Weighted power variation of integrals with respect to a Gaussian process

RIMAS NORVAIŠA

Institute of Mathematics and Informatics, Vilnius University, Akademijos 4, Vilnius, Lithuania.
E-mail: rimas.norvaisa@mii.vu.lt

We consider a stochastic process Y defined by an integral in quadratic mean of a deterministic function f with respect to a Gaussian process X, which need not have stationary increments. For a class of Gaussian processes X, it is proved that sums of properly weighted powers of increments of Y over a sequence of partitions of a time interval converge almost surely. The conditions of this result are expressed in terms of the p-variation of the covariance function of X. In particular, the result holds when X is a fractional Brownian motion, a subfractional Brownian motion and a bifractional Brownian motion.

Keywords: covariance; double Riemann–Stieltjes integral; Gaussian process; locally stationary increments; Orey index; power variation; p-variation; quadratic mean integral

1. Introduction

Let $X = \{X(t) : t \in [0, T]\}$ be a Gaussian process and let $f : [0, T] \to \mathbb{R}$ be a real-valued function for some $0 < T < \infty$. We consider a stochastic process $Y = \{Y(t) : t \in [0, T]\}$, given by an integral

$$Y(t) = \text{q.m.} \int_0^t f \, dX, \qquad 0 \le t \le T, \tag{1}$$

defined as a limit of Riemann–Stieltjes sums converging in *quadratic mean*. According to the main result of this paper (Theorem 21), under suitable hypotheses on the covariance of X and the p-variation of f, there exists a stochastic process Y defined by (1) and with probability one

$$\lim_{n \to \infty} \sum_{i=1}^{m_n} \frac{|Y(t_i^n) - Y(t_{i-1}^n)|^r}{[\rho(t_i^n - t_{i-1}^n)]^r} (t_i^n - t_{i-1}^n) = E|\eta|^r \int_0^T |f|^r, \tag{2}$$

where η is a standard normal random variable, ρ is a function equivalent to $(E[X(s + \cdot) - X(s)]^2)^{1/2}$ near zero uniformly in $s \in [\varepsilon, T)$ for each $\varepsilon > 0$, and $((t_i^n)_{i=0}^{m_n})$ is a sequence of partitions of $[0, T]$ such that the sequence $(\max_i (t_i^n - t_{i-1}^n))$ tends to zero as $n \to \infty$ sufficiently fast.

In the case $f \equiv 1$, X is a real centered Brownian motion B, $\rho(h) = \sqrt{h}$, $T = 1$ and $t_i^n = i2^{-n}$, $i \in \{0, 1, \ldots, 2^n\}$, (2) gives the result of Lévy [15]

$$\lim_{n \to \infty} \sum_{i=1}^{2^n} \left[B(i2^{-n}) - B((i-1)2^{-n}) \right]^2 = 1 \qquad \text{a.s.} \tag{3}$$

This result has been extended in many directions. The Brownian motion B has been replaced by a general Gaussian procces X under suitable hypotheses on the covariance of X ([1,9,20]). The sequence of dyadic partitions of $[0, 1]$ in (3) have been replaced by a sequence $((t_i^n)_{i=0}^{m_n})$ of partitions of $[0, 1]$ such that $\max_i (t_i^n - t_{i-1}^n) = o(1/\log n)$ as $n \to \infty$ ([6]), and this is the best possible rate ([5]). The second power of increments of a Brownian motion B in (3) has been replaced by rth power of increments of a Gaussian process X with stationary increments by Marcus and Rosen [19] when $r \geq 2$ and by Shao [23] when $r > 1$. A different strand of research led to similar results for a power variation of a general stochastic process with convergence in probability in place of convergence with probability one (see [4], and references there). In fact, the present paper is an attempt to prove the almost sure convergence for a weighted power variation of integral process (1) like in [4] (but different) and keeping the framework of the above mentioned results.

In the rest of this section, we formulate and discuss in more detail the stated main result of the paper. In Section 2, we give conditions for the existence of the integral process Y given by (1). In Section 3, we give conditions for a Gaussian process to have positively or negatively correlated increments in terms of p-variation of its covariance function. The general results are proved to hold for a fractional Brownian motion, a subfractional Brownian motion and a bifractional Brownian motion. The main result is proved in Section 4.

We consider the integral process Y given by (1) with a Gaussian process X having "locally stationary increments" defined next (Definition 2). While in this paper we consider only Gaussian processes, the following concept makes sense for any stochastic process whose finite-dimensional distributions have finite moments of the first and second orders. Such a process with mean zero will be called a second order stochastic process as in [16], Chapter 37.

Definition 1. *Let $T > 0$ and let $R[0, T]$ be a set of functions $\rho : [0, T] \to \mathbb{R}_+$ such that $\rho(0) = 0$, ρ is continuous at zero, and for each $\delta \in (0, T)$,*

$$0 < \inf\{\rho(u) : u \in [\delta, T]\} \leq \sup\{\rho(u) : u \in [\delta, T]\} < \infty. \tag{4}$$

Let $X = \{X(t) : t \in [0, T]\}$ be a second order stochastic process with the incremental variance function σ_X^2 defined on $[0, T]^2 := [0, T] \times [0, T]$ with values

$$\sigma_X^2(s, t) := E[X(t) - X(s)]^2, \qquad (s, t) \in [0, T]^2.$$

We say that X has a local variance *if there is a function $\rho \in R[0, T]$ such that* (A1) *and* (A2) *hold, where*

(A1) *there is a finite constant L such that for all $(s, t) \in [0, T]^2$*

$$\sigma_X(s, t) \leq L\rho(|t - s|);$$

(A2) *for each $\varepsilon \in (0, T)$*

$$\limsup_{\delta \downarrow 0}\left\{\left|\frac{\sigma_X(s, s + h)}{\rho(h)} - 1\right| : s \in [\varepsilon, T), h \in (0, \delta \wedge (T - s)]\right\} = 0. \tag{5}$$

In this case, we say that X has a local variance with $\rho \in R[0, T]$.

Let $X = \{X(t) : t \in [0, T]\}$, $T > 0$, be a mean zero Gaussian process with stationary incre-
ments, and let $\rho_X(u) := \sigma_X(u, 0)$ for each $u \in [0, T]$. Then (A1) and (A2) for $\rho = \rho_X$ hold
trivially. Also, if X is such that ρ_X is continuous at zero, and (4) holds for $\rho = \rho_X$ and each
$\delta \in (0, T)$, then $\rho_X \in R[0, T]$, and so X has a local variance with ρ_X.

Suppose X is a second order stochastic process such that $\sigma_X(s, t) \neq 0$ for each $(s, t) \in [0, T]^2$.
If X has a local variance with two elements ρ_1 and ρ_2 in $R[0, T]$, then by (A2) we have

$$\lim_{u \downarrow 0} \rho_1(u)/\rho_2(u) = 1.$$

This property defines a binary relation in the set $R[0, T]$, which is an equivalence relation. Let
us denote this relation by $\sim$. If X has a local variance with $\rho_1 \in R[0, T]$, and if $\rho_2 \in R[0, T]$ is
such that $\rho_1 \sim \rho_2$, then X has a local variance with ρ_2. Therefore, the property of X having a
local variance is a class invariant under the binary relation $\sim$.

Definition 2. *Let X be a second order stochastic process. We say that X has* locally stationary
increments *if X has a local variance with some $\rho \in R[0, T]$. Any element in the equivalence
class $\{\rho' \in R[0, T] : \rho' \sim \rho\}$ will be called a* local variance function. *We write $X \in \mathcal{LSI}(\rho(\cdot))$ if
X has local variance with $\rho \in R[0, T]$.*

So far as we are aware, a similar concept was suggested by [2], Section 8, under the name
of *local stationarity*. We show that a subfractional Brownian motion $G_H = \{G_H(t) : t \in [0, T]\}$
with index $H \in (0, 1)$ and covariance function (26) has local variance function $\rho_H(u) = u^H$,
$u \in [0, T]$ (see Proposition 17). Also, we show that a bifractional Brownian motion $B_{H,K} =
\{B_{H,K}(t) : t \in [0, T]\}$ with parameters $(H, K) \in (0, 1) \times (0, 1)$ and covariance function (30) has
local variance function $\rho_{H,K}(u) = 2^{(1-K)/2}u^{HK}$, $u \in [0, T]$ (see Proposition 18).

For $\rho \in R[0, T]$ let

$$\gamma_*(\rho) := \inf\{\gamma > 0 : u^\gamma/\rho(u) \to 0 \text{ as } u \downarrow 0\} = \limsup_{u \downarrow 0} \frac{\log \rho(u)}{\log u}$$

and

$$\gamma^*(\rho) := \sup\{\gamma > 0 : u^\gamma/\rho(u) \to +\infty \text{ as } u \downarrow 0\} = \liminf_{u \downarrow 0} \frac{\log \rho(u)}{\log u}.$$

By definition, we have $0 \leq \gamma^*(\rho) \leq \gamma_*(\rho) \leq \infty$. Clearly, $\gamma_*(\rho)$ and $\gamma^*(\rho)$ do not change when
ρ is replaced by $\rho' \sim \rho$. If a second order stochastic process X has a local variance with $\rho \in
R[0, T]$ and if $0 < \gamma^*(\rho) = \gamma_*(\rho) < \infty$ then we will say that X has the *Orey index* $\gamma_X :=
\gamma^*(\rho) = \gamma_*(\rho)$. Clearly this notion extends the one suggested by Orey [22] (see also [21]) for a
Gaussian stochastic process with stationary increments. We will be interested in the case in which
a Gaussian stochastic process $X \in \mathcal{LSI}(\rho(\cdot))$ has the Orey index $\gamma_X \in (0, 1)$. In this case, X is
equivalent to a stochastic process whose almost all sample functions satisfy a Hölder condition
of order α for each $\alpha < \gamma_X$.

Now we can formulate the main result of the paper with more details. Suppose that a mean zero
Gaussian process X has locally stationary increments with a local variance ρ and has the Orey

index $\gamma_X(\rho) = \gamma \in (0, 1)$. Suppose that a function $f : [0, T] \to \mathbb{R}$ is regulated if $\gamma \geq 1/2$ and has bounded q-variation for some $q < 1/(1 - 2\gamma)$ if $\gamma < 1/2$, and let $1 < r < 2/\max\{(2\gamma - 1), 0\}$. Under the further hypotheses of Theorem 21 on the covariance of X, a stochastic process Y defined by (1) exists, and (2) holds with probability one. The proof of the main result (Theorem 21) use the ideas of Marcus and Rosen [19] and Shao [23].

Gladyshev [9] considered a stochastic process $X = \{X(t) : t \in [0, 1]\}$ with Gaussian increments, mean zero and a covariance function Γ_X such that the expression

$$\sigma_X^2(t, t - h)/h^{2\gamma} = \left[\Gamma_X(t, t) - 2\Gamma_X(t, t - h) + \Gamma_X(t - h, t - h)\right]/h^{2\gamma} \tag{6}$$

converges uniformly to a function g on $[0, 1]$ as $h \to 0$, Γ_X is continuous, twice differentiable outside the diagonal and

$$\left|\frac{\partial^2 \Gamma_X(t, s)}{\partial t\, \partial s}\right| \leq \frac{C}{|t - s|^{2(1-\gamma)}} \tag{7}$$

(here $\gamma = 1 - \tilde{\gamma}/2$ for $\tilde{\gamma}$ in [9]). Under these assumptions, E. G. Gladyshev proved (2) with the right-hand side replaced by $\int_0^1 g$ when $f \equiv 1$, $\rho(u) = u^\gamma$, $r = 2$, $t_i^n = i2^{-n}$ for $i \in \{1, \ldots, 2^n\}$ and each $n \geq 1$. In [20], we showed that hypothesis (6) does not hold when X is a subfractional Brownian motion and a bifractional Brownian motion, but the conclusion of Theorem 1 in [9] (with $g \equiv 1$) still holds for these processes. Malukas [17] further extended this result to arbitrary sequences of partitions using the ideas of Klein and Giné [12], and proved a central limit theorem in his setting.

As compared to previous results, in the present paper a class of Gaussian processes is defined by conditions (A1) and (A2) which seem to fit perfectly Gladyshev's theorem for the mean convergence (see Corollary 20 below), and are weaker than hypothesis (6) with $g \equiv 1$. Instead of hypothesis (7), we use the following assumption on a Gaussian process X having locally stationary increments and the Orey index $\gamma \in (0, 1)$: there is a constant C_2 such that the inequality

$$\sum_{j=1}^m \left|E\left[X(t_i) - X(t_{i-1})\right]\left[X(t_j) - X(t_{j-1})\right]\right| \leq C_2(t_i - t_{i-1})^{1 \wedge (2\gamma)}$$

holds for each partition $(t_j)_{j \in \{0,\ldots,m\}}$ of $[0, T]$ and each $i \in \{1, \ldots, m\}$ (see Corollary 23). Finally, in place of X, we consider a stochastic process Y defined by (1). In this case the preceding assumption on X is replaced by the following one: there is a constant C_2 such that the inequality

$$\sum_{j=1}^m V_p\left(\Gamma_X; [t_{i-1}, t_i] \times [t_{j-1}, t_j]\right) \leq C_2(t_i - t_{i-1})^{1 \wedge (2\gamma)}$$

holds for each partition $(t_j)_{j \in \{0,\ldots,m\}}$ of $[0, T]$ and each $i \in \{1, \ldots, m\}$, where $V_p(\cdot)$ is the p-variation seminorm defined by (11) below and $p = \max\{1, 1/(2\gamma)\}$ (see Theorem 21). The two assumptions are shown to be easily verified using the properties of negative or positive correlation of X (see Section 3).

The following is a consequence of Theorem 21, Proposition 15 when $K = 1$ and Proposition 18 when $K \in (0, 1)$.

Corollary 3. *Let $T > 0$, $H \in (0, 1)$, $K \in (0, 1]$, $r \in (1, 2/\max\{(2HK - 1), 0\})$, and let $B_{H,K} = \{B_{H,K}(t) : t \in [0, T]\}$ be a bifractional Brownian motion with parameters (H, K). Let $f : [0, T] \to \mathbb{R}$ be regulated if $HK \geq 1/2$ and of bounded q-variation for some $q < 1/(1 - 2HK)$ if $HK < 1/2$. Let (κ_n) be a sequence of partitions $\kappa_n = (t_i^n)_{i \in \{0, \dots, m_n\}}$ of $[0, T]$ such that*

$$\lim_{n \to \infty} |\kappa_n|^{(1 \wedge 2/r) + (0 \wedge (1 - 2HK))} \log n = 0.$$

Then with probability one

$$\lim_{n \to \infty} \sum_{i=1}^{m_n} \left| \text{q.m.} \int_{t_{i-1}^n}^{t_i^n} f \, dB_{H,K} \right|^r \left(t_i^n - t_{i-1}^n\right)^{1 - rHK} = 2^{r(1-K)/2} E|\eta|^r \int_0^T |f|^r,$$

where η is a standard normal random variable.

A similar result holds for a subfractional Brownian motion due to Theorem 21 and Proposition 17. Corollary 3 when $K = 1$ (the case of fractional Brownian motion B_H) may be compared with Theorem 1 of [4] where f is a stochastic process, the integral $\int_0^t f \, dB_H$, $t \in [0, 1]$, is defined pathwise as the Riemann–Stieltjes integral, partition $\kappa_n = (i/n)_{i \in \{0, \dots, n\}}$, convergence holds in probability and with no restrictions on r.

Notation. For $n \in \mathbb{N} := \{0, 1, \dots\}$ let $[n] := \{0, 1, \dots, n\}$ and $(n) := \{1, \dots, n\}$. An interval $[a, b]$ is a closed set of real numbers r such that $a \leq r \leq b$. A partition of an interval $[a, b]$ is a finite sequence of real numbers $\kappa = (t_i)_{i \in [n]}$ such that $a = t_0 < t_1 < \cdots < t_n = b$. The set of all partitions of $[a, b]$ is denoted by $\Pi[a, b]$. Given a partition $\kappa = (t_i)_{i \in [n]}$, for each $i \in (n)$, let $J_i^\kappa := [t_{i-1}, t_i]$ and $\Delta_i^\kappa := t_i - t_{i-1}$. The mesh of a partition κ is $|\kappa| := \max_i \Delta_i^\kappa$. Given a function $g : [a, b] \to \mathbb{R}$ and a sequence (κ_n) of partitions $\kappa_n = (t_i^n)_{i \in [m_n]}$ of $[a, b]$, for each $i \in (m_n)$, let $\Delta_i^n := \Delta_i^{\kappa_n} = t_i^n - t_{i-1}^n$ and $\Delta_i^n g := g(t_i^n) - g(t_{i-1}^n)$.

2. Riemann–Stieltjes integrals

In this section, the double Riemann–Stieltjes integral and the quadratic mean Riemann–Stieltjes integral are defined, and several their properties to be used are given.

A double Riemann–Stieltjes integral

Let F and G be real-valued functions defined on a rectangle $R := [a, b] \times [c, d]$ in $\mathbb{R}^2$ defined by real numbers $a < b$ and $c < d$. We recall a definition of the Riemann–Stieltjes integral of F with respect to G over R. A partition of $[a, b] \times [c, d]$ is a finite double sequence of pairs of real numbers $\tau = \{(s_i, t_j) : (i, j) \in [n] \times [m]\}$ such that $(s_i)_{i \in [n]} \in \Pi[a, b]$ and $(t_j)_{j \in [m]} \in \Pi[c, d]$.

The set of all partitions of a rectangle R is denoted by $\Pi(R)$. Thus $\tau \in \Pi(R)$ if and only if $\tau = \kappa \times \lambda$ for some $\kappa \in \Pi[a, b]$ and $\lambda \in \Pi[c, d]$. The *mesh* of $\tau = \kappa \times \lambda \in \Pi(R)$ is $|\tau| := \max\{|\kappa|, |\lambda|\}$. Given such τ, for each $i \in (n)$ and $j \in (m)$, the double increment of G over the rectangle $Q_{i,j} = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$ is defined by

$$\Delta_{i,j}^{\tau} G := \Delta^{Q_{i,j}} G := G(s_i, t_j) - G(s_{i-1}, t_j) - G(s_i, t_{j-1}) + G(s_{i-1}, t_{j-1}). \tag{8}$$

Also if $(u_i, v_j) \in Q_{i,j}$ for $(i, j) \in (n] \times (m]$, then (u_i, v_j) is called a *tag* and the collection $\dot{\tau} := \{((u_i, v_j), Q_{i,j}) : (i, j) \in (n] \times (m]\}$ is called a *tagged partition* of R. The *Riemann–Stieltjes sum* of F with respect to G and based on a tagged partition $\dot{\tau}$ is

$$S_{\mathrm{RS}}(F, \Delta^2 G; \dot{\tau}) := \sum_{i=1}^{n} \sum_{j=1}^{m} F(u_i, v_j) \Delta_{i,j}^{\tau} G.$$

We say that the *double Riemann–Stieltjes* integral over $[a, b] \times [c, d]$ of F with respect to G exists and equals $A \in \mathbb{R}$, if for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\left| S_{\mathrm{RS}}(F, \Delta^2 G; \dot{\tau}) - A \right| < \varepsilon$$

for each tagged partition $\dot{\tau}$ of $[a, b] \times [c, d]$ with the mesh $|\tau| < \delta$. Clearly, if such A exists then it is unique and is denoted by

$$\int_a^b \int_c^d F \, \mathrm{d}^2 G = \int_a^b \int_c^d F(s, t) \, \mathrm{d}^2 G(s, t) := A.$$

Since in this paper we work with the quadratic mean Riemann–Stieltjes integral $\int f \, \mathrm{d}X$ of a deterministic function f it is enough to treat double Riemann–Stieltjes integral for integrands $F = f \otimes f$, where $f \otimes f(s, t) = f(s) f(t)$ for $(s, t) \in R$.

First, we give sufficient conditions for the existence of a double Riemann–Stieltjes when the integrator has bounded total variation. Let $R = [a, b] \times [c, d]$ be a rectangle and $G : R \to \mathbb{R}$. For a partition $\tau = \{(s_i, t_j) : (i, j) \in [n] \times [m]\} \in \Pi(R)$ let

$$s_1(G; \tau) := \sum_{i=1}^{n} \sum_{j=1}^{m} |\Delta_{i,j}^{\tau} G|,$$

where $\Delta_{i,j}^{\tau} G$ is defined by (8).

Definition 4. *Let R be a rectangle in $\mathbb{R}^2$ and $G : R \to \mathbb{R}$.*

$$V_1(G; R) := \sup\{s_1(G; \tau) : \tau \in \Pi(R)\}.$$

If $V_1(G, R) < \infty$ then one says that G is of bounded variation in the sense of Vitali–Lebesgue–Fréchet–de la Vallée Poussin and write $G \in \mathcal{W}_1(R)$.

We say that a function $G : R \to \mathbb{R}$ is *separately continuous* if its sections $x \mapsto G(x, y)$ and $y \mapsto G(x, y)$ are continuous for each fixed y and x, respectively. A function $f : [a, b] \to \mathbb{R}$ is regulated if for each $x \in (a, b]$ it has left limits $f(x-)$ and for each $x \in [a, b)$ it has right limits $f(x+)$. The set of all regulated functions on $[a, b]$ is denoted by $\mathcal{W}_\infty[a, b]$. Each regulated function is bounded and for such a function f we write

$$\|f\|_{\sup} := \sup\{|f(x)| : x \in [a, b]\}, \qquad Osc(f) := \sup\{|f(x) - f(y)| : x, y \in [a, b]\}.$$

Theorem 5. *Let $R = [a, b] \times [c, d]$ for some real numbers $a < b$ and $c < d$. Let $G \in \mathcal{W}_1(R)$ be separately continuous, $f \in \mathcal{W}_\infty[a, b]$ and $g \in \mathcal{W}_\infty[c, d]$. Then the double Riemann–Stieltjes integral $\int_a^b \int_c^d f \otimes g \, \mathrm{d}^2 G$ is defined and we have the bounds*

$$\left| \int_a^b \int_c^d f \otimes g \, \mathrm{d}^2 G \right| \leq \|f\|_{\sup} \|g\|_{\sup} V_1(G; R), \tag{9}$$

$$\left| \int_a^b \int_c^d \left[f \otimes g - f(a)g(c) \right] \mathrm{d}^2 G \right| \leq \left[\|g\|_{\sup} Osc(f) + \|f\|_{\sup} Osc(g) \right] V_1(G; R). \tag{10}$$

The proof is standard for such statements about existence of Riemann–Stieltjes integrals when the integrand is a regulated function and the integrator has bounded variation (see, e.g., Theorem 2.17 in [7] when functions have single variable). Namely, one needs to compare a difference between two Riemann–Stieltjes sums corresponding to sufficiently fine partitions and one of them is a refinement of the other. The sum of terms corresponding to subrectangles containing a jump of either f or g can be made small due to separate continuity of G and since G is a difference of two quasi-monotone functions as shown in [10], page 345. The details are omitted.

Next, we give sufficient conditions for the existence of a double Riemann–Stieltjes integral in terms of p-variation of the integrand and integrator. Let $p \geq 1$ and let $f : [a, b] \to \mathbb{R}$. For a partition $\kappa = (s_i)_{i \in [n]}$ of $[a, b]$, let

$$s_p(f; \kappa) := \sum_{i=1}^n |f(s_i) - f(s_{i-1})|^p.$$

The p-variation seminorm of f on $[a, b]$ is the quantity

$$V_p(f) = V_p(f; [a, b]) := \sup\{[s_p(f; \kappa)]^{1/p} : \kappa \in \Pi[a, b]\}.$$

One says that f has bounded p-variation or $f \in \mathcal{W}_p[a, b]$ if $V_p(f; [a, b]) < \infty$. We also use the p-variation norm defined by

$$\|f\|_{[p]} = \|f\|_{[p], [a,b]} := \|f\|_{\sup} + V_p(f; [a, b]).$$

Recalling that $\mathcal{W}_\infty[a, b]$ is the set of regulated functions on $[a, b]$, $\mathcal{W}_p[a, b]$ is defined for $1 \leq p \leq \infty$.

Again, let $R = [a, b] \times [c, d]$ be a rectangular and $G : R \to \mathbb{R}$. For $p \geq 1$ and a partition $\tau = \{(s_i, t_j) : (i, j) \in [n] \times [m]\}$ of R let

$$s_p(G; \tau) := \sum_{i=1}^{n} \sum_{j=1}^{m} |\Delta_{i,j}^{\tau} G|^p,$$

where $\Delta_{i,j}^{\tau} G$ is defined by (8). The p-variation seminorm of G is

$$V_p(G; R) := \sup\{[s_p(G; \tau)]^{1/p} : \tau \in \Pi(R)\}. \tag{11}$$

Let $\mathcal{W}_p(R)$ be the set of all functions $G : R \to \mathbb{R}$ such that $V_p(G; R)$ is bounded, which extends Definition 4 when $p = 1$.

The following is an elaboration on the statements 3.7(ii) and 4.3 of [14]. In the present case, we do not assume $f(a) = 0$ and $g(c) = 0$.

Theorem 6. *Let $R = [a, b] \times [c, d]$ for some real numbers $a < b$ and $c < d$. Let $p > 1$ and $q > 1$ be such that $p^{-1} + q^{-1} > 1$. Let $f \in \mathcal{W}_q[a, b], g \in \mathcal{W}_q[c, d]$ and let $G \in \mathcal{W}_p(R)$ be continuous. There exists the double Riemann–Stieltjes integral $\int_a^b \int_c^d f \otimes g \, d^2 G$ and*

$$\left| \int_a^b \int_c^d [f \otimes g - f(a)g(c)] \, d^2 G \right| \leq 8 K_{p,q} [\|f\|_{[q]} V_q(g) + \|g\|_{[q]} V_q(f)] V_p(G; R), \tag{12}$$

where $K_{p,q} := (1 + \zeta(p^{-1} + q^{-1}))^2$ and $\zeta(s) := \sum_{k=1}^{\infty} k^{-s}$ for $s > 1$.

Proof. The functions Φ_p and Φ_q with values $\Phi_p(x) := x^p/p$ and $\Phi_q(x) := x^q/q$ for $x \geq 0$ are the N-functions. We apply the results of Leśniewicz and Leśniewicz [14] for $\Phi = \Psi = \Phi_p$ and $\tilde{\Phi} = \tilde{\Psi} = \Phi_q$. The integral $\int_a^b \int_c^d f \otimes g \, d^2 G$ exists by Theorem 4.3 in [14], page 57. (We note that continuity of the functions f and g is not used in the proof there.) To obtain the bound (12), it is enough to bound the Riemann–Stieltjes sums. Let $\dot{\tau} = \{(u_i, v_j), [s_{i-1}, s_i] \times [t_{j-1}, t_j] : (i, j) \in (n] \times (m]\}$ be a tagged partition of R. Letting $u_0 := a$ and $v_0 := c$ we have the identity

$$S(f \otimes g - f(a)g(c), \Delta^2 G; \dot{\tau}) = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{i} \sum_{l=1}^{j} \Delta_k f \Delta_l g \Delta_{i,j}^{\tau} G + f(a) \sum_{j=1}^{m} \sum_{l=1}^{j} \Delta_l g \sum_{i=1}^{n} \Delta_{i,j}^{\tau} G$$

$$+ g(c) \sum_{i=1}^{n} \sum_{k=1}^{i} \Delta_k f \sum_{j=1}^{m} \Delta_{i,j}^{\tau} G,$$

where $\Delta_k f := f(u_k) - f(u_{k-1})$ and $\Delta_l g := g(v_l) - g(v_{l-1})$. Using the bounds 3.5 in [14], page 53, and (5.1) in [24], page 254, we get

$$|S(f \otimes g - f(a)g(c), \Delta^2 G; \dot{\tau})|$$

$$\leq 16 \left(1 + \zeta\left(\frac{1}{p} + \frac{1}{q}\right)\right)^2 V_q(f; [a, b]) V_q(g; [a, b]) V_p(G; R)$$

$$+ \left(1 + \zeta\left(\frac{1}{p} + \frac{1}{q}\right)\right) [|f(a)| V_q(g; [c, d]) + |g(c)| V_q(f; [a, b])] V_p(G; R),$$

and so (12) follows. Instead of the bound 3.3 in [14], page 51, we used the bound of L.C. Young since it gives a smaller constant in (12) in the present setting.   □

We use the following two versions of the preceding inequality (12) adapted to subrectangles of a rectangle $[0, T]^2$.

Corollary 7. *Let $p > 1$ and $q > 1$ be such that $p^{-1} + q^{-1} > 1$. Let $T > 0$, let $f \in \mathcal{W}_q[0, T]$ and let $G \in \mathcal{W}_p[0, T]^2$ be continuous. There exists the double Riemann–Stieltjes integral $\int_0^T \int_0^T f \otimes f \, d^2 G$. Also, letting $K_{p,q} := 16(1 + \zeta(p^{-1} + q^{-1}))^2$,*

(i) *the inequality*

$$\left| \int_s^t \int_s^t [f \otimes f - f^2(s)] \, d^2 G \right| \le K_{p,q} \|f\|_{[q],[0,T]} V_q(f; [s,t]) V_p(G; [s,t]^2) \qquad (13)$$

holds for any $0 \le s < t \le T$;

(ii) *the inequality*

$$\left| \int_s^t \int_u^v f \otimes f \, d^2 G \right| \le K_{p,q} \|f\|_{[q],[0,T]}^2 V_p(G; [s,t] \times [u,v]) \qquad (14)$$

holds for any $0 \le s < t \le T$ and $0 \le u < v \le T$.

The quadratic mean Riemann–Stieltjes integral

This integral is defined for a (deterministic) function with respect to a stochastic process in the present paper. Let $X = \{X(t) : t \ge 0\}$ be a second order stochastic process on a probability space $(\Omega, \mathcal{F}, \mathrm{Pr})$, which is a family of random variables $X(t)$ having mean zero and finite second moment. The covariance function of X is the function Γ_X defined on $\mathbb{R}_+^2 = [0, \infty) \times [0, \infty)$ with values

$$\Gamma_X(s, t) := E[X(s)X(t)], \qquad (s, t) \in \mathbb{R}_+^2.$$

Let $f : [0, \infty) \to \mathbb{R}$ be a function and let $0 \le a < b < \infty$. For a tagged partition $\dot\kappa = \{(u_i, [t_{i-1}, t_i]) : i \in (n)\}$ of the interval $[a, b]$ the Riemann–Stieltjes sum is

$$S_{\mathrm{RS}}(f, \Delta X; \dot\kappa) := \sum_{i=1}^n f(u_i)[X(t_i) - X(t_{i-1})],$$

and so it is a random variable in $L^2(\Omega, \mathcal{F}, \mathrm{Pr})$. We say that the *quadratic mean Riemann–Stieltjes integral* over $[a, b]$ of f with respect to X exists and equals $I \in L^2(\Omega, \mathcal{F}, \mathrm{Pr})$, if for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$E[S_{\mathrm{RS}}(f, \Delta X; \dot\kappa) - I]^2 < \varepsilon$$

for each tagged partition $\acute{\kappa}$ of $[a, b]$ with the mesh $|\kappa| < \delta$. If such I exists, then it is unique in L^2 and is denoted by

$$\int_a^b f \, dX = \text{q.m.} \int_a^b f(t) \, dX(t) := I.$$

Next, is the integration in quadratic mean criterion of Loève [16], page 138.

Proposition 8. *Let X be a second order stochastic process and $f : [0, \infty) \to \mathbb{R}$. For $0 \le a < b < \infty$, the quadratic mean Riemann–Stieltjes integral*

$$\int_a^b f \, dX \text{ exists} \quad \text{if and only if} \quad \int_a^b \int_a^b f \otimes f \, d^2 \Gamma_X \text{ exists}$$

as the double Riemann–Stieltjes integral. Moreover, for any $0 \le s < t < \infty$ and $0 \le u < v < \infty$ if the two integrals $\int_s^t f \, dX$ and $\int_u^v f \, dX$ exist then so does $\int_s^t \int_u^v f \otimes f \, d^2 \Gamma_X$ and the equality

$$E\left[\int_s^t f \, dX \int_u^v f \, dX\right] = \int_s^t \int_u^v f \otimes f \, d^2 \Gamma_X \tag{15}$$

holds.

Formal properties of Riemann–Stieltjes integrals such as (finite) additivity and linearity hold almost surely for corresponding integrals in quadratic mean.

We shall write $\mathcal{Q}_p := [1, p/(p-1))$ if $p > 1$ and $\mathcal{Q}_1 := \{\infty\}$. The following theorem holds by Theorem 5, Corollary 7 and Proposition 8.

Theorem 9. *Let X be a second order stochastic process with the continuous covariance function $\Gamma_X \in \mathcal{W}_p[0, T]^2$ for some $p \ge 1$ and $0 < T < \infty$, and let $f \in \mathcal{W}_q[0, T]$ with $q \in \mathcal{Q}_p$. Then for each $t \in [0, T]$ there exists the q.m. Riemann–Stieltjes integral $\int_0^t f \, dX$ and there is a finite constant $K = K(p, f)$ (depending on p and f) such that the inequality*

$$E\left[\text{q.m.} \int_s^t f \, dX\right]^2 \le K V_p\left(\Gamma_X; [s, t]^2\right)$$

holds for any $0 \le s < t \le T$.

Given a second order stochastic process X, a class of functions f such that $\int_0^T f \, dX$ is defined as the quadratic mean Riemann–Stieltjes integral can be larger than the class of functions f such that $\int_0^T f \, dX$ is defined as the pathwise Riemann–Stieltjes integral. Indeed, let X be a fractional Brownian motion B_H with the Hurst index $H \in (0, 1)$. By Proposition 15 below, B_H has the continuous covariance function $\Gamma_{B_H} \in \mathcal{W}_p[0, T]^2$ with $p = \max\{1, 1/(2H)\}$. Therefore, the q.m. Riemann–Stieltjes integral $\int_0^T f \, dB_H$ is defined for each $f \in \mathcal{W}_q[0, T]$, where

$$q < \frac{1}{1 - 2H} \quad \text{if } H \in (0, 1/2) \quad \text{and} \quad q = \infty \quad \text{if } H \in [1/2, 1)$$

by the preceding theorem. While the pathwise Riemann–Stieltjes integral $\int_0^T f\, dB_H$ is defined for each $f \in \mathcal{W}_q[0, T]$ with $q < 1/(1 - H)$ if $H \in (0, 1)$ by the result of Young [24], and these results are best possible in terms of p-variation (see [7], Section 3.7).

The preceding comment suggests that a family of random variables

$$\text{q.m.} \int_0^t f\, dX, \qquad t \in [0, T], \tag{16}$$

need not be a stochastic process with well-behaved sample functions. The following is a standard approach to deal with such cases.

Theorem 10. *Suppose that the hypotheses of the preceding theorem hold. Suppose that for each $t \in (0, T]$*

$$\lim_{s \uparrow t} V_p\big(\Gamma_X; [s, t]^2\big) = 0.$$

Then a measurable and separable stochastic process $Y = \{Y(t) : t \in [0, T]\}$ exists on $(\Omega, \mathcal{F}, \Pr)$ such that

$$\Pr\left(\left\{Y(t) = \text{q.m.} \int_0^t f\, dX\right\}\right) = 1$$

for each $t \in [0, T]$.

Throughout the paper, we assume that the q.m. Riemann–Stieltjes integrals (16) are given by the stochastic process Y from the preceding theorem, to be called the *q.m. integral process*.

3. *p*-variation of the covariance function

We start with a simple fact concerning the boundedness of variation of the covariance functions of stochastic processes with positively or negatively correlated disjoint increments (meaning hypothesis (17) or (18), respectively).

Proposition 11. *Let $0 < T < \infty$ and let $X = \{X(t) : t \in [0, T]\}$ be a second order stochastic process with the covariance function Γ_X on $[0, T]^2$.*

  (i) *If for any $0 \le u < v \le s < t \le T$,*

$$E\big[X(v) - X(u)\big]\big[X(t) - X(s)\big] \ge 0, \tag{17}$$

  then for any $0 \le a < b \le T$ and $0 \le c < d \le T$

$$V_1\big(\Gamma_X; [a, b] \times [c, d]\big) = E\big[X(b) - X(a)\big]\big[X(d) - X(c)\big].$$

  (ii) *If for any $0 \le u < v \le s < t \le T$,*

$$E\big[X(v) - X(u)\big]\big[X(t) - X(s)\big] \le 0, \tag{18}$$

then for any $0 \le a < b \le c < d \le T$

$$V_1\left(\Gamma_X; [a, b] \times [c, d]\right) = \left|E\left[X(b) - X(a)\right]\left[X(d) - X(c)\right]\right|. \tag{19}$$

Proof. To prove (i) note that (17) holds for any pairs of closed intervals $[u, v]$ and $[s, t]$ in $[0, T]$ provided (17) holds for such intervals having at most a common endpoint, as assumed. Then the conclusion follows using the relation

$$\left|\Delta^{[u, v] \times [s, t]} \Gamma_X\right| = E\left[X(v) - X(u)\right]\left[X(t) - X(s)\right].$$

In the case (ii), the conclusion follows using the relation

$$\left|\Delta^{[u, v] \times [s, t]} \Gamma_X\right| = -E\left[X(v) - X(u)\right]\left[X(t) - X(s)\right]$$

for nonoverlapping intervals $[u, v]$ and $[s, t]$ in $[0, T]$. □

By the second part of the preceding proposition the covariance function of a stochastic process with negatively correlated disjoint increments has bounded variation over rectangles which do not contain a diagonal. The following result for such a process, with an additional assumption (20), gives a bound of the p-variation of the covariance function over rectangles containing a diagonal.

Theorem 12. *Let $0 < T < \infty$, let $p \ge 1$ and let $X = \{X(t) : t \ge 0\}$ be a second order stochastic process with the covariance function Γ_X such that (18) holds for any $0 \le u < v \le s < t \le T$, and*

$$E\left[X(v) - X(u)\right]\left[X(t) - X(s)\right] \ge 0, \tag{20}$$

holds for any $0 \le s \le u < v \le t \le T$. Then for any $0 \le a < b \le T$

$$V_p\left(\Gamma_X; [a, b]^2\right) \le 2V_{2p}\left(\psi_X; [a, b]\right)^2, \tag{21}$$

where $\psi_X : [0, T] \to L^2(\Omega, \mathcal{F}, \Pr)$ defined by $\psi_X(t) := X(t, \cdot)$ for $t \in [0, T]$.

Remark 13. The theorem is meaningful provided the right side of (21) is finite. In addition to the hypotheses of Theorem 12, suppose that X and $p \ge 1$ are such that for a finite constant L the inequality

$$E\left[X(t) - X(s)\right]^2 \le L(t - s)^{1/p}$$

holds for each $0 \le s < t \le T$. Then for any $0 \le a < b \le T$ we have $V_{2p}(\psi_X; [a, b]) \le \sqrt{L}(b - a)^{1/(2p)}$, and so by Theorem 12

$$V_p\left(\Gamma_X; [a, b]^2\right) \le 2L(b - a)^{1/p}.$$

Proof of Theorem 12. Let $0 \leq a < b \leq T$. Without loss of generality, we can assume that the right side of (21) is finite. Let $\lambda \times \kappa = \{(s_i, t_j) : i \in [n], j \in [m]\}$ be a partition of $[a, b]^2$ with $n \geq 1$ and $m \geq 1$. If $n = 1$ then, since $p \geq 1$ and (20) holds, we have

$$s_p(\Gamma_X; \lambda \times \kappa) \leq \left(\sum_{j=1}^{m} E[X(b) - X(a)] \Delta_j^{\kappa} X \right)^p = \left(E[X(b) - X(a)]^2 \right)^p$$

$$= \| \psi_X(b) - \psi_X(a) \|_{L_2}^{2p} \leq V_{2p}(\psi_X; [a, b])^{2p}.$$

(22)

Let $n \geq 2$, let $1 \leq i \leq n$, and let $A_i := \{j \in (m-1) : t_j \in (s_{i-1}, s_i)\}$. If A_i is the empty set, then there is a $j_0 \in (m)$ such that $[s_{i-1}, s_i] \subset [t_{j_0-1}, t_{j_0}]$. In this case, we have

$$\left| E \Delta_i^{\lambda} X \Delta_{j_0}^{\kappa} X \right| \leq E[\Delta_i^{\lambda} X]^2$$

$$+ \left| E \Delta_i^{\lambda} X [X(t_{j_0}) - X(s_i)] \right| + \left| E \Delta_i^{\lambda} X [X(s_{i-1}) - X(t_{j_0-1})] \right|.$$

If A_i is not the empty set then let j_1 be the minimal element in A_i and let j_2 be the maximal element in A_i. In this case, we have

$$\left| E \Delta_i^{\lambda} X \Delta_{j_1}^{\kappa} X \right| \leq \left| E \Delta_i^{\lambda} X [X(t_{j_1}) - X(s_{i-1})] \right| + \left| E \Delta_i^{\lambda} X [X(s_{i-1}) - X(t_{j_1-1})] \right|$$

and

$$\left| E \Delta_i^{\lambda} X \Delta_{j_2+1}^{\kappa} X \right| \leq \left| E \Delta_i^{\lambda} X [X(t_{j_2+1}) - X(s_i)] \right| + \left| E \Delta_i^{\lambda} X [X(s_i) - X(t_{j_2})] \right|.$$

Therefore to bound $\sum_{j=1}^{m} |E \Delta_i^{\lambda} X \Delta_j^{\kappa} X|$, we can and do assume that in the partition κ we have $t_{j_1} = s_{i-1}$ and $t_{j_2} = s_i$ for some $j_1 < j_2$ in $(m-1)$. Using this assumption and negative correlation for disjoint increments it follows that

$$\sum_{j=1}^{m} \left| E \Delta_i^{\lambda} X \Delta_j^{\kappa} X \right| = 2E[\Delta_i^{\lambda} X]^2 - E \Delta_i^{\lambda} X [X(b) - X(a)] \leq 2E[\Delta_i^{\lambda} X]^2,$$

where the last inequality holds by (20). Finally, since $p \geq 1$, we have

$$s_p(\Gamma_X; \lambda \times \kappa) \leq \sum_{i=1}^{n} \left(\sum_{j=1}^{m} \left| E \Delta_i^{\lambda} X \Delta_j^{\kappa} X \right| \right)^p \leq 2^p \sum_{i=1}^{n} \left(E[\Delta_i^{\lambda} X]^2 \right)^p$$

$$= 2^p \sum_{i=1}^{n} \| \psi_X(s_i) - \psi_X(s_{i-1}) \|_{L_2}^{2p} \leq 2^p V_{2p}(\psi_X; [a, b])^{2p}.$$

Recalling the bound (22) in the case $n = 1$, the conclusion (21) follows. □

Next, we show that for several classes of stochastic processes including fractional Brownian motion, subfractional Brownian motion and bifractional Brownian motion one has positively or negatively correlated increments.

Stochastic processes with stationary increments

First, consider real-valued stochastic processes X with mean zero, finite second moments $E[X(t)]^2$ and (weakly) stationary increments. Then the incremental variance function $\sigma_X^2(t, t + r)$ does not depend on t, and so it is a function of r. The following fact is known (see [18], page 32); we sketch a proof for completeness.

Lemma 14. *Let* $X = \{X(t) : t \geq 0\}$ *be a mean zero second order stochastic process with stationary increments, and let* $\phi : [0, \infty) \to [0, \infty)$ *be the function with values*

$$\phi(r) := \sigma_X^2(t, t + r) = E\big[X(t + r) - X(t)\big]^2$$

for each $r \geq 0$.

  (i) *If* ϕ *is convex on* $[0, T]$, *then* (17) *holds for any* $0 \leq u < v \leq s < t \leq T$.
  (ii) *If* ϕ *is concave on* $[0, T]$, *then* (18) *holds for any* $0 \leq u < v \leq s < t \leq T$.

Proof. To prove (i) let ϕ be convex on $[0, T]$, and let $0 \leq u < v \leq s < t \leq T$. Using an expression of ϕ in terms of the covariance function Γ_X, it follows that

$$2E\big[X(v) - X(u)\big]\big[X(t) - X(s)\big] = \big[\phi(t - u) - \phi(t - v)\big] - \big[\phi(s - u) - \phi(s - v)\big].$$

Inserting additional points in the interval $[u, v]$ if necessary, one can suppose that $v - u < t - s$. Then letting $x_1 := s - v$, $x_2 := s - u$, $x_3 := t - v$ and $x_4 := t - u$, we have $0 \leq x_1 < x_2 < x_3 < x_4 \leq T$ and

$$\frac{\phi(x_4) - \phi(x_3)}{x_4 - x_3} \geq \frac{\phi(x_3) - \phi(x_2)}{x_3 - x_2} \geq \frac{\phi(x_2) - \phi(x_1)}{x_2 - x_1},$$

by convexity of ϕ. This proves (17), and so (i). The proof of (ii) is symmetric. $\square$

We apply this fact to a fractional Brownian motion $B_H = \{B_H(t) : t \in [0, T]\}$ with the Hurst index $H \in (0, 1)$, which is a Gaussian stochastic process with mean zero and the covariance function

$$F_H(s, t) := \Gamma_{B_H}(s, t) = \tfrac{1}{2}\big(t^{2H} + s^{2H} - |t - s|^{2H}\big)$$

for $(s, t) \in [0, T]^2$.

Proposition 15. *Let* B_H *be a fractional Brownian motion with the Hurst index* $H \in (0, 1)$, *let* $\rho_H(u) := u^H$ *for each* $u \in [0, T]$, *and let* $p := \max\{1, 1/(2H)\}$. *Then* $B_H \in \mathcal{LSI}(\rho_H(\cdot))$. *Also, the inequality*

$$V_p\big(F_H; [s, t]^2\big) \leq C_1\big[\rho_H(t - s)\big]^2 \equiv C_1(t - s)^{2H} \tag{23}$$

with $C_1 = 1$ *if* $2H \geq 1$ *and* $C_1 = 2$ *if* $2H < 1$, *holds for any* $0 \leq s < t \leq T$, *and the inequality*

$$\sum_{j=1}^m V_p\big(F_H; J_i^\kappa \times J_j^\kappa\big) \leq \sum_{j=1}^m \big|E\big[\Delta_i^\kappa B_H \Delta_j^\kappa B_H\big]\big| \leq C_2\big(\Delta_i^\kappa\big)^{1 \wedge (2H)} \tag{24}$$

with $C_2 = 2HT^{2H-1}$ if $2H \geq 1$ and $C_2 = 3$ if $2H < 1$, holds for any partition $\kappa = (t_j)_{j \in [m]}$ of $[0, T]$ and for any $i \in (m)$.

Proof. The incremental variance function $\sigma^2_{B_H}(s, t) = |t - s|^{2H}$ for $(s, t) \in [0, T]$. Clearly, $B_H \in \mathcal{LSI}(\rho_H(\cdot))$. By Lemma 14, disjoint increments of B_H are positively correlated if $2H \geq 1$ and negatively correlated if $2H < 1$. If $2H \geq 1$, then by part (i) of Proposition 11,

$$V_1\big(F_H; [a, b]^2\big) = E\big[B_H(b) - B_H(a)\big]^2 = (b - a)^{2H}$$

for any $0 \leq a < b \leq T$, proving (23) with $C_1 = 1$ in this case. If $2H < 1$ then (23) holds with $C_1 = 2$ by Remark 13 and Theorem 12 since its hypothesis (20) holds due to the relation

$$
\begin{aligned}
2E\big[B_H(v) &- B_H(u)\big]\big[B_H(t) - B_H(s)\big] \\
&= (v - s)^{2H} - (u - s)^{2H} + (t - u)^{2H} - (t - v)^{2H} > 0
\end{aligned}
\tag{25}
$$

for any $0 \leq s \leq u < v \leq t \leq T$.

To prove (24) let $\kappa = (t_j)_{j \in [m]}$ be a partition of $[0, T]$, and let $i \in (m)$. Due to (23) one can suppose that $m > 1$. First let $H \in [1/2, 1)$. Then $p = 1$. By part (i) of Lemma 14 and by part (i) of Proposition 11 we have

$$
\begin{aligned}
\sum_{j=1}^{m} V_1\big(F_H; [t_{i-1}, t_i] \times [t_{j-1}, t_j]\big) &= \sum_{j=1}^{m} E\big[\Delta_i^\kappa B_H \Delta_j^\kappa B_H\big] \\
&= E\big[B_H(t_i) - B_H(t_{i-1})\big]\big[B_H(T) - B_H(0)\big] \\
&= \frac{1}{2}\big[t_i^{2H} - t_{i-1}^{2H} + (T - t_{i-1})^{2H} - (T - t_i)^{2H}\big] \\
&\leq 2HT^{2H-1}(t_i - t_{i-1}),
\end{aligned}
$$

where the last inequality holds by the mean value theorem. Now, let $H \in (0, 1/2)$. Then $p = 1/(2H) > 1$. Since $V_p(\cdot) \leq V_1(\cdot)$, by part (ii) of Lemma 14 and by part (ii) of Proposition 11 we have

$$
\begin{aligned}
\sum_{j \in (m) \setminus \{i\}} V_p\big(F_H; J_i^\kappa \times J_j^\kappa\big) &\leq \sum_{j \in (m) \setminus \{i\}} V_1\big(F_H; J_i^\kappa \times J_j^\kappa\big) \\
&= \sum_{j \in (m) \setminus \{i\}} \big|E\big(\Delta_i^\kappa B_H \Delta_j^\kappa B_H\big)\big| \\
&= E\big(\Delta_i^\kappa B_H\big)^2 - E\Delta_i^\kappa B_H\big[B_H(T) - B_H(0)\big] \\
&\leq E\big(\Delta_i^\kappa B_H\big)^2 = (t_i - t_{i-1})^{2H},
\end{aligned}
$$

where the last inequality holds by (25). This together with (23) gives (24), completing the proof. $\qquad\square$

The inequality (23) in the case $H \in (0, 1/2)$ and with a different constant is the same as the one stated by Proposition 13 in [8]. The proofs seem to be also different.

Sub-fractional Brownian motion

Let $H \in (0, 1)$ and $0 < T < \infty$. The function $R_H : [0, T]^2 \to \mathbb{R}$ with values

$$R_H(s, t) := s^{2H} + t^{2H} - \tfrac{1}{2}\big[(s + t)^{2H} + |s - t|^{2H}\big], \tag{26}$$

$(s, t) \in [0, T]^2$, is positive definite as shown in [3]. A *sub-fractional Brownian motion* with index H is a mean zero Gaussian stochastic process $G_H = \{G_H(t) : t \in [0, T]\}$ with the covariance function R_H and with the incremental variance function

$$\sigma_{G_H}^2(s, t) = |s - t|^{2H} + (s + t)^{2H} - 2^{2H-1}\big[t^{2H} + s^{2H}\big]$$

for $s, t \in [0, T]$. In the case $H = 1/2$, $G_{1/2}$ is a Brownian motion. A subfractional Brownian motion G_H with index H is H-self-similar but does not have stationary increments if $H \neq 1/2$.

Proposition 16. *Let* $G_H = \{G_H(t) : t \in [0, T]\}$ *be a sub-fractional Brownian motion with* $H \in (0, 1)$. *The following properties hold*:

(i) *for any* $0 \leq s \leq t \leq T$

$$(t - s)^{2H} \leq \sigma_{G_H}^2(s, t) \leq \big(2 - 2^{2H-1}\big)(t - s)^{2H}, \qquad \text{if } 0 < H < 1/2,$$

$$\big(2 - 2^{2H-1}\big)(t - s)^{2H} \leq \sigma_{G_H}^2(s, t) \leq (t - s)^{2H}, \qquad \text{if } 1/2 < H < 1;$$

(ii) *for any* $0 \leq u < v \leq s < t \leq T$

$$E\big[G_H(v) - G_H(u)\big]\big[G_H(t) - G_H(s)\big] \begin{cases} < 0, & \text{if } 0 < H < 1/2, \\ > 0, & \text{if } 1/2 < H < 1; \end{cases}$$

(iii) *for any* $0 \leq s \leq u < v \leq t \leq T$

$$C(u, v, s, t) := E\big[G_H(v) - G_H(u)\big]\big[G_H(t) - G_H(s)\big] > 0.$$

Proof. Statements (i) and (ii) are proved in [3], Theorems (3), (5). To prove (iii), let $0 \leq s \leq u < v \leq t \leq T$. Since the pairs of intervals $[s, u]$, $[u, v]$ and $[v, t]$, $[u, v]$ do not intersect (except for the endpoints), by (ii) in the case $1/2 < H < 1$, we have

$$\begin{aligned} C(u, v, s, t) = {} & E\big[G_H(v) - G_H(u)\big]^2 \\ & + E\big[G_H(v) - G_H(u)\big]\big[G_H(u) - G_H(s)\big] \\ & + E\big[G_H(v) - G_H(u)\big]\big[G_H(t) - G_H(v)\big] \\ & > 0. \end{aligned}$$

Thus we can suppose that $0 < H < 1/2$. Using the values of the covariance function (26) it follows that

$$C(u, v, s, t) = \tfrac{1}{2}\big\{-(t+v)^{2H} - (t-v)^{2H} + (t+u)^{2H} + (t-u)^{2H}$$
$$+ (v+s)^{2H} + (v-s)^{2H} - (u+s)^{2H} - (u-s)^{2H}\big\}. \tag{27}$$

Let $f_t(x) := (t+x)^{2H} + (t-x)^{2H}$ for each $x \in [0, t]$. Since $0 < H < 1/2$, then $f_t'(x) < 0$ for $x \in (0, t)$, and so for $0 \le u < v \le t$ we have $-f_t(v) + f_t(u) > 0$ by the mean value theorem. Let $g_s(x) := (x+s)^{2H} + (x-s)^{2H}$ for each $x \ge s$. Since $g_s'(x) > 0$ for $x > s$ and $v > u \ge s$, we have $g_s(v) - g_s(u) > 0$ by the mean value theorem again. Therefore

$$C(u, v, s, t) = \tfrac{1}{2}\big\{-f_t(v) + f_t(u) + g_s(v) - g_s(u)\big\} > 0,$$

as claimed. $\square$

The following proposition shows that the hypotheses (36) and (44) of the main result (Theorem 21) hold true for a sub-fractional Brownian motion.

Proposition 17. *Let $G_H = \{G_H(t) : t \in [0, T]\}$ be a sub-fractional Brownian motion with $H \in (0, 1)$, let $\rho_H(u) := u^H$ for each $u \in [0, T]$, and let $p := \max\{1, 1/(2H)\}$. Then $G_H \in \mathcal{LSI}(\rho_H(\cdot))$. Also, there is a finite constant C_1 such that the inequality*

$$V_p\big(R_H; [s, t]^2\big) \le C_1\big[\rho_H(t-s)\big]^2 \equiv C_1(t-s)^{2H} \tag{28}$$

holds for any $0 \le s < t \le T$, and there is a finite constant C_2 such that the inequality

$$\sum_{j=1}^m V_p\big(R_H; J_i^\kappa \times J_j^\kappa\big) \le \sum_{j=1}^m \big|E\big[\Delta_i^\kappa G_H \Delta_j^\kappa G_H\big]\big| \le C_2\big(\Delta_i^\kappa\big)^{1\wedge(2H)} \tag{29}$$

holds for any partition $(t_i)_{i\in[m]}$ of $[0, T]$ and for any $i \in (m)$.

Proof. Condition (A1) of Definition 1 holds by part (i) of Proposition 16. To prove condition (A2) suppose that $H \in (0, 1/2) \cup (1/2, 1)$ and let $\varepsilon \in (0, T)$. For each $s \in [\varepsilon, T)$ and $t \in [-\varepsilon, T-s]$, let

$$f_s(t) := (2s+t)^{2H} - 2^{2H-1}\big[s^{2H} + (s+t)^{2H}\big].$$

Then $f_s(0) = f_s'(0) = 0$ and

$$b(s, s+h) := \left(\frac{\sigma_{G_H}(s, s+h)}{\rho_H(h)}\right)^2 - 1 = h^{-2H} f_s(h).$$

Let $s \in [\varepsilon, T)$ and $h \in (0, T-s]$. By Taylor's theorem with the Lagrange remainder applied to the function f_s, there exists $u = u(s, h) \in (0, h)$ such that

$$b(s, s+h) = 2^{-1} h^{2(1-H)} f_s''(u),$$

where

$$f_s''(u) = 2H(2H-1)\big[(2s+u)^{2H-2} - 2^{2H-1}(s+u)^{2H-2}\big].$$

Then there is a finite constant $C = C(\varepsilon, H)$ such that the inequality

$$\big|b(s, s+h)\big| \le Ch^{2(1-H)}$$

holds for each $s \in [\varepsilon, T)$ and $h \in (0, T-s]$. Since $H < 1$ the preceding bound yields that condition (A2) holds, and so $G_H \in \mathcal{LSI}(\rho_H(\cdot))$.

To prove (28), first let $H \in (1/2, 1)$. Then $p = 1$ and hypothesis (17) holds for $X = G_H$ by part (ii) of Proposition 16. Therefore in this case by part (i) of Proposition 11 and by part (i) of Proposition 16, for any $0 \le s < t \le T$ we have

$$V_1\big(R_H; [s,t]^2\big) = E\big[G_H(t) - G_H(s)\big]^2 \le (t-s)^{2H}.$$

Therefore, (28) holds with $C_1 = 1$ in the case $H \in (1/2, 1)$. Now let $H \in (0, 1/2)$. Then $p = 1/(2H) > 1$ and the hypotheses of Theorem 12 hold by Proposition 16. By part (i) of Proposition 16 and Remark 13 with $L = 2 - 2^{2H-1}$, (28) holds with $C_1 = 4 - 2^{2H}$ in the case $H \in (0, 1/2)$.

To prove (29), let $\kappa = (t_j)_{j\in[m]}$ be a partition of $[0, T]$ and $i \in (m)$. Due to (28), one can suppose that $m > 1$. First, let $H \in (1/2, 1)$. Then $p = 1$. As for fractional Brownian motion (Proposition 15), in the present case by part (ii) of Proposition 16 and by part (i) of Proposition 11, we have

$$\sum_{j=1}^m V_1\big(R_H; J_i^\kappa \times J_j^\kappa\big) = \sum_{j=1}^m E\big[\Delta_i^\kappa G_H \Delta_j^\kappa G_H\big] = -\frac{1}{2}\big[f_T(t_i) - f_T(t_{i-1})\big] + t_i^{2H} - t_{i-1}^{2H},$$

where $f_T(t) = (T+t)^{2H} + (T-t)^{2H}$ for $t \in [0, T]$ and the last equality is the special case of (27). Since $1/2 < H < 1$ the function f_T is increasing, and so

$$\sum_{j=1}^m E\big[\Delta_i^\kappa G_H \Delta_j^\kappa G_H\big] \le t_i^{2H} - t_{i-1}^{2H} \le 2H T^{2H-1}(t_i - t_{i-1})$$

by the mean value theorem. Now let $H \in (0, 1/2)$. Then $p = 1/(2H) > 1$. Again as for fractional Brownian motion (Proposition 15), in the present case by part (ii) of Proposition 11 and by parts (ii), (iii) of Proposition 16, we have

$$\sum_{j\in(m)\setminus\{i\}} V_p\big(R_H; J_i^\kappa \times J_j^\kappa\big) \le \sum_{j\in(m)\setminus\{i\}} \big|E\big[\Delta_i^\kappa G_H \Delta_j^\kappa G_H\big]\big| \le E\big(\Delta_i^\kappa G_H\big)^2.$$

Then by part (i) of Proposition 16, the inequality

$$\sum_{j=1}^m V_p\big(R_H; J_i^\kappa \times J_j^\kappa\big) \le \sum_{j=1}^m \big|E\big[\Delta_i^\kappa G_H \Delta_j^\kappa G_H\big]\big| \le 2E\big(\Delta_i^\kappa G_H\big)^2 \le C_2\big(\Delta_i^\kappa\big)^{1/p},$$

holds with $c_2 = 2H T^{2H-1}$ if $2H > 1$ and $C_2 = 4 - 2^{2H}$ if $2H < 1$, completing the proof. $\quad\square$

Bifractional Brownian motion

Let $0 < T < \infty$, $0 < H < 1$ and $0 < K \leq 1$. The function $C_{H,K} : [0,T]^2 \to \mathbb{R}$ with values

$$C_{H,K}(s,t) := 2^{-K}\big\{\big(t^{2H} + s^{2H}\big)^K - |t-s|^{2HK}\big\}, \tag{30}$$

$(s,t) \in [0,T]^2$, is positive definite as shown in [11]. A *bifractional Brownian motion* with parameters (H, K) is a mean zero Gaussian stochastic process $B_{H,K} = \{B_{H,K}(t) : t \in [0,T]\}$ with the covariance function $C_{H,K}$. When $K = 1$, $B_{H,1}$ is the fractional Brownian motion B_H with the Hurst index $H \in (0,1)$. The Gaussian process $B_{H,K}$ is a self-similar stochastic process of order $HK \in (0,1)$, the increments are not stationary and its incremental variance function is

$$\sigma^2_{B_{H,K}}(s,t) = 2^{1-K}\big[|t-s|^{2HK} - \big(t^{2H} + s^{2H}\big)^K\big] + t^{2HK} + s^{2HK}$$

for each $s, t \geq 0$. By Proposition 3.1 of [11], for every $s, t \geq 0$,

$$2^{-K}|t-s|^{2HK} \leq \sigma^2_{B_{H,K}}(s,t) \leq 2^{1-K}|t-s|^{2HK}. \tag{31}$$

This suggests that the incremental variance function $\sigma^2_{B_{H,K}}$ is dominated by a single variable function $u \mapsto const|u|^{2HK}$, $u \in \mathbb{R}$. A more precise property is proved next.

Proposition 18. *Let* $0 < T < \infty$, $0 < H < 1$, $0 < K < 1$ *and* $B_{H,K} = \{B_{H,K}(t) : t \in [0,T]\}$ *be a bifractional Brownian motion with parameters* (H, K). *Let* $\rho_{H,K}(u) := 2^{(1-K)/2}u^{HK}$ *for each* $u \in [0,T]$, *and let* $p := \max\{1, 1/(2HK)\}$. *Then* $B_{H,K} \in \mathcal{LSI}(\rho_{H,K}(\cdot))$. *Also, there is a finite constant* C_1 *such that the inequality*

$$V_p\big(C_{H,K}; [a,b]^2\big) \leq C_1(b-a)^{2HK} \tag{32}$$

holds for any $0 \leq a < b \leq T$, *and there is a finite constant* C_2 *such that the inequality*

$$\sum_{j=1}^m V_p\big(C_{H,K}; J_i^\kappa \times J_j^\kappa\big) \leq C_2\big(\Delta_i^\kappa\big)^{1\wedge(2HK)} \tag{33}$$

holds for any partition $(t_j)_{j\in[m]}$ *of* $[0,T]$ *and for any* $i \in (m)$.

Proof. Concerning the property of local stationarity of increments of $B_{H,K}$ with the local variance function $\rho = \rho_{H,K}(\cdot)$ note that condition (A1) in Definition 1 holds with $L = 2^{1-K}$ by (31). To prove condition (A2) let $\varepsilon > 0$. For each $s \in [\varepsilon, T)$ and $t \in (-\varepsilon, T-s]$ let

$$f_s(t) := 2^{1-K}\big[s^{2H} + (s+t)^{2H}\big]^K - s^{2HK} - (s+t)^{2HK}.$$

Then $f_s(0) = f_s'(0) = 0$ and

$$b(s, s+h) := \left(\frac{\sigma_{B_{H,K}}(s, s+h)}{\rho_{H,K}(h)}\right)^2 - 1 = -2^{K-1}h^{-2HK}f_s(h).$$

Let $s \in [\varepsilon, T)$ and $h \in (0, T - s]$. By Taylor's theorem with the Lagrange remainder applied to the function f_s, there exists $u = u(s, h) \in (0, h)$ such that

$$b(s, s + h) = -2^{K-2} h^{2(1-HK)} f_s''(u),$$

where

$$
\begin{aligned}
f_s''(u) &= 2^{3-K} K(K-1) H^2 [(s+u)^{2H} + s^{2H}]^{K-2} (s+u)^{2(2H-1)} \\
&\quad + 2^{2-K} K H (2H-1) [(s+u)^{2H} + s^{2H}]^{K-1} (s+u)^{2H-2} \\
&\quad - 2HK(2HK-1)(s+u)^{2HK-2}.
\end{aligned}
$$

Then there is a finite constant $C = C(\varepsilon, H, K)$ such that

$$\big| b(s, s+h) \big| \leq C h^{2(1-HK)}$$

for each $s \in [\varepsilon, T)$ and $h \in (0, T - s]$. Since $HK < 1$, the preceding bound yields that condition (A2) holds, and so $B_{H,K} \in \mathcal{LSI}(\rho_{H,K}(\cdot))$.

To prove (32) and (33), we use a decomposition in distribution of a fractional Brownian motion B_{HK} with the Hurst index HK into a linear combination of a bifractional Brownian motion $B_{H,K}$ and a Gaussian process $Y_{H,K}$ with the covariance function

$$D_{H,K}(s, t) := \frac{\Gamma(1-K)}{K} \big[t^{2HK} + s^{2HK} - (t^{2H} + s^{2H})^K \big], \tag{34}$$

$(s, t) \in [0, T]^2$, due to [13], Proposition 1. Letting $A := 2^{-K} K / \Gamma(1 - K)$ and $B := 2^{1-K}$, by the decomposition we have the relation

$$C_{H,K} = -A D_{H,K} + B F_{HK} \tag{35}$$

between the covariance functions of $B_{H,K}$, $Y_{H,K}$ and B_{HK}, respectively. For any $0 \leq u < v \leq T$ and $0 \leq s < t \leq T$, if $Q = [u, v] \times [s, t]$ and $f(r) = f_{u,v}(r) := (u^{2H} + r^{2H})^K - (v^{2H} + r^{2H})^K$ for $r \geq 0$, then

$$\Delta^Q D_{H,K} = \frac{\Gamma(1-K)}{K} \big[f(t) - f(s) \big] > 0,$$

since $f'(r) > 0$ for each $r > 0$, and so $Y_{H,K}$ has positively correlated increments. Let $0 \leq a < b \leq T$. Since $V_p(\cdot) \leq V_1(\cdot)$, by part (i) of Proposition 11, it follows that

$$
\begin{aligned}
V_p\big(C_{H,K}; [a, b]^2\big) &\leq A V_1\big(D_{H,K}; [a, b]^2\big) + B V_p\big(F_{HK}; [a, b]^2\big) \\
&= A E\big[Y_{H,K}(b) - Y_{H,K}(a)\big]^2 + B V_p\big(F_{HK}; [a, b]^2\big).
\end{aligned}
$$

Using (34) we have

$$
\begin{aligned}
A E\big[Y_{H,K}(b) - Y_{H,K}(a)\big]^2 &= 2^{-K} \big[2(b^{2H} + a^{2H})^K - 2^K b^{2HK} - 2^K a^{2HK} \big] \\
&\leq 2^{-K} (b - a)^{2HK}
\end{aligned}
$$

by the left inequality in (31). Using inequality (23) for the fractional Brownian motion with the Hurst index HK, the first desired bound (32) with $C_1 = 5 \cdot 2^{-K}$ follows.

To prove the second desired bound (33) let $(t_j)_{j \in [m]}$ be a partition of $[0, T]$ and let $i \in (m)$. Again, since $Y_{H,K}$ has positively correlated increments and using (34) it follows that

$$A \sum_{j=1}^{m} \left| E \Delta_i^\kappa Y_{H,K} \Delta_j^\kappa Y_{H,K} \right| = A E \left[Y_{H,K}(t_i) - Y_{H,K}(t_{i-1}) \right] Y_{H,K}(T)$$

$$= 2^{-K} \left[t_i^{2HK} - t_{i-1}^{2HK} + \left(t_{i-1}^{2H} + T^{2H} \right)^K - \left(t_i^{2H} + T^{2H} \right)^K \right]$$

$$\leq \begin{cases} 2^{-K} (t_i - t_{i-1})^{2HK}, & \text{if } 2HK < 1, \\ 2^{1-K} HK T^{2HK-1} (t_i - t_{i-1}), & \text{if } 2HK \geq 1. \end{cases}$$

Since $V_p(\cdot) \leq V_1(\cdot)$, using (35), (24) with HK in place of H, and the preceding inequality, it follows that

$$\sum_{j=1}^{m} V_p \left(C_{H,K} ; J_i^\kappa \times J_j^\kappa \right)$$

$$\leq A \sum_{j=1}^{m} \left| E \left[\Delta_i^\kappa Y_{H,K} \Delta_j^\kappa Y_{H,K} \right] \right| + B \sum_{j=1}^{m} \left| E \left[\Delta_i^\kappa B_{HK} \Delta_j^\kappa B_{HK} \right] \right|$$

$$\leq C_2 (t_i - t_{i-1})^{1 \wedge (2HK)},$$

where $C_2 = 7 \cdot 2^{-K}$ if $2HK < 1$ and $C_2 = 6HK 2^{-K} T^{2HK-1}$ if $2HK \geq 1$. This completes the proof of the proposition. $\qquad \square$

4. Proof of the main result

The main result is Theorem 21 below dealing with almost sure convergence of sums of properly normalized powers of increments of the q.m. integral process (1). First, we prove a convergence of the mean of such sums under less restrictive assumptions.

Theorem 19. *Let $r > 0$ and $T > 0$. Let $X = \{X(t) : t \in [0, T]\}$ be a mean zero Gaussian process from the class $\mathcal{LSI}(\rho(\cdot))$ with the covariance function Γ_X such that for a constant C_1 and a number $p \geq 1$ the inequality*

$$V_p \left(\Gamma_X ; [s, t]^2 \right) \leq C_1 \left[\rho(t - s) \right]^2 \tag{36}$$

holds for all $0 \leq s < t \leq T$. Let $f \in \mathcal{W}_q[0, T]$ with $q \in \mathcal{Q}_p$ and let (κ_n) be a sequence of partitions $\kappa_n = (t_i^n)_{i \in [m_n]}$ of $[0, T]$ such that $|\kappa_n| \to 0$ as $n \to \infty$. Then there exists the q.m. integral process $Y(t) = \text{q.m.} \int_0^t f \, dX$, $t \in [0, T]$, and

$$\lim_{n \to \infty} \sum_{i=1}^{m_n} \frac{E |\Delta_i^n Y|^r}{[\rho(\Delta_i^n)]^r} \Delta_i^n = E |\eta|^r \int_0^T |f|^r, \tag{37}$$

where η is a standard normal random variable.

Proof. Since $\rho(\cdot)$ is continuous at zero, by (A1) of Definition 1, it follows that Γ_X is a continuous function. Then the q.m. integral process Y exists by Theorems 9 and 10. We shall prove (37). Since Y is a Gaussian process, for $0 \le s < t \le T$ we have

$$E\big|Y(t) - Y(s)\big|^r = E|\eta|^r \left(E\left[\int_s^t f \, dX\right]^2 \right)^{r/2}.$$

By (15), we have

$$E\left[\int_s^t f \, dX\right]^2 = \int_s^t \int_s^t f \otimes f \, d^2\Gamma_X$$

$$= \left| \int_s^t \int_s^t \big[f \otimes f - f^2(s)\big] \, d^2\Gamma_X + f^2(s) E\big[X(t) - X(s)\big]^2 \right|.$$

For $(s, t) \in [0, T]^2$ let

$$b(s, t) := \frac{\sigma_X^2(s, t)}{[\rho(|t - s|)]^2} - 1, \tag{38}$$

if $s \ne t$, and let $b(s, t) := 0$ if $s = t$. Then

$$R_n := \big(E|\eta|^r\big)^{-1} \sum_{i=1}^{m_n} \frac{E|\Delta_i^n Y|^r}{[\rho(\Delta_i^n)]^r} \Delta_i^n$$

$$= \sum_{i=1}^{m_n} \left| \frac{1}{[\rho(\Delta_i^n)]^2} \int_{t_{i-1}^n}^{t_i^n} \int_{t_{i-1}^n}^{t_i^n} \big[f \otimes f - f^2(t_{i-1}^n)\big] \, d^2\Gamma_X + f^2(t_{i-1}^n)\big[1 + b(t_{i-1}^n, t_i^n)\big] \right|^{r/2} \Delta_i^n$$

for each n. Also for each integer $n \ge 1$, let

$$T_n := \sum_{i=1}^{m_n} \big\{f^2(t_{i-1}^n)\big\}^{r/2} \Delta_i^n, \qquad U_n := \sum_{i=1}^{m_n} \big\{f^2(t_{i-1}^n)\big|b(t_{i-1}^n, t_i^n)\big|\big\}^{r/2} \Delta_i^n$$

and

$$W_n := \sum_{i=1}^{m_n} \left\{ \frac{1}{[\rho(\Delta_i^n)]^2} \left| \int_{t_{i-1}^n}^{t_i^n} \int_{t_{i-1}^n}^{t_i^n} \big[f \otimes f - f^2(t_{i-1}^n)\big] \, d^2\Gamma_X \right| \right\}^{r/2} \Delta_i^n.$$

If $r < 2$ then using the inequality $\big||A|^{r/2} - |B|^{r/2}\big| \le |A - B|^{r/2}$ it follows that

$$|R_n - T_n| \le U_n + W_n \tag{39}$$

for each n. If $r \ge 2$, then using the Minkowskii inequality for weighted sums, it follows that

$$\big|R_n^{2/r} - T_n^{2/r}\big| \le U_n^{2/r} + W_n^{2/r} \tag{40}$$

for each n. Recall that the mesh $|\kappa_n| \to 0$ as $n \to \infty$. Therefore since f is regulated, and so $|f|^r$ is Riemann integrable, it follows that

$$\lim_{n\to\infty} T_n = \lim_{n\to\infty} \sum_{i=1}^{m_n} |f(t_{i-1}^n)|^r \Delta_i^n = \int_0^T |f|^r. \tag{41}$$

We will show that U_n and W_n tend to zero as $n \to \infty$. Assuming this, by (39) if $r < 2$, by (40) if $r \geq 2$ and (41), the conclusion (37) follows.

To prove convergence of U_n let $\varepsilon > 0$. Recalling notation (38) and using condition (A1) of Definition 1, we have

$$|b(t_{i-1}^n, t_i^n)| \leq \frac{E[\Delta_i^n X]^2}{[\rho(\Delta_i^n)]^2} + 1 \leq L^2 + 1$$

for each $n \in \mathbb{N}$ and $i \in (m_n]$. For each $\delta > 0$ letting

$$\phi_\varepsilon(\delta) := \sup\{|b(s, s+h)| : s \in [\varepsilon, T), h \in (0, \delta \wedge (T-s)]\} \tag{42}$$

it follows that

$$\sum_{i:t_{i-1}^n \geq \varepsilon} \{f^2(t_{i-1}^n)|b(t_{i-1}^n, t_i^n)|\}^{r/2} \Delta_i^n \leq \{\phi_\varepsilon(|\kappa_n|)\}^{r/2} \sum_{i=1}^{m_n} |f(t_{i-1}^n)|^r \Delta_i^n$$

for all $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$ we have

$$U_n \leq \left(\sum_{i:t_{i-1}^n \geq \varepsilon} + \sum_{i:t_{i-1}^n < \varepsilon} \right) \{f^2(t_{i-1}^n)|b(t_{i-1}^n, t_i^n)|\}^{r/2} \Delta_i^n$$

$$\leq \{\phi_\varepsilon(|\kappa_n|)\}^{r/2} \sum_{i=1}^{m_n} |f(t_{i-1}^n)|^r \Delta_i^n + (\varepsilon + |\kappa_n|)\|f\|_{\sup}^r (L^2+1)^{r/2}.$$

By conditions (A1) and (A2) of Definition 1, and since the Riemann sums are bounded as $|\kappa_n| \to 0$ with $n \to \infty$, U_n tends to zero as $n \to \infty$.

We prove convergence of W_n first assuming that $p = 1$. In this case, by (10) and (36), we have

$$W_n \leq (2\|f\|_{\sup})^{r/2} \sum_{i=1}^{m_n} \left\{ \frac{V_1(\Gamma_X; [t_{i-1}^n, t_i^n]^2)}{[\rho(\Delta_i^n)]^2} Osc(f; [t_{i-1}^n, t_i^n]) \right\}^{r/2} \Delta_i^n$$

$$\leq (2C_1\|f\|_{\sup})^{r/2} \sum_{i=1}^{m_n} \{Osc(f; [t_{i-1}^n, t_i^n])\}^{r/2} \Delta_i^n. \tag{43}$$

Let $\varepsilon > 0$. Since f is a regulated function there is a partition $\{s_j\}_{j=0}^k$ of $[0, T]$ such that $Osc(f; (s_{j-1}, s_j)) < \varepsilon$ for each $j \in (k]$ by Theorem 2.1 in [7]. Since $|\kappa_n| \to 0$ as $n \to \infty$ there is an $N \in \mathbb{N}$ such that $|\kappa_n| < \varepsilon/(2k)$ for each $n \geq N$. For each n let J_n be the set of indices $i \in (m_n]$

such that $s_j \in [t_{i-1}^n, t_i^n]$ for some $j \in [k]$ and let $J_n^c := (m_n) \setminus J_n$. Then the cardinality of J_n does not exceed $2k$, and continuing (43), we have for each $n \geq N$

$$W_n \leq \left(4C_1 \|f\|_{\sup}^2\right)^{r/2} \sum_{i \in J_n} \Delta_i^n + \left(2\varepsilon C_1 \|f\|_{\sup}\right)^{r/2} \sum_{i \in J_n^c} \Delta_i^n$$

$$\leq \varepsilon \left(4C_1 \|f\|_{\sup}^2\right)^{r/2} + \left(2\varepsilon C_1 \|f\|_{\sup}\right)^{r/2} T,$$

since the mesh $|\kappa_n| < \varepsilon/(2k)$.

Now suppose that $p > 1$. Let $q' > q$ be such that $\frac{1}{p} + \frac{1}{q} > \frac{1}{p} + \frac{1}{q'} > 1$. By (13), we have

$$W_n \leq \left(K_{p,q'} \|f\|_{[q']}\right)^{r/2} \sum_{i=1}^{m_n} \left\{ \frac{V_p(\Gamma_X; [t_{i-1}^n, t_i^n]^2)}{[\rho(\Delta_i^n)]^2} V_{q'}\left(f; [t_{i-1}^n, t_i^n]\right) \right\}^{r/2} \Delta_i^n$$

$$\leq \left(K_{p,q'} \|f\|_{[q']} C_1 V_q(f)^{q/q'}\right)^{r/2} \sum_{i=1}^{m_n} \left\{ Osc\left(f; [t_{i-1}^n, t_i^n]\right)^{1-q/q'} \right\}^{r/2} \Delta_i^n.$$

Since a function of bounded q-variation is regulated the arguments used in the preceding case $p = 1$ apply and show that W_n tends to zero as $n \to \infty$. The theorem is proved. □

In the case $f \equiv 1$, we have the following conclusion.

Corollary 20. *Let $r > 0$ and $T > 0$. Let $X = \{X(t) : t \in [0, T]\}$ be a mean zero Gaussian process from the class $\mathcal{LSI}(\rho(\cdot))$, and let (κ_n) be a sequence of partitions $\kappa_n = (t_i^n)_{i \in [m_n]}$ of $[0, T]$ such that $|\kappa_n| \to 0$ as $n \to \infty$. Then*

$$\lim_{n \to \infty} \sum_{i=1}^{m_n} \frac{E|\Delta_i^n X|^r}{[\rho(\Delta_i^n)]^r} \Delta_i^n = E|\eta|^r T,$$

where η is a standard normal random variable.

Proof. In the proof of Theorem 19 taking $f \equiv 1$ it follows that for each $n \geq 1$, in the present case we have $T_n = T$, $W_n = 0$,

$$R_n = \sum_{i=1}^{m_n} \left[1 + b\left(t_{i-1}^n, t_i^n\right)\right]^{r/2} \Delta_i^n \quad \text{and} \quad U_n = \sum_{i=1}^{m_n} \left[b\left(t_{i-1}^n, t_i^n\right)\right]^{r/2} \Delta_i^n.$$

The argument used in the proof of Theorem 19 gives that $U_n \to 0$ as $n \to \infty$, and so $R_n \to T$ as $n \to \infty$, proving the corollary. □

Next, is the main result.

Theorem 21. *Let $T > 0$, let $\rho \in R[0, T]$ be such that $\gamma_*(\rho) = \gamma$ for some $\gamma \in (0, 1)$, let $p := \max\{1, 1/(2\gamma)\}$, and let $1 < r < 2/\max\{(2\gamma - 1), 0\}$. Let X be a mean zero Gaussian process*

from the class $\mathcal{LSI}(\rho(\cdot))$ with the covariance function Γ_X. Suppose that there is a constant C_1 such that (36) holds for all $0 \le s < t \le T$, and there is a constant C_2 such that the inequality

$$\sum_{j=1}^{m} V_p\big(\Gamma_X; J_i^\kappa \times J_j^\kappa\big) \le C_2\big(\Delta_i^\kappa\big)^{1 \wedge (2\gamma)} \tag{44}$$

holds for each partition $\kappa = (t_j)_{j \in [m]}$ of $[0, T]$ and each $i \in (m]$. Let $f \in \mathcal{W}_q[0, T]$ with $q \in \mathcal{Q}_p$, and let (κ_n) be a sequence of partitions $\kappa_n = (t_i^n)_{i \in [m_n]}$ of $[0, T]$ such that

$$\sup\Big\{\alpha : \lim_{n \to \infty} |\kappa_n|^\alpha \log n = 0\Big\} = \Big(1 \wedge \frac{2}{r}\Big) + \big(0 \wedge (1 - 2\gamma)\big). \tag{45}$$

Then there exists the q.m. integral process $Y(t) = $ q.m. $\int_0^t f \, dX$, $t \in [0, T]$, and with probability one

$$\lim_{n \to \infty} \sum_{i=1}^{m_n} \frac{|\Delta_i^n Y|^r}{[\rho(\Delta_i^n)]^r} \Delta_i^n = E|\eta|^r \int_0^T |f|^r, \tag{46}$$

where η is a standard normal random variable.

Remark 22. The right side of (45) is less than or equal to 1. Also it is positive for any $1 < r < \infty$ if $\gamma \le 1/2$, and for any $1 < r < 2/(2\gamma - 1)$ if $\gamma > 1/2$. It follows from the proof of the theorem that if the local variance $\rho(u) = u^\gamma$ then the hypothesis (45) can be replaced by the following one

$$\lim_{n \to \infty} |\kappa_n|^{(1 \wedge 2/r) + (0 \wedge (1 - 2\gamma))} \log n = 0.$$

It is known that this condition with $r = 2$ is best possible ([5] and [19], Theorem 2.6).

Proof of Theorem 21. The q.m. integral process $Y(t) = $ q.m. $\int_0^t f \, dX$, $t \in [0, T]$, exists due to reasons stated in the proof of Theorem 19. For each $n \ge 1$, let

$$Z_n := \left(\sum_{i=1}^{m_n} c_{i,n}\big|\Delta_i^n Y\big|^r\right)^{1/r}, \qquad \text{where } c_{i,n} := \big[\rho\big(\Delta_i^n\big)\big]^{-r} \Delta_i^n.$$

Denoting the median of a real random variable Z by med(Z), by Lemma 2.2 of [19], for each $\varepsilon > 0$

$$\Pr\big(\{\big|Z_n - \text{med}(Z_n)\big| > \varepsilon\}\big) \le 2 \exp\Big\{-\frac{\varepsilon^2}{2\sigma_n^2}\Big\}, \tag{47}$$

where

$$\sigma_n^2 := \sup\left\{E\left(\sum_{i=1}^{m_n} b_i c_{i,n}^{1/r} \Delta_i^n Y\right)^2 : (b_i)_{i \in (m_n]} \in \mathbb{R}^{m_n}, \sum_{i=1}^{m_n} |b_i|^{r'} \le 1\right\}$$

and $1/r + 1/r' = 1$. For each $n \geq 1$ and $i \in (m_n]$, by (15) we have

$$M_{i,n} := \sum_{j=1}^{m_n} \left| E\left(\Delta_i^n Y \Delta_j^n Y\right) \right| = \sum_{j=1}^{m_n} \left| \int_{t_{i-1}^n}^{t_i^n} \int_{t_{j-1}^n}^{t_j^n} f \otimes f \, d^2 \Gamma_X \right|.$$

For each $n \geq 1$ and each vector $(b_i) \in \mathbb{R}^{m_n}$, by Lemma 2.2 of [23]

$$E\left(\sum_{i=1}^{m_n} b_i c_{i,n}^{1/r} \Delta_i^n Y \right)^2 = \left| \sum_{i=1}^{m_n} \sum_{j=1}^{m_n} b_i b_j (c_{i,n} c_{j,n})^{1/r} E\left(\Delta_i^n Y \Delta_j^n Y\right) \right|$$

$$\leq \sum_{i=1}^{m_n} b_i^2 c_{i,n}^{2/r} M_{i,n}$$

$$\leq \begin{cases} \displaystyle \max_{1 \leq i \leq m_n} c_{i,n}^{2/r} M_{i,n} \left(\sum_{j=1}^{m_n} |b_j|^{r'} \right)^{2/r'}, & \text{if } 2 \leq r < \infty, \\[4mm] \displaystyle \left(\sum_{i=1}^{m_n} c_{i,n}^{2/(2-r)} M_{i,n}^{r/(2-r)} \right)^{(2-r)/r} \left(\sum_{j=1}^{m_n} |b_j|^{r'} \right)^{2/r'}, & \text{if } 1 < r < 2. \end{cases}$$

It then follows that for each $n \geq 1$

$$\sigma_n^2 \leq \begin{cases} \displaystyle \max_{1 \leq i \leq m_n} \left(\frac{\Delta_i^n}{[\rho(\Delta_i^n)]^r} \right)^{2/r} M_{i,n}, & \text{if } 2 \leq r < \infty, \\[4mm] \displaystyle \left(\sum_{i=1}^{m_n} \left(\frac{\Delta_i^n}{[\rho(\Delta_i^n)]^r} \right)^{2/(2-r)} M_{i,n}^{r/(2-r)} \right)^{(2-r)/r}, & \text{if } 1 < r < 2. \end{cases}$$

By (14) if $p > 1$ and by (9) if $p = 1$, and then by (44), for each i

$$M_{i,n} \leq K_{p,q} \|f\|_{[q]}^2 \sum_{j=1}^{m_n} V_p\left(\Gamma_X; J_i^{\kappa_n} \times J_j^{\kappa_n}\right) \leq C_2 K_{p,q} \|f\|_{[q]}^2 [\Delta_i^n]^{1/p},$$

where $K_{1,\infty} := 1$ and $\|f\|_{[\infty]} := \|f\|_{\sup}$. By the hypothesis on the local variance ρ, we have

$$\gamma_*(\rho) = \inf \left\{ \alpha > 0 : \sup_{u > 0} \frac{u^\alpha}{\rho(u)} < \infty \right\} = \gamma.$$

In the case $p > 1$, we have $1 - 2\gamma > 0$, and so for each $\delta \in (0, 1/r)$,

$$\sigma_n^2 \leq C_2 K_{p,q} \|f\|_{[q]}^2 \begin{cases} \displaystyle |\kappa_n|^{2/r - 2\delta} \max_{1 \leq i \leq m_n} \left(\frac{(\Delta_i^n)^{\gamma+\delta}}{\rho(\Delta_i^n)} \right)^2, & \text{if } 2 \leq r < \infty, \\[4mm] \displaystyle T^{(2-r)/r} |\kappa_n|^{1-2\delta} \max_{1 \leq i \leq m_n} \left(\frac{(\Delta_i^n)^{\gamma+\delta}}{\rho(\Delta_i^n)} \right)^2, & \text{if } 1 < r < 2, \end{cases}$$

$$= o\left(1/(\log n)\right),$$

as $n \to \infty$ by the hypothesis (45). In the case $p = 1$, we have $1 - 2\gamma \leq 0$, and so for each $\delta > 0$,

$$
\sigma_n^2 \leq C_2 \|f\|_{\sup}^2
\begin{cases}
|\kappa_n|^{2/r + 1 - 2\gamma - 2\delta} \max\limits_{1 \leq i \leq m_n} \left(\dfrac{(\Delta_i^n)^{\gamma + \delta}}{\rho(\Delta_i^n)} \right)^2, & \text{if } 2 \leq r < \infty, \\[4mm]
T^{(2-r)/r} |\kappa_n|^{2 - 2\gamma - 2\delta} \max\limits_{1 \leq i \leq m_n} \left(\dfrac{(\Delta_i^n)^{\gamma + \delta}}{\rho(\Delta_i^n)} \right)^2, & \text{if } 1 < r < 2,
\end{cases}
$$

$$
= o\big(1/(\log n)\big),
$$

as $n \to \infty$ by the hypothesis (45). By (47) and Borel–Cantelli lemma it then follows that

$$
\lim_{n \to \infty} \big| Z_n - \operatorname{med}(Z_n) \big| = 0
$$

with probability one. Using our Theorem 19 and the argument of [19], Theorem 2.3, it follows that (46) holds with probability one. $\qquad\square$

In the case $f \equiv 1$, we have the following conclusion.

Corollary 23. *Let $T > 0$, let $\rho \in R[0, T]$ be such that $\gamma_*(\rho) = \gamma$ for some $\gamma \in (0, 1)$, and let $1 < r < 2/\max\{(2\gamma - 1), 0\}$. Let $X = \{X(t) : t \in [0, T]\}$ be a mean zero Gaussian process from the class $\mathcal{LSI}(\rho(\cdot))$. Suppose there is a constant C_2 such that the inequality*

$$
\sum_{j=1}^{m} \big| E\big[\Delta_i^\kappa X \Delta_j^\kappa X \big] \big| \leq C_2 \big(\Delta_i^\kappa \big)^{1 \wedge (2\gamma)} \tag{48}
$$

holds for each partition $\kappa = (t_j)_{j \in [m]}$ of $[0, T]$ and each $i \in (m)$. Let (κ_n) be a sequence of partitions $\kappa_n = (t_i^n)_{i \in [m_n]}$ of $[0, T]$ such that (45) holds. Then with probability one

$$
\lim_{n \to \infty} \sum_{i=1}^{m_n} \frac{|\Delta_i^n X|^r}{[\rho(\Delta_i^n)]^r} \Delta_i^n = E|\eta|^r T, \tag{49}
$$

where η is a standard normal random variable.

Proof. The proof is the same as of Theorem 21 except that now Corollary 20 is used in place of Theorem 19 and the bound (48) is used in place of (44). $\qquad\square$

Acknowledgements

References

[1] Baxter, G. (1956). A strong limit theorem for Gaussian processes. *Proc*. *Amer*. *Math*. *Soc*. **7** 522–527. MR0090920

[2] Berman, S.M. (1974). Sojourns and extremes of Gaussian processes. *Ann*. *Probab*. **2** 999–1026. Corrections: *Ann*. *Probab*. **8** (1980) 999, **12** (1984) 281. MR0372976

[3] Bojdecki, T., Gorostiza, L.G. and Talarczyk, A. (2004). Sub-fractional Brownian motion and its relation to occupation times. *Statist*. *Probab*. *Lett*. **69** 405–419. MR2091760

[4] Corcuera, J.M., Nualart, D. and Woerner, J.H.C. (2006). Power variation of some integral fractional processes. *Bernoulli* **12** 713–735. MR2248234

[5] de la Vega, F.W. (1974). On almost sure convergence of quadratic Brownian variation. *Ann*. *Probab*. **2** 551–552. MR0359029

[6] Dudley, R.M. (1973). Sample functions of the Gaussian process. *Ann*. *Probab*. **1** 66–103. MR0346884

[7] Dudley, R.M. and Norvaiša, R. (2011). *Concrete Functional Calculus*. *Springer Monographs in Mathematics*. New York: Springer. MR2732563

[8] Friz, P. and Victoir, N. (2010). Differential equations driven by Gaussian signals. *Ann*. *Inst*. *Henri Poincaré Probab*. *Stat*. **46** 369–413. MR2667703

[9] Gladyshev, E.G. (1961). A new limit theorem for stochastic processes with Gaussian increments. *Teor*. *Verojatnost*. *i Primenen* **6** 57–66. MR0145574

[10] Hobson, E.W. (1927). *The Theory of Functions of a Real Variable and the Theory of Fourier's Series*, *Vol*. *I*. Cambridge: Cambridge Univ. Press.

[11] Houdré, C. and Villa, J. (2003). An example of infinite dimensional quasi-helix. In *Stochastic Models* (*Mexico City*, 2002). *Contemp*. *Math*. **336** 195–201. Providence, RI: Amer. Math. Soc. MR2037165

[12] Klein, R. and Giné, E. (1975). On quadratic variation of processes with Gaussian increments. *Ann*. *Probab*. **3** 716–721. MR0378070

[13] Lei, P. and Nualart, D. (2009). A decomposition of the bifractional Brownian motion and some applications. *Statist*. *Probab*. *Lett*. **79** 619–624. MR2499385

[14] Leśniewicz, M. and Leśniewicz, R. (1985). On double Riemann–Stieltjes integrals for functions of finite generalized variations. *Fasc*. *Math*. **15** 37–61. MR0819811

[15] Lévy, P. (1940). Le mouvement brownien plan. *Amer*. *J*. *Math*. **62** 487–550. MR0002734

[16] Loève, M. (1978). *Probability Theory II*, 4th ed. *Graduate Texts in Mathematics* **46**. New York: Springer. MR0651018

[17] Malukas, R. (2011). Limit theorems for a quadratic variation of Gaussian processes. *Nonlinear Anal*. *Model*. *Control* **16** 435–452. MR2885721

[18] Marcus, M.B. (1968). Hölder conditions for Gaussian processes with stationary increments. *Trans*. *Amer*. *Math*. *Soc*. **134** 29–52. MR0230368

[19] Marcus, M.B. and Rosen, J. (1992). *p*-variation of the local times of symmetric stable processes and of Gaussian processes with stationary increments. *Ann*. *Probab*. **20** 1685–1713. MR1188038

[20] Norvaiša, R. (2011). A complement to Gladyshev's theorem. *Lithuanian Math*. *J*. **51** 26–35. MR2784375

[21] Norvaiša, R. and Salopek, D.M. (2000). Estimating the Orey index of a Gaussian stochastic process with stationary increments: An application to financial data set. In *Stochastic Models* (*Ottawa*, *ON*, 1998). *CMS Conf*. *Proc*. **26** 353–374. Providence, RI: Amer. Math. Soc. MR1765020

[22] Orey, S. (1970). Gaussian sample functions and the Hausdorff dimension of level crossings. *Z*. *Wahrsch*. *Verw*. *Gebiete* **15** 249–256. MR0279882

[23] Shao, Q.-M. (1996). *p*-variation of Gaussian processes with stationary increments. *Studia Sci. Math. Hungar.* **31** 237–247. MR1367709

[24] Young, L.C. (1936). An inequality of the Hölder type, connected with Stieltjes integration. *Acta Math.* **67** 251–282. MR1555421