{k_n\}$ satisfies

$$k_n \to \infty, k_n \le k_{n+1}, \qquad \lim_{n\to\infty} \frac{k_{n+1}}{k_n} = c \in [1, \infty). \tag{2.5}$$

Moreover, if S_n^* converges to a nontrivial limit (semi-stable distribution), the distribution F of X_j is said to be in the domain of attraction of the limiting semi-stable law. In this case and supposing the limiting law is non-Gaussian semi-stable, it is known that the normalizing sequence $\{A_{k_n}\}$ necessarily satisfies

$$A_{k_n} \to \infty, A_{k_n} \le A_{k_{n+1}}, \qquad \lim_{n\to\infty} \frac{A_{k_{n+1}}}{A_{k_n}} = c^{1/\alpha} \qquad \text{where } \alpha \in (0, 2). \tag{2.6}$$

Megyesi [13], Grinevich and Khokhlov [8] gave necessary and sufficient conditions for a distribution to be in the domain of attraction of a semi-stable distribution.

Theorem 2.2 (Megyesi [13], Corollary 3). *Distribution F is in the domain of attraction of a non-Gaussian semi-stable distribution with the characteristic function (2.3) along the subsequence k_n with normalizing constants A_{k_n} satisfying (2.5) and (2.6) if and only if for all $x > 0$ large enough,*

$$F_-(-x) = x^{-\alpha} l^*(x)\big(M_L\big(-\delta(x)\big) + h_L(x)\big), \tag{2.7}$$

$$1 - F(x) = x^{-\alpha} l^*(x)\big(M_R\big(\delta(x)\big) + h_R(x)\big), \tag{2.8}$$

where l^* is a right-continuous function, slowly varying at ∞, $\alpha \in (0, 2)$, F_- is the left-continuous version of F and the error functions h_R and h_L are such that

$$h_K(A_{k_n} x_0) \to 0 \qquad \text{as } n \to \infty, \tag{2.9}$$

for every continuity point x_0 of M_R, if $K = R$, and $-x_0$ of M_L, if $K = L$. M_K, $K \in \{L, R\}$, are two periodic functions with common multiplicative period $c^{1/\alpha}$ and for all large enough x, $\delta(x)$ is defined as

$$\delta(x) = \frac{x}{a(x)} \in \big[1, c^{1/\alpha} + \varepsilon\big], \tag{2.10}$$

where $\varepsilon > 0$ is any fixed number, with

$$a(x) = A_{k_n} \qquad \text{if } A_{k_n} \le x < A_{k_{n+1}}. \tag{2.11}$$

Grinevich and Khokhlov [8] also showed that, in the sufficiency part of the theorem above, k_n can be chosen as follows. First, choose a sequence $\{\tilde{A}_n\}$ such that

$$\lim_{n\to\infty} n\tilde{A}_n^{-\alpha} l^*(\tilde{A}_n) = 1 \tag{2.12}$$

and

$$\tilde{A}_n \to \infty, \qquad \tilde{A}_n \le \tilde{A}_{n+1} \quad \text{and} \quad \lim_{n\to\infty} \frac{\tilde{A}_{n+1}}{\tilde{A}_n} = 1. \tag{2.13}$$

Define a new sequence $\{a_n\}$ by setting $a_n = A_{k_n}$ for every n, where A_{k_n} appears in (2.11). Then, the natural numbers k_n can be chosen as

$$\tilde{A}_{k_n} \le a_n < \tilde{A}_{k_{n+1}}. \tag{2.14}$$

The centering constants B_{k_n} in (2.4) can be chosen as (Csörgö and Megyesi [6])

$$B_{k_n} = k_n \int_{1/k_n}^{1-1/k_n} Q(s)\,\mathrm{d}s, \tag{2.15}$$

where, for $0 \le s \le 1$,

$$Q(s) = \inf_y \bigl\{F(y) \ge s\bigr\}. \tag{2.16}$$

The location parameter η of the limiting semi-stable law in (2.3) is then given by

$$\eta = \Theta(\psi_1) - \Theta(\psi_2), \tag{2.17}$$

where

$$\Theta(\psi_i) = \int_0^1 \frac{\psi_i(s)}{1+\psi_i^2(s)}\,\mathrm{d}s - \int_1^\infty \frac{\psi_i^3(s)}{1+\psi_i^2(s)}\,\mathrm{d}s, \qquad i = 1, 2, \tag{2.18}$$

and

$$\psi_1(s) = \inf_{x<0}\bigl\{L(x) > s\bigr\}, \qquad \psi_2(s) = \inf_{x<0}\bigl\{-R(-x) > s\bigr\}. \tag{2.19}$$

It is also worth mentioning that the slowly varying function $l^*(x)$ entering in (2.7) and (2.8) can be replaced by two different, asymptotically equivalent slowly varying functions $l_1^*(x)$ and $l_2^*(x)$. The proof of this result is given in Lemma A.5 in the Appendix.

3. General results concerning semi-stable domain of attraction

The next theorem is the main result of this work. We use the following notation throughout this work:

$$\lceil x\rceil = \text{ the smallest integer larger than or equal to } x,$$

$$\lceil x\rceil_+ = \text{ the smallest integer strictly larger than } x.$$

For example, $\lceil 2.47 \rceil = \lceil 2.47 \rceil_+ = 3$ but $\lceil 3 \rceil = 3$ and $\lceil 3 \rceil_+ = 4$. The function $\lceil x \rceil_+$ is the right-continuous version of the function $\lceil x \rceil$. Also note that $\lceil x \rceil_+ = [x] + 1$, where $[x]$ is the integer part of x (i.e., the largest integer smaller than or equal to x).

Theorem 3.1. *Let W_q be an integer-valued random variable taking values in $0, 1, 2, \ldots$ such that, for all $x > 0$,*

$$P\left(\frac{W_q}{2} \geq x, W_q \text{ is even}\right) = \sum_{n=\lceil x \rceil}^{\infty} P\left(\frac{W_q}{2} = n\right) = h_1(\lceil x \rceil) e^{-\nu \lceil x \rceil}, \tag{3.1}$$

$$P\left(\frac{W_q - 1}{2} \geq x, W_q \text{ is odd}\right) = \sum_{n=\lceil x \rceil}^{\infty} P\left(\frac{W_q - 1}{2} = n\right) = h_2(\lceil x \rceil) e^{-\nu \lceil x \rceil}, \tag{3.2}$$

where $\nu > 0$ and the functions h_1 and h_2 satisfy

$$\frac{h_2(x)}{h_1(x)} \to c_1 \qquad \text{as } x \to \infty, \tag{3.3}$$

for some fixed $c_1 \geq 0$, and

$$\frac{h_1(ax)}{h_1(x)} \to 1 \qquad \text{as } x \to \infty, a \to 1. \tag{3.4}$$

Let also

$$X = L\left(e^{W_q}\right) e^{\beta W_q} (-1)^{W_q}, \tag{3.5}$$

where $\beta > 0$ and L is a slowly varying function at ∞ such that $L(e^n)$ is ultimately monotonically increasing. Suppose that

$$\alpha := \frac{\nu}{2\beta} < 2. \tag{3.6}$$

Then, X is attracted to the domain of a semi-stable distribution in the following sense. If $X, X_1, X_2, \ldots$ are i.i.d. random variables, then as $n \to \infty$, the partial sums

$$\frac{1}{A_{k_n}} \left\{ \sum_{j=1}^{k_n} X_j - B_{k_n} \right\} \tag{3.7}$$

converge to a semi-stable distribution with

$$k_n = \left\lceil \frac{e^{(n-1)\nu}}{h_1(n-1)} \right\rceil, \qquad A_{k_n} = L\left(e^{2n-2}\right) e^{2\beta(n-1)} \tag{3.8}$$

and B_{k_n} given by (2.15). The limiting semi-stable distribution is non-Gaussian, has location parameter given in (2.17) and is characterized by

$$\alpha = \frac{\nu}{2\beta}, \tag{3.9}$$

$$\begin{aligned} M_L(-x) &= c_1 \mathrm{e}^{-\nu([1/2+(1/(2\beta))\log x]-(1/(2\beta))\log x)}, \\ M_R(x) &= \mathrm{e}^{-\nu(\lceil(1/(2\beta))\log x\rceil_+ -(1/(2\beta))\log x)}, \qquad x > 0. \end{aligned} \tag{3.10}$$

Proof. The result will be proved by verifying the sufficient conditions (2.7)–(2.8) of Theorem 2.2. We break the proof into two cases dealing with (2.7) and (2.8) separately. The final part of the proof shows that the sequence k_n can be chosen as in (3.8).

Step 1 (showing (2.8)): Fix $x > 0$ large enough. In view of (3.5), we are interested in

$$\bar{F}(x) := 1 - F(x) = P\big(L\big(\mathrm{e}^{W_q}\big)\mathrm{e}^{\beta W_q}(-1)^{W_q} > x\big). \tag{3.11}$$

Let $Z_2 = \frac{W_q}{2}$. Note that (3.11) can be written as

$$\begin{aligned} \bar{F}(x) &= P\big(L\big(\mathrm{e}^{2Z_2}\big)\mathrm{e}^{2\beta Z_2} > x, Z_2 \text{ is integer}\big) \\ &= P\big(L\big(\mathrm{e}^{2Z_2}\big)\mathrm{e}^{2\beta Z_2} > x\big) \\ &= P\left(Z_2 + \frac{1}{2\beta}\log L\big(\mathrm{e}^{2Z_2}\big) > \frac{1}{2\beta}\log x\right), \end{aligned} \tag{3.12}$$

where, in view of (3.1),

$$P(Z_2 \ge x) = h_1\big(\lceil x\rceil\big)\mathrm{e}^{-\nu\lceil x\rceil}. \tag{3.13}$$

We next want to write $\bar{F}(x)$ in (3.12) as

$$\bar{F}(x) = P\left(Z_2 \ge g\left(\frac{1}{2\beta}\log x\right)\right) \tag{3.14}$$

for some function g.

There are many choices for g in (3.14). One natural choice is to take

$$g_0(y) = n \qquad \text{if } (n-1) + \frac{1}{2\beta}\log L\big(\mathrm{e}^{2n-2}\big) \le y < n + \frac{1}{2\beta}\log L\big(\mathrm{e}^{2n}\big). \tag{3.15}$$

The function g_0, however, turns out not to be suitable for our purpose. It will be used below only for reference and comparison to other related functions. We will use a related function g_1 defined, for integer $n \ge 2$, as

$$g_1(y) = \begin{cases} n-1, \\ \quad \text{if } n-1+\dfrac{1}{2\beta}\log L\big(\mathrm{e}^{2n-2}\big) \le y < n-1+\dfrac{1}{2\beta}\log L\big(\mathrm{e}^{2n}\big), \\ y - \dfrac{1}{2\beta}\log L\big(\mathrm{e}^{2n}\big), \\ \quad \text{if } n-1+\dfrac{1}{2\beta}\log L\big(\mathrm{e}^{2n}\big) \le y < n+\dfrac{1}{2\beta}\log L\big(\mathrm{e}^{2n}\big). \end{cases} \tag{3.16}$$

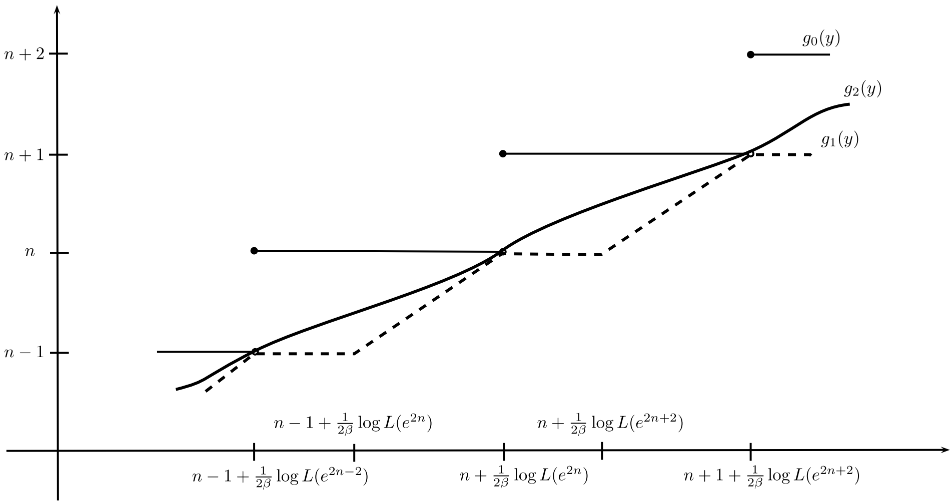

Figure 1. Plot of $g_0(y)$, $g_1(y)$ and $g_2(y)$.

We will also use the function

$$g_2(y) = f^{-1}(y) = \inf\{z\colon\ f(z) \geq y\} \tag{3.17}$$

defined as an inverse of the function

$$f(z) = z + \frac{1}{2\beta}\log L\left(\mathrm{e}^{2z}\right). \tag{3.18}$$

Note that

$$\lceil g_0(y)\rceil = \lceil g_1(y)\rceil_+ = \lceil g_2(y)\rceil_+ = \lceil g(y)\rceil, \tag{3.19}$$

where g is any function satisfying (3.14). The functions g_0, g_1 and g_2 are plotted in Figure 1.

We shall use another function $\tilde{g}_1$ which modifies g_1 in the following way: for $n \geq 2$,

$$\begin{aligned} &\tilde{g}_1(y) = y - \frac{1}{2\beta}\log L\left(\mathrm{e}^{2n-2}\right) \\ &\text{if } n-1+\frac{1}{2\beta}\log L\left(\mathrm{e}^{2n-2}\right) \leq y < n + \frac{1}{2\beta}\log L\left(\mathrm{e}^{2n}\right). \end{aligned} \tag{3.20}$$

One relationship between the functions g_1 and $\tilde{g}_1$ can be found in Lemma A.1 in the Appendix, and will be used in the proof below. Note that $\tilde{g}_1(y)$ can be expressed as

$$\tilde{g}_1(y) = y - \tilde{g}_1^*(y), \tag{3.21}$$

where, for $n \geq 2$,

$$\tilde{g}_1^*(y) = \frac{1}{2\beta} \log L\big(\mathrm{e}^{2n-2}\big) \qquad \text{if } n-1+\frac{1}{2\beta}\log L\big(\mathrm{e}^{2n-2}\big) \leq y < n + \frac{1}{2\beta}\log L\big(\mathrm{e}^{2n}\big). \tag{3.22}$$

See Lemma A.2 in the Appendix for a property of $\tilde{g}_1^*$ which will be used in the proof below.

We need few properties of the function g_2. Since g_2 is the inverse of the function f, we have $\mathrm{e}^{g_2(\log x)}$ as the inverse of $\mathrm{e}^{f(\log x)}$. Indeed,

$$\mathrm{e}^{g_2(\log \mathrm{e}^{f(\log x)})} = \mathrm{e}^{g_2(f(\log x))} = \mathrm{e}^{\log x} = x.$$

Note now from (3.18) that

$$\mathrm{e}^{f(\log x)} = \mathrm{e}^{\log x + (1/(2\beta))\log L(x^2)} = x\big(L\big(x^2\big)\big)^{1/(2\beta)}.$$

Since $(L(x^2))^{1/(2\beta)}$ is a slowly varying function, $\mathrm{e}^{f(\log x)}$ is a regularly varying function. So, by Theorem 1.5.13 of Bingham, Goldie and Teugels [4],

$$\mathrm{e}^{g_2(\log x)} = x l(x),$$

where $l(x)$ is a slowly varying function. Hence,

$$g_2(\log x) = \log x + \log l(x) = \log x + g_2^*(\log x),$$

where

$$g_2^*(\log x) = \log l(x)$$

or replacing $\log x$ by y,

$$g_2(y) = y + g_2^*(y). \tag{3.23}$$

Note also that for any $A > 0$, we have

$$g_2^*(\log Ax) - g_2^*(\log x) = \log l(Ax) - \log l(x) = \log\frac{l(Ax)}{l(x)} \to 0 \qquad \text{as } x \to \infty. \tag{3.24}$$

Continuing with (3.14) now, note that, by using (3.13) and (3.19),

$$\begin{aligned}\bar{F}(x) &= P\left(Z_2 \geq g\left(\frac{1}{2\beta}\log x\right)\right)\\ &= h_1\left(\left\lceil g\left(\frac{1}{2\beta}\log x\right)\right\rceil\right)\mathrm{e}^{-\nu\lceil g((1/(2\beta))\log x)\rceil}\\ &= h_1\left(\left\lceil g_2\left(\frac{1}{2\beta}\log x\right)\right\rceil_+\right)\mathrm{e}^{-\nu\lceil g_1((1/(2\beta))\log x)\rceil_+}.\end{aligned} \tag{3.25}$$

By using (3.21), note further that

$$
\begin{aligned}
\bar{F}(x) &= h_1\left(\left\lceil g_2\left(\frac{1}{2\beta}\log x\right)\right\rceil_+\right)\mathrm{e}^{-\nu\tilde{g}_1((1/(2\beta))\log x)} \\
&\quad\times \mathrm{e}^{-\nu(g_1((1/(2\beta))\log x)-\tilde{g}_1((1/(2\beta))\log x))}\mathrm{e}^{-\nu(\lceil g_1((1/(2\beta))\log x)\rceil_+-g_1((1/(2\beta))\log x))} \\
&= h_1\left(\left\lceil g_2\left(\frac{1}{2\beta}\log x\right)\right\rceil_+\right)\mathrm{e}^{-\nu((1/(2\beta))\log x-\tilde{g}_1^*((1/(2\beta))\log x))} \\
&\quad\times \mathrm{e}^{-\nu(g_1((1/(2\beta))\log x)-\tilde{g}_1((1/(2\beta))\log x))}\mathrm{e}^{-\nu(\lceil g_1((1/(2\beta))\log x)\rceil_+-g_1((1/(2\beta))\log x))} \\
&= x^{-\alpha}l_1^*(x)\bigl(M_R\bigl(\delta(x)\bigr)+h_R(x)\bigr),
\end{aligned}
\tag{3.26}
$$

where $\alpha=\frac{\nu}{2\beta}$ as given in (3.9),

$$
\begin{aligned}
l_1^*(x) &= h_1\left(\left\lceil g_2\left(\frac{1}{2\beta}\log x\right)\right\rceil_+\right)\mathrm{e}^{\nu\tilde{g}_1^*((1/(2\beta))\log x)} \\
&\quad\times \mathrm{e}^{-\nu(g_1((1/(2\beta))\log x)-\tilde{g}_1((1/(2\beta))\log x))},
\end{aligned}
\tag{3.27}
$$

$$
M_R\bigl(\delta(x)\bigr)=\mathrm{e}^{-\nu(\lceil\tilde{g}_1((1/(2\beta))\log x)\rceil_+-\tilde{g}_1((1/(2\beta))\log x))}
\tag{3.28}
$$

and

$$
\begin{aligned}
h_R(x) &= \mathrm{e}^{-\nu(\lceil g_1((1/(2\beta))\log x)\rceil_+-g_1((1/(2\beta))\log x))} \\
&\quad- \mathrm{e}^{-\nu(\lceil\tilde{g}_1((1/(2\beta))\log x)\rceil_+-\tilde{g}_1((1/(2\beta))\log x))}.
\end{aligned}
\tag{3.29}
$$

We next show that the functions l_1^*, M_R and h_R satisfy the conditions of Theorem 2.2 with suitable choices of $\delta(x)$ and A_{k_n}.

By Lemma A.3 in the Appendix, $l_1^*(x)$ is a right-continuous slowly varying function and hence it satisfies the conditions of Theorem 2.2. For the function $M_R(\delta(x))$, note from (3.28) that

$$
\begin{aligned}
M_R\bigl(\delta(x)\bigr) &= \mathrm{e}^{-\nu(\lceil 2\beta\tilde{g}_1((1/(2\beta))\log x)/(2\beta)\rceil_+-2\beta\tilde{g}_1((1/(2\beta))\log x)/(2\beta))} \\
&= M_R\bigl(\mathrm{e}^{2\beta\tilde{g}_1((1/(2\beta))\log x)}\bigr)
\end{aligned}
\tag{3.30}
$$

with

$$
M_R(x)=\mathrm{e}^{-\nu(\lceil\log x/(2\beta)\rceil_+-\log x/(2\beta))}.
\tag{3.31}
$$

The function $M_R(x)$ is periodic with multiplicative period $\mathrm{e}^{2\beta}$, and is right-continuous as required in Theorem 2.2. Since the period $\mathrm{e}^{2\beta}$ is also $c^{1/\alpha}$, this yields

$$
c=\mathrm{e}^{\nu}.
\tag{3.32}
$$

To choose $\delta(x)$, note from (3.30) that

$$
M_R\bigl(\delta(x)\bigr)=M_R\bigl(\mathrm{e}^{2\beta\tilde{g}_1((1/(2\beta))\log x)-2\beta(n-1)}\bigr),
$$

for any $n \geq 1$, since M_R has multiplicative period $e^{2\beta}$. We can set

$$\delta(x) = e^{2\beta \tilde{g}_1((1/(2\beta))\log x) - 2\beta(n-1)} \qquad \text{if } e^{2\beta(n-1)} L\bigl(e^{2n-2}\bigr) \leq x < e^{2n\beta} L\bigl(e^{2n}\bigr). \tag{3.33}$$

From (3.20), we have

$$\begin{aligned} \delta(x) &= e^{2\beta((1/(2\beta))\log x - (1/(2\beta))\log L(e^{2n-2})) - 2\beta(n-1)} \\ &= \frac{x}{e^{2\beta(n-1)} L(e^{2n-2})} \qquad \text{if } e^{2\beta(n-1)} L\bigl(e^{2n-2}\bigr) \leq x < e^{2n\beta} L\bigl(e^{2n}\bigr). \end{aligned} \tag{3.34}$$

Thus, $\delta(x)$ has the required form (2.10)–(2.11) with

$$A_{k_n} = e^{2\beta(n-1)} L\bigl(e^{2n-2}\bigr) \tag{3.35}$$

and

$$a(x) = e^{2\beta(n-1)} L\bigl(e^{2n-2}\bigr) = A_{k_n} \qquad \text{if } A_{k_n} \leq x < A_{k_{n+1}}. \tag{3.36}$$

Note also from (3.34) that

$$\begin{aligned} 1 \leq \delta(x) &< \frac{e^{2\beta n} L(e^{2n})}{e^{2\beta(n-1)} L(e^{2n-2})} \\ &= e^{2\beta} \frac{L(e^{2n})}{L(e^{-2} e^{2n})} \to e^{2\beta} = c^{1/\alpha}, \end{aligned}$$

so that $\delta(x) \in [1, c^{1/\alpha} + \varepsilon]$ for large enough x when $\varepsilon > 0$ is fixed.

To complete step 1, we need to prove that $h_R(A_{k_n} x_0) \to 0$ as $n \to \infty$ for every continuity point x_0 of $M_R(x)$. The discontinuity points of M_R are

$$x = e^{2k\beta}, \qquad k \in \mathbb{Z}. \tag{3.37}$$

To show $h_R(A_{k_n} x_0) \to 0$, note that, by Lemma A.1, it is enough to prove that $\tilde{h}_R(A_{k_n} x_0) \neq 0$ for finitely many values of n, where

$$\tilde{h}_R(x) = e^{-\nu \lceil g_1((1/(2\beta))\log x)\rceil_+} - e^{-\nu \lceil \tilde{g}_1((1/(2\beta))\log x)\rceil_+}.$$

This holds only if for some integer $m \geq 2$,

$$m + \log L\bigl(e^{2m-2}\bigr) \leq \frac{1}{2\beta} \log A_{k_n} x_0 < m + \log L\bigl(e^{2m}\bigr). \tag{3.38}$$

By Lemma A.4, (3.38) holds for infinitely many values of n only if $x_0 = e^{2r\beta}$, $r \in \mathbb{Z}$, which is a discontinuity point of $M_R(x)$ in (3.37). Hence, $h_R(A_{k_n} x_0) \to 0$ as $n \to \infty$ for every continuity point x_0 of $M_R(x)$.

Step 2 (showing (2.7)): In view of (3.5), we are now interested in

$$F_-(-x) = P\bigl(L\bigl(e^{W_q}\bigr) e^{\beta W_q} (-1)^{W_q} < -x\bigr). \tag{3.39}$$

Let $Z_2 = \frac{W_q}{2}$ as in step 1. Note that (3.39) can be written as

$$\begin{aligned} F_-(-x) &= P\left(L\left(\mathrm{e}^{2Z_2}\right)\mathrm{e}^{2\beta Z_2} > x, Z_2 - \frac{1}{2} \text{ is integer}\right) \\ &= P\left(L\left(\mathrm{e}\mathrm{e}^{2(Z_2-1/2)}\right)\mathrm{e}^{\beta}\mathrm{e}^{2\beta(Z_2-1/2)} > x, Z_2 - \frac{1}{2} \text{ is integer}\right) \\ &= P\left(L\left(\mathrm{e}\mathrm{e}^{2Z_1}\right)\mathrm{e}^{\beta}\mathrm{e}^{2\beta Z_1} > x\right) \\ &= P\left(Z_1 + \frac{1}{2} + \frac{1}{2\beta}\log L\left(\mathrm{e}^{2Z_1+1}\right) > \frac{1}{2\beta}\log x\right), \end{aligned} \tag{3.40}$$

where, in view of (3.2),

$$P(Z_1 \geq x) = h_2(\lceil x \rceil)\mathrm{e}^{-\nu\lceil x \rceil}. \tag{3.41}$$

Writing (3.40) as

$$F_-(-x) = P\left(Z_1 + \frac{1}{2\beta}\log L\left(\mathrm{e}^{2Z_1+1}\right) > \frac{1}{2\beta}\log x - \frac{1}{2}\right),$$

the right-hand side has the form (3.11) where $L(\mathrm{e}^{2Z_2})$ is replaced by $L(\mathrm{e}\mathrm{e}^{2Z_1})$ and $\frac{1}{2\beta}\log x$ is replaced by $\frac{1}{2\beta}\log x - \frac{1}{2}$. Thus, as in (3.14)–(3.15), one can write

$$F_-(-x) = P\left(Z_1 \geq \tilde{g}\left(\frac{1}{2\beta}\log x - \frac{1}{2}\right)\right), \tag{3.42}$$

where

$$\tilde{g}(y) = n \qquad \text{if } n - 1 + \frac{1}{2\beta}\log L\left(\mathrm{e}\mathrm{e}^{2n-2}\right) \leq y < n + \frac{1}{2\beta}\log L\left(\mathrm{e}\mathrm{e}^{2n}\right). \tag{3.43}$$

The expression (3.42) can also be written as

$$F_-(-x) = P\left(Z_1 \geq \tilde{g}_0\left(\frac{1}{2\beta}\log x\right)\right), \tag{3.44}$$

where $\tilde{g}_0(y) = \tilde{g}(y - \frac{1}{2})$ or, for $n \geq 2$,

$$\tilde{g}_0(y) = n \qquad \text{if } n - \frac{1}{2} + \frac{1}{2\beta}\log L\left(\mathrm{e}^{2n-1}\right) \leq y < n + \frac{1}{2} + \frac{1}{2\beta}\log L\left(\mathrm{e}^{2n+1}\right). \tag{3.45}$$

We want to work with the intervals $[n - 1 + \frac{1}{2\beta}\log L(\mathrm{e}^{2n-2}), n + \frac{1}{2\beta}\log L(\mathrm{e}^{2n}))$ appearing in step 1, and use the results of that step. Note that, on the interval $[n - 1 + \frac{1}{2\beta}\log L(\mathrm{e}^{2n-2}), n +$

$\frac{1}{2\beta}\log L(\mathrm{e}^{2n}))$, the function $\tilde{g}_0$ has the form

$$\tilde{g}_0(y)=\begin{cases} n-1, & \text{if } n-1+\dfrac{1}{2\beta}\log L\bigl(\mathrm{e}^{2n-2}\bigr)\le y<n-\dfrac{1}{2}+\dfrac{1}{2\beta}\log L\bigl(\mathrm{e}^{2n-1}\bigr),\\ n, & \text{if } n-\dfrac{1}{2}+\dfrac{1}{2\beta}\log L\bigl(\mathrm{e}^{2n-1}\bigr)\le y<n+\dfrac{1}{2\beta}\log L\bigl(\mathrm{e}^{2n}\bigr).\end{cases} \tag{3.46}$$

Defining

$$I_0(y)=\begin{cases} -1, & \text{if } n-1+\dfrac{1}{2\beta}\log L\bigl(\mathrm{e}^{2n-2}\bigr)\le y<n-\dfrac{1}{2}+\dfrac{1}{2\beta}\log L\bigl(\mathrm{e}^{2n-1}\bigr),\\ 0, & \text{if } n-\dfrac{1}{2}+\dfrac{1}{2\beta}\log L\bigl(\mathrm{e}^{2n-1}\bigr)\le y<n+\dfrac{1}{2\beta}\log L\bigl(\mathrm{e}^{2n}\bigr),\end{cases} \tag{3.47}$$

and combining (3.15), (3.46) and (3.47), we have

$$\tilde{g}_0(y)=g_0(y)+I_0(y). \tag{3.48}$$

Continuing with (3.44), note further that, by using (3.41) and (3.48),

$$\begin{aligned} F_-(-x)&=h_2\biggl(\tilde{g}_0\biggl(\frac{1}{2\beta}\log x\biggr)\biggr)\mathrm{e}^{-\nu\tilde{g}_0((1/(2\beta))\log x)}\\ &=\mathrm{e}^{-\nu I_0((1/(2\beta))\log x)}h_2\biggl(g_0\biggl(\frac{1}{2\beta}\log x\biggr)+I_0\biggl(\frac{1}{2\beta}\log x\biggr)\biggr)\mathrm{e}^{-\nu g_0((1/(2\beta))\log x)}. \end{aligned} \tag{3.49}$$

We want to write $F_-(-x)$ as in (2.7) of Theorem 2.2 (where by Lemma A.5, we can take a slowly varying function l_2^* which is asymptotically equivalent to l_1^*). We need the notation for the intervals appearing in (3.46)–(3.47), namely, for $n\ge 1$,

$$\begin{aligned} D_n&=\biggl[n-1+\frac{1}{2\beta}\log L\bigl(\mathrm{e}^{2n-2}\bigr),n-\frac{1}{2}+\frac{1}{2\beta}\log L\bigl(\mathrm{e}^{2n-1}\bigr)\biggr),\\ E_n&=\biggl[n-\frac{1}{2}+\frac{1}{2\beta}\log L\bigl(\mathrm{e}^{2n-1}\bigr),n+\frac{1}{2\beta}\log L\bigl(\mathrm{e}^{2n}\bigr)\biggr). \end{aligned}$$

We also need a similar notation without the slowly varying function L, that is, for $n\ge 1$,

$$D'_n=\bigl[n-1,n-\tfrac{1}{2}\bigr),\qquad E'_n=\bigl[n-\tfrac{1}{2},n\bigr).$$

Set also

$$\begin{aligned} D&=\bigcup_{n=1}^{\infty}D_n, & E&=\bigcup_{n=1}^{\infty}E_n,\\ D'&=\bigcup_{n=1}^{\infty}D'_n, & E'&=\bigcup_{n=1}^{\infty}E'_n. \end{aligned} \tag{3.50}$$

As in (3.26), we can now write (3.49) as

$$F_-(-x) = x^{-\alpha} \frac{h_2(g_0((1/(2\beta))\log x) + I_0((1/(2\beta))\log x))}{c_1 h_1(g_0((1/(2\beta))\log x))} l_1^*(x) c_1 \mathrm{e}^{-\nu I_0((1/(2\beta))\log x)} \times \mathrm{e}^{-\nu(\lceil g_1((1/(2\beta))\log x)\rceil_+ - g_1((1/(2\beta))\log x))},$$

where $\alpha = \frac{\nu}{2\beta}$ and $l_1^*(x)$ is given in (3.27). This can also be written as

$$F_-(-x) = x^{-\alpha} l_2^*(x)\big(M_L\big(-\delta(x)\big) + h_L(x)\big),$$

where

$$l_2^*(x) = \frac{h_2(g_0((1/(2\beta))\log x) + I_0((1/(2\beta))\log x))}{c_1 h_1(g_0((1/(2\beta))\log x))} l_1^*(x), \tag{3.51}$$

$$M_L\big(-\delta(x)\big) = c_1 \mathrm{e}^{-\nu([1/2+\tilde{g}_1((1/(2\beta))\log x)] - \tilde{g}_1((1/(2\beta))\log x))}, \tag{3.52}$$

$$\begin{aligned} h_L(x) &= c_1 \mathrm{e}^{-\nu I_0((1/(2\beta))\log x)} \mathrm{e}^{-\nu(\lceil g_1((1/(2\beta))\log x)\rceil_+ - g_1((1/(2\beta))\log x))} \\ &\quad - c_1 \mathrm{e}^{-\nu([1/2+\tilde{g}_1((1/(2\beta))\log x)] - \tilde{g}_1((1/(2\beta))\log x))}. \end{aligned} \tag{3.53}$$

By using (3.3)–(3.4), we have

$$\frac{h_2(g_0((1/(2\beta))\log x) + I_0((1/(2\beta))\log x))}{c_1 h_1(g_0((1/(2\beta))\log x))} \to 1 \qquad \text{as } x \to \infty.$$

Hence, $\frac{l_2^*(x)}{l_1^*(x)} \to 1$, as $x \to \infty$, that is, $l_2^*(x)$ and $l_1^*(x)$ are two asymptotically equivalent functions. By the definition of I_0 and using Lemma A.3, $l_2^*(x)$ is right-continuous and slowly varying.

The function $\delta(x)$ appearing in (3.52) is the same as in (3.33)–(3.34) of step 1, while the function $M_L(-x)$ is defined as

$$M_L(-x) = c_1 \mathrm{e}^{-\nu([1/2+(1/(2\beta))\log x] - (1/(2\beta))\log x)}, \qquad x > 0. \tag{3.54}$$

It is left-continuous when $x > 0$, and also periodic with multiplicative period $\mathrm{e}^{2\beta} = c^{1/\alpha}$. Thus, $M_L(x)$ for $x < 0$ is left-continuous as required in Theorem 2.2. The discontinuity points of $M_L(-x)$ are

$$x = \mathrm{e}^{\beta(2k+1)}, \qquad k \in \mathbb{Z}. \tag{3.55}$$

To conclude the proof of step 2, we need to show that $h_L(A_{k_n} x_0) \to 0$ as $n \to \infty$ for every continuity point x_0 of $M_L(-x)$, that is, x_0 different from (3.55). For this, we rewrite $h_L(x)$ as follows. Observe that

$$\mathrm{e}^{-\nu I_0(y)} = \mathrm{e}^{\nu} 1_D(y) + 1_E(y)$$

and

$$\big(\mathrm{e}^{\nu} 1_{D'}(y) + 1_{E'}(y)\big)\mathrm{e}^{-\nu(\lceil y\rceil_+ - y)} = \mathrm{e}^{-\nu([1/2+y]-y)},$$

where after taking the logs, using $\lceil y\rceil_+ = [y]+1$ and simplification, the last identity is equivalent to $[y]1_{D'}(y) + ([y]+1)1_{E'}(y) = [\frac{1}{2}+y]$ and can be seen easily by drawing a picture. By using these identities and (3.53), we can write

$$\begin{aligned} c_1^{-1}h_L(x) &= \left(\mathrm{e}^{\nu}1_D\left(\frac{1}{2\beta}\log x\right) + 1_E\left(\frac{1}{2\beta}\log x\right)\right)\mathrm{e}^{-\nu(\lceil g_1((1/(2\beta))\log x)\rceil_+ - g_1((1/(2\beta))\log x))} \\ &\quad - \mathrm{e}^{-\nu([1/2+\tilde g_1((1/(2\beta))\log x)] - \tilde g_1((1/(2\beta))\log x))} \\ &= h_{1,L}(x)\mathrm{e}^{-\nu(\lceil g_1((1/(2\beta))\log x)\rceil_+ - g_1((1/(2\beta))\log x))} + h_{2,L}(x), \end{aligned}$$

where

$$\begin{aligned} h_{1,L}(x) &= \mathrm{e}^{\nu}1_D\left(\frac{1}{2\beta}\log x\right) + 1_E\left(\frac{1}{2\beta}\log x\right) - \mathrm{e}^{\nu}1_{D'}\left(g_1\left(\frac{1}{2\beta}\log x\right)\right) \\ &\quad - 1_{E'}\left(g_1\left(\frac{1}{2\beta}\log x\right)\right), \\ h_{2,L}(x) &= \mathrm{e}^{-\nu([1/2+g_1((1/(2\beta))\log x)] - g_1((1/(2\beta))\log x))} - \mathrm{e}^{-\nu([1/2+\tilde g_1((1/(2\beta))\log x)] - \tilde g_1((1/(2\beta))\log x))}. \end{aligned}$$

It is therefore enough to show that $h_{1,L}(A_{k_n}x_0) \to 0$ and $h_{2,L}(A_{k_n}x_0) \to 0$, as $n \to \infty$. From (3.16), (3.20) and (3.50), $h_{1,L}(A_{k_n}x_0) \neq 0$ if, for some integer $m \geq 1$,

$$m - \frac{1}{2} + \log L\bigl(\mathrm{e}^{2m-1}\bigr) \leq \frac{1}{2\beta}\log A_{k_n}x_0 < m - \frac{1}{2} + \log L\bigl(\mathrm{e}^{2m}\bigr). \tag{3.56}$$

(To see this, partition $[m-1+\frac{1}{2\beta}\log L(\mathrm{e}^{2m-2}), m+\frac{1}{2\beta}\log L(\mathrm{e}^{2m}))$ into four subintervals $[m-1+\frac{1}{2\beta}\log L(\mathrm{e}^{2m-2}), m-1+\frac{1}{2\beta}\log L(\mathrm{e}^{2m}))$, $[m-1+\frac{1}{2\beta}\log L(\mathrm{e}^{2m}), m-\frac{1}{2}+\frac{1}{2\beta}\log L(\mathrm{e}^{2m-1}))$, $[m-\frac{1}{2}+\frac{1}{2\beta}\log L(\mathrm{e}^{2m-1}), m-\frac{1}{2}+\frac{1}{2\beta}\log L(\mathrm{e}^{2m}))$, $[m-\frac{1}{2}+\frac{1}{2\beta}\log L(\mathrm{e}^{2m}), m+\frac{1}{2\beta}\log L(\mathrm{e}^{2m}))$ and check that the function is nonzero only on the third subinterval as given in (3.56).) By Lemma A.4, (3.56) holds for infinitely many values of n only if $x_0 = \mathrm{e}^{\beta(2r+1)}$ which is a discontinuity point of $M_L(-x)$ in (3.55). To show $h_{2,L}(A_{k_n}x_0) \to 0$, note that, by Lemma A.1, it is enough to prove that $\tilde h_{2,L}(A_{k_n}x_0) \neq 0$ for finitely many values of n, where

$$\tilde h_{2,L}(x) = \mathrm{e}^{-\nu[1/2+g_1((1/(2\beta))\log x)]} - \mathrm{e}^{-\nu[1/2+\tilde g_1((1/(2\beta))\log x)]}.$$

By using (3.16) and (3.20), the relation $\tilde h_{2,L}(A_{k_n}x_0) = 0$ holds only if, for some integer $m \geq 1$,

$$m - \frac{1}{2} + \log L\bigl(\mathrm{e}^{2m-2}\bigr) \leq \frac{1}{2\beta}\log A_{k_n}x_0 < m - \frac{1}{2} + \log L\bigl(\mathrm{e}^{2m}\bigr). \tag{3.57}$$

(To see this, draw a plot of $g_1(y)$ and $\tilde g_1(y)$ for y in $[m-1+\frac{1}{2\beta}\log L(\mathrm{e}^{2m-2}), m-\frac{1}{2}\log L(\mathrm{e}^{2m}))$, and note that $\tilde g_1(y) = m - \frac{1}{2}$ at $y = m - \frac{1}{2} + \frac{1}{2\beta}\log L(\mathrm{e}^{2m-2})$ and $g_1(y) = m - \frac{1}{2} + \frac{1}{2\beta}\log L(\mathrm{e}^{2m})$.) By Lemma A.4, (3.57) holds for infinitely many values of n only if $x_0 = \mathrm{e}^{\beta(2r+1)}$ which is a discontinuity point of $M_L(-x)$ in (3.55). Hence, $h_L(A_{k_n}x_0) \to 0$ as $n \to \infty$ for every continuity point x_0 of $M_L(-x)$.

Step 3 (Deriving subsequence k_n): We conclude the proof of the theorem by showing that k_n is given by (3.8). In view of the discussion following Theorem 2.2, we want to choose a sequence $\tilde{A}_n$ satisfying (2.12)–(2.13) such that k_n given by (3.8) now satisfies (2.14). We define such sequence $\tilde{A}_n$ as

$$\begin{aligned}\log \tilde{A}_n &= 2\beta(m-1) + \log L\bigl(\mathrm{e}^{2m-2}\bigr) \\ &\quad + \frac{(\log n - \log k_m)(2\beta + \log L(\mathrm{e}^{2m}) - \log L(\mathrm{e}^{2m-2}))}{\log k_{m+1} - \log k_m} \qquad \text{if } k_m \le n < k_{m+1}, m \ge 1.\end{aligned} \tag{3.58}$$

The sequence $\tilde{A}_n$ satisfies (2.13). If $k_m \le n < k_{m+1} - 1$, the last limit in (2.13) follows from

$$\log \tilde{A}_{n+1} - \log \tilde{A}_n = \frac{(\log n - \log(n+1))(2\beta + \log L(\mathrm{e}^{2m}) - \log L(\mathrm{e}^{2m-2}))}{\log k_{m+1} - \log k_m} \to 0.$$

If $n = k_{m+1} - 1$, the limit follows from

$$\begin{aligned}&\log \tilde{A}_{n+1} - \log \tilde{A}_n \\ &\quad = 2\beta + \log L\bigl(\mathrm{e}^{2m}\bigr) - \log L\bigl(\mathrm{e}^{2m-2}\bigr) \\ &\qquad - \frac{(\log(k_{m+1}-1) - \log k_m)(2\beta + \log L(\mathrm{e}^{2m}) - \log L(\mathrm{e}^{2m-2}))}{\log k_{m+1} - \log k_m} \to 0\end{aligned}$$

since $\log L(\mathrm{e}^{2m}) - \log L(\mathrm{e}^{2m-2}) \to 0$, and

$$\frac{\log(k_{m+1}-1) - \log k_m}{\log k_{m+1} - \log k_m} \to 1.$$

Next we show (2.12), that is, $n\tilde{A}_n^{-\alpha} l_1^*(\tilde{A}_n) \to 1$, as $n \to \infty$, where $\alpha = \frac{\nu}{2\beta}$ and l_1^* is as defined in (3.27). When $k_m \le n < k_{m+1}$, observe that

$$\begin{aligned}\log n\tilde{A}_n^{-\alpha} l_1^*(\tilde{A}_n) &= \log \frac{n l_1^*(\tilde{A}_n)}{\tilde{A}_n^{\nu/2\beta}} \\ &= \log \frac{n l_1^*(\tilde{A}_n)}{\mathrm{e}^{(m-1)\nu} L(\mathrm{e}^{2m-2})^{\nu/2\beta}} \\ &\quad + \frac{\nu + (\nu/(2\beta))\log L(\mathrm{e}^{2m}) - (\nu/(2\beta))\log L(\mathrm{e}^{2m-2})}{\log k_{m+1} - \log k_m} \log\biggl(\frac{k_m}{n}\biggr) \qquad (3.59) \\ &\sim \log n + \log \frac{l_1^*(\tilde{A}_n)}{h_1(m-1) L(\mathrm{e}^{2m-2})^{\nu/2\beta}} - \log k_m \\ &\quad + \frac{\nu + (\nu/(2\beta))\log L(\mathrm{e}^{2m}) - (\nu/(2\beta))\log L(\mathrm{e}^{2m-2})}{\log k_{m+1} - \log k_m} \log\biggl(\frac{k_m}{n}\biggr).\end{aligned}$$

Now observe that as $n \to \infty$, we have $m \to \infty$, and thus $\frac{k_m}{n}$ is bounded and

$$\frac{\nu + (\nu/(2\beta)) \log L(\mathrm{e}^{2m}) - (\nu/(2\beta)) \log L(\mathrm{e}^{2m-2})}{\log k_{m+1} - \log k_m} \to 1.$$

Thus, (3.59) is asymptotically equivalent to

$$\log \frac{l_1^*(\tilde{A}_n)}{h_1(m-1) L(\mathrm{e}^{2m-2})^{\nu/2\beta}}. \tag{3.60}$$

By the relation (A.4) in the Appendix, $l_1^*(\tilde{A}_n) \sim h_1(g_2(\frac{1}{2\beta} \log \tilde{A}_n))\mathrm{e}^{\nu \tilde{g}_1^*((1/(2\beta)) \log \tilde{A}_n)}$ and hence (3.59) is also asymptotically equivalent to

$$\log \frac{h_1(g_2((1/(2\beta)) \log \tilde{A}_n))\mathrm{e}^{\nu \tilde{g}_1^*((1/(2\beta)) \log \tilde{A}_n)}}{h_1(m-1) L(\mathrm{e}^{2m-2})^{\nu/2\beta}}. \tag{3.61}$$

Since $k_m \le n < k_{m+1}$, we have

$$2\beta(m-1) + \log L\bigl(\mathrm{e}^{2m-2}\bigr) \le \log \tilde{A}_n < 2\beta m + \log L\bigl(\mathrm{e}^{2m}\bigr)$$

and, by (3.22), $\frac{\mathrm{e}^{\nu \tilde{g}_1^*((1/(2\beta)) \log \tilde{A}_n)}}{L(\mathrm{e}^{2m-2})^{\nu/2\beta}} = 1$. Hence, (3.61) simplifies to $\log \frac{h_1(m-1+\kappa)}{h_1(m-1)}$, where $0 \le \kappa < 1$. But as $n \to \infty$, we have $m \to \infty$ and thus $\frac{h_1(m-1+\kappa)}{h_1(m-1)} \to 1$ by using (3.4). This proves that $\log n \tilde{A}_n^{-\alpha} l_1^*(\tilde{A}_n) \to 0$ and thus $n \tilde{A}_n^{-\alpha} l_1^*(\tilde{A}_n) \to 1$, as $n \to \infty$.

Finally, we show that k_n defined in (3.8) satisfies (2.14). Define $a_n = A_{k_n} = \mathrm{e}^{2\beta(n-1)} L(\mathrm{e}^{2n-2})$. Hence,

$$\log a_n = \log A_{k_n} = 2\beta(n-1) + \log L\bigl(\mathrm{e}^{2n-2}\bigr).$$

Now observe that $\tilde{A}_{k_n} = a_n$ and thus (2.14) is satisfied. $\square$

The partial sums (3.7) involve centering constants B_{k_n} defined in (2.15). As in the stable case, one can expect to replace B_{k_n} by $k_n EX$ when $1 < \alpha < 2$, and to show the convergence of (3.7) without B_{k_n} when $0 < \alpha < 1$. The next result shows that this is indeed the case.

Proposition 3.2. *Suppose that the assumptions of Theorem 3.1 hold. Let*

$$\begin{aligned}\zeta = &-\frac{1-\mathrm{e}^{-\nu}}{1-\mathrm{e}^{2\beta-\nu}} - \mathrm{e}^{\beta(2\lceil(1/\nu)\log c_1\rceil - 1)}\bigl(c_1 \mathrm{e}^{-\nu(\lceil(1/\nu)\log c_1\rceil - 1)} - 1\bigr) \\ &+ c_1 \frac{(1-\mathrm{e}^{-\nu})\mathrm{e}^{\nu-\beta}}{1-\mathrm{e}^{2\beta-\nu}} \mathrm{e}^{(2\beta-\nu)\lceil(1/\nu)\log c_1\rceil}.\end{aligned} \tag{3.62}$$

If $0 < \alpha < 1$, *then*

$$\frac{B_{k_n}}{A_{k_n}} \to \zeta, \qquad \frac{1}{A_{k_n}} \sum_{j=1}^{k_n} X_j \xrightarrow{d} Y + \zeta$$

and if $1 < \alpha < 2$, *then*

$$\frac{k_n EX - B_{k_n}}{A_{k_n}} \to -\zeta, \qquad \frac{1}{A_{k_n}}\left\{\sum_{j=1}^{k_n} X_j - k_n EX\right\} \xrightarrow{d} Y + \zeta,$$

where Y follows the semi-stable law characterized by (3.9) *and* (3.10).

Proof. *Case* $0 < \alpha < 1$: It is enough to show the convergence of $\frac{B_{k_n}}{A_{k_n}} = \frac{k_n}{A_{k_n}} \int_{1/k_n}^{1-1/k_n} Q(s)\,\mathrm{d}s$ to ζ, where $Q(s)$ is defined in (2.16). For fixed s_1 and s_2, write

$$\begin{aligned} &\frac{k_n}{A_{k_n}} \int_{1/k_n}^{1-1/k_n} Q(s)\,\mathrm{d}s \\ &\quad = \frac{k_n}{A_{k_n}} \int_{1/k_n}^{s_1} Q(s)\,\mathrm{d}s + \frac{k_n}{A_{k_n}} \int_{s_1}^{s_2} Q(s)\,\mathrm{d}s + \frac{k_n}{A_{k_n}} \int_{s_2}^{1-1/k_n} Q(s)\,\mathrm{d}s. \end{aligned} \tag{3.63}$$

Observe first that, for fixed s_1 and s_2, the second term in (3.63) converges to zero. Indeed, this follows from the fact that $\frac{k_n}{A_{k_n}} \to 0$. For the latter convergence, note from (3.8) that

$$\frac{k_n}{A_{k_n}} \sim \frac{\mathrm{e}^{(n-1)\nu}}{h_1(n-1)} \frac{1}{L(\mathrm{e}^{2n-2})\mathrm{e}^{2\beta(n-1)}}. \tag{3.64}$$

For arbitrarily small $\delta > 0$, by using Potter's bounds for L and Lemma A.6 for h_1, the right-hand side of (3.64) is bounded by $C\mathrm{e}^{(\nu-2\beta+\delta)(n-1)} \to 0$, as long as $\nu - 2\beta + \delta < 0$.

Consider now the third term in (3.63), involving the function $Q(s)$ for values of s close to 1. The function $Q(s)$ is defined as the inverse of the distribution function $F(x) = P(L(\mathrm{e}^{W_q})\mathrm{e}^{\beta W_q}(-1)^{W_q} \le x)$. Since we are interested in $Q(s)$ for s close to 1, it is enough to look at the function for $x > 0$. For $x > 0$, the function $F(x)$ has jumps at points $x = L(\mathrm{e}^{2n})\mathrm{e}^{2\beta n}$ of size

$$P(W_q = 2n) = P\left(\frac{W_q}{2} \ge n, W_q \text{ is even}\right) - P\left(\frac{W_q}{2} \ge n+1, W_q \text{ is even}\right).$$

This means that, for s close to 1, the inverse function $Q(s)$ has jumps at points $s = 1 - P(\frac{W_q}{2} \ge n, W_q \text{ is even})$ of size $L(\mathrm{e}^{2n})\mathrm{e}^{2\beta n} - L(\mathrm{e}^{2n-2})\mathrm{e}^{2\beta(n-1)}$. Moreover, $Q(s) = L(\mathrm{e}^{2n})\mathrm{e}^{2\beta n}$ when $1 - P(\frac{W_q}{2} \ge n, W_q \text{ is even}) \le s < 1 - P(\frac{W_q}{2} \ge n+1, W_q \text{ is even})$. (If this step is unclear, the reader may want to draw a picture.) Note that the jump points satisfy

$$1 - s = P\left(\frac{W_q}{2} \ge n, W_q \text{ is even}\right) = h_1(n)\mathrm{e}^{-\nu n}$$

by (3.1).

Assuming for simplicity that $\frac{e^{\nu(n-1)}}{h_1(n-1)}$ are integers so that $k_n = \frac{e^{\nu(n-1)}}{h_1(n-1)}$ and taking $s_2 = 1 - h_1(n_1)e^{-\nu n_1}$, we can write,

$$\begin{aligned}
&\frac{k_n}{A_{k_n}} \int_{s_2}^{1-h_1(n-1)e^{-\nu(n-1)}} Q(s)\,\mathrm{d}s \\
&= \frac{k_n}{A_{k_n}} \sum_{m=n_1}^{n-2} L\big(e^{2m}\big)e^{2\beta m}\big(h_1(m)e^{-\nu m} - h_1(m+1)e^{-\nu(m+1)}\big) \\
&= \frac{e^{\nu(n-1)}}{h_1(n-1)e^{2\beta(n-1)}L(e^{2n-2})} \sum_{m=n_1}^{n-2} L\big(e^{2m}\big)e^{2\beta m} h_1(m)e^{-\nu m}\left(1 - \frac{h_1(m+1)}{h_1(m)}e^{-\nu}\right) \\
&=: I_1 + I_2,
\end{aligned}$$

where, for fixed K,

$$\begin{aligned}
I_1 &= \frac{e^{\nu(n-1)}}{h_1(n-1)e^{2\beta(n-1)}L(e^{2n-2})} \sum_{m=n_1}^{n-K} L\big(e^{2m}\big)e^{2\beta m} h_1(m)e^{-\nu m}\left(1 - \frac{h_1(m+1)}{h_1(m)}e^{-\nu}\right), \\
I_2 &= \frac{e^{\nu(n-1)}}{h_1(n-1)e^{2\beta(n-1)}L(e^{2n-2})} \sum_{m=n-K}^{n-2} L\big(e^{2m}\big)e^{2\beta m} h_1(m)e^{-\nu m}\left(1 - \frac{h_1(m+1)}{h_1(m)}e^{-\nu}\right).
\end{aligned}$$

For the term I_2, note that, after changing m to $n-j$ in the sum,

$$I_2 = e^{2\beta-\nu} \sum_{j=2}^{K} \frac{L(e^{2(n-j)})}{L(e^{2(n-1)})} e^{-(2\beta-\nu)j} \frac{h_1(n-j)}{h_1(n-1)}\left(1 - \frac{h_1(n-j+1)}{h_1(n-j)}e^{-\nu}\right).$$

By using (3.4), we get that

$$I_2 \to e^{2\beta-\nu}\big(1-e^{-\nu}\big)\sum_{j=2}^{K} e^{-(2\beta-\nu)j} = \big(1-e^{-\nu}\big)\frac{e^{\nu-2\beta}}{1-e^{\nu-2\beta}}\big(1-e^{-(K-1)(2\beta-\nu)}\big), \tag{3.65}$$

as $n \to \infty$. For the term I_1, we have similarly

$$I_1 = e^{2\beta-\nu} \sum_{j=K}^{n-n_1} \frac{L(e^{2(n-j)})}{L(e^{2(n-2)})} e^{-(2\beta-\nu)j} \frac{h_1(n-j)}{h_1(n-1)}\left(1 - \frac{h_1(n-j+1)}{h_1(n-j)}e^{-\nu}\right).$$

For arbitrarily small $\delta > 0$, by using Potter's bounds and Lemma A.6, we can write

$$|I_1| \le C \sum_{j=K}^{n-n_1} e^{-(2\beta-\nu-\delta)j}. \tag{3.66}$$

When $2\beta-\nu-\delta>0$, the last bound is arbitrarily small for large enough K. Together with (3.65), this shows that

$$\frac{k_n}{A_{k_n}}\int_{s_2}^{1-h_1(n-1)\mathrm{e}^{-(n-1)\nu}} Q(s)\,\mathrm{d}s = I_1+I_2 \to \bigl(1-\mathrm{e}^{-\nu}\bigr)\frac{\mathrm{e}^{\nu-2\beta}}{1-\mathrm{e}^{\nu-2\beta}} = -\frac{1-\mathrm{e}^{-\nu}}{1-\mathrm{e}^{2\beta-\nu}},$$

as $n\to\infty$.

Consider now the first term in (3.63), involving the function $Q(s)$ for values of s close to 0. Here we need to examine the function $F(x)=P(L(\mathrm{e}^{W_q})\mathrm{e}^{\beta W_q}(-1)^{W_q}\le x)$ for $x<0$. For $x<0$, the function $F(x)$ has jumps at $x=-L(\mathrm{e}^{2n+1})\mathrm{e}^{\beta(2n+1)}$ of size

$$P(W_q=2n+1)=P\biggl(\frac{W_q-1}{2}\ge n,\, W_q \text{ is odd}\biggr)-P\biggl(\frac{W_q-1}{2}\ge n+1,\, W_q \text{ is odd}\biggr).$$

Moreover, $Q(s)=-L(\mathrm{e}^{2n+1})\mathrm{e}^{\beta(2n+1)}$ when $P(\frac{W_q-1}{2}\ge n+1,\, W_q \text{ is odd})<s\le P(\frac{W_q-1}{2}\ge n,\, W_q \text{ is odd})$. Note that, by (3.2), the jump points satisfy

$$s=P\biggl(\frac{W_q-1}{2}\ge n,\, W_q \text{ is odd}\biggr)=h_2(n)\mathrm{e}^{-\nu n}.$$

Write the first term in (3.63) as

$$\frac{k_n}{A_{k_n}}\int_{h_1(n-1)\mathrm{e}^{-\nu(n-1)}}^{h_2(l(n)-1)\mathrm{e}^{-\nu(l(n)-1)}} Q(s)\,\mathrm{d}s+\frac{k_n}{A_{k_n}}\int_{h_2(l(n)-1)\mathrm{e}^{-\nu(l(n)-1)}}^{s_1} Q(s)\,\mathrm{d}s =: I_1^*+I_2^*, \tag{3.67}$$

where $l(n)$ is the integer such that

$$h_2\bigl(l(n)\bigr)\mathrm{e}^{-\nu l(n)}\le h_1(n-1)\mathrm{e}^{-\nu(n-1)}<h_2\bigl(l(n)-1\bigr)\mathrm{e}^{-\nu(l(n)-1)}$$

or

$$h_2\bigl(l(n)\bigr)\mathrm{e}^{-\nu l(n)}\le h_2(n-1)\mathrm{e}^{-\nu(n-1+(1/\nu)\log(h_2(n-1)/(h_1(n-1))))}<h_2\bigl(l(n)-1\bigr)\mathrm{e}^{-\nu(l(n)-1)}.$$

Note that, when $\frac{h_2(x)}{h_1(x)}\to c_1$ and $\frac{1}{\nu}\log c_1$ is not an integer, or when $\frac{1}{\nu}\log c_1$ is an integer and $\frac{h_2(x)}{h_1(x)}\uparrow c_1$, for large values of n one can take $l(n)=n-1+\lceil\frac{1}{\nu}\log c_1\rceil$. Indeed, this follows from

$$\mathrm{e}^{-\nu}<\mathrm{e}^{-\nu(\lceil(1/\nu)\log(h_2(n-1)/(h_1(n-1)))\rceil-(1/\nu)\log(h_2(n-1)/(h_1(n-1))))}\le 1 \tag{3.68}$$

and the fact that

$$\frac{h_2(n-1+\lceil(1/\nu)\log(h_2(n-1)/(h_1(n-1)))\rceil)}{h_2(n-1)}\to 1, \tag{3.69}$$

as $n\to\infty$.

Now, taking $s_1 = h_2(n_2)\mathrm{e}^{-\nu n_2}$, we can write I_2^* in (3.67) as

$$I_2^* = -\frac{k_n}{A_{k_n}} \sum_{m=n_2}^{l(n)-2} L\big(\mathrm{e}^{2m+1}\big)\mathrm{e}^{\beta(2m+1)}\big(h_2(m)\mathrm{e}^{-\nu m} - h_2(m+1)\mathrm{e}^{-\nu(m+1)}\big).$$

Following a similar calculation as done for the third term in (3.63), we get, as $n \to \infty$,

$$\begin{aligned} I_2^* &\to -c_1 \frac{(1-\mathrm{e}^{-\nu})\mathrm{e}^{2(\nu-2\beta)}\mathrm{e}^{\beta}}{1-\mathrm{e}^{\nu-2\beta}} \mathrm{e}^{-(\nu-2\beta)\lceil(1/\nu)\log c_1\rceil} \\ &= c_1 \frac{(1-\mathrm{e}^{-\nu})\mathrm{e}^{\nu-\beta}}{1-\mathrm{e}^{2\beta-\nu}} \mathrm{e}^{(2\beta-\nu)\lceil(1/\nu)\log c_1\rceil}. \end{aligned}$$

One can write I_1^* in (3.67) as

$$\begin{aligned} I_1^* &= -\frac{\mathrm{e}^{\nu(n-1)}}{h_1(n-1)\mathrm{e}^{2\beta(n-1)}L(\mathrm{e}^{2n-2})} L\big(\mathrm{e}^{2l(n)-1}\big)\mathrm{e}^{\beta(2l(n)-1)} \\ &\quad \times \big(h_2\big(l(n)-1\big)\mathrm{e}^{-\nu(l(n)-1)} - h_1(n-1)\mathrm{e}^{-\nu(n-1)}\big) \\ &= -\frac{L(\mathrm{e}^{2l(n)-1})\mathrm{e}^{\beta(2l(n)-1)}}{L(\mathrm{e}^{2n-2})\mathrm{e}^{2\beta(n-1)}} \left(\frac{h_2(l(n)-1)}{h_1(n-1)} \mathrm{e}^{-\nu(l(n)-n)} - 1\right) \\ &= -\frac{L(\mathrm{e}^{2l(n)-1})}{L(\mathrm{e}^{2n-2})} \mathrm{e}^{\beta(2\lceil(1/\nu)\log c_1\rceil-1)} \left(\frac{h_2(l(n)-1)}{h_1(n-1)} \mathrm{e}^{-\nu(\lceil(1/\nu)\log c_1\rceil-1)} - 1\right). \end{aligned}$$

Now, by using (3.3) and (3.4), it can be seen that

$$I_1^* \to -\mathrm{e}^{\beta(2\lceil(1/\nu)\log c_1\rceil-1)}\big(c_1\mathrm{e}^{-\nu(\lceil(1/\nu)\log c_1\rceil-1)} - 1\big),$$

as $n \to \infty$.

Now we consider the case when $\frac{1}{\nu}\log c_1$ is an integer and $\frac{h_2(x)}{h_1(x)} \downarrow c_1$. We want to find $l(n)$ such that (3) holds. Hence, we want

$$\lim_{n\to\infty} \frac{h_2(n-1)}{h_2(l(n))} \mathrm{e}^{-\nu(n-1+(1/\nu)\log(h_2(n-1)/(h_1(n-1)))-l(n))} \geq 1.$$

Take $l(n) = n - 2 + \lceil \frac{1}{\nu} \log \frac{h_2(n-1)}{h_1(n-1)} \rceil$. Then, $\lim_{n\to\infty} \frac{h_2(n-1)}{h_1(l(n))} \to 1$. Now,

$$\begin{aligned} &\lim_{n\to\infty} \mathrm{e}^{-\nu(n-1+(1/\nu)\log(h_2(n-1)/(h_1(n-1)))-n+2-\lceil(1/\nu)\log(h_2(n-1)/(h_1(n-1)))\rceil)} \\ &\quad = \mathrm{e}^{-\nu(1+(1/\nu)\log c_1-(1/\nu)\log c_1-1)} = \mathrm{e}^0 = 1. \end{aligned}$$

We also need

$$\frac{h_2(l(n)-1)}{h_2(n-1)} \mathrm{e}^{-\nu(l(n)-1-n+1-(1/\nu)\log(h_2(n-1)/(h_1(n-1))))} > 1$$

for large n. For this, observe that $\frac{h_2(l(n)-1)}{h_2(n-1)} \to 1$ and

$$\begin{aligned}
&\lim_{n\to\infty} \mathrm{e}^{-\nu(l(n)-1-n+1-(1/\nu)\log(h_2(n-1)/(h_1(n-1))))} \\
&\quad = \lim_{n\to\infty} \mathrm{e}^{-\nu(n-3+\lceil(1/\nu)\log(h_2(n-1)/(h_1(n-1)))\rceil-n+1-(1/\nu)\log(h_2(n-1)/(h_1(n-1))))} \\
&\quad = \lim_{n\to\infty} \mathrm{e}^{-\nu(-2+1+(1/\nu)\log c_1-(1/\nu)\log c_1)} = \mathrm{e}^{-\nu}.
\end{aligned}$$

Hence, when $\frac{1}{\nu}\log c_1$ is an integer, we have $\frac{h_2(x)}{h_1(x)} \downarrow c_1$ and $\lceil \frac{1}{\nu}\log\frac{h_2(x)}{h_1(x)}\rceil \downarrow \frac{1}{\nu}\log c_1 + 1$, and as in the previous calculations,

$$I_1^* \to -\mathrm{e}^{\beta(2\lceil(1/\nu)\log c_1\rceil-1)}\bigl(c_1\mathrm{e}^{-\nu(\lceil(1/\nu)\log c_1\rceil-1)} - 1\bigr)$$

and

$$I_2^* \to c_1\frac{(1-\mathrm{e}^{-\nu})\mathrm{e}^{\nu-\beta}}{1-\mathrm{e}^{2\beta-\nu}}\mathrm{e}^{(2\beta-\nu)\lceil(1/\nu)\log c_1\rceil}.$$

Finally, gathering the results above, we deduce the convergence to the constant ζ given by (3.62).

Case $1 < \alpha < 2$: It is enough to show the convergence of $\frac{k_nEX-B_{k_n}}{A_{k_n}}$ to $-\zeta$. Using the fact that $EX = \int_0^1 Q(s)\,\mathrm{d}s$, observe that

$$\frac{k_nEX-B_{k_n}}{A_{k_n}} = \frac{k_n}{A_{k_n}}\int_0^{1/k_n} Q(s)\,\mathrm{d}s + \frac{k_n}{A_{k_n}}\int_{1-1/k_n}^1 Q(s)\,\mathrm{d}s.$$

For simplicity, we assume that $\frac{\mathrm{e}^{(n-1)\nu}}{h_1(n-1)}$ is an integer. To evaluate $\int_{1-1/k_n}^1 Q(s)\,\mathrm{d}s$, one follows a similar procedure as in the case $0 < \alpha < 1$ to obtain

$$\begin{aligned}
&\frac{k_n}{A_{k_n}}\int_{1-h_1(n-1)\mathrm{e}^{-\nu(n-1)}}^1 Q(s)\,\mathrm{d}s \\
&\quad = \frac{k_n}{A_{k_n}}\sum_{m=n-1}^{\infty} L\bigl(\mathrm{e}^{2m}\bigr)\mathrm{e}^{2\beta m}\bigl(h_1(m)\mathrm{e}^{-\nu m} - h_1(m+1)\mathrm{e}^{-\nu(m+1)}\bigr) =: \tilde{I}_1 + \tilde{I}_2,
\end{aligned}$$

where, for fixed K,

$$\begin{aligned}
\tilde{I}_1 &= \frac{\mathrm{e}^{\nu(n-1)}}{h_1(n-1)\mathrm{e}^{2\beta(n-1)}L(\mathrm{e}^{2n-2})}\sum_{m=n-1}^{n+K} L\bigl(\mathrm{e}^{2m}\bigr)\mathrm{e}^{2\beta m}h_1(m)\mathrm{e}^{-\nu m}\biggl(1-\frac{h_1(m+1)}{h_1(m)}\mathrm{e}^{-\nu}\biggr), \\
\tilde{I}_2 &= \frac{\mathrm{e}^{\nu(n-1)}}{h_1(n-1)\mathrm{e}^{2\beta(n-1)}L(\mathrm{e}^{2n-2})}\sum_{m=n+K}^{\infty} L\bigl(\mathrm{e}^{2m}\bigr)\mathrm{e}^{2\beta m}h_1(m)\mathrm{e}^{-\nu m}\biggl(1-\frac{h_1(m+1)}{h_1(m)}\mathrm{e}^{-\nu}\biggr).
\end{aligned}$$

Similar to the case $0 < \alpha < 1$, one can show that

$$\frac{k_n}{A_{k_n}} \int_{1-h_1(n-1)\mathrm{e}^{-(n-1)\nu}}^{1} Q(s)\,\mathrm{d}s = \tilde{I}_1 + \tilde{I}_2 \to \frac{1-\mathrm{e}^{-\nu}}{1-\mathrm{e}^{2\beta-\nu}}.$$

Similarly, one can write

$$\begin{aligned}\int_0^{1/k_n} Q(s)\,\mathrm{d}s &= \frac{k_n}{A_{k_n}} \int_0^{h_2(l(n)-1)\mathrm{e}^{-\nu(l(n)-1)}} Q(s)\,\mathrm{d}s - \frac{k_n}{A_{k_n}} \int_{h_1(n-1)\mathrm{e}^{-\nu(n-1)}}^{h_2(l(n)-1)\mathrm{e}^{-\nu(l(n)-1)}} Q(s)\,\mathrm{d}s \\ &:= \tilde{I}_2^* - \tilde{I}_1^*.\end{aligned}$$

As shown in the case $0 < \alpha < 1$, we again use two different representations of $l(n)$ for two different cases. Note that $\tilde{I}_1^*$ is exactly I_1^* considered in that case.

Observe that

$$\tilde{I}_2^* = -\frac{k_n}{A_{k_n}} \sum_{m=l(n)-1}^{\infty} L\bigl(\mathrm{e}^{2m+1}\bigr)\mathrm{e}^{(2m+1)\beta}\bigl(h_2(m)\mathrm{e}^{-\nu m} - h_2(m+1)\mathrm{e}^{-\nu(m+1)}\bigr).$$

As $n \to \infty$,

$$\tilde{I}_2^* \to -c_1 \frac{(1-\mathrm{e}^{-\nu})\mathrm{e}^{\nu-\beta}}{1-\mathrm{e}^{2\beta-\nu}} \mathrm{e}^{(2\beta-\nu)\lceil(1/\nu)\log c_1\rceil}$$

and, from the case $0 < \alpha < 1$,

$$\tilde{I}_1^* \to -\mathrm{e}^{\beta(2\lceil(1/\nu)\log c_1\rceil-1)}\bigl(c_1\mathrm{e}^{-\nu\lceil(1/\nu)\log c_1\rceil} - 1\bigr).$$

Finally, gathering the results above, we deduce the convergence to $-\zeta$ where ζ is given by (3.62). □

Theorem 3.1 concerns the partial sums $\sum_{j=1}^{n} X_j$ along a subsequence k_n of n. The following result describes the behavior of the partial sums across all n. The result is a direct consequence of Lemma 5 of Meerschaert and Scheffler [12]. Recall that a collection of random variables $\{Y_n\}_{n\geq 1}$ is called *stochastically compact* if every subsequence $\{n'\}$ has a further subsequence $\{n''\} \subset \{n'\}$ for which $\{Y_{n''}\}$ converges in distribution. The following notation will also be used. For a semi-stable distribution τ with characteristic function $\psi(t)$, τ^{λ} will denote the semi-stable distribution with the characteristic function $\psi(t)^{\lambda}$.

Proposition 3.3. *Let $X, X_1, X_2, \ldots$ be i.i.d. random variables such that*

$$\frac{1}{A_{k_n}} \left\{ \sum_{j=1}^{k_n} X_j - B_{k_n} \right\} \xrightarrow{d} Y, \tag{3.70}$$

where Y follows a semi-stable distribution τ with $0 < \alpha < 2$ and k_n, A_{k_n}, B_{k_n} are given in (2.5), (2.6) and (2.15). Then, there exist a_n and b_n such that a_n is regularly varying with index $\frac{1}{\alpha}$,

$a_{k_n} = A_{k_n}$ and $a_n^{-1}(X_1 + X_2 + \cdots + X_n) - b_n$ is stochastically compact, with every limit point of the form $\lambda^{-1/\alpha}\tau^\lambda$ for some $\lambda \in [1, c]$. Moreover, one can take

$$a_n = \lambda_n^{1/\alpha} A_{k_{p_n}} \quad \text{and} \quad b_n = \lambda_n^{1-1/\alpha} \frac{B_{k_{p_n}}}{A_{k_{p_n}}}, \tag{3.71}$$

where $\lambda_n = \frac{n}{k_{p_n}}$ and p_n, k_{p_n} are chosen so that $k_{p_n} \le n < k_{p_{n+1}}$ for every $n \ge 1$.

Proof. The proposition follows directly from Lemma 5 and its proof in Meerschaert and Scheffler [12]. The left-hand side of (3.70) appears in (2.9) of Meerschaert and Scheffler [12] as

$$\tilde{a}_n^{-1}(X_1 + X_2 + \cdots + X_{k_n}) - \tilde{b}_n.$$

The existence of a regularly varying a_n with $a_{k_n} = \tilde{a}_n$ is part of the statement of Lemma 5 of Meerschaert and Scheffler [12]. The expressions in (3.71) can be found in the proof of that Lemma 5. □

Corollary 3.4. *Under the assumptions of Proposition 3.3,*

$$\limsup_n P\big(a_n^{-1}(X_1 + X_2 + \cdots + X_n) - b_n > x\big) \le \sup_{1 \le \lambda \le c} P(Y_\lambda > x) \tag{3.72}$$

and

$$\limsup_n P\big(a_n^{-1}(X_1 + X_2 + \cdots + X_n) - b_n < x\big) \le \sup_{1 \le \lambda \le c} P(Y_\lambda < x), \tag{3.73}$$

where Y_λ has the distribution of the form $\lambda^{-1/\alpha}\tau^\lambda$.

Proof. Along a subsequence $\{n(k)\}$ of $\{n\}$, we have

$$\begin{aligned} &\limsup_n P\big(a_n^{-1}(X_1 + X_2 + \cdots + X_n) - b_n > x\big) \\ &\quad = \lim_k P\big(a_{n(k)}^{-1}(X_1 + X_2 + \cdots + X_{n(k)}) - b_{n(k)} > x\big). \end{aligned} \tag{3.74}$$

Now, by Proposition 3.3, there exists a further subsequence $\{n(k_m)\}$ of $\{n(k)\}$ such that

$$\begin{aligned} &\lim_m P\big(a_{n(k_m)}^{-1}(X_1 + X_2 + \cdots + X_{n(k_m)}) - b_{n(k_m)} > x\big) \\ &\quad = P(Y_\lambda > x), \end{aligned} \tag{3.75}$$

where Y_λ follows the distribution $\lambda^{-1/\alpha}\tau^\lambda$. The relation (3.75) holds for all x as long as the semi-stable distribution τ^λ is continuous. By Huff [10], the continuity of τ^λ is equivalent to $\int_{-\infty}^0 \mathrm{d}L_\lambda(x) + \int_0^\infty \mathrm{d}R_\lambda(x) = \infty$, where L_λ and R_λ define the Lévy measure of τ^λ. By the definition of τ^λ, $L_\lambda = \lambda L$ and $R_\lambda = \lambda R$. Denote the multiplicative period of $M_L(x)$ and $M_R(x)$

by $p > 1$. Then, after the change of variables $x = p^k y$ in the integrals below,

$$\int_{-\infty}^{0} \mathrm{d}L(x) + \int_{0}^{\infty} \mathrm{d}R(x) = \sum_{k=-\infty}^{\infty} \int_{-p^{k+1}}^{-p^k} \mathrm{d}\frac{M_L(x)}{|x|^\alpha} + \sum_{k=-\infty}^{\infty} \int_{p^k}^{p^{k+1}} \mathrm{d}\frac{(-M_R(x))}{x^\alpha}$$
$$= \sum_{k=-\infty}^{\infty} p^{-k\alpha} \int_{-p}^{-1} \mathrm{d}\frac{M_L(y)}{|y|^\alpha} + \sum_{k=-\infty}^{\infty} p^{-k\alpha} \int_{1}^{p} \mathrm{d}\frac{(-M_R(y))}{y^\alpha} = \infty,$$

unless $M_L \equiv 0$ and $M_R \equiv 0$. Combining (3.74) and (3.75), we have (3.72) for all $x \in \mathbb{R}$. The relation (3.73) can be obtained similarly. □

We will use Corollary 3.4 to provide a conservative confidence interval for $f_W(w)$ in Section 4.

4. Application to sampling of finite point processes

We now turn back to the context of sampling of finite point processes. The following result restates Theorem 3.1 and Proposition 3.2 for the nonparametric estimator $\widehat{f_W}(w)$ of $f_W(w)$ given in (1.4) or (1.11)–(1.12).

Theorem 4.1. *Suppose conditions* (3.1)–(3.4) *hold and* k_n *is given in* (3.8). *Let*

$$\alpha = \frac{\nu}{2\log(q^{-1} - 1)}. \tag{4.1}$$

If $\alpha \in (1, 2)$, *then*

$$d_N\big(\widehat{f}(w) - f(w)\big) \xrightarrow{d} (-1)^{-w}(Y + \zeta),$$

and if $\alpha \in (0, 1)$, *then*

$$d_N \widehat{f}(w) \xrightarrow{d} (-1)^{-w}(Y + \zeta),$$

along the sample sizes $N = k_n$, *where* $d_N = \frac{k_n}{A_{k_n}}$ *with*

$$A_{k_n} = \binom{2n-2}{w} (1-q)^{-w}\big(q^{-1} - 1\big)^{2n-2}, \tag{4.2}$$

and ζ *defined in* (3.62) *and* Y *is a semi-stable distribution characterized by* (3.10) *with*

$$\beta = \log\big(q^{-1} - 1\big). \tag{4.3}$$

Proof. In view of (1.11)–(1.12), we are interested in the distribution of

$$X = \binom{W_q}{w} (-1)^{W_q - w} \frac{(1-q)^{W_q - w}}{q^{W_q}} 1_{\{W_q \ge w\}},$$

where $w > 0$ is fixed and W_q follows a p.m.f. satisfying (3.1)–(3.4). For $W_q > w$ large enough, one can write $(-1)^w X = L(\mathrm{e}^{W_q})\mathrm{e}^{\beta W_q}(-1)^{W_q}$ as given in Theorem 3.1 with

$$L(x) = \binom{\log x}{w}(1-q)^{-w} = (1-q)^{-w}\frac{\prod_{i=0}^{w-1}(\log x - i)}{w!} \tag{4.4}$$

and $\beta = \log\frac{1-q}{q} = \log(q^{-1} - 1)$. Observe that $L(x)$ is an ultimately increasing slowly varying function. Hence, when $\alpha \in (1, 2)$, by using (1.11)–(1.12) and applying Theorem 3.1 and Proposition 3.2,

$$\frac{k_n}{A_{k_n}}\bigl(\widehat{f}_W(w) - f_W(w)\bigr) = d_N\bigl(\widehat{f}_W(w) - f_W(w)\bigr)$$

converges to a semi-stable distribution $(-1)^{-w}(Y + \zeta)$ with α in (4.1) and A_{k_n} in (4.2). When $\alpha \in (0, 1)$,

$$\frac{k_n}{A_{k_n}}\widehat{f}_W(w) = d_N\widehat{f}_W(w)$$

converges to a semi-stable distribution $(-1)^{-w}(Y + \zeta)$ with α in (4.1) and A_{k_n} in (4.2). □

The next result provides a conservative confidence interval for $f(w)$ based on $\widehat{f}(w)$ when $1 < \alpha < 2$. The finite-sample performance of the confidence interval and related issues are considered in Chaudhuri and Pipiras [5].

Proposition 4.2. *Under the assumptions and notation of Theorem 4.1, suppose $\alpha \in (1, 2)$. For $\gamma \in (0, 1)$, set*

$$\mathcal{C} = \bigl[\widehat{f}_W(w) - \tilde{b}_N x_{1-\gamma/2}, \widehat{f}_W(w) - \tilde{b}_N x_{\gamma/2}\bigr], \tag{4.5}$$

where

$$\tilde{b}_N = N^{1/\alpha-1}A_{k_{p_N}}k_{p_N}^{-1/\alpha} \tag{4.6}$$

with p_N such that $k_{p_N} \le N < k_{p_{N+1}}$ and

$$\sup_{1\le\lambda\le c} P\bigl(Y_\lambda^\zeta < x_{\gamma/2}\bigr) = \frac{\gamma}{2}, \qquad \sup_{1\le\lambda\le c} P\bigl(Y_\lambda^\zeta > x_{1-\gamma/2}\bigr) = \frac{\gamma}{2}, \tag{4.7}$$

where Y_λ^ζ has the distribution of the form $\lambda^{-1/\alpha}\tau^\lambda$ and τ is the distribution of $Y + \zeta$. Then,

$$\liminf_{N\to\infty} P\bigl(f_W(w) \in \mathcal{C}\bigr) \ge 1 - \gamma, \tag{4.8}$$

that is, $\mathcal{C}$ is a conservative $100(1-\gamma)\%$ confidence interval for $f_W(w)$.

Proof. When $\alpha \in (1, 2)$, by using Corollary 3.4 and Theorem 4.1, we get

$$\limsup_{N\to\infty} P\biggl(\frac{N\lambda_N^{-1/\alpha}}{A_{k_{p_N}}}\widehat{f}_W(w) - \lambda_N^{1-1/\alpha}\frac{k_{p_N}}{A_{k_{p_N}}}f_W(w) < x_{\gamma/2}\biggr) \le \sup_{1\le\lambda\le c} P\bigl(Y_\lambda^\eta < x_{\gamma/2}\bigr) = \frac{\gamma}{2}$$

$$\Leftrightarrow \quad \limsup_{N\to\infty} P\left(\frac{N}{\lambda_N k_{p_N}}\widehat{f}_W(w) - \frac{\lambda_N^{1/\alpha-1} A_{k_{p_N}}}{k_{p_N}} x_{\gamma/2} < f_W(w)\right) \le \frac{\gamma}{2}.$$

Using $\lambda_N = \frac{N}{k_{p_N}}$, we get

$$\limsup_{N\to\infty} P\big(\widehat{f}_W(w) - N^{1/\alpha-1} A_{k_{p_N}} k_{p_N}^{-1/\alpha} x_\gamma/2 < f_W(w)\big) \le \frac{\gamma}{2}. \tag{4.9}$$

Similarly for the right tail, we get

$$\begin{aligned}
&\limsup_{N\to\infty} P\left(\frac{N\lambda_N^{-1/\alpha}}{A_{k_{p_N}}}\widehat{f}_W(w) - \lambda_N^{1-1/\alpha}\frac{k_{p_N}}{A_{k_{p_N}}} f_W(w) > x_{1-\gamma/2}\right) \le \sup_{1\le\lambda\le c} P\big(Y_\lambda^\eta > x_{1-\gamma/2}\big) = \frac{\gamma}{2}\\
&\Leftrightarrow \quad \limsup_{N\to\infty} P\left(\frac{N}{\lambda_N k_{p_N}}\widehat{f}_W(w) - \frac{\lambda_N^{1/\alpha-1} A_{k_{p_N}}}{k_{p_N}} x_{1-\gamma/2} > f_W(w)\right) \le \frac{\gamma}{2} \qquad (4.10)\\
&\Leftrightarrow \quad \limsup_{N\to\infty} P\big(\widehat{f}_W(w) - N^{1/\alpha-1} A_{k_{p_N}} k_{p_N}^{-1/\alpha} x_{1-\gamma/2} > f_W(w)\big) \le \frac{\gamma}{2}.
\end{aligned}$$

Combining (4.9) and (4.10), we get (4.8). □

We conclude with two examples illustrating Theorem 4.1.

Example 4.3. Consider the case where W follows a geometric distribution, that is, $f_W(w) = c^{w-1}(1-c)$, $w = 1, 2, 3, \ldots$ and $0 < c < 1$. Substituting this into (1.1) leads to

$$f_{W_q}(s) = \sum_{w=s}^{\infty} \binom{w}{s} q^s (1-q)^{w-s} c^{w-1}(1-c). \tag{4.11}$$

When $s = 0$, we get

$$f_{W_q}(0) = \sum_{w=1}^{\infty} (1-q)^w c^{w-1}(1-c) = \frac{(1-q)(1-c)}{1-c(1-q)}. \tag{4.12}$$

When $s \ge 1$, on the other hand, we have

$$\begin{aligned}
f_{W_q}(s) &= q^s c^{s-1}(1-c) \sum_{w=s}^{\infty} \binom{w}{s} \big(c(1-q)\big)^{w-s}\\
&= \frac{q^s c^{s-1}(1-c)}{(1-c(1-q))^{s+1}} = \frac{c_q}{c} c_q^{s-1}(1-c_q),
\end{aligned} \tag{4.13}$$

where $c_q = \frac{qc}{1-c(1-q)}$, by using the identity $\sum_{w=s}^{\infty} \binom{w}{s} x^{w-s} = \sum_{r=0}^{\infty} \binom{s+r}{r} x^r = (1-x)^{-(s+1)}$. Hence, for $x \geq 1$,

$$P\left(\frac{W_q}{2} \geq x,\, W_q \text{ is even}\right) = \sum_{s=\lceil x \rceil}^{\infty} \frac{c_q}{c} c_q^{2s-1}(1-c_q) = \frac{1}{c} \frac{c_q^{2\lceil x \rceil}}{1+c_q} \tag{4.14}$$

and

$$P\left(\frac{W_q - 1}{2} \geq x,\, W_q \text{ is odd}\right) = \sum_{s=\lceil x \rceil}^{\infty} \frac{c_q}{c} c_q^{2s}(1-c_q) = \frac{c_q}{c} \frac{c_q^{2\lceil x \rceil}}{1+c_q}. \tag{4.15}$$

Thus, the conditions (3.1)–(3.4) in Theorem 3.1 are satisfied with $\nu = 2\log\frac{1}{c_q}$, $h_1(\lceil x \rceil) = \frac{1}{c(1+c_q)}$, $h_2(\lceil x \rceil) = \frac{c_q}{c(1+c_q)}$ with $\frac{h_2(x)}{h_1(x)} = c_q$. By using the expression of β in (4.3), the parameter α appearing in (3.9) or (4.1) is given by

$$\alpha = \frac{\log(1/c_q)}{\log(q^{-1}-1)} = \frac{\log\,(1-c(1-q))/(cq)}{\log(q^{-1}-1)}.$$

Note that $c_q < 1$ and hence $\log\frac{1}{c_q} > 0$. Then, $\alpha > 0$ is possible only when $q \in (0, 0.5)$. In particular, for $q \in (0, 0.5)$,

$$1 < \alpha < 2 \quad \Leftrightarrow \quad \frac{q}{1-q} < c < \frac{1}{2(1-q)}, \tag{4.16}$$

$$0 < \alpha < 1 \quad \Leftrightarrow \quad \frac{1}{2(1-q)} < c < 1. \tag{4.17}$$

Theorem 4.1 can now be applied in these two cases with

$$A_{k_n} = \binom{2n-2}{w}(1-q)^{-w}\left(q^{-1}-1\right)^{2n-2} \quad \text{and} \quad k_n = \left\lceil \frac{c(1+c_q)}{c_q^{2n-2}} \right\rceil.$$

Remark. Under (4.16) or (4.17), and $q \in (0, 0.5)$, the limit of $\widehat{f}(w)$ involves a semi-stable distribution. On the other hand, as proved in Antunes and Pipiras [1], $\widehat{f}(w)$ is asymptotically normal if $R_{q,w} < \infty$, where $R_{q,w}$ is given in (1.9). This condition obviously holds when $q \in (0.5, 1)$ (and also for $q = 0.5$ by recalling from Example 4.3 above that $f_{W_q}(s) \sim C c_q^s$ as $s \to \infty$). To understand when $R_{q,w} < \infty$ for $q \in (0, 0.5)$, observe that

$$\begin{aligned} R_{q,w} &= \sum_{k=w}^{\infty} f_W(k)(1-q)^{k-2w} \binom{k}{w} \sum_{s=w}^{k} \binom{s}{w} \binom{k-w}{s-w} \left(\frac{1}{q} - 1\right)^s \\ &= \sum_{s=w}^{\infty} \binom{s}{w} \left(q^{-1}-1\right)^s \sum_{k=s}^{\infty} c^{k-1}(1-c)(1-q)^{k-2w} \binom{k}{w} \binom{k-w}{s-w}. \end{aligned} \tag{4.18}$$

Since

$$\binom{k}{w}\binom{k-w}{s-w} = \frac{k!}{w!(k-w)!}\frac{(k-w)!}{(s-w)!(k-s)!} = \frac{k!}{(k-s)!s!}\frac{s!}{w!(s-w)!} = \binom{k}{s}\binom{s}{w},$$

we have

$$\begin{aligned}
R_{q,w} &= (1-c)\sum_{s=w}^{\infty}\binom{s}{w}^2 \left(q^{-1}-1\right)^s \sum_{k=s}^{\infty}\binom{k}{s} c^{k-1}(1-q)^{k-2w} \\
&= (1-c)\sum_{s=w}^{\infty}\binom{s}{w}^2 \left(q^{-1}-1\right)^s \sum_{k=s}^{\infty}\binom{k}{s} \left(c(1-q)\right)^{k-s} c^{s-1}(1-q)^{s-2w} \\
&= (1-c)\sum_{s=w}^{\infty}\binom{s}{w}^2 \left(q^{-1}-1\right)^s c^{s-1}(1-q)^{s-2w} \sum_{k=s}^{\infty}\binom{k}{k-s} \left(c(1-q)\right)^{k-s} \\
&= (1-c)\sum_{s=w}^{\infty}\binom{s}{w}^2 \left(q^{-1}-1\right)^s c^{s-1}(1-q)^{s-2w}\left(1-c(1-q)\right)^{-(s+1)} \\
&= \left(\frac{1-c}{c}\right)\sum_{s=w}^{\infty}\binom{s}{w}^2 \left(q^{-1}-1\right)^s \left(c(1-q)\right)^s (1-q)^{-2w}\left(1-c(1-q)\right)^{-(s+1)} \\
&= d_w \sum_{s=w}^{\infty}\binom{s}{w}^2 \left(\left(q^{-1}-1\right)\frac{c(1-q)}{1-c(1-q)}\right)^s,
\end{aligned} \tag{4.19}$$

where $d_w = (\frac{1-c}{c})(1-q)^{-2w}\frac{1}{1-c(1-q)}$. Thus, $R_{q,w} < \infty$ if and only if

$$\left(q^{-1}-1\right)\frac{c(1-q)}{1-c(1-q)} < 1 \quad \Leftrightarrow \quad c < \frac{q}{1-q}. \tag{4.20}$$

Apart from the boundary cases $c = \frac{q}{1-q}$ and $c = \frac{1}{2(1-q)}$, the ranges of c given in (4.16), (4.17) and (4.20) now cover the whole permissible interval $c \in (0, 1)$.

Example 4.4. Consider the case where W follows a negative binomial distribution, that is, $f_W(w) = \binom{w-1}{r-1}c^{w-r}(1-c)^r$, $w = r, r+1, \ldots, 0 < c < 1$. We first compute $f_{W_q}(s)$. One can write $W = G_1 + G_2 + \cdots + G_r$, where $G_1, G_2, \ldots, G_r$ are i.i.d. geometric random variables with p.m.f. $f_{G_1}(w) = c^{w-1}(1-c)$, $w \geq 1$, and hence $W_q = G_1' + G_2' + \cdots + G_r'$, where $G_1', G_2', \ldots, G_r'$ are i.i.d. random variables following the distribution given in (4.12)–(4.13). Hence,

$$f_{W_q}(0) = \left\{\frac{(1-q)(1-c)}{1-c(1-q)}\right\}^r. \tag{4.21}$$

For $s \geq 1$, we have

$$f_{W_q}(s) = \sum_{i_1, i_2, \ldots, i_r \geq 0, i_1 + i_2 + \cdots + i_r = s} P\big(G_1' = i_1\big) P\big(G_2' = i_2\big) \cdots P\big(G_r' = i_r\big).$$

To evaluate this quantity, let

$$p_j^r = \sum_{i_{j+1}, i_{j+2}, \ldots, i_r \geq 1, i_{j+1} + i_{j+2} + \cdots + i_r = s} P\big(G_{j+1}' = i_{j+1}\big) P\big(G_{j+2}' = i_{j+2}\big) \cdots P\big(G_r' = i_r\big), \quad (4.22)$$

for $0 \leq j < r$. Then, by using (4.12),

$$f_{W_q}(s) = \sum_{j=0}^{r-1} \binom{r}{j} \left\{ \frac{(1-q)(1-c)}{1-c(1-q)} \right\}^j p_j^r.$$

Now, by using (4.13),

$$\begin{aligned} p_j^r &= \left(\frac{c_q(1-c_q)}{c} \right)^{r-j} c_q^{s-(r-j)} \sum_{i_{j+1}, i_{j+2}, \ldots, i_r \geq 1, i_{j+1} + i_{j+2} + \cdots + i_r = s} 1 \\ &= \left(\frac{1-c_q}{c} \right)^{r-j} c_q^s \binom{s-1}{r-j-1}. \end{aligned}$$

Hence, for $s \geq 1$,

$$\begin{aligned} f_{W_q}(s) &= c_q^s \sum_{j=0}^{r-1} \left\{ \frac{(1-q)(1-c)}{1-c(1-q)} \right\}^j \left(\frac{1-c_q}{c} \right)^{r-j} \binom{r}{j} \binom{s-1}{r-j-1} \\ &= c_q^{s-1} p^*(s), \end{aligned}$$

where $p^*(s)$ is a polynomial given as

$$p^*(s) = \sum_{i=1}^{r-1} a_i^* s^i.$$

This implies that for $x > 1$,

$$P\left(\frac{W_q}{2} \geq x, W_q \text{ is even} \right) = \sum_{s=\lceil x \rceil}^{\infty} c_q^{2s-1} p^*(2s) = c_q^{2\lceil x \rceil} \sum_{s=\lceil x \rceil}^{\infty} c_q^{2s-2\lceil x \rceil - 1} p^*(2s) \quad (4.23)$$

and

$$P\left(\frac{W_q - 1}{2} \geq x, W_q \text{ is odd} \right) = \sum_{s=\lceil x \rceil}^{\infty} c_q^{2s} p^*(2s+1) = c_q^{2\lceil x \rceil} \sum_{s=\lceil x \rceil}^{\infty} c_q^{2s-2\lceil x \rceil} p^*(2s+1). \quad (4.24)$$

Thus the conditions (3.1)–(3.2) in Theorem 3.1 are satisfied with $\nu = 2\log\frac{1}{c_q}$, $h_1(x) = \sum_{k=0}^{\infty} c_q^{2k-1} p^*(2x+2k)$, $h_2(x) = \sum_{k=0}^{\infty} c_q^{2k} p^*(2x+1+2k)$. The conditions (3.3)–(3.4) also hold with $c_1 = c_q$. The parameter α appearing in (3.9) is given by

$$\alpha = \frac{\log(1/c_q)}{\log(q^{-1}-1)} = \frac{\log((1-c(1-q))/(cq))}{\log(q^{-1}-1)}.$$

Note that $c_q < 1$ and hence $\log\frac{1}{c_q} > 0$. Then, $\alpha > 0$ is possible only when $q \in (0, 0.5)$. In particular, for $q \in (0, 0.5)$, the two cases (4.16)–(4.17) can be considered. Theorem 4.1 can now be applied in these two cases with

$$A_{k_n} = \binom{2n-2}{w}(1-q)^{-w}\left(q^{-1}-1\right)^{2n-2} \quad \text{and} \quad k_n = \left\lceil \frac{1}{c_q^{2n-2} h_1(n-1)} \right\rceil.$$

Appendix: Auxiliary results

We state and prove here a number of auxiliary results used in Section 3.

Lemma A.1. *Let g_1 and $\tilde{g}_1$ be defined in* (3.16) *and* (3.20), *respectively. Then,* $\tilde{g}_1(y) - g_1(y) \to 0$, *as $y \to \infty$.*

Proof. For $n \geq 2$, if

$$n - 1 + \frac{1}{2\beta}\log L\left(e^{2n-2}\right) \leq y < n - 1 + \frac{1}{2\beta}\log L\left(e^{2n}\right),$$

then

$$0 \leq \tilde{g}_1(y) - g_1(y) < \frac{1}{2\beta}\log\frac{L(e^{2n})}{L(e^{2n-2})} \to 0 \qquad \text{as } y \to \infty \ (n \to \infty), \tag{A.1}$$

since L is a slowly varying function. If

$$n - 1 + \frac{1}{2\beta}\log L\left(e^{2n}\right) \leq y < n + \frac{1}{2\beta}\log L\left(e^{2n}\right),$$

then similarly

$$\tilde{g}_1(y) - g_1(y) = \frac{1}{2\beta}\log\frac{L(e^{2n})}{L(e^{2n-2})} \to 0 \qquad \text{as } y \to \infty \ (n \to \infty). \tag{A.2}$$

□

Lemma A.2. *Let $\tilde{g}_1^*$ be defined in* (3.22). *Then, for any $A > 0$,*

$$\tilde{g}_1^*(\log Ax) - \tilde{g}_1^*(\log x) \to 0 \qquad \textit{as } x \to \infty.$$

Proof. Suppose without loss of generality that $A > 1$. First, note that

$$\begin{aligned}\tilde{g}_1^*(\log Ax) - \tilde{g}_1^*(\log x) &= \frac{1}{2\beta}\big(\log L\big(\mathrm{e}^{2n_{Ax}-2}\big) - \log L\big(\mathrm{e}^{2n_x-2}\big)\big) \\ &= \frac{1}{2\beta}\log\frac{L(\mathrm{e}^{n_{Ax}-2})}{L(\mathrm{e}^{n_x-2})} \\ &= \frac{1}{2\beta}\log\frac{L(\mathrm{e}^{2n_{Ax}-2n_x}\mathrm{e}^{2n_x-2})}{L(\mathrm{e}^{n_x-2})},\end{aligned} \tag{A.3}$$

where, for y $(= x$ or $Ax)$,

$$n_y - 1 + \frac{1}{2\beta}\log L\big(\mathrm{e}^{2n_y-2}\big) \le \log y < n_y + \frac{1}{2\beta}\log L\big(\mathrm{e}^{2n_y}\big).$$

Observe that $n_{Ax} - n_x$ takes only positive integer values, and that

$$0 \le n_{Ax} - n_x \le \lceil \log A \rceil.$$

Hence, by Theorem 1.2.1 of Bingham, Goldie and Teugels [4],

$$\frac{L(\mathrm{e}^{n_{Ax}-n_x}\mathrm{e}^{n_x-1})}{L(\mathrm{e}^{n_x-1})} \to 1 \qquad \text{as } \mathrm{e}^{n_x-1} \to \infty \text{ (or } x \to \infty).$$

This yields the result. □

Lemma A.3. *The function $l_1^*(x)$ defined in (3.27) is right-continuous and slowly varying at ∞.*

Proof. To show that $l_1^*(x)$ is slowly varying, write

$$\begin{aligned}l_1^*(x) = {}& \frac{h_1(\lceil g_2((1/(2\beta))\log x)\rceil_+)}{h_1(g_2((1/(2\beta))\log x))} \\ &\times h_1\left(g_2\left(\frac{1}{2\beta}\log x\right)\right)\mathrm{e}^{\nu\tilde{g}_1^*((1/(2\beta))\log x)}\mathrm{e}^{-\nu(g_1((1/(2\beta))\log x)-\tilde{g}_1((1/(2\beta))\log x))}.\end{aligned}$$

Note that

$$\begin{aligned}&\frac{h_1(\lceil g_2((1/(2\beta))\log x)\rceil_+)}{h_1(g_2((1/(2\beta))\log x))} \\ &\quad = \frac{h_1((\lceil g_2((1/(2\beta))\log x)\rceil_+/(g_2((1/(2\beta))\log x)))g_2((1/(2\beta))\log x))}{h_1(g_2((1/(2\beta))\log x))} \to 1\end{aligned}$$

by using (3.4), since $g_2(\frac{1}{2\beta}\log x) \to \infty$ and

$$\frac{\lceil g_2((1/(2\beta))\log x)\rceil_+}{g_2((1/(2\beta))\log x)} \to 1 \qquad \text{as } x \to \infty.$$

By Lemma A.1, we also have

$$\mathrm{e}^{-\nu(g_1((1/(2\beta))\log x)-\tilde{g}_1((1/(2\beta))\log x))} \to 1 \qquad \text{as } x \to \infty.$$

Hence, $l_1^*(x)$ is asymptotically equivalent to

$$h_1\left(g_2\left(\frac{1}{2\beta}\log x\right)\right)\mathrm{e}^{\nu\tilde{g}_1^*((1/(2\beta))\log x)}. \tag{A.4}$$

It is enough to show that the function (A.4) is slowly varying. By using Lemma A.2, we have

$$\frac{\mathrm{e}^{\nu\tilde{g}_1^*((1/(2\beta))\log Ax)}}{\mathrm{e}^{\nu\tilde{g}_1^*((1/(2\beta))\log x)}} \to 1 \qquad \text{as } x \to \infty. \tag{A.5}$$

It remains to show that $h_1(g_2(\frac{1}{2\beta}\log x))$ is a slowly varying function. For $A > 0$,

$$\begin{aligned} &\frac{h_1(g_2((1/(2\beta))\log Ax))}{h_1(g_2((1/(2\beta))\log x))} \\ &\quad = \frac{h_1((g_2((1/(2\beta))\log Ax)/(g_2((1/(2\beta))\log x)))g_2((1/(2\beta))\log x))}{h_1(g_2((1/(2\beta))\log x))}. \end{aligned} \tag{A.6}$$

Now, by using (3.23),

$$\begin{aligned} &\frac{g_2((1/(2\beta))\log Ax)}{g_2((1/(2\beta))\log x)} \\ &\quad = \frac{(1/(2\beta))\log Ax + g_2^*((1/(2\beta))\log Ax)}{(1/(2\beta))\log x + g_2^*((1/(2\beta))\log x)} \\ &\quad = 1 + \frac{(1/(2\beta))\log Ax + g_2^*((1/(2\beta))\log Ax) - (1/(2\beta))\log x - g_2^*((1/(2\beta))\log x)}{(1/(2\beta))\log x + g_2^*((1/(2\beta))\log x)} \\ &\quad = 1 + \frac{(1/(2\beta))\log A + g_2^*((1/(2\beta))\log Ax) - g_2^*((1/(2\beta))\log x)}{g_2((1/(2\beta))\log x)} \to 1, \end{aligned}$$

since $g_2(\frac{1}{2\beta}\log x) \to \infty$ and by using (3.24), $g_2^*(\frac{1}{2\beta}\log Ax) - g_2^*(\frac{1}{2\beta}\log x) \to 0$. Thus, by using (3.4) and (A.6), we have

$$\frac{h_1(g_2((1/(2\beta))\log Ax))}{h_1(g_2(\frac{1}{2\beta}\log x))} \to 1 \qquad \text{as } x \to \infty.$$

This completes the proof that $l_1^*(x)$ is a slowly varying function.

The function $l_1^*(x)$ is right-continuous since $h_1(x)$ can be defined to be continuous, g_2 is continuous (as the inverse of a continuous increasing function) and g_1, $\tilde{g}_1$ and $\tilde{g}_1^*$ are right-continuous functions. $\square$

Lemma A.4. *Let L be a slowly varying function. Then, for any fixed $x_0 \neq \mathrm{e}^{2\beta(r+1-b_1)}$, $r \in \mathbb{Z}$, $\beta > 0$, there are only finitely many integer values of n for which*

$$m - b_1 + \frac{1}{2\beta} \log L\big(\mathrm{e}^{2m-b_2}\big) \leq \frac{1}{2\beta} \log(A_{k_n} x_0) < m - b_1 + \frac{1}{2\beta} \log L\big(\mathrm{e}^{2m-b_3}\big), \tag{A.7}$$

where $A_{k_n} = \mathrm{e}^{(n-1)2\beta} L(\mathrm{e}^{2n-2})$, m takes positive integer values, b_1, b_2 and b_3 are fixed positive constants with $b_2 > b_3$.

Proof. Suppose $m = n + r_n$, where r_n is a sequence of integers. We first show that if (A.7) is satisfied for infinitely many values of n, then $\sup_{n \geq 1} |r_n| < \infty$. Arguing by contradiction, for example, assume $r_n \to \infty$ as $n \to \infty$. From (A.7), we need to have

$$\mathrm{e}^{2\beta(r_n+1-b_1)} \frac{L(\mathrm{e}^{2n+2r_n-b_2})}{L(\mathrm{e}^{2n-2})} \leq x_0 < \mathrm{e}^{2\beta(r_n+1-b_1)} \frac{L(\mathrm{e}^{2n+2r_n-b_3})}{L(\mathrm{e}^{2n-2})}. \tag{A.8}$$

A standard argument using Potter's bounds for L shows that $\mathrm{e}^{2\beta(r_n+1-b_1)} \frac{L(\mathrm{e}^{2n+2r_n-b})}{L(\mathrm{e}^{2n-2})} \to \infty$ ($b = b_2$ or b_3) when $r_n \to \infty$. Since x_0 is fixed, this leads to a contradiction. A similar argument can be applied when $r_n \to -\infty$.

Next we show that m is necessarily of the form $m = n + r$ where r is a fixed integer for large enough n. We prove this by contradiction. First, observe that r_n can only take finitely many integer values. Now if r_n has a subsequence $r_{n_k} \to r$, then letting $n \to \infty$ in (A.8), we have $\mathrm{e}^{2\beta(r+1-b_1)} = x_0$. Thus, r is determined uniquely and since r_n are integers, we have that $r_n = r$ for large enough n.

Finally, if $m = n + r$, then (A.7) cannot hold for infinitely many values of n unless $x_0 = \mathrm{e}^{2\beta(r+1-b_1)}$. This proves the lemma. □

Lemma A.5. *Let (2.7)–(2.8) hold for a random variable X with $l^*(x)$ replaced by a right-continuous slowly varying function $l_1^*(x)$ in (2.7). Then, $l^*(x)$ in (2.8) can be replaced by another right-continuous function $l_2^*(x)$ if $\frac{l_2^*(x)}{l_1^*(x)} \to 1$ as $x \to \infty$.*

Proof. Observe that

$$\begin{aligned}
1 - F(x) &= x^{-\alpha} l_2^*(x)\big(M_R\big(\delta(x)\big) + h_R(x)\big) \\
&= x^{-\alpha} l_1^*(x)\left(M_R\big(\delta(x)\big) + h_R(x) + \left(\frac{l_2^*(x)}{l_1^*(x)} - 1\right)\big(M_R\big(\delta(x)\big) + h_R(x)\big)\right) \\
&= x^{-\alpha} l_1^*(x)\big(M_R\big(\delta(x)\big) + h_R(x) + \tilde{h}_R(x)\big),
\end{aligned} \tag{A.9}$$

where

$$\tilde{h}_R(x) = \left(\frac{l_2^*(x)}{l_1^*(x)} - 1\right)\big(M_R\big(\delta(x)\big) + h_R(x)\big). \tag{A.10}$$

Since $\frac{l_2^*(x)}{l_1^*(x)} \to 1$ as $x \to \infty$, M_R is a bounded periodic function from (2.2) and $h_R(A_{k_n}x) \to 0$, as $n \to \infty$, we have $\tilde{h}_R(A_{k_n}x) \to 0$ for every continuity point x of $M_R(x)$. Hence, in (A.9), one can take the new error function to be $h_R(x) + \tilde{h}_R(x)$. Hence, the result is proved. □

Lemma A.6. *Let h_1 be the function defined in Theorem 3.1 and satisfying (3.4). For every $\delta > 0$, there is M_δ such that, for all $n > M_\delta$,*

$$h_1(M_\delta + 1)e^{M_\delta+1}e^{\delta n} < h_1(n) < \frac{h_1(M_\delta + 1)}{e^{\delta(M_\delta+1)}}e^{\delta n}.$$

Proof. Fix any $\delta = \delta_0 \in (0, 1)$. By using (3.4), there exists M_{δ_0} such that for all $m > M_{\delta_0}$, $1 - \delta_0 < \frac{h_1(m+1)}{h_1(m)} < 1 + \delta_0$. Take any $n > M_{\delta_0}$. Then,

$$\begin{aligned} h_1(n) &= \frac{h_1(n)}{h_1(n-1)} \frac{h_1(n-1)}{h_1(n-2)} \cdots \frac{h_1(M_{\delta_0}+2)}{h_1(M_{\delta_0}+1)} h_1(M_{\delta_0}+1) \\ &< h_1(M_{\delta_0}+1)(1+\delta_0)^{n-M_{\delta_0}-1} < h_1(M_{\delta_0}+1)e^{\delta_0(n-M_{\delta_0}-1)}. \end{aligned}$$

Similarly,

$$h_1(n) > h_1(M_{\delta_0}+1)(1-\delta_0)^{n-M_{\delta_0}-1} > h_1(M_{\delta_0}+1)e^{-\delta_0(n-M_{\delta_0}-1)}. \qquad \square$$

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