

A new class of large claim size distributions: Definition, properties, and ruin theory

SERGEJ BECK*, JOCHEN BLATH** and MICHAEL SCHEUTZOW†

Institut für Mathematik, Technische Universität Berlin, Straße des 17. Juni 136, 10623 Berlin, Germany.
E-mail: *sergej.beck@hotmail.de; **blath@math.tu-berlin.de; †ms@math.tu-berlin.de

We investigate a new natural class \mathcal{J} of probability distributions modeling large claim sizes, motivated by the ‘principle of one big jump’. Though significantly more general than the (sub-)class of subexponential distributions \mathcal{S} , many important and desirable structural properties can still be derived. We establish relations to many other important large claim distribution classes (such as \mathcal{D} , \mathcal{S} , \mathcal{L} , \mathcal{K} , \mathcal{OS} and \mathcal{OL}), discuss the stability of \mathcal{J} under tail-equivalence, convolution, convolution roots, random sums and mixture, and then apply these results to derive a partial analogue of the famous Pakes–Veraverbeke–Embrechts theorem from ruin theory for \mathcal{J} . Finally, we discuss the (weak) tail-equivalence of infinitely-divisible distributions in \mathcal{J} with their Lévy measure.

Keywords: heavy-tailed; random walks; ruin theory; subexponential

1. Introduction

Large claim size distributions play an important role in many areas of probability theory and related fields, in particular insurance and finance. They often describe ‘extreme events’ and are typically ‘heavy-tailed’ (see, e.g., [9] for an overview). However, the class of heavy-tailed random variables \mathcal{K} (defined in Section 2.3 below) has a very rich structure, and the identification and discussion of relevant sub-classes is still an area of active research (see, e.g., [12] for a recent account). While this makes it difficult to formulate general statements for \mathcal{K} , for example regarding ruin probabilities, such results can be achieved for certain important subclasses, most importantly the *subexponential distributions* \mathcal{S} . Recall that the distribution F of i.i.d. nonnegative random variables X_1, X_2, \dots is called subexponential, iff

$$\lim_{x \rightarrow \infty} \frac{\mathbb{P}(\max(X_1, \dots, X_n) > x)}{\mathbb{P}(X_1 + \dots + X_n > x)} = 1 \quad (1.1)$$

for every $n \geq 2$. This means that the tail of the distribution of the maximum of n such random variables is asymptotically equivalent to the tail of the distribution of their sum. Hence, this sum is typically dominated by its largest element in the case of an extreme event.

The class \mathcal{S} of subexponential distributions has several important stability properties, and in particular allows an elegant characterization of the asymptotic behaviour of the ruin probability in the Cramér–Lundberg model (and in a weaker form also for more general renewal models). Indeed, the corresponding ruin function Ψ is asymptotically equivalent, for large initial capital, to the so-called tail-integrated distribution F_I associated with F (suitably normalized), iff $F_I \in \mathcal{S}$ (e.g., [10], see also Theorem 18 below).

From an intuitive point of view, one might ask whether condition (1.1) on the tail behaviour of the X_1, X_2, \dots might be too restrictive and could be weakened. For example, one could require that the maximum is sufficiently close to, but not quite at the same level as, the sum of the claim sizes x , say greater than $x - K$ for some constant K . This appears to be a natural definition of the folklore ‘principle of one big jump’ and leads to the following definition.

Definition 1 (Distributions of class \mathcal{J}). Let X_1, X_2, \dots be a sequence of i.i.d. nonnegative (essentially) unbounded random variables. Let F denote their common distribution function and let \mathcal{F} denote the set of all distribution functions of nonnegative random variables with unbounded support. We define the class $\mathcal{J} \subset \mathcal{F}$ as the set of all distribution functions $F \in \mathcal{F}$, such that for all $n \geq 2$,

$$\lim_{K \rightarrow \infty} \liminf_{x \rightarrow \infty} \frac{\mathbb{P}(\max(X_1, \dots, X_n) > x - K, X_1 + \dots + X_n > x)}{\mathbb{P}(X_1 + \dots + X_n > x)} = 1. \tag{1.2}$$

Just as in the case of subexponential distributions, it is enough that (1.2) holds for $n = 2$ (see Proposition 3). We will provide several equivalent formulations of (1.2) below. Of course, such a definition raises immediately a variety of questions. First, one certainly needs to clarify whether this definition produces a nontrivial new class of distributions at all. We will answer this affirmatively in Section 2.3 where we will also discuss the relation of \mathcal{J} to other distribution classes (it is obvious from the definition that $\mathcal{S} \subset \mathcal{J}$). At first sight rather surprisingly, it turns out that our definition also admits some light-tailed distributions to \mathcal{J} , see Example 6 (which, as an element of $\mathcal{S}(\gamma)$ with $\gamma = 1$, is also known to obey a ‘principle of one big jump’). Given this last fact, a second natural question is whether \mathcal{J} is still sufficiently coherent to exhibit convenient closure properties. It turns out that \mathcal{J} is closed under weak tail-equivalence (in contrast to \mathcal{S}), see Proposition 8 below, and has good properties with respect to closure under convolution and, importantly, convolution roots (Proposition 10), as well as mixture (Proposition 12). The same holds true for random sums (Propositions 14 and 16).

These rather remarkable properties will then be applied in Section 3, where we provide a partial analogue of the Pakes–Veraverbeke–Embrechts theorem for \mathcal{J} (Theorem 19), establishing weak tail-equivalence among classical risk quantities from ruin theory. This result is new and appears quite striking, given that the class \mathcal{J} is far richer than \mathcal{S} .

Finally, for infinitely-divisible elements of \mathcal{J} , we prove their weak tail-equivalence with their normalized Lévy measure, in the spirit of earlier results of Goldie *et al.* [8] and Shimura and Watanabe [18].

Remark 2. Regarding the rationale behind (1.2) one might wonder whether one should also consider distribution functions F with the property that

$$\lim_{x \rightarrow \infty} \frac{\mathbb{P}(\max(X_1, \dots, X_n) > (1 - \varepsilon)x, X_1 + \dots + X_n > x)}{\mathbb{P}(X_1 + \dots + X_n > x)} = 1, \tag{1.3}$$

for all $n \geq 2$, and for all $\varepsilon \in (0, 1)$. Indeed, this natural condition gives rise to an even larger class of distributions, denoted by \mathcal{A} , with $\mathcal{S} \subset \mathcal{J} \subset \mathcal{A}$. Some results for the class \mathcal{A} can be found in the dissertation of Beck [2].

2. Basic properties of the class \mathcal{J}

2.1. Notation and set-up

Throughout Section 2, we let X_1, X_2, \dots be a sequence of i.i.d. random variables on some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with values in $[0, \infty)$. Let $F \in \mathcal{F}$ denote their common distribution function. We denote by S_n the sum of the first n random variables, that is

$$S_n = \sum_{i=1}^n X_i.$$

Further, let $\bar{F}(x) := 1 - F(x)$ be the *tail* of F . If ν is a probability measure on $[0, \infty)$, then we define $\bar{\nu}(t) := \nu((t, \infty))$. Let $F * G$ be the convolution of two distribution functions $F, G \in \mathcal{F}$ and F^{n*} , for $n \geq 0$, the n -fold convolution of F with itself, where $F^{1*} := F$ and F^{0*} is the distribution corresponding to the Dirac measure at 0. Let f and g be two positive functions on $[0, \infty)$. We write that $f \sim g$ if

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 1,$$

that is, f and g are (strongly) asymptotically equivalent (as $x \rightarrow \infty$), and $f \asymp g$ in the case

$$0 < \liminf_{x \rightarrow \infty} \frac{f(x)}{g(x)} \leq \limsup_{x \rightarrow \infty} \frac{f(x)}{g(x)} < \infty.$$

The latter relation will be called *weak asymptotic equivalence*. Whenever $\bar{\mathcal{F}} \subseteq \mathcal{F}$, we freely write $X \in \bar{\mathcal{F}}$ or $\mu \in \bar{\mathcal{F}}$ for a nonnegative random variable X or a probability measure μ on $[0, \infty)$ when the associated distribution function belongs to $\bar{\mathcal{F}}$. Let \mathcal{G} denote the set of nonnegative, unbounded and nondecreasing functions. Finally, for all $n \in \mathbb{N}$ and for all $k \in \{1, \dots, n\}$, $x_{k,n}$ denotes the k th largest among x_1, \dots, x_n .

2.2. Equivalent characterizations of the class \mathcal{J}

It is interesting to see that the defining relation (1.2) is only one of many ways to characterize the class \mathcal{J} . Define, for $n \geq 2$,

$$\begin{aligned} \mathcal{J}_1^{(n)} &:= \left\{ F \in \mathcal{F} : \lim_{K \rightarrow \infty} \liminf_{x \rightarrow \infty} \mathbb{P}(X_{2,n} \leq K | S_n > x) = 1 \right\}, \\ \mathcal{J}_2^{(n)} &:= \left\{ F \in \mathcal{F} : \lim_{K \rightarrow \infty} \inf_{x \geq 0} \mathbb{P}(X_{2,n} \leq K | S_n > x) = 1 \right\}, \\ \mathcal{J}_3^{(n)} &:= \left\{ F \in \mathcal{F} : \lim_{K \rightarrow \infty} \liminf_{x \rightarrow \infty} \mathbb{P}(X_{1,n} > x - K | S_n > x) = 1 \right\}, \\ \mathcal{J}_4^{(n)} &:= \left\{ F \in \mathcal{F} : \lim_{K \rightarrow \infty} \liminf_{x \rightarrow \infty} \mathbb{P}(X_{1,n} > S_n - K | S_n > x) = 1 \right\}, \\ \mathcal{J}^{(n)} &:= \left\{ F \in \mathcal{F} : \lim_{x \rightarrow \infty} \mathbb{P}(X_{2,n} > g(x) | S_n > x) = 0 \forall g \in \mathcal{G} \right\}. \end{aligned} \tag{2.1}$$

Note that, by definition, $\mathcal{J} = \bigcap_{n \geq 2} \mathcal{J}_3^{(n)}$. However, it turns out that all of the above subclasses are equal to the class \mathcal{J} . Indeed, we have

Proposition 3. For all $n \geq 2$,

- (a) $\mathcal{J}^{(n)} = \mathcal{J}_1^{(n)} = \mathcal{J}_2^{(n)} = \mathcal{J}_3^{(n)} = \mathcal{J}_4^{(n)}$,
- (b) $\mathcal{J}^{(n)} = \mathcal{J}^{(n+1)}$,
- (c) $\mathcal{J} = \mathcal{J}^{(2)}$.

A proof can be found in Section 4. Note that a term reminiscent to the one in the definition of class $\mathcal{J}^{(2)}$ appears implicitly in [1], Proposition 2.

Remark 4. An elegant probabilistic way to think about the condition giving rise to class $\mathcal{J}_2^{(n)}$ is to interpret it as tightness condition of the conditional laws of $X_{2,n}$, given $S_n > x$.

2.3. Relation to other classes of claim size distributions and heavy tails

Recall that a claim size distribution $F \in \mathcal{F}$ is called *heavy-tailed*, if it has no exponential moments, that is,

$$\int_0^\infty e^{\lambda x} dF(x) = \infty \quad \text{for all } \lambda > 0.$$

In this case, we write $F \in \mathcal{K}$. Following the definition (but not the notation) of [20], we write $F \in \mathcal{K}^*$ if $\lim_{x \rightarrow \infty} e^{\lambda x} \overline{F}(x) = \infty$ holds for all $\lambda > 0$. Note that $\mathcal{K}^* \subsetneq \mathcal{K}$, see, for example, [19], and thus we call elements of \mathcal{K}^* ‘strongly heavy tailed’.

Three of the most important and well-studied subclasses of heavy-tailed distributions are the class \mathcal{S} of *subexponential* distributions, the class of *long-tailed* distributions \mathcal{L} and the class \mathcal{D} of *dominatedly varying* distributions. Recall that a distribution $F \in \mathcal{F}$ is subexponential if for all $n \geq 2$,

$$\lim_{x \rightarrow \infty} \frac{\overline{F^{n*}}(x)}{\overline{F}(x)} = n$$

(it is actually enough to require this condition for $n = 2$ only, see, e.g., [9]), and that F is long-tailed if

$$\lim_{x \rightarrow \infty} \frac{\overline{F}(x + y)}{\overline{F}(x)} = 1$$

for every $y \in \mathbb{R} \setminus \{0\}$ (or equivalently for some). Further, F has a dominatedly varying tail, if

$$\limsup_{x \rightarrow \infty} \frac{\overline{F}(xu)}{\overline{F}(x)} < \infty$$

for all (or equivalently for some) $0 < u < 1$. It is well known that

$$\mathcal{S} \subset \mathcal{L} \subset \mathcal{K}^* \subset \mathcal{K}, \quad \mathcal{D} \subset \mathcal{K}^* \subset \mathcal{K}, \quad \mathcal{L} \cap \mathcal{D} \subset \mathcal{S} \quad \text{and} \quad \mathcal{D} \not\subset \mathcal{S}, \mathcal{S} \not\subset \mathcal{D},$$

see [9] for most of these inclusions (the remaining ones are easy to check).

A generalization of the class of subexponential distributions is given by Shimura and Watanabe [18]. They systematically investigate the class \mathcal{OS} of ‘*O-subexponential*’ distributions, which was introduced by Klüppelberg in [14], where $F \in \mathcal{OS}$ if

$$c_F := \limsup_{x \rightarrow \infty} \frac{\overline{F^{2*}}(x)}{\overline{F}(x)} < \infty. \tag{2.2}$$

In a similar way, it is possible to generalize the class \mathcal{L} . Let \mathcal{OL} be the class of all distributions such that

$$\limsup_{x \rightarrow \infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} < \infty$$

for every $y \in \mathbb{R}$. The generalizations \mathcal{OL} and \mathcal{OS} of the classes \mathcal{L} and \mathcal{S} contain some light-tailed distributions, so that $\mathcal{OL}, \mathcal{OS} \not\subset \mathcal{K}$. Further, it can be shown [18], Proposition 2.1, that

$$\mathcal{OS} \subset \mathcal{OL}.$$

Finally, we recall the light-tailed distribution classes $\mathcal{S}(\gamma)$ and $\mathcal{L}(\gamma)$, for $\gamma \geq 0$: We say that a distribution $F \in \mathcal{F}$ belongs to $\mathcal{S}(\gamma)$ for some $\gamma \geq 0$, if for any $y \in \mathbb{R}$,

$$\lim_{x \rightarrow \infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = \exp(-\gamma y), \tag{2.3}$$

and for some constant $c \in (0, \infty)$,

$$\frac{\overline{F^{2*}}(x)}{\overline{F}(x)} = 2c < \infty.$$

A distribution $F \in \mathcal{F}$ belongs to $\mathcal{L}(\gamma)$, iff it satisfies (2.3). These classes were introduced independently by Chistyakov [3] and Chover, Ney and Wainger [4,5], see also [7]. Note that $\mathcal{L}(0) = \mathcal{L}$ and $\mathcal{S}(0) = \mathcal{S}$.

For our new class \mathcal{J} , we have the following results.

Proposition 5.

- (a) $\mathcal{J} \subset \mathcal{OS}$,
- (b) $\mathcal{J} \cap \mathcal{L} = \mathcal{S}$,
- (b') $\mathcal{J} \cap \mathcal{L}(\gamma) = \mathcal{S}(\gamma), \gamma > 0$,
- (c) $\mathcal{D} \subset \mathcal{J}$,
- (d) $\mathcal{J} \not\subset \mathcal{K}$.

A proof can be found in Section 4. Part (b') has been suggested to us by Sergey Foss. Note that (b') already implies (d) (so that the latter is in principle redundant), but we think that the fact that \mathcal{J} includes some light-tailed functions is important and thus we end this subsection with a concrete example (still obeying a ‘principle of one big jump’).

Example 6. Consider the distribution function $F \in \mathcal{F}$ with density

$$f(x) = e^{-x} \frac{C}{1+x^2}, \quad x \geq 0,$$

for $C > 0$ such that $\int_0^\infty f(x) dx = 1$. Note that there seems to be no closed-form expression for C , but it can be evaluated numerically to $C \approx 1.609$. Obviously $F \notin \mathcal{K}$ and thus $F \notin \mathcal{S}$. Since

$$\lim_{K \rightarrow \infty} \lim_{x \rightarrow \infty} \frac{\int_K^{x-K} f(y)f(x-y) dy}{\int_0^x f(y)f(x-y) dy} = 0,$$

it follows that

$$\lim_{K \rightarrow \infty} \limsup_{z \rightarrow \infty} \frac{\int_z^\infty \int_K^{x-K} f(y)f(x-y) dy dx}{\int_z^\infty \int_0^x f(y)f(x-y) dy dx} = \lim_{K \rightarrow \infty} \limsup_{z \rightarrow \infty} \frac{\mathbb{P}(X_{2,2} > K, S_2 > z)}{\mathbb{P}(S_2 > z)} = 0$$

and hence $F \in \mathcal{J}$. From Proposition 5(a), we also have that $F \in \mathcal{OS}$. Indeed, we can compute c_F from (2.2) and obtain

$$c_F = C\pi. \tag{2.4}$$

Note that $f(x)$ is obtained from the (subexponential) density $2/(\pi(1+x^2))$ by multiplication with a negative exponential and a suitable constant. This is a typical way to construct distributions of the distribution class $\mathcal{S}(\gamma)$, $\gamma \geq 0$, and indeed we have $F \in \mathcal{S}(\gamma)$ with index $\gamma = 1$. This class consists of light-tailed functions and has a well-studied ruin theory, obeys the ‘principle of one big jump’, and is outside the classical Lundberg framework.

Remark 7. It seems natural to ask ‘how many’ or ‘which kind of’ light-tailed functions can be found in \mathcal{J} . As a first result in this direction note that since $\mathcal{J} \subset \mathcal{OL}$, it follows from Proposition 2.2 in [18] that each light-tailed distribution $F \in \mathcal{J}$ exhibits at least some infinite exponential moments, that is, there exists a $\lambda_F > 0$ such that

$$\int_0^\infty e^{\lambda_F x} dF(x) = \infty.$$

Hence, the class \mathcal{J} in some sense ‘touches the boundary’ of the class of light-tailed functions. In view of (b'), the conjecture $\mathcal{J} = (\mathcal{J} \cap \mathcal{K}) \cup (\bigcup_{\gamma > 0} \mathcal{S}(\gamma))$ seems attractive.

2.4. Closure properties

As a first result, we show that our new class \mathcal{J} is closed under weak asymptotic tail-equivalence (in contrast to \mathcal{S} and \mathcal{L} , which require (strong) asymptotic tail-equivalence for closure).

Proposition 8. *If $F \in \mathcal{J}$ and $\overline{F} \asymp \overline{G}$, then $G \in \mathcal{J}$.*

Example 9. Neither \mathcal{L} nor \mathcal{S} are closed under weak tail-equivalence. Indeed, let

$$F(x) := \left(1 - \frac{1}{x}\right)^+, \quad x \geq 0,$$

be a Pareto distribution with index 1, so that F is subexponential and long-tailed. Let G be the ‘Peter-and-Paul’ distribution, that is, $\overline{G}(x) = 2^{-k}$ for $x \in [2^k, 2^{k+1})$, $k \in \mathbb{N}_0$. Then F and G are weakly tail-equivalent, that is, $\overline{F} \asymp \overline{G}$, but $G \notin \mathcal{L}$ and hence $G \notin \mathcal{S}$.

Although we will see that \mathcal{J} is not closed under convolution, we will find below that we have closure for ‘convolution powers’ and for weakly tail-equivalent distributions. Further, we have closure for ‘convolution roots’, in contrast to \mathcal{OS} (cf. [18]) – this property is highly desirable as we will see in the sequel.

We say that a distribution class \mathcal{C} is *closed under convolution*, if $F_1 * F_2 \in \mathcal{C}$ for any $F_1, F_2 \in \mathcal{C}$. It is well known that the class \mathcal{L} is closed under convolution, see [6], Theorem 3(b).

Proposition 10.

- (a) *If $F \in \mathcal{J}$, then $\overline{F} \asymp \overline{F^{n*}}$ and hence $F^{n*} \in \mathcal{J}$.*
- (b) *If $F \in \mathcal{J}$ and $\overline{F} \asymp \overline{G}$, then $F * G \in \mathcal{J}$.*
- (c) *If $F^{n*} \in \mathcal{J}$, then $\overline{F} \asymp \overline{F^{n*}}$ and hence $F \in \mathcal{J}$.*

Example 11. The classes \mathcal{S} and \mathcal{J} are not closed under convolution. A counterexample for the class \mathcal{S} is given in [16], Section 3. Since $\mathcal{S} \subset \mathcal{L}$, by the counterexample from [16] and Theorem 3(b) from [6] (convolution closure of \mathcal{L}) we know there exist two distributions F_1, F_2 such that $F_1, F_2 \in \mathcal{S}$ and $F_1 * F_2 \in \mathcal{L}$ but $F_1 * F_2 \notin \mathcal{S}$. Since we have $\mathcal{J} \cap \mathcal{L} = \mathcal{S}$ from Proposition 5(b), $F_1, F_2 \in \mathcal{J}$ but $F_1 * F_2 \notin \mathcal{J}$.

We now turn to mixture properties of the class \mathcal{J} . Let X, Y be two random variables with distribution functions $F, G \in \mathcal{F}$. Recall that $X \vee Y$ (resp. $X \wedge Y$) denotes the pointwise maximum (resp. minimum) of X and Y . We call a random variable Z *mixture* of X and Y with parameter $p \in (0, 1)$, if its distribution function is given by

$$pF + (1 - p)G \in \mathcal{F}.$$

It is easy to see that if X and Y are independent, we have for all mixtures Z with $p \in (0, 1)$,

$$(X \vee Y) \asymp Z. \tag{2.5}$$

Proposition 12. *Let $X, Y \in \mathcal{F}$ be independent.*

- (a) *If $X, Y \in \mathcal{J}$, then the following are equivalent:*
 - (i) $(X \vee Y) \in \mathcal{J}$;
 - (ii) $(X + Y) \in \mathcal{J}$;

- (iii) $Z \in \mathcal{J}$.
- (b) If $X, Y \in \mathcal{J}$, then $(X \wedge Y) \in \mathcal{J}$.

The previous statement remains true when \mathcal{J} is replaced by \mathcal{S} (see [23], Theorem 1, [13], Theorem 1, and [12], Theorem 3.33).

Remark 13. Concerning part (b) of the previous proposition one may ask if $X, Y \in \mathcal{J}$ even implies that $X \vee Y$ and $X + Y$ are weakly tail equivalent which would immediately imply the equivalence of (i) and (ii). Perhaps surprisingly, this is not true in general – not even under the stronger assumption that $X, Y \in \mathcal{S}$ as the example in [16] shows.

2.5. Random sums

As before, let $F \in \mathcal{F}$ be the common distribution function of the i.i.d. random variables $\{X_i\}$. Recall the notation

$$c_F = \limsup_{x \rightarrow \infty} \frac{\overline{F^{2*}}(x)}{\overline{F}(x)}.$$

Denote by N a discrete random variable with values in \mathbb{N}_0 , independent of the $\{X_i\}$, with probability weights $p_n := \mathbb{P}\{N = n\}$, $n \geq 0$ and $p_0 < 1$. Denote by $N^{(1)}$ and $N^{(2)}$ two independent copies of N , and write

$$(p * p)_n := \mathbb{P}(N^{(1)} + N^{(2)} = n), \quad n \geq 0.$$

We now consider the random sum

$$S_N := \sum_{i=1}^N X_i$$

with distribution function F_N . Under a suitable decay condition on the (p_n) , we obtain the following stability property of \mathcal{J} for a random number of convolutions and convolution roots:

Proposition 14.

- (a) If $F \in \mathcal{J}$ and $\sum_{k=1}^{\infty} p_k (c_F + \varepsilon - 1)^k < \infty$ for some $\varepsilon > 0$, then $\overline{F_N} \asymp \overline{F}$ and hence $F_N \in \mathcal{J}$.
- (b) If $F_N \in \mathcal{J}$ and $\sum_{k=1}^{\infty} p_k (c_{F_N} + \varepsilon - 1)^k < \infty$ for some $\varepsilon > 0$, then $\overline{F} \asymp \overline{F_N}$ and hence $F \in \mathcal{J}$.

Remark 15. Note that one cannot infer $F_N \in \mathcal{J}$ or $\overline{F} \asymp \overline{F_N}$ from $F \in \mathcal{J}$ without additional conditions on N . This is true even if N is a geometric random variable, say with parameter $p \in (0, 1)$ and probability weights $p_k = p^k(1 - p)$, $k \geq 0$. Indeed, while it is obvious that the condition of the Proposition is satisfied for all $p \in (0, (c_F + \varepsilon - 1)^{-1})$, a counterexample is given by the distribution $F \in \mathcal{J}$ from Example 6 with geometric N that has a parameter p close enough

to 1. To see this, consider a sequence of i.i.d. random variables X_1, X_2, X_3, \dots with distribution function $F \in \mathcal{J}$ from Example 6. Let $\alpha > 0$ be such that $\alpha \mathbb{E}X_1 > 1$. For all $m \in \mathbb{N}$ we have

$$\mathbb{P}(X_1 + \dots + X_N > m) \geq \mathbb{P}(N \geq \lfloor \alpha m \rfloor) \mathbb{P}(X_1 + \dots + X_{\lfloor \alpha m \rfloor} > m)$$

and

$$\mathbb{P}(X_1 > m) = \int_m^\infty \frac{C e^{-x}}{1+x^2} dx \leq C e^{-m}.$$

If p is close enough to 1 we obtain by our choice of α and the law of large numbers,

$$\frac{\mathbb{P}(X_1 + \dots + X_N > m)}{\mathbb{P}(X_1 > m)} \geq \frac{e^m}{C} p^{\lfloor \alpha m \rfloor} \mathbb{P}(X_1 + \dots + X_{\lfloor \alpha m \rfloor} > m) \rightarrow \infty,$$

for $m \rightarrow \infty$, so $\overline{F} \asymp \overline{F}_N$ does not hold. It follows from part (b) of Proposition 16 below that $F_N \notin \mathcal{J}$.

A related result for random sums in \mathcal{J} and \mathcal{OS} can be obtained under the following condition on N and c_F .

Proposition 16. *Suppose*

$$\liminf_{n \rightarrow \infty} \frac{\mathbb{P}(N_1 + N_2 > n)}{\mathbb{P}(N_1 > n)} > c_{F_N} = \limsup_{x \rightarrow \infty} \frac{\overline{F_N^{2*}}(x)}{F_N(x)},$$

then the following assertions hold.

- (a) If $F_N \in \mathcal{OS}$, then there exists $m \in \mathbb{N}$ such that $\overline{F^{m*}} \asymp \overline{F_N}$.
- (b) If $F_N \in \mathcal{J}$, then $\overline{F} \asymp \overline{F_N}$ and hence $F \in \mathcal{J}$.

In [21] it is pointed out that $\lim_{n \rightarrow \infty} \frac{p_{n+1}}{p_n} = 0$ implies $\lim_{n \rightarrow \infty} \frac{(p^*p)_n}{p_n} = \infty$, which in turn implies $\liminf_{n \rightarrow \infty} \frac{\mathbb{P}(N_1 + N_2 > n)}{\mathbb{P}(N_1 > n)} = \infty$. From there, we also recall some examples for N .

Example 17. The following distributions satisfy the condition $\liminf_{n \rightarrow \infty} \frac{(p^*p)_n}{p_n} = \infty$:

- (1) Poisson distribution: $p_n = \frac{c^n}{n!} e^{-c}$, $c > 0$.
- (2) Geometric distribution: $p_n = (1-p)p^n$, $p \in (0, 1)$.
- (3) Negative Binomial distribution: $p_n = \binom{n+r-1}{n} p^n (1-p)^r$, $p \in (0, 1)$, $r > 0$.

3. Applications

3.1. Ruin theory and maximum of a random walk

Let X_1, X_2, \dots be a family of strictly positive i.i.d. random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with distribution function F_X and finite expectation μ_X . Let $N = \{N(t), t \geq 0\}$ be a

renewal process with i.i.d. strictly positive waiting times W_1, W_2, \dots . We assume that the W_i are independent of the X_i , and with finite expectation $1/\lambda$, for some $\lambda > 0$. We then define the *total claim amount process* as

$$S(t) := \sum_{i=1}^{N(t)} X_i, \quad t \geq 0.$$

Let $T_n := W_1 + \dots + W_n, n \geq 1$ be the arrival times of the claims, where we set $T_0 := 0$. By $c > 0$ we denote the *premium rate* and by $u \geq 0$ the *initial capital*. Finally, we define the risk process, for $u \geq 0$, by

$$Z(t) := u + ct - S(t), \quad t \geq 0.$$

If the above *claim arrival process* N is a Poisson process, we are in the classical *Cramér–Lundberg* model, otherwise, we are in the more general *Sparre Andersen* model. By

$$\tau := \inf\{k \geq 1 : Z(T_k) < 0\}$$

we denote the *ruin time* (with the usual convention that $\inf \emptyset = \infty$), and by

$$\Psi(u) := \mathbb{P}\{\tau < \infty | Z(0) = u\}$$

we denote the *ruin probability*. This quantity is the central object of study in *ruin theory*. We will be interested in obtaining asymptotic results for $\Psi(u)$ for large u if $F_X \in \mathcal{J}$. To this end, we first reformulate the classical ruin problem into a question about the maximum of an associated random walk with negative drift. We follow the exposition of [24]. Let

$$\bar{X}_k := X_k - cW_k = -(Z(T_k) - Z(T_{k-1})), \quad k \geq 1.$$

Note that the \bar{X}_i are i.i.d. with values in \mathbb{R} . We denote their distribution function by $F_{\bar{X}}$. By the strong law of large numbers, we have $\Psi(u) \equiv 1$ if $\mathbb{E}[\bar{X}_k] \geq 0$ (unless $\bar{X}_1 \equiv 0$). Otherwise, we say that the *net profit condition* holds, and we denote $a := \mathbb{E}[\bar{X}_k] < 0$. Let

$$\bar{S}_n := \bar{X}_1 + \dots + \bar{X}_n, \quad n \geq 1, \bar{S}_0 := 0.$$

Under the net profit condition, this is a discrete-time random walk with negative drift. For the ruin probability, we obtain

$$\Psi(u) = \mathbb{P}\left\{\sup_{n \geq 0} \bar{S}_n > u\right\}, \quad u \geq 0.$$

Hence we have expressed the probability of ruin in terms of the distribution of the supremum of a random walk with negative drift, which is the object that we will now investigate. Let

$$M := \sup_{n \geq 0} \bar{S}_n$$

be the supremum of the random walk and denote its distribution by F_M . With this notation, we have $\Psi(u) = \overline{F_M}(u), u \geq 0$.

Denote by

$$\tau_+ := \inf\{n \geq 1 : \bar{S}_n > 0\},$$

the first passage time over zero, with the convention $\inf \emptyset = \infty$. Then, the first ascending ladder height is given by \bar{S}_{τ_+} , which is a defective random variable (since τ_+ may be infinite). Set

$$\mathbb{P}(\tau_+ < \infty) =: p < 1.$$

We assume that $p > 0$ which only excludes uninteresting cases and is automatically satisfied in the Cramér–Lundberg model. Further, let

$$\bar{G}(x) := \mathbb{P}(\bar{S}_{\tau_+} > x | \tau_+ < \infty), \quad x \geq 0.$$

It is well known (see [11], Chapter XII) that the tail of the distribution M can be calculated by the formula

$$\bar{F}_M(x) = (1 - p) \sum_{n=0}^{\infty} p^n \bar{G}^{n*}(x), \quad x \geq 0. \tag{3.1}$$

Finally, denote by

$$F_I(x) := 1 - \min\left\{1, \int_x^{\infty} \bar{F}_{\bar{X}}(y) dy\right\}, \quad x \geq 0,$$

the tail-integrated distribution of $F_{\bar{X}}$. Then, the classical Pakes–Veraverbeke–Embrechts theorem can be stated as follows (see [24]).

Theorem 18 (Pakes–Veraverbeke–Embrechts). *With the above notation and assumptions, recalling $a := \mathbb{E}[\bar{X}_k] < 0$, the following assertions are equivalent:*

- (1) $F_I \in \mathcal{S}$;
- (2) $G \in \mathcal{S}$;
- (3) $F_M \in \mathcal{S}$;
- (4) $\bar{F}_M \sim -\frac{1}{a} \bar{F}_I$.

Our main goal in this section is to (partially) extend this result from the class \mathcal{S} to \mathcal{J} . Recall the notation

$$c_G = \limsup_{x \rightarrow \infty} \frac{\bar{G}^{2*}(x)}{\bar{G}(x)},$$

and from [24], Lemma 2.2, that if $F_I \in \mathcal{OL}$, then $\bar{G} \asymp \bar{F}_I$.

Theorem 19. *With the above notation and $a < 0$, assume additionally that $F_I \in \mathcal{OL}$ and that one of the following conditions holds:*

- (i) $p(c_G + \varepsilon - 1) < 1$ for some $\varepsilon > 0$,
- (ii) $F_M \in \mathcal{OS}$,

(iii) $F_I \in \mathcal{J} \cap \mathcal{K}^*$.

Then the following assertions are equivalent:

- (1) $F_I \in \mathcal{J}$;
- (2) $G \in \mathcal{J}$;
- (3) $F_M \in \mathcal{J}$.

Each one of (1), (2) or (3) combined with each one of (i), (ii) or (iii) implies

(4) $\overline{F_M} \asymp \overline{G} \asymp \overline{F_I}$.

For a (nontrivial) example of a distribution $F_I \in \mathcal{OL} \cap \mathcal{J} \cap \mathcal{K}^*$, but $F \notin \mathcal{L}$, see the recent article [22].

Corollary 20. *If $a < 0$ and $\overline{F_I} \asymp \overline{H}$ for some $H \in \mathcal{S}$, then $\overline{F_M} \asymp \overline{G} \asymp \overline{F_I}$.*

Note that the analogous weak tail-equivalence does not hold if F_I is an exponential distribution with parameter $\lambda > 0$, while one has strong asymptotic tail-equivalence (up to a constant) in the case $F_I \in \mathcal{S}(\gamma)$, see [15].

Theorem 19 is inspired by and should be compared with the recent partial generalization of Theorem 18 to the even larger class \mathcal{OS} by Yang and Wang in [24], Theorems 1.2 and 1.3, which is however considerably weaker. In particular, it does not cover Example 3.2 in [22]. One reason is that the class \mathcal{OS} is not closed under convolution roots, as opposed to \mathcal{S} and \mathcal{J} .

Let H^+ denote the positive part of a distribution function H .

Theorem 21. *With the above notation and $a < 0$, if $F_I \in \mathcal{OL}$, then*

- (a) $\limsup_{x \rightarrow \infty} \frac{\overline{F_I}(x)}{\overline{F_M}(x)} < \infty$;
- (b) (i) $F_I \in \mathcal{OS}$ and
(ii) $G \in \mathcal{OS}$ are equivalent;
- (c) (iii) $\overline{F_M} \asymp \overline{G} \asymp \overline{F_I}$ yields (i) or (ii).

If $F_I \in \mathcal{L}$ and $(c_{F_I^+} - 1) < a$, then (i) or (ii) yields (iii). In this case, (i) ((ii) or (iii)) implies $F_M \in \mathcal{OS}$.

Note that the weak asymptotic tail equivalence (iii) requires $F_I \in \mathcal{L}$ as opposed to the situation in Theorem 19. Further, (a) gives only a lower asymptotic bound for $\overline{F_M}$ in terms of $\overline{F_I}$.

3.2. Infinitely divisible laws

In this section, we consider the relation between the asymptotic tail behaviour of infinitely divisible laws and their Lévy measures. Following [18], we denote by \mathcal{ID}_+ the class of all infinitely divisible distributions μ on $[0, \infty)$ with Laplace transform

$$\hat{\mu}(s) = \exp \left\{ \int_0^\infty (e^{-st} - 1) \nu(dt) \right\},$$

where the Lévy measure ν satisfies $\bar{\nu}(t) > 0$ for every $t > 0$, and

$$\int_0^\infty (1 \wedge t)\nu(dt) < \infty.$$

Define the normalized Lévy measure ν_1 as $\nu_1 = 1_{\{x>1\}}\nu/\nu(1, \infty)$. Embrechts *et al.* proved in [8], Theorem 1, the following classical result.

Theorem 22. *Let μ be a distribution in \mathcal{ID}_+ with Lévy measure ν . Then the following assertions are equivalent:*

- (1) $\mu \in \mathcal{S}$;
- (2) $\nu_1 \in \mathcal{S}$;
- (3) $\bar{\mu} \sim \bar{\nu}$.

Shimura and Watanabe partially extended the result of Embrechts from the class \mathcal{S} to the class \mathcal{OS} in [18], Theorem 1.1:

Theorem 23. *Let μ be a distribution in \mathcal{ID}_+ with Lévy measure ν .*

- (a) *The following are equivalent:*
 - (1) $\nu_1 \in \mathcal{OS}$;
 - (2) $\bar{\mu} \asymp \bar{\nu}_1$.
- (b) *The following are equivalent:*
 - (1) $\mu \in \mathcal{OS}$;
 - (2) $\nu_1^{n*} \in \mathcal{OS}$ for some $n \geq 1$;
 - (3) $\bar{\mu} \asymp \bar{\nu}_1^{n*}$ for some $n \geq 1$.
- (c) *If ν_1 is in \mathcal{OS} , then μ is in \mathcal{OS} . The converse does not hold.*

Since the class \mathcal{J} is closed under convolution roots, one expects to be able to improve the result for \mathcal{OS} to class \mathcal{J} significantly. Indeed this is possible.

Theorem 24. *Let μ be a distribution in \mathcal{ID}_+ with Lévy measure ν .*

- (a) *Then the following assertions are equivalent:*
 - (1) $\mu \in \mathcal{J}$;
 - (2) $\nu_1 \in \mathcal{J}$.
- (b) *If (1) or (2) holds, then $\bar{\mu} \asymp \bar{\nu}_1$.*

Since the proof is simple, we refrain from postponing it to the next section and state it here.

Proof of Theorem 24. (a) From Theorem 23(b), $\mathcal{J} \subseteq \mathcal{OS}$, Proposition 8, $\mu \in \mathcal{J}$ we infer $\bar{\mu} \asymp \bar{\nu}_1^{n*}$ and $\nu_1^{n*} \in \mathcal{J}$ for some $n \geq 1$. The equivalence $\mu \in \mathcal{J} \Leftrightarrow \nu_1 \in \mathcal{J}$ follows immediately from Proposition 10(c).

(b) If (1) holds the assertion follows from Theorem 23(b), Propositions 8 and 10(c). If (2) holds, then the assertion follows from Theorem 23(a) and $\mathcal{J} \subset \mathcal{OS}$. □

4. Proofs

Throughout the proofs, we will use the following notation. Denote by X, X_1, X_2, \dots i.i.d. random variables with common distribution function $F \in \mathcal{F}$, and by Y, Y_1, Y_2, \dots i.i.d. random variables with common distribution function $G \in \mathcal{F}$. By $X_{k,n}$ we denote the k th largest element (pointwise) out of $X_1, \dots, X_n, 1 \leq k \leq n$, and by $X_{k,(l,\dots,m)}$ the k th largest element (pointwise) out of $X_l, \dots, X_m, 1 \leq l \leq m, 1 \leq k \leq m - l + 1$. Further, let

$$S_n := \sum_{k=1}^n X_k \quad \text{and} \quad \hat{S}_n := \sum_{k=1}^n Y_k.$$

Finally, denote by $S_n^{(i)}$, respectively $\hat{S}_n^{(i)}, i = 1, \dots, 4$, independent identically distributed copies of S_n respectively \hat{S}_n . We begin with several technical lemmas, which we collect here for reference.

Lemma 25. *Let $F, G, H, I \in \mathcal{F}$. Suppose $\bar{F} \asymp \bar{G}$ and $\bar{H} \asymp \bar{I}$. Then $\overline{F * H} \asymp \overline{G * I}$.*

A proof can be found in [18], Proposition 2.7. Recall from Section 4.1 that \mathcal{G} denotes the set of nonnegative, unbounded and nondecreasing real functions.

Lemma 26. *Suppose $g \in \mathcal{G}$. Then:*

$$\limsup_{x \rightarrow \infty} \frac{\mathbb{P}(S_2 > x, X_1 \wedge X_2 > g(x))}{\mathbb{P}(\hat{S}_2 > x, Y_1 \wedge Y_2 > g(x))} \leq \left(\limsup_{x \rightarrow \infty} \frac{\bar{F}(x)}{\bar{G}(x)} \right)^2.$$

A proof can be found in [12], Lemma 2.36.

Lemma 27. *For each $F \in \mathcal{F}, c \geq 0$ and $n \geq 2$, we have*

$$\lim_{K \rightarrow \infty} \limsup_{x \rightarrow \infty} \mathbb{P}(X_{1,n} > x - c, X_{2,n} > K | S_n > x) = 0.$$

Proof. For $x - c \geq K \geq c$, we have

$$\begin{aligned} & \mathbb{P}(X_{1,n} > x - c, X_{2,n} > K | S_n > x) \\ & \leq \frac{\mathbb{P}(X_{1,n} > x - c, X_{2,n} > K)}{\mathbb{P}(X_{1,n} > x - c, X_{2,n} > c)} = \frac{1 - (1 - \bar{F}(x - c))^n - n\bar{F}(x - c)(F(K))^{n-1}}{1 - (1 - \bar{F}(x - c))^n - n\bar{F}(x - c)(F(c))^{n-1}} \\ & = \frac{n\bar{F}(x - c)(1 - F(K)^{n-1}) + o(\bar{F}(x - c))}{n\bar{F}(x - c)(1 - F(c)^{n-1}) + o(\bar{F}(x - c))} = \frac{1 - F(K)^{n-1} + o(1)}{1 - F(c)^{n-1} + o(1)}, \end{aligned}$$

and the result follows by passing to the limit. □

Lemma 28. Let $\gamma \geq 0$ and $F \in \mathcal{L}(\gamma)$. Then $F \in \mathcal{S}(\gamma)$ if and only if

$$\mathbb{P}(X_1 + X_2 > x, \min(X_1, X_2) > h(x)) = o(\overline{F}(x)) \quad \text{as } x \rightarrow \infty \quad (4.1)$$

for all $h \in \mathcal{G}$.

The case $\gamma = 0$ is shown in [1], Proposition 2, and the case $\gamma > 0$ is analogous. In some proofs, we will need the dominated convergence theorem and for its application an upper bound for $\overline{F^{n*}}(x)/\overline{F}(x)$ is required. One such is given by the lemma below, known as Kesten’s lemma.

Lemma 29. If $F \in \mathcal{OS}$ then, for every $\varepsilon > 0$, there exists $c > 0$ such that for all $n \geq 1$ and $x \geq 0$:

$$\frac{\overline{F^{n*}}(x)}{\overline{F}(x)} \leq c(c_F + \varepsilon - 1)^n.$$

A proof can be found in [18], Proposition 2.4.

4.1. Proof of Proposition 3

Proof of Proposition 3. (a) We show $\mathcal{J}^{(n)} = \mathcal{J}_1^{(n)} = \mathcal{J}_2^{(n)} = \mathcal{J}_3^{(n)} = \mathcal{J}_4^{(n)}$. The inclusions $\mathcal{J}^{(n)} \supseteq \mathcal{J}_1^{(n)} \supseteq \mathcal{J}_2^{(n)}$ and $\mathcal{J}_4^{(n)} \subseteq \mathcal{J}_3^{(n)}$ are obvious. The inclusion $\mathcal{J}_1^{(n)} \subseteq \mathcal{J}_4^{(n)}$ follows immediately from

$$\left\{ X_{2,n} < \frac{K}{n-1}, S_n > x \right\} \subseteq \{X_{1,n} > S_n - K, S_n > x\},$$

for all $F \in \mathcal{F}$, $K > 0$ and $x > 0$. It remains to show $\mathcal{J}^{(n)} \subseteq \mathcal{J}_1^{(n)} \subseteq \mathcal{J}_2^{(n)}$ and $\mathcal{J}_1^{(n)} \supseteq \mathcal{J}_3^{(n)}$.

First, we prove the inclusion $\mathcal{J}^{(n)} \subseteq \mathcal{J}_1^{(n)}$. Suppose $F \in \mathcal{J}^{(n)}$ and $F \notin \mathcal{J}_1^{(n)}$, then there is $\tau > 0$ such that for any $m \geq 1$:

$$\liminf_{x \rightarrow \infty} \mathbb{P}(X_{2,n} \leq m | S_n > x) \leq 1 - \tau.$$

For every $m \geq 1$, we choose an unbounded and strictly increasing sequence $(x_k^m)_{k \in \mathbb{N}}$ with $\lim_{m \rightarrow \infty} x_m^m = \infty$ such that for all $k \in \mathbb{N}$:

$$\mathbb{P}(X_{2,n} \leq m | S_n > x_k^m) \leq 1 - \frac{\tau}{2}.$$

Hence, we obtain

$$\limsup_{m \rightarrow \infty} \mathbb{P}(X_{2,n} \leq m | S_n > x_m^m) \leq 1 - \frac{\tau}{2},$$

contradicting the fact that

$$\lim_{x \rightarrow \infty} \mathbb{P}(X_{2,n} > g(x) | S_n > x) = 0 \quad \text{for all } g \in \mathcal{G},$$

so $\mathcal{J}^{(n)} \subseteq \mathcal{J}_1^{(n)}$.

Next, we show the inclusion $\mathcal{J}_1^{(n)} \subseteq \mathcal{J}_2^{(n)}$. Suppose $F \in \mathcal{J}_1^{(n)}$. By definition we know that for every $\varepsilon > 0$ there are constants x_0 and K_0 such that for all $x \geq x_0$ and $K \geq K_0$:

$$\mathbb{P}(X_{2,n} \leq K | S_n > x) \geq 1 - \varepsilon. \tag{4.2}$$

Let $\delta > 0$. Increasing K_0 if necessary, we can assume that $\mathbb{P}(X_{2,n} \leq K) \geq 1 - \delta$ for all $K \geq K_0$. Hence, we obtain for $x \leq x_0$ and $K \geq K_0$:

$$\mathbb{P}(X_{2,n} \leq K | S_n > x) \geq \frac{\mathbb{P}(X_{2,n} \leq K) + \mathbb{P}(S_n > x) - 1}{\mathbb{P}(S_n > x)} \geq 1 - \frac{\delta}{\mathbb{P}(S_n > x)}. \tag{4.3}$$

By (4.2) and (4.3) we see that $F \in \mathcal{J}_2^{(n)}$, since $\delta > 0$ and $\varepsilon > 0$ are arbitrary.

Finally, we show $\mathcal{J}_1^{(n)} \supseteq \mathcal{J}_3^{(n)}$. Suppose $F \in \mathcal{J}_3^{(n)}$ and $F \notin \mathcal{J}_1^{(n)}$. Then there exists some $\delta > 0$ such that for any $m \geq 1$:

$$\liminf_{x \rightarrow \infty} \mathbb{P}(X_{2,n} \leq m | S_n > x) \leq 1 - 2\delta.$$

For every $m \geq 1$, we choose an unbounded and strictly increasing sequence $(x_k^m)_{k \in \mathbb{N}}$ such that for all $k \in \mathbb{N}$:

$$\mathbb{P}(X_{2,n} \leq m | S_n > x_k^m) \leq 1 - \delta.$$

Since $F \in \mathcal{J}_3^{(n)}$ we know there exist $c > 0$ and $\bar{x} > 0$ such that for all $x \geq \bar{x}$:

$$\mathbb{P}(X_{1,n} > x - c | S_n > x) \geq 1 - \frac{\delta}{3}.$$

Hence, we obtain for any $m \geq 1$ and all $k \geq 1$, $x_k^m \geq \bar{x}$:

$$\begin{aligned} &\mathbb{P}(X_{1,n} > x_k^m - c, X_{2,n} > m | S_n > x_k^m) \\ &= \mathbb{P}(X_{1,n} > x_k^m - c | S_n > x_k^m) - \mathbb{P}(X_{1,n} > x_k^m - c, X_{2,n} \leq m | S_n > x_k^m) \\ &\geq \mathbb{P}(X_{1,n} > x_k^m - c | S_n > x_k^m) - \mathbb{P}(X_{2,n} \leq m | S_n > x_k^m) \\ &\geq 1 - \frac{\delta}{3} - 1 + \delta = \frac{2}{3}\delta. \end{aligned}$$

We get

$$\lim_{m \rightarrow \infty} \limsup_{x \rightarrow \infty} \mathbb{P}(X_{1,n} > x - c, X_{2,n} > m | S_n > x) > 0,$$

which contradicts Lemma 27.

(b) We show $\mathcal{J}^{(n+1)} = \mathcal{J}^{(n)}$. It suffices to prove the inclusion $\mathcal{J}_2^{(n+1)} \supseteq \mathcal{J}_2^{(n)}$. Then, we can conclude from (a) that $\mathcal{J}^{(n+1)} \supseteq \mathcal{J}^{(n)}$.

Suppose $F \in \mathcal{J}_2^{(n)}$. By definition of $\mathcal{J}_2^{(n)}$ we know for all $\varepsilon > 0$ there exists a constant $K_0 > 0$ such that for all $x \geq 0$:

$$\mathbb{P}(X_{2,n} \leq K_0, S_n > x) \geq \left(1 - \frac{\varepsilon}{n+1}\right) \overline{F^{n*}}(x).$$

Hence, we obtain for $K \geq K_0$ and $x \geq 0$:

$$\begin{aligned} \mathbb{P}(X_{2,(1,\dots,n)} \leq K | S_{n+1} > x) &= \frac{\int_0^\infty \mathbb{P}(X_{2,n} \leq K, S_n > x - t) dF(t)}{F^{(n+1)*}(x)} \\ &\geq \frac{(1 - \varepsilon/(n + 1)) \int_0^\infty \overline{F}^{n*}(x - t) dF(t)}{F^{(n+1)*}(x)} = 1 - \frac{\varepsilon}{n + 1}. \end{aligned}$$

Thus, we see that for all $K \geq K_0$ and $x \geq 0$:

$$\begin{aligned} \mathbb{P}(X_{2,n+1} \leq K | S_{n+1} > x) &= \mathbb{P}(X_{2,(1,\dots,n)} \leq K, X_{2,(1,\dots,n-1,n+1)} \leq K, \\ &\quad \dots, X_{2,(2,\dots,n,n+1)} \leq K | S_{n+1} > x) \\ &\geq (n + 1)\mathbb{P}(X_{2,n} \leq K | S_{n+1} > x) - n \\ &\geq 1 - \varepsilon, \end{aligned}$$

where we used the inequality

$$\mathbb{P}\left(\bigcap_{i=1}^{n+1} A_i\right) \geq \sum_{i=1}^{n+1} \mathbb{P}(A_i) - n.$$

We obtain $F \in \mathcal{J}_2^{(n+1)}$. □

In the second part of the proof of Proposition 3(b) we will use Proposition 5(a), for that reason we give the proof of Proposition 5(a) already here.

Proof of Proposition 5(a). We prove $\mathcal{J}^{(n)} \subseteq \mathcal{O}\mathcal{S}$. Let $n \geq 3$. Suppose that $F \in \mathcal{J}^{(n)} = \mathcal{J}_1^{(n)}$. Then,

$$\begin{aligned} 1 &= \lim_{K \rightarrow \infty} \liminf_{x \rightarrow \infty} \mathbb{P}\left(X_{2,n} < \frac{K}{n - 1} \mid S_n > x\right) \\ &\leq \lim_{K \rightarrow \infty} \liminf_{x \rightarrow \infty} \mathbb{P}(X_{1,n} > x | S_n > x + K) \\ &\leq \lim_{K \rightarrow \infty} \liminf_{x \rightarrow \infty} \frac{n\mathbb{P}(X_n > x)}{\mathbb{P}(S_{n-1} > x)\mathbb{P}(X_n > K)} \\ &\leq n \lim_{K \rightarrow \infty} \left(\frac{1}{\mathbb{P}(X_n > K)} \liminf_{x \rightarrow \infty} \frac{\mathbb{P}(X_n > x)}{\mathbb{P}(S_{n-1} > x)}\right). \end{aligned}$$

Hence, we have $\liminf_{x \rightarrow \infty} \frac{\mathbb{P}(X_1 > x)}{\mathbb{P}(S_{n-1} > x)} > 0$ and thus $\limsup_{x \rightarrow \infty} \frac{\mathbb{P}(S_{n-1} > x)}{\mathbb{P}(X_1 > x)} < \infty$.

In the case $n = 2$ we use the inclusion $\mathcal{J}^{(3)} \supseteq \mathcal{J}^{(2)}$, which was already shown above, to get $F \in \mathcal{J}^{(3)}$. □

We now resume the second part of the proof of Proposition 3.

Proof of Proposition 3. We begin with the inclusion $\mathcal{J}^{(n+1)} \subseteq \mathcal{J}^n$. Suppose $F \in \mathcal{J}^{(n+1)}$ and $F \notin \mathcal{J}^{(n)}$. Then there exists $g \in \mathcal{G}$ such that

$$\limsup_{x \rightarrow \infty} \mathbb{P}(X_{2,n} > g(x) | S_n > x) > 0$$

and

$$\lim_{x \rightarrow \infty} \mathbb{P}(X_{2,n+1} > g(x) | S_{n+1} > x) = 0.$$

Thus, we have:

$$\limsup_{x \rightarrow \infty} \frac{\mathbb{P}(X_{2,n} > g(x), S_n > x)}{\mathbb{P}(X_{2,n+1} > g(x), S_{n+1} > x)} \frac{\mathbb{P}(S_{n+1} > x)}{\mathbb{P}(S_n > x)} \leq \limsup_{x \rightarrow \infty} \frac{\mathbb{P}(S_{n+1} > x)}{\mathbb{P}(S_n > x)} = \infty. \tag{4.4}$$

By Proposition 5(a), we obtain $F \in \mathcal{J}^{(n+1)} \Rightarrow F \in \mathcal{OS}$. From the identical convolution closure of \mathcal{OS} (see also [18], page 452, Proposition 2.5(iv)), we see that $F \in \mathcal{OS} \Rightarrow F^{n*} \in \mathcal{OS} \Leftrightarrow \limsup_{x \rightarrow \infty} \frac{\mathbb{P}(S_{2n} > x)}{\mathbb{P}(S_n > x)} < \infty$, by (4.4) we obtain a contradiction. \square

4.2. Proof of Proposition 5

Proof of Proposition 5. (a) This was already shown above as part of the proof of Proposition 3.

(b) and (b') Let $\gamma \geq 0$. If $F \in \mathcal{S}(\gamma)$, then $F \in \mathcal{L}(\gamma)$ and, by Lemma 28, $\mathcal{F} \in \mathcal{J}$, so $\mathcal{S}(\gamma) \subseteq \mathcal{J} \cap \mathcal{L}(\gamma)$.

Conversely, let $F \in \mathcal{J} \cap \mathcal{L}(\gamma)$. From $F \in \mathcal{J} \subset \mathcal{OS}$, it follows that F satisfies (4.1) and therefore Lemma 28 implies that $F \in \mathcal{S}(\gamma)$.

(c) We show $\mathcal{D} \subseteq \mathcal{J}$. Let $F \in \mathcal{D}$ and

$$\gamma := \sup_{x \geq 0} \frac{\overline{F}(x/2)}{\overline{F}(x)}.$$

Let $\varepsilon > 0$. There exists $K_0 > 0$ such that for all $K \geq K_0$

$$\mathbb{P}(X_2 > K) \gamma < \frac{\varepsilon}{2}.$$

For $K \geq K_0$ and x such that $x \geq 2K$, we get

$$\begin{aligned} \mathbb{P}(X_1 \wedge X_2 > K | S_2 > x) &\leq 2\mathbb{P}\left(X_1 > \frac{x}{2}, X_2 > K | S_2 > x\right) \\ &\leq 2\mathbb{P}\left(X_1 > \frac{x}{2}\right) \frac{\mathbb{P}(X_2 > K)}{\mathbb{P}(S_2 > x)} \\ &\leq 2\mathbb{P}(X_2 > K) \gamma < \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the assertion follows. \square

4.3. Proof of Proposition 8

Next we prove Proposition 8, which establishes tail closure property of \mathcal{J} .

Proof of Proposition 8. Suppose $F \in \mathcal{J}$, $\overline{F} \asymp \overline{G}$ and $G \notin \mathcal{J}$. There exists $h \in \mathcal{G}$ such that

$$\limsup_{x \rightarrow \infty} \mathbb{P}(Y_{2,2} > h(x) | \hat{S}_2 > x) > 0.$$

Thus, we have by the definition of \mathcal{J} :

$$\limsup_{x \rightarrow \infty} \frac{\mathbb{P}(Y_{2,2} > h(x), \hat{S}_2 > x)}{\mathbb{P}(\hat{S}_2 > x)} \frac{\mathbb{P}(S_2 > x)}{\mathbb{P}(X_{2,2} > h(x), S_2 > x)} = \infty.$$

By Lemmas 25 and 26, we get a contradiction. □

4.4. Proof of Proposition 10

We prove the convolution closure properties of the class \mathcal{J} .

Proof of Proposition 10. (a) We prove closure under convolution powers of \mathcal{J} , that is, if $F \in \mathcal{J}$ then $F^{n*} \in \mathcal{J}$. Suppose $F^{n*} \in \mathcal{J}$. We show $\overline{F^{n*}} \asymp \overline{F^{(n+1)*}}$ and hence $F^{(n+1)*} \in \mathcal{J}$. From $S_n \in \mathcal{J} \subset \mathcal{OS}$ we obtain

$$\limsup_{x \rightarrow \infty} \frac{\mathbb{P}(S_{n+1} > x)}{\mathbb{P}(S_n > x)} \frac{\mathbb{P}(S_{2n} > x)}{\mathbb{P}(S_{2n} > x)} \leq c_{F^{n*}} \limsup_{x \rightarrow \infty} \frac{\mathbb{P}(S_{n+1} > x)}{\mathbb{P}(S_{2n} > x)} \leq c_{F^{n*}}.$$

(b) We prove closure under convolution for tail-equivalent random variables from the class \mathcal{J} , that is, if $F \in \mathcal{J}$ and $\overline{F} \asymp \overline{G}$, then $F * G \in \mathcal{J}$. Suppose $F \in \mathcal{J}$, $\overline{F} \asymp \overline{G}$ and $F * G \notin \mathcal{J}$. Then, there exists an $h \in \mathcal{G}$ such that

$$\limsup_{x \rightarrow \infty} \mathbb{P}((X_1 + Y_1) \wedge (X_2 + Y_2) > h(x) | \hat{S}_2 + S_2 > x) > 0. \tag{4.5}$$

By $F \in \mathcal{J}$ and (a) we have that $F^{2*} \in \mathcal{J}$ and it follows by definition that

$$\limsup_{x \rightarrow \infty} \mathbb{P}((X_1 + X_2) \wedge (X_3 + X_4) > h(x) | S_4 > x) = 0. \tag{4.6}$$

Combining (4.5) and (4.6) yields

$$\limsup_{x \rightarrow \infty} \frac{\mathbb{P}((X_1 + Y_1) \wedge (X_2 + Y_2) > h(x), \hat{S}_2 + S_2 > x)}{\mathbb{P}((X_1 + X_2) \wedge (X_3 + X_4) > h(x), S_4 > x)} \frac{\mathbb{P}(S_4 > x)}{\mathbb{P}(\hat{S}_2 + S_2 > x)} = \infty. \tag{4.7}$$

By Lemma 25, we obtain $\overline{(F^{2*}) * (G^{2*})} \asymp \overline{F^{4*}}$, that is, $\limsup_{x \rightarrow \infty} \frac{\mathbb{P}(S_4 > x)}{\mathbb{P}(\hat{S}_2 + S_2 > x)} < \infty$. Hence, by Lemma 26 and (4.7) we get a contradiction.

(c) We show root convolution closure for \mathcal{J} , that is, if $F^{n*} \in \mathcal{J}$ then $F \in \mathcal{J}$. Let $n = 2^m$, $m \in \mathbb{N}$. Suppose $F^{2^m*} \in \mathcal{J}$. Since $\mathcal{J} \subset \mathcal{OS}$ we have $F^{2^m*} \in \mathcal{OS}$ and hence there exists a constant c_{2^m} such that

$$\liminf_{x \rightarrow \infty} \frac{\mathbb{P}(S_{2^{m-1}} > x)}{\mathbb{P}(S_{2^m} > x)} > c_{2^m} > 0.$$

We obtain by definition for all $h \in \mathcal{G}$

$$\begin{aligned} 0 &= \limsup_{x \rightarrow \infty} \mathbb{P}(S_{2^m}^{(1)} \wedge S_{2^m}^{(2)} > h(x) | S_{2^m}^{(1)} + S_{2^m}^{(2)} > x) \\ &\geq c_{2^m} \limsup_{x \rightarrow \infty} \mathbb{P}(S_{2^{m-1}}^{(1)} \wedge S_{2^{m-1}}^{(2)} > h(x) | S_{2^{m-1}}^{(1)} + S_{2^{m-1}}^{(2)} > x). \end{aligned} \tag{4.8}$$

Thus, we have $F^{2^{m-1}*} \in \mathcal{J}$. We repeat the argument leading to (4.8) for $(m - 1)$ times and arrive at

$$\begin{aligned} 0 &\geq c_{2^m} \limsup_{x \rightarrow \infty} \mathbb{P}(S_{2^{m-1}}^{(1)} \wedge S_{2^{m-1}}^{(2)} > h(x) | S_{2^{m-1}}^{(1)} + S_{2^{m-1}}^{(2)} > x) \\ &\geq c_{2^m} \cdot c_{2^{m-1}} \limsup_{x \rightarrow \infty} \mathbb{P}(S_{2^{m-2}}^{(1)} \wedge S_{2^{m-2}}^{(2)} > h(x) | S_{2^{m-2}}^{(1)} + S_{2^{m-2}}^{(2)} > x) \\ &\vdots \\ &\geq c_{2^m} \cdots c_2 \limsup_{x \rightarrow \infty} \mathbb{P}(X_1 \wedge X_2 > h(x) | S_2 > x), \end{aligned} \tag{4.9}$$

which gives $F \in \mathcal{J}$. In case $n \neq 2^m$ for all $m \in \mathbb{N}$, we take $\tilde{m} := \min\{m \in \mathbb{N} : n < 2^m\}$. Denote by $k := 2^{\tilde{m}}$. By the argument in the proof of (a), we know that $F^n \in \mathcal{J} \Rightarrow F^k \in \mathcal{J}$. From (4.9), we obtain $F \in \mathcal{J}$. □

4.5. Proof of Proposition 12

Proof of Proposition 12. (a) The equivalence (i) \Leftrightarrow (iii) follows from (2.5) and Lemma 8. Next, we show the equivalence (i) \Leftrightarrow (ii). Let $(X + Y) \in \mathcal{J}$ (with our usual slight abuse of notation). To show that $(X \vee Y) \in \mathcal{J}$, abbreviate $V_i := X_i \vee Y_i$ for $i \in \{1, 2, 3, 4\}$. Then, for every $g \in \mathcal{G}$ and $x \geq 0$,

$$\begin{aligned} \frac{\mathbb{P}(V_1 \wedge V_2 > g(x), V_1 + V_2 > x)}{\mathbb{P}(V_1 + V_2 > x)} &\leq \frac{\mathbb{P}(X_1 \wedge X_2 > g(x), X_1 + X_2 > x)}{\mathbb{P}(V_1 + V_2 > x)} \\ &\quad + \frac{\mathbb{P}(Y_1 \wedge Y_2 > g(x), Y_1 + Y_2 > x)}{\mathbb{P}(V_1 + V_2 > x)} \\ &\quad + 2 \frac{\mathbb{P}(X_1 \wedge Y_2 > g(x), X_1 + Y_2 > x)}{\mathbb{P}(V_1 + V_2 > x)}. \end{aligned} \tag{4.10}$$

From $X \in \mathcal{J}$, we obtain for the first term on the right-hand side of (4.10):

$$\limsup_{x \rightarrow \infty} \frac{\mathbb{P}(X_1 \wedge X_2 > g(x), S_2 > x)}{\mathbb{P}(V_1 + V_2 > x)} \leq \limsup_{x \rightarrow \infty} \frac{\mathbb{P}(X_1 \wedge X_2 > g(x), S_2 > x)}{\mathbb{P}(S_2 > x)} = 0.$$

Analogously for the second term:

$$\limsup_{x \rightarrow \infty} \frac{\mathbb{P}(Y_1 \wedge Y_2 > g(x), Y_1 + Y_2 > x)}{\mathbb{P}(V_1 + V_2 > x)} = 0.$$

From $(X + Y) \in \mathcal{J}$, we obtain for the third term on the right-hand side of (4.10):

$$\begin{aligned} & \limsup_{x \rightarrow \infty} 2 \frac{\mathbb{P}(X_1 \wedge Y_2 > g(x), X_1 + Y_2 > x)}{\mathbb{P}(V_1 + V_2 > x)} \\ & \leq 2 \limsup_{x \rightarrow \infty} \left(\frac{\mathbb{P}(X_1 \wedge Y_2 > g(x), S_2 + \hat{S}_2 > x)}{\mathbb{P}(S_2 + \hat{S}_2 > x)} \frac{\mathbb{P}(S_2 + \hat{S}_2 > x)}{\mathbb{P}(X_1 + Y_1 > x)} \right) = 0. \end{aligned}$$

Altogether, we arrive at

$$\begin{aligned} & \limsup_{x \rightarrow \infty} \mathbb{P}((X_1 \vee Y_1) \wedge (X_2 \vee Y_2) > g(x) | (X_1 \vee Y_1) + (X_2 \vee Y_2) > x) \\ & = \limsup_{x \rightarrow \infty} \frac{\mathbb{P}(V_1 \wedge V_2 > g(x), V_1 + V_2 > x)}{\mathbb{P}(V_1 + V_2 > x)} = 0 \end{aligned}$$

for all $g \in \mathcal{G}$, that is, by (2.1), $(X \vee Y) \in \mathcal{J}$.

For the opposite implication $(X \vee Y) \in \mathcal{J} \Rightarrow (X + Y) \in \mathcal{J}$ abbreviate $W_i := X_i + Y_i$ for $i \in \{1, 2\}$. From $(X \vee Y) \in \mathcal{J}$ and $V_1 + V_2 \in \mathcal{J} \subset \mathcal{OS}$, we obtain for all $g \in \mathcal{G}$

$$\begin{aligned} & \limsup_{x \rightarrow \infty} \frac{\mathbb{P}(W_1 \wedge W_2 > g(x), W_1 + W_2 > x)}{\mathbb{P}(W_1 + W_2 > x)} \\ & \leq \limsup_{x \rightarrow \infty} \left(\frac{\mathbb{P}((V_1 + V_2) \wedge (V_3 + V_4) > g(x), V_1 + \dots + V_4 > x)}{\mathbb{P}(V_1 + \dots + V_4 > x)} \frac{\mathbb{P}(V_1 + \dots + V_4 > x)}{\mathbb{P}(V_1 + V_2 > x)} \right) \\ & = 0, \end{aligned}$$

hence by (2.1) $(X + Y) \in \mathcal{J}$, which completes the proof of the equivalence (i) \Leftrightarrow (ii).

Next, we prove Proposition 12(b). The proof is analogous to the proof of the same assertion for the class \mathcal{OS} , see [17], Lemma 3.1. Let $L_i := X_i \wedge Y_i$ for $i \in \{1, 2\}$. For all $g \in \mathcal{G}$ we have

$$\begin{aligned} & \frac{\mathbb{P}(L_1 \wedge L_2 > g(x), L_1 + L_2 > x)}{\mathbb{P}(L_1 + L_2 > x)} \\ & \leq \frac{\int_{g(x)}^{\infty} \mathbb{P}(X_1 > (x - y) \vee g(x)) \mathbb{P}(Y_1 > (x - y) \vee g(x)) dF_{L_2}(y)}{\mathbb{P}(L_1 > x)} \end{aligned}$$

$$\begin{aligned} &\leq \frac{\int_{g(x)}^{\infty} \mathbb{P}(X_1 > (x - y) \vee g(x)) \mathbb{P}(Y_1 > (x - y) \vee g(x)) \mathbb{P}(Y_2 \geq y) \, dF_{X_2}(y)}{\mathbb{P}(X_1 > x) \mathbb{P}(Y_1 > x)} \\ &\quad + \frac{\int_{g(x)}^{\infty} \mathbb{P}(X_1 > (x - y) \vee g(x)) \mathbb{P}(Y_1 > (x - y) \vee g(x)) \mathbb{P}(X_2 \geq y) \, dF_{Y_2}(y)}{\mathbb{P}(X_1 > x) \mathbb{P}(Y_1 > x)}. \end{aligned}$$

Using the inequality

$$\mathbb{P}(Y_1 > (x - y) \vee g(x)) \mathbb{P}(Y_2 \geq y) \leq \mathbb{P}(Y_1 + Y_2 > x)$$

we obtain

$$\begin{aligned} &\limsup_{x \rightarrow \infty} \frac{\mathbb{P}(L_1 \wedge L_2 > g(x), L_1 + L_2 > x)}{\mathbb{P}(L_1 + L_2 > x)} \\ &\leq \limsup_{x \rightarrow \infty} \frac{\mathbb{P}(Y_1 + Y_2 > x)}{\mathbb{P}(Y_1 > x)} \frac{\int_{g(x)}^{\infty} \mathbb{P}(X_1 > (x - y) \vee g(x)) \, dF_{X_2}(y)}{\mathbb{P}(X_1 > x)} \\ &\quad + \limsup_{x \rightarrow \infty} \frac{\mathbb{P}(X_1 + X_2 > x)}{\mathbb{P}(X_1 > x)} \frac{\int_{g(x)}^{\infty} \mathbb{P}(Y_1 > (x - y) \vee g(x)) \, dF_{Y_2}(y)}{\mathbb{P}(Y_1 > x)} \\ &= \limsup_{x \rightarrow \infty} \frac{\mathbb{P}(\hat{S}_2 > x)}{\mathbb{P}(Y_1 > x)} \frac{\mathbb{P}(S_2 > x, X_1 \wedge X_2 > g(x))}{\mathbb{P}(X_1 > x)} \\ &\quad + \limsup_{x \rightarrow \infty} \frac{\mathbb{P}(S_2 > x)}{\mathbb{P}(X_1 > x)} \frac{\mathbb{P}(\hat{S}_2 > x, Y_1 \wedge Y_2 > g(x))}{\mathbb{P}(Y_1 > x)} \\ &= 0, \end{aligned}$$

since $F, G \in \mathcal{J} \subset \mathcal{OS}$. The proof is complete. □

4.6. Proofs of Propositions 14 and 16

We begin with Proposition 14.

Proofs of Proposition 14. (a) Suppose $F \in \mathcal{J}$ and $\sum_{k=1}^{\infty} p_k (c_F + \varepsilon - 1)^k < \infty$ for some $\varepsilon > 0$. Recall that we need to show that $\overline{F}_N \asymp \overline{F}$. From Lemma 29 (Kesten’s) and $F \in \mathcal{J} \subset \mathcal{OS}$ we obtain for some suitable $c_1 \in (0, \infty)$ and all $x \geq 0$,

$$\begin{aligned} \overline{F}_N(x) &= \sum_{k=1}^{\infty} p_k \overline{F}^{k*}(x) \leq \sum_{k=1}^{\infty} p_k c_1 (c_F + \varepsilon - 1)^k \overline{F}(x) \\ &= \overline{F}(x) \sum_{k=1}^{\infty} p_k c_1 (c_F + \varepsilon - 1)^k. \end{aligned}$$

Hence, we see that $\limsup_{x \rightarrow \infty} \overline{F_N}(x)/\overline{F}(x) < \infty$. For the lower bound, pick some $k \geq 1$ with $p_k > 0$. Then, for all $x \geq 0$,

$$\overline{F_N}(x) \geq p_k \mathbb{P}(S_k > x) \geq p_k \overline{F}(x).$$

We obtain $\overline{F_N} \asymp \overline{F}$ and therefore $F_N \in \mathcal{J}$.

(b) Suppose $F_N \in \mathcal{J}$ and that $\sum_{k=1}^{\infty} p_k (c_{F_N} + \varepsilon - 1)^k < \infty$ for some $\varepsilon > 0$. Again, we need to prove that $\overline{F_N} \asymp \overline{F}$. To this end, by means of contradiction, suppose that for every integer $n \geq 2$,

$$\liminf_{x \rightarrow \infty} \frac{\overline{F^{n*}}(x)}{\overline{F_N}(x)} = 0.$$

Our proof then splits into two cases:

Case 1: $p_0 = 0$. For every $n \geq 1$, we choose an unbounded and strictly increasing sequence $(x_k^n)_{k \in \mathbb{N}}$ such that for all $n \in \mathbb{N}$

$$\lim_{m \rightarrow \infty} \frac{\overline{F^{n*}}(x_m^n)}{\overline{F_N}(x_m^n)} = 0 \quad \text{and in particular} \quad \lim_{m \rightarrow \infty} \frac{\overline{F^{m*}}(x_m^m)}{\overline{F_N}(x_m^m)} = 0.$$

From Lemma 29 (Kesten's) and $p_0 = 0$ we conclude that, for some suitable $c_2 \in (0, \infty)$, for all $n, m \in \mathbb{N}$

$$\frac{\overline{F^{n*}}(x_m^n)}{\overline{F_N}(x_m^n)} \leq \frac{\overline{F^{n*}}(x_m^n)}{\overline{F_N}(x_m^n)} \leq c_2 (c_{F_N} + \varepsilon - 1)^n.$$

Since by assumption the right-hand side is summable in n , we can use the dominated convergence theorem to arrive at the desired contradiction:

$$\begin{aligned} 1 &= \lim_{m \rightarrow \infty} \frac{\overline{F_N}(x_m^m)}{\overline{F_N}(x_m^m)} = \lim_{m \rightarrow \infty} \sum_{k=1}^{\infty} p_k \frac{\overline{F^{k*}}(x_m^m)}{\overline{F_N}(x_m^m)} \\ &= \sum_{k=1}^{\infty} p_k \lim_{m \rightarrow \infty} \frac{\overline{F^{k*}}(x_m^m)}{\overline{F_N}(x_m^m)} \leq \sum_{k=1}^{\infty} p_k \lim_{m \rightarrow \infty} \frac{\overline{F^{m*}}(x_m^m)}{\overline{F_N}(x_m^m)} = 0. \end{aligned}$$

Case 2: $p_0 > 0$. This can be reduced to Case 1 by switching to the reweighted random variable \hat{N} with probabilities

$$\hat{p}_n := \mathbb{P}(\hat{N} = n) := \frac{p_n}{1 - p_0},$$

for $n > 0$ and $\hat{p}_0 = \mathbb{P}(\hat{N} = 0) := 0$. Thanks to Case 1 we have that $F_{\hat{N}} \in \mathcal{J}$. Further, observe that

$$\lim_{x \rightarrow \infty} \frac{\overline{F_N}(x)}{\overline{F_{\hat{N}}}(x)} = \lim_{x \rightarrow \infty} \frac{\sum_{n=0}^{\infty} p_n \overline{F^{n*}}(x)}{\sum_{n=0}^{\infty} \hat{p}_n \overline{F^{n*}}(x)} = \frac{1 - p_0}{1}.$$

From Proposition 8 and $\overline{F_N} \asymp \overline{F_{\hat{N}}}$, we conclude that $F_N \in \mathcal{J}$. □

Next, we prove Proposition 16 using the arguments of the proof of Theorem 1.5 of Watanabe [21].

Proof of Proposition 16. (a) To infer that $\overline{F_N} \asymp \overline{F^{m^*}}$ for some $m \in \mathbb{N}$, we again argue by contradiction. So suppose that for every integer $m \geq 2$

$$\liminf_{x \rightarrow \infty} \frac{\overline{F^{m^*}}(x)}{\overline{F_N}(x)} = 0.$$

From $F_N \in \mathcal{OS}$, we know that $c_{F_N} < \infty$ and from our assumption $\liminf_{n \rightarrow \infty} \frac{\mathbb{P}(N_1 + N_2 > n)}{\mathbb{P}(N_1 > n)} > c_{F_N}$ we infer that there exists a $\delta > 0$ and an integer $m_0 = m_0(\delta)$ such that, for every $k \geq m_0 + 1$,

$$\frac{\mathbb{P}(N_1 + N_2 > k)}{\mathbb{P}(N_1 > k)} > c_{F_N} + \delta. \tag{4.11}$$

Let $(x_n)_{n \in \mathbb{N}}$ be a strictly increasing sequence with $\lim_{n \rightarrow \infty} x_n = \infty$ such that

$$\lim_{n \rightarrow \infty} \frac{\overline{F^{m_0^*}}(x_n)}{\overline{F_N}(x_n)} = 0. \tag{4.12}$$

Since $\overline{F^{m_0^*}}(x) \geq \overline{F^{k^*}}(x)$ for $1 \leq k \leq m_0$, we have

$$\lim_{n \rightarrow \infty} \frac{\overline{F^{k^*}}(x_n)}{\overline{F_N}(x_n)} = 0$$

for $1 \leq k \leq m_0$. As in [21], define $I_j(n)$ and $J_j(n)$ for $j = 1, 2$ as

$$\begin{aligned} J_1(n) &= \sum_{k=0}^{m_0} (p * p)_k \overline{F^{k^*}}(x_n), & I_1(n) &= \sum_{k=0}^{m_0} p_k \overline{F^{k^*}}(x_n), \\ J_2(n) &= \sum_{k=m_0+1}^{\infty} (p * p)_k \overline{F^{k^*}}(x_n), & I_2(n) &= \sum_{k=m_0+1}^{\infty} p_k \overline{F^{k^*}}(x_n). \end{aligned}$$

We see from (4.12) that

$$\lim_{n \rightarrow \infty} \frac{I_1(n)}{\overline{F_N}(x_n)} = \lim_{n \rightarrow \infty} \frac{J_1(n)}{\overline{F_N}(x_n)} = 0, \tag{4.13}$$

and since $F_N^{2^*} = \sum_{k=0}^{\infty} (p * p)_k F^{k^*}$, (4.11) and (4.13) give

$$c_{F_N} \geq \limsup_{n \rightarrow \infty} \frac{\overline{F_N^{2^*}}(x_n)}{\overline{F_N}(x_n)} = \limsup_{n \rightarrow \infty} \frac{(J_1(n) + J_2(n))/\overline{F_N}(x_n)}{(I_1(n) + I_2(n))/\overline{F_N}(x_n)} = \limsup_{n \rightarrow \infty} \frac{J_2(n)}{I_2(n)}.$$

To arrive at the desired contradiction, define $h_{m_0+1}(x_n) := \overline{F^{(m_0+1)*}}(x_n)$ and $h_j(x_n) := \overline{F^{j*}}(x_n) - \overline{F^{(j-1)*}}(x_n)$ for $j > m_0 + 1$. We obtain

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{J_2(n)}{I_2(n)} &= \limsup_{n \rightarrow \infty} \frac{\sum_{k=m_0+1}^{\infty} (p * p)_k \sum_{j=m_0+1}^k h_j(x_n)}{\sum_{k=m_0+1}^{\infty} p_k \sum_{j=m_0+1}^k h_j(x_n)} \\ &= \limsup_{n \rightarrow \infty} \frac{\sum_{j=m_0+1}^{\infty} h_j(x_n) \sum_{k=j}^{\infty} \mathbb{P}(N_1 + N_2 = k)}{\sum_{j=m_0+1}^{\infty} h_j(x_n) \sum_{k=j}^{\infty} \mathbb{P}(N_1 = k)} \\ &= \limsup_{n \rightarrow \infty} \frac{\sum_{j=m_0+1}^{\infty} h_j(x_n) \mathbb{P}(N_1 + N_2 > j - 1)}{\sum_{j=m_0+1}^{\infty} h_j(x_n) \mathbb{P}(N_1 > j - 1)} \\ &> c_{F_N} + \delta. \end{aligned}$$

This is a contradiction. Since $\overline{F^{m*}}(x) \leq \overline{F_N}(x) \frac{1}{p_m}$ with $p_m > 0$ for sufficiently large integers m , it follows that $\overline{F_N} \asymp \overline{F^{m*}}$.

(b) The assertion follows from (a), Proposition 10 and $\mathcal{J} \subset \mathcal{OS}$. □

4.7. Proof of Theorem 19

We prepare the proof by recalling two results due to Yang and Wang [24] for reference.

Lemma 30 ([24], Lemma 2.2). *With the notation of Theorem 18, if $a < 0$ and $F_I \in \mathcal{OL}$ then $\overline{G} \asymp \overline{F_I}$.*

Theorem 31 ([24], Theorem 1.4). *Let F be a distribution function on $(-\infty, \infty)$ such that \overline{F} is integrable and $F_I \in \mathcal{OS} \cap \mathcal{DK}$. Further, let α and β be two fixed positive constants. Consider any sequence $\{X_i : i \geq 1\}$ of independent random variables such that, for each $i \geq 1$, the distribution F_i of X_i satisfies the conditions*

$$\overline{F_i}(x) \leq \overline{F}(x), \quad \text{for all } x \in (-\infty, \infty) \quad \text{and} \quad \int_{-\infty}^{\infty} (y \vee -\beta) dF_i(y) \leq -\alpha.$$

Then there exists a positive constant r , depending only on F , α and β , such that for all sequences $\{X_i : i \geq 1\}$ as above,

$$\overline{F_M}(x) \leq r \overline{F_I}(x)$$

for all $x \in (-\infty, \infty)$.

Now the proof of the P–V–E theorem for class \mathcal{J} can simply be reduced to previously stated results.

Proof of Theorem 19. Since by assumption $F_I \in \mathcal{OL}$ and $a := \mathbb{E}[\bar{X}_k] < 0$ we obtain from Lemma 30 that $\bar{F}_I \asymp \bar{G}$. Hence, the equivalence of (1) and (2) follows from the weak tail-equivalence closure of the class \mathcal{J} (Proposition 8).

Now additionally assume $p(c_G + \varepsilon - 1) < 1$ holds for some $\varepsilon > 0$ (condition (i)). As we know from (3.1), we can write F_M as a random sum

$$\bar{F}_M = (1 - p) \sum_{n=0}^{\infty} p^n \bar{G}^{n*}(x). \quad (4.14)$$

Hence, we obtain $\bar{F}_M \asymp \bar{G}$ by application of Proposition 14. Now, applying the weak tail-equivalence closure of the class \mathcal{J} we conclude the equivalence of (2) and (3).

Next, assume additionally that $F_M \in \mathcal{OS}$ (condition (ii)) holds. Again, by using the expression (4.14) and Proposition 16(b) we obtain $\bar{F}_M \asymp \bar{G}$ and hence the equivalence of (2) and (3).

Finally, under condition $F_I \in \mathcal{J} \cap \mathcal{DK}$ we can use Theorem 31. By choosing $F_i = F_{\bar{X}}$ it is easy to see that we can find appropriate constants α, β such that $\int_{-\infty}^{\infty} (y \vee -\beta) dF_{\bar{X}}(y) \leq -\alpha$ holds. Hence, there exists a constant r such that $\bar{F}_M(x) \leq r \bar{F}_I(x)$ for all $x \in (-\infty, \infty)$. By (a) of Theorem 21 we obtain $\bar{F}_M \asymp \bar{F}_I$ and hence $F_M, G \in \mathcal{J}$. Now, $\bar{F}_M \asymp \bar{G}$ follows from $\bar{F}_I \asymp \bar{G}$ and $\bar{F}_M \asymp \bar{F}_I$. \square

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