Testing equality of spectral densities using randomization techniques

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In this paper, we investigate the testing problem that the spectral density matrices of several, not necessarily independent, stationary processes are equal. Based on an L_2-type test statistic, we propose a new nonparametric approach, where the critical values of the tests are calculated with the help of randomization methods. We analyze asymptotic exactness and consistency of these randomization tests and show in simulation studies that the new procedures posses very good size and power characteristics.

Keywords: multivariate time series; nonparametric tests; periodogram matrix; randomization tests; spectral density matrix

1. Introduction and motivation

Suppose, one observes two stretches of time series data $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ generated by some unknown stationary time series processes. The question whether the models where these two stretches come from are identical is of considerable interest and has been investigated elaborately in statistical literature. For instance, by imposing certain parametric assumptions on the data generation process (DGP), it becomes possible to compare suitable estimated parameters to judge how "near" the related models are. The assumption of underlying autoregressive processes of some finite order, say, allows to compare estimated coefficients to quantify the nearness of both DGPs. But these models are usually not valid if the parametric assumption is not fulfilled. Therefore, nonparametric methods are desired that can be justified theoretically for a broader class of stochastic processes.

If one is interested in second order properties, it seems obvious to focus on autocovariances $\gamma(h)$, autocorrelations $\rho(h)$ or spectral densities $f(\omega)$, where the latter choice has some advantages over the others, see, for example, Paparoditis [21]. Over the years, several test statistics have been introduced that could be used for testing the null hypothesis

$$\left\{ f_X(\omega) = f_Y(\omega) \text{ for all } \omega \in [-\pi, \pi] \right\} \tag{1.1}$$

versus the alternative $\{f_X \neq f_Y$ on a set of positive λ-measure$\}$. These may be periodogram-based, see, for example, Coates and Diggle [6], Pötscher and Reschenhofer [24], Diggle and Fisher [10], Caiado *et al.* [4,5] and Preuss and Hildebrandt [25], or based on kernel spectral density estimators, see, for example, Maharaj [20], Eichler [11] or Dette and Paparoditis [9].

Jentsch and Pauly [16] have investigated the asymptotic properties of test statistics closely related to "periodogram-based distances" proposed by Caiado *et al.* [5]. They show that contrary to integrated periodograms these tests are usually not consistent which is caused by the use of non-consistent estimators for the spectral densities, the periodograms. In contrast to that, L_2-type statistics based on the smoothed periodogram are shown to be consistent, see Paparoditis [21, 22], Eichler [11], Dette and Paparoditis [9] and Jentsch [15]. A possible test statistic for testing equality of spectral densities is therefore a suitable inflated version of

$$\int_{-\pi}^{\pi} \left(\widehat{f_X}(\omega) - \widehat{f_Y}(\omega) \right)^2 d\omega, \tag{1.2}$$

where $\widehat{f_X}$ and $\widehat{f_Y}$ are nonparametric kernel estimators of the spectras. Central limit theorems for such statistics can be found in the above mentioned literature. For instance, Paparoditis [21] has emphasized that statistics of this kind converge very slowly to a normal distribution, which results in a poor performance of the corresponding asymptotic test, particularly for small sample sizes. Therefore, other approaches as bootstrap or resampling techniques in general are needed to improve the small sample behavior of these tests. Paparoditis [21,22] has used a (parametric) ARMA-bootstrap for goodness-of-fit testing and Dette and Paparoditis [9] have proposed a different approach based on the asymptotic Wishart-distribution of periodogram matrices.

In this paper, we consider the more general testing problem of comparing the spectral density matrices of several, not necessarily independent, stationary processes. As motivated above, we will use an L_2-type statistic of the form (1.2) which is adjusted for this multiple sample testing problem. To overcome the slow convergence speed of our test statistic, we propose a resampling method to compute critical values, the randomization technique, that exploits the well-known asymptotic independence of periodograms $I_n(\omega)$ and $I_n(\lambda)$ for different frequencies $\omega \neq \lambda$ with $\omega, \lambda \in [0, \pi]$. A nice feature of this approach is that only one tuning parameter, the bandwidth of the involved kernel spectral densities, has to be selected. This can be done automatically by selection procedures based on cross validation as has been proposed by Beltrão and Bloomfield [2] and Robinson [28]. In comparison to other mainstream resampling techniques as for example, the autoregressive sieve or block bootstrap techniques, it has the advantage that no other tuning parameter as order or block length has to be assessed in addition to the bandwidth. It is worth noting that these choices appear to be very crucial in applications and the performance of the corresponding tests usually reacts very sensitive on the choice of these quantities. Nevertheless, these procedures are not applicable (at least) directly, because they do not mimic the desired null distribution under the alternative automatically. Moreover, the usage of randomization methods is very natural and has also been proposed by Diggle and Fisher [10] for the special (and included) testing problem of comparing the spectral densities of two independent univariate stationary time series. This idea has been adopted by Maharaj [20] for comparing evolutionary spectra of univariate nonstationary time series. In both papers, the authors (only) account for the application of this procedure with the help of simulations and the following nice heuristic. Recall that a randomization test holds the prescribed level α exact for finite sample sizes if the data is invariant under the corresponding randomization group, see, for example, Lehmann and Romano [18]. Since $I_{n,X}(\omega)$ and $I_{n,Y}(\omega)$ are asymptotically exchangeable in the case of independent time series, one would suggest that a randomization test, say φ_n^*, for (1.1) based on periodograms or

functions of periodograms is at least asymptotically exact, that is, $E(\varphi_n^*) \to \alpha$ holds as $n \to \infty$. However, this is only a conjecture and so far the randomization technique has not been analyzed in this context from a mathematical point of view in the literature. This is without question meaningful since it is offhand not clear in which situations it is applicable. We will close this gap in the next sections by investigating more general and in mathematical detail whether at all and, if true, under which possibly needed additional assumptions randomization-based testing procedures lead to success, that is, asymptotically exact and consistent tests. Amongst others it will turn out that this can be done effectively for testing the special null hypothesis (1.1).

The paper is organized as follows. In the next Section 1.1, we introduce our notation, state the general testing problem and define the L_2-type test statistic. Section 2 presents the main assumptions on the process and states asymptotic results for the test statistic which lead to an asymptotically exact and consistent benchmark test. The randomization procedure together with a corresponding discussion about its consistency is described in Section 3. Based on our theoretical results we compare the proposed tests in the special situation of testing for (1.1) in an extensive simulation study in Section 4, where the effects of dependence and bandwidth choice on the size and power behavior of both types of tests are investigated. All proofs are deferred to Section 6.

1.1. Notation and formulation of the problem

We consider a d-dimensional zero mean stationary process $(\underline{X}_t, t \in \mathbb{Z})$ with $d = pq$, $p, q \in \mathbb{N}$ and $q \geq 2$. Under suitable assumptions, the process $(\underline{X}_t, t \in \mathbb{Z})$ posses a continuous $(d \times d)$ spectral density matrix $\mathbf{f}$ given by

$$\mathbf{f}(\omega) := \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \mathbf{\Gamma}(h) \exp(-ih\omega), \qquad \omega \in [-\pi, \pi],$$

where $\mathbf{\Gamma}(h) := E(\underline{X}_{t+h}\underline{X}_t^T)$, $h \in \mathbb{Z}$, are the corresponding autocovariance matrices. To introduce our testing problem of interest, we write

$$\mathbf{f}(\omega) = \begin{pmatrix} f_{11}(\omega) & f_{12}(\omega) & \cdots & f_{1d}(\omega) \\ f_{21}(\omega) & f_{22}(\omega) & & f_{2d}(\omega) \\ \vdots & & \ddots & \vdots \\ f_{d1}(\omega) & f_{d2}(\omega) & \cdots & f_{dd}(\omega) \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{F}_{11}(\omega) & \mathbf{F}_{12}(\omega) & \cdots & \mathbf{F}_{1q}(\omega) \\ \mathbf{F}_{21}(\omega) & \mathbf{F}_{22}(\omega) & & \mathbf{F}_{2q}(\omega) \\ \vdots & & \ddots & \vdots \\ \mathbf{F}_{q1}(\omega) & \mathbf{F}_{q2}(\omega) & \cdots & \mathbf{F}_{qq}(\omega) \end{pmatrix}, \qquad \omega \in [-\pi, \pi],$$

where $\mathbf{F}_{mn}(\omega)$ are $(p \times p)$ block matrices with $\mathbf{F}_{mn}(\omega) = \overline{\mathbf{F}_{nm}(\omega)}^T$ for all m, n. Here and throughout the paper, all matrix-valued quantities are written as bold letters, all vector-valued

quantities are underlined, $\overline{\mathbf{A}}$ denotes the complex conjugation and $\mathbf{A}^T$ the transposition of a complex matrix $\mathbf{A}$.

In this general setup, we want to test whether all $(p \times p)$ spectral density matrices $\mathbf{F}_{kk}$ of the q (sub-)processes $(\underline{X}_{t,k}, t \in \mathbb{Z})$, $k = 1, \ldots, q$ with

$$\underline{X}_{t,k} = (X_{t,(k-1)p+1}, \ldots, X_{t,kp})^T$$

are identical. Precisely, suppose we have observations $\underline{X}_1, \ldots, \underline{X}_n$ at hand and we want to test the null hypothesis

$$H_0 \colon \left\{ \mathbf{F}_{11}(\omega) = \mathbf{F}_{22}(\omega) = \cdots = \mathbf{F}_{qq}(\omega) \text{ for all } \omega \in [-\pi, \pi] \right\} \tag{1.3}$$

against the alternative

$$H_1 \colon \left\{ \exists k_1, k_2 \in \{1, \ldots, q\}, A \subset \mathcal{B} \text{ with } \lambda(A) > 0 \text{ such that } \mathbf{F}_{k_1 k_1}(\omega) \neq \mathbf{F}_{k_2 k_2}(\omega) \; \forall \omega \in A \right\},$$

where λ denotes the one-dimensional Lebesgue-measure on the Borel σ-algebra $\mathcal{B}$. Observe that this general framework includes (1.1).

Consider now the periodogram matrix $\mathbf{I}(\omega) := \underline{J}(\omega)\overline{\underline{J}(\omega)}^T$ based on $\underline{X}_1, \ldots, \underline{X}_n$, where

$$\underline{J}(\omega) := \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} \underline{X}_t e^{-it\omega}, \qquad \omega \in [-\pi, \pi] \tag{1.4}$$

is the corresponding d-variate discrete Fourier transform (DFT). For a better comparison with the spectral density matrix $\mathbf{f}$, we write as above

$$\mathbf{I}(\omega) = \begin{pmatrix} I_{11}(\omega) & I_{12}(\omega) & \cdots & I_{1d}(\omega) \\ I_{21}(\omega) & I_{22}(\omega) & & I_{2d}(\omega) \\ \vdots & & \ddots & \vdots \\ I_{d1}(\omega) & I_{d2}(\omega) & \cdots & I_{dd}(\omega) \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{I}_{11}(\omega) & \mathbf{I}_{12}(\omega) & \cdots & \mathbf{I}_{1q}(\omega) \\ \mathbf{I}_{21}(\omega) & \mathbf{I}_{22}(\omega) & & \mathbf{I}_{2q}(\omega) \\ \vdots & & \ddots & \vdots \\ \mathbf{I}_{q1}(\omega) & \mathbf{I}_{q2}(\omega) & \cdots & \mathbf{I}_{qq}(\omega) \end{pmatrix}, \qquad \omega \in [-\pi, \pi],$$

where $\mathbf{I}_{mn}(\omega)$ are $(p \times p)$ block matrices with $\mathbf{I}_{mn}(\omega) = \overline{\mathbf{I}_{nm}(\omega)}^T$ for all m, n. Moreover, we define its pooled block diagonal periodogram matrix by

$$\widetilde{\mathbf{I}}(\omega_k) := \frac{1}{q} \sum_{j=1}^{q} \mathbf{I}_{jj}(\omega_k).$$

With that we can introduce the kernel estimator

$$\widehat{\mathbf{f}}(\omega) := \frac{1}{n} \sum_{k=-\lfloor(n-1)/2\rfloor}^{\lfloor n/2 \rfloor} K_h(\omega - \omega_k)\mathbf{I}(\omega_k), \qquad \omega \in [-\pi, \pi]$$

for the spectral density matrix $\mathbf{f}(\omega)$, where $\lfloor x \rfloor$ is the integer part of $x \in \mathbb{R}$, $\omega_k := 2\pi\frac{k}{n}, k = -\lfloor\frac{n-1}{2}\rfloor, \ldots, \lfloor\frac{n}{2}\rfloor$ are the Fourier frequencies, K is a nonnegative symmetric kernel function satisfying $\int K(x)\,dx = 2\pi$, h is the bandwidth and $K_h(\cdot) := \frac{1}{h}K(\frac{\cdot}{h})$. Note that the periodogram is understood as a 2π-periodically extended function on the real line throughout the paper. Moreover, we write as above

$$\widehat{\mathbf{f}}(\omega) = \begin{pmatrix} \widehat{f}_{11}(\omega) & \widehat{f}_{12}(\omega) & \cdots & \widehat{f}_{1d}(\omega) \\ \widehat{f}_{21}(\omega) & \widehat{f}_{22}(\omega) & & \widehat{f}_{2d}(\omega) \\ \vdots & & \ddots & \vdots \\ \widehat{f}_{d1}(\omega) & \widehat{f}_{d2}(\omega) & \cdots & \widehat{f}_{dd}(\omega) \end{pmatrix}$$

$$= \begin{pmatrix} \widehat{\mathbf{F}}_{11}(\omega) & \widehat{\mathbf{F}}_{12}(\omega) & \cdots & \widehat{\mathbf{F}}_{1q}(\omega) \\ \widehat{\mathbf{F}}_{21}(\omega) & \widehat{\mathbf{F}}_{22}(\omega) & & \widehat{\mathbf{F}}_{2q}(\omega) \\ \vdots & & \ddots & \vdots \\ \widehat{\mathbf{F}}_{q1}(\omega) & \widehat{\mathbf{F}}_{q2}(\omega) & \cdots & \widehat{\mathbf{F}}_{qq}(\omega) \end{pmatrix}, \qquad \omega \in [-\pi, \pi],$$

where $\widehat{\mathbf{F}}_{mn}(\omega)$ are uniformly consistent estimators of the $(p \times p)$ block matrices $\mathbf{F}_{mn}(\omega)$ for all m, n, that is, convergence $\sup_\omega \|\widehat{\mathbf{F}}_{mn}(\omega) - \mathbf{F}_{mn}(\omega)\| \to 0$ holds almost surely. Here and throughout the paper, $\| \cdot \|$ denotes the Frobenius norm, that is, for a matrix $\mathbf{A} = (a_{ij})_{1 \le i, j \le p} \in \mathbb{C}^{p \times p}$ we set

$$\|\mathbf{A}\|^2 := \text{tr}(\mathbf{A}\overline{\mathbf{A}}^T) = \sum_{i,j=1}^{p} |a_{ij}|^2,$$

where $\text{tr}(\cdot)$ stands for the trace of a matrix, i.e. $\text{tr}(\mathbf{A}) := \sum_{i=1}^{p} a_{ii}$.

For testing H_0, we now propose the L_2-type test statistic

$$T_n := nh^{1/2} \int_{-\pi}^{\pi} \sum_{r=1}^{q} \left\| \frac{1}{n} \sum_{k=-\lfloor(n-1)/2\rfloor}^{\lfloor n/2 \rfloor} K_h(\omega - \omega_k)\big(\mathbf{I}_{rr}(\omega_k) - \widetilde{\mathbf{I}}(\omega_k)\big) \right\|^2 d\omega$$

$$= nh^{1/2} \int_{-\pi}^{\pi} \sum_{r=1}^{q} \left\| \widehat{\mathbf{F}}_{rr}(\omega) - \widetilde{\mathbf{F}}(\omega) \right\|^2 d\omega,$$

(1.5)

where $\widetilde{\mathbf{F}}(\omega) := \frac{1}{q} \sum_{j=1}^{q} \widehat{\mathbf{F}}_{jj}(\omega)$ is the pooled spectral density estimator.

For a better understanding of the asymptotic results in the following sections, it is interesting to note that

$$\mathbf{I}_{rr}(\omega_k) - \widetilde{\mathbf{I}}(\omega_k) = -\frac{1}{q}\sum_{j=1}^{q}(1 - q\delta_{jr})\mathbf{I}_{jj}(\omega_k)$$

holds, where $\delta_{jr} = 1$ if $j = r$ and $\delta_{jr} = 0$ otherwise.

2. The unconditional test

For deriving an asymptotically exact test for (1.3), we will in the sequel apply a result of Eichler [11] for a general class of L_2-type statistics that includes the form (1.5). Therefore, we have to impose some assumptions.

2.1. Assumptions

First, we assume some strong mixing conditions for the DGP (compare Assumption 3.1 in Eichler [11]).

Assumption 2.1. $(\underline{X}_t, t \in \mathbb{Z})$ *is a zero mean d-variate* (*strictly*) *stationary stochastic process defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$. *Furthermore, for any $k > 0$, the kth order cumulants of* $(\underline{X}_t, t \in \mathbb{Z})$ *satisfy the mixing conditions*

$$\sum_{u_1,\ldots,u_{k-1}\in\mathbb{Z}} \left(1 + |u_j|^2\right)\left|c_{a_1,\ldots,a_k}(u_1,\ldots,u_{k-1})\right| < \infty \tag{2.1}$$

for all $j = 1,\ldots,k-1$ and $a_1,\ldots,a_k = 1,\ldots,d$, where

$$c_{a_1,\ldots,a_k}(u_1,\ldots,u_{k-1}) = \mathrm{cum}(X_{u_1,a_1},\ldots,X_{u_{k-1},a_{k-1}},X_{0,a_k})$$

is the kth order joint cumulant of $X_{u_1,a_1},\ldots,X_{u_{k-1},a_{k-1}},X_{0,a_k}$ (*cf. Brillinger [3] for the definition*).

Note that the above assumption requires the existence of the moments of all orders. However, in the case that we presume that our process has a linear structure, it can be weakened. See Remark 3.2 below.

Our second assumption on the kernel function K and the bandwidth h ensures the consistency of the kernel density estimators. It is similar to and implies Assumption 3.3 of Eichler [11].

Assumption 2.2.

(i) *The kernel K is a bounded, symmetric, nonnegative and Lipschitz-continuous function with compact support $[-\pi, \pi]$ and*

$$\int_{-\pi}^{\pi} K(\omega)\, d\omega = 2\pi.$$

Furthermore, $K(\omega)$ has continuous Fourier transform $k(u)$ such that

$$\int k^2(u)\, du < \infty \quad \text{and} \quad \int k^4(u)\, du < \infty. \tag{2.2}$$

(ii) *The bandwidth $h = h(n)$ is such that $h^{9/2}n \to 0$ and $h^2 n \to \infty$ as $n \to \infty$.*

In comparison to the notation in Eichler [11], remark that a factor 2π is incorporated in K. Moreover, for a better lucidity we define the positive constants

$$A_K := \frac{1}{2\pi} \int_{-\pi}^{\pi} K^2(v)\, dv \quad \text{and} \quad B_K := \frac{1}{\pi^2} \int_{-2\pi}^{2\pi} \left(\int_{-\pi}^{\pi} K(v) K(v+z)\, dv \right)^2 dz. \tag{2.3}$$

Furthermore, we like to point out that the optimal rate of $h \approx n^{-1/5}$ (by means of an averaged mean integrated squared error type criterion) for estimating the spectral density is not covered by our assumptions. However, assumptions of such kind are commonly imposed in the literature in order to reduce the bias of the smoothed kernel estimator leading to a central limit theorem for T_n, see Theorem 2.1 below. Compare for instance Taniguchi and Kondo [30], Taniguchi *et al.* [31], Taniguchi and Kakizawa [29], Eichler [11] or Dette and Paparoditis [9], who used similar (non-optimal) assumptions on the rate of the bandwidth.

2.2. Asymptotic results for T_n

We are now ready to state a CLT for the test statistic T_n given in (1.5).

Theorem 2.1 (Asymptotic null distribution of T_n). *Suppose that Assumptions 2.1 and 2.2 are fulfilled. If H_0 is true, it holds*

$$T_n - \frac{\mu_0}{\sqrt{h}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \tau_0^2)$$

as $n \to \infty$, where

$$\mu_0 = A_K \int_{-\pi}^{\pi} \left(\frac{1}{q} \sum_{j_1, j_2=1}^{q} (-1 + q \delta_{j_1 j_2}) \left| \mathrm{tr}(\mathbf{F}_{j_1 j_2}(\omega)) \right|^2 \right) d\omega \tag{2.4}$$

and

$$\tau_0^2 = B_K \int_{-\pi}^{\pi} \left(\frac{1}{q^2} \sum_{j_1, j_2, j_3, j_4=1}^{q} (-1 + q\delta_{j_1 j_2})(-1 + q\delta_{j_3 j_4}) \right.$$

$$\left. \times \left| \text{tr}\big(\mathbf{F}_{j_1 j_3}(\omega) \overline{\mathbf{F}_{j_2 j_4}(\omega)}^T \big) \right|^2 \right) d\omega, \tag{2.5}$$

where $\mathbf{F}_{jj}(\omega) = \mathbf{F}_{11}(\omega)$ *holds for all* ω *and* $j = 1, \ldots, q$. *Here and throughout the paper,* $\xrightarrow{\mathcal{D}}$ *denotes convergence in distribution.*

Remark that we have chosen the above representation of μ_0 and τ_0^2 for notational convenience. It is of course possible to split the sums above and to rewrite the expressions by using $\mathbf{F}_{jj} = \mathbf{F}_{11}$.

It is interesting to note that we do not need the assumption of positive definite spectral density block matrices since we do not work with ratio-type test statistics, see, for example, Eichler [11] for $p = 1$ and $q = 2$ or Dette and Paparoditis [9] for the case $p = 1$.

We give one important example which is possibly of most relevant interest, where we compare two real-valued spectral densities.

Example 2.1 (*The case* $p = 1$ *and* $q = 2$). For $p = 1$ and $q = 2$, the quantities μ_0 and τ_0^2 defined in Theorem 2.1 become

$$\mu_0 = A_K \int_{-\pi}^{\pi} f_{11}^2(\omega)\big(1 - C_{12}(\omega)\big) \, d\omega$$

and

$$\tau_0^2 = B_K \int_{-\pi}^{\pi} f_{11}^4(\omega)\big(1 - C_{12}(\omega)\big)^2 \, d\omega,$$

where $C_{jk}(\omega) = \frac{|f_{jk}(\omega)|^2}{f_{jj}(\omega) f_{kk}(\omega)} 1(f_{jj}(\omega) f_{kk}(\omega) > 0)$ *is the squared coherence between the two components* j *and* k *of* $(\underline{X}_t, t \in \mathbb{Z})$ (cf. Hannan [12], page 43).

Since μ_0 and τ_0^2 are in general unknown we have to estimate them before we can apply the above results for testing (1.3). Following Eichler [11], Remark 3.7, we can estimate both quantities by substituting $\mathbf{F}_{jk}$ in their expressions by the corresponding consistent estimators $\widehat{\mathbf{F}}_{jk}$. Rewriting the right-hand side of the equations (2.4) and (2.5) under H_0 by means of the equality $\mathbf{F}_{jj} = \frac{1}{q} \sum_{i=1}^{q} \mathbf{F}_{ii}$ this leads to

$$\widehat{\mu} := A_K \int_{-\pi}^{\pi} \left((q-1) \big| \text{tr}\big(\widetilde{\mathbf{F}}(\omega) \big) \big|^2 - \frac{1}{q} \sum_{\substack{j_1, j_2=1 \\ j_1 \neq j_2}}^{q} \big| \text{tr}\big(\widehat{\mathbf{F}}_{j_1 j_2}(\omega) \big) \big|^2 \right) d\omega \tag{2.6}$$

as an adequate estimator for μ_0. Similarly, we construct a consistent estimator $\widehat{\tau}^2$ for τ_0^2.

Note that it is in applications more convenient to use a discretized version of $\widehat{\mu}$, where the last integral is approximated by its Riemann sum.

Now we are ready to define the unconditional test

$$\varphi_n := \mathbf{1}_{(u_{1-\alpha},\infty)}\big((T_n - h^{-1/2}\widehat{\mu})/\widehat{\tau}\big),$$

where $u_{1-\alpha}$ denotes the $(1-\alpha)$-quantile of the standard normal distribution. Its properties are summarized in the following theorem.

Theorem 2.2. *Suppose that the Assumptions* 2.1 *and* 2.2 *hold. Then the test φ_n is not only of asymptotic level α, that is, $E(\varphi_n) \to \alpha$ holds if H_0 is true, but also consistent for testing H_0 versus H_1, that is, $E(\varphi_n) \to 1$ under H_1 as $n \to \infty$.*

2.3. Power under local alternatives

When studying the behavior of the test φ_n under local alternatives we have to consider observations $\underline{X}_1^n, \ldots, \underline{X}_n^n$ that come from sequences of d-dimensional zero-mean processes $(\underline{X}_t^n, t \in \mathbb{Z})$ with spectral density matrices given by

$$\mathbf{f}^n(\omega) = \mathbf{f}(\omega) + \alpha_n \mathbf{g}(\omega). \tag{2.7}$$

Following Eichler [11] we suppose that $\mathbf{f}$ is nonnegative definite, Hermitian and satisfies H_0 and that $\alpha_n \to 0$ as $n \to \infty$. Moreover, $\mathbf{f}$ and $\mathbf{g}$ are assumed to be twice continuously differentiable. In addition, for every $n \in \mathbb{N}$ the sequence of processes $(\underline{X}_t^n, t \in \mathbb{Z})$ satisfies the strong mixing condition (2.1) uniformly in n, where we replace $c_{a_1,\ldots,a_k}(u_1, \ldots, u_{k-1})$ by $\mathrm{cum}(X_{u_1,a_1}^n, \ldots, X_{u_{k-1},a_{k-1}}^n, X_{0,a_k}^n)$ (the kth order joint cumulant of $X_{u_1,a_1}^n, \ldots, X_{u_{k-1},a_{k-1}}^n$, X_{0,a_k}^n). Under these assumptions it can be deduced from Theorem 5.4 in Eichler [11] that the test φ_n can detect local alternatives up to an order of $\alpha_n = h^{-1/4}n^{-1/2}$. To be concrete, we have the following result.

Theorem 2.3. *Under the conditions stated above the asymptotic power of the test φ_n under local alternatives* (2.7) *with $\alpha_n = h^{-1/4}n^{-1/2}$ is given by $\lim_{n\to\infty} E(\varphi_n) = 1 - \Phi(u_{1-\alpha} - \nu/\tau_0)$, where Φ denotes the c.d.f. of the standard normal distribution and the detection shift ν is defined as*

$$\nu = \int_{-\pi}^{\pi} \mathrm{vec}\big(\overline{\mathbf{g}(\omega)}\big)^T \mathbf{\Gamma} \, \mathrm{vec}\big(\mathbf{g}(\omega)\big) \, \mathrm{d}\omega.$$

Here the $(d^2 \times d^2)$ matrix $\mathbf{\Gamma} = (\Gamma_{r,s})_{r,s}$ has the entries

$$\Gamma_{i+(j-1)d,k+(\ell-1)d} = \begin{cases} \dfrac{q-1}{q} & \text{for } |i-k| = |j-l| = 0 \text{ and } (i,j) \in \Xi, \\[2mm] -\dfrac{1}{q} & \text{for } |i-k| = |j-l| \in p\mathbb{N} \text{ and } (i,j) \in \Xi, \\[2mm] 0 & \text{otherwise}, \end{cases}$$

with $\Xi = \bigcup_{c=1}^{q}\{(r,s)\colon 1 + (c-1)p \le r, s \le cp\}$.

Note, that the same detection order has also been found by Paparoditis [21].

3. The randomization tests

Although the asymptotic performance of the unconditional test proposed above seems to be satisfactory, it is well known that the convergence speed of L_2-type statistics like (1.5) is rather slow (see Härdle and Mammen [13] as well as Paparoditis ([21,22]) for more details). For small or moderate sample sizes, we therefore expect that the above Gaussian approximation will not perform pretty well in applications. Our simulations in Section 4 support this presumption. To overcome this hitch, we propose in the following a suitable randomization technique for approximating the finite sample distribution of our test statistic. This method leads to so called randomization tests that use more appropriate, data-dependent critical values. Let $\pi_k : (\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}) \to \mathcal{S}_q, k = 0, \ldots, \lfloor \frac{n}{2} \rfloor$, with

$$\pi_k = \big(\pi_k(1), \ldots, \pi_k(q)\big)^T \tag{3.1}$$

be a sequence of independent and uniformly distributed random variables on the symmetric group $\mathcal{S}_q$ (the set of all permutations of $(1, \ldots, q)$) defined on some further probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$. In what follows, we assume that $(\pi_k)_k$ and $(\underline{X}_t)_t$ are independent (defined as random variables on the joint probability space $(\Omega \times \widetilde{\Omega}, \mathcal{F} \otimes \widetilde{\mathcal{F}}, \mathbb{P} \otimes \widetilde{\mathbb{P}})$). Conditional on the observations $\underline{X}_1, \ldots, \underline{X}_n$ the corresponding randomization test statistic is given by

$$T_n^* = n h^{1/2} \int_{-\pi}^{\pi} \sum_{r=1}^{q} \left\| \frac{1}{n} \sum_{k=-\lfloor (n-1)/2 \rfloor}^{\lfloor n/2 \rfloor} K_h(\omega - \omega_k) \big(\mathbf{I}_{\pi_k(r), \pi_k(r)}(\omega_k) - \widetilde{\mathbf{I}}(\omega_k)\big) \right\|^2 d\omega,$$

where we set $\pi_{-k} := \pi_k$ and $\pi_{k+sn} := \pi_k$ for $s \in \mathbb{Z}$ to maintain the symmetry properties of a spectral density matrix. Note that similar to the unconditional case, it holds

$$\mathbf{I}_{\pi_k(r), \pi_k(r)}(\omega_k) - \widetilde{\mathbf{I}}(\omega_k) = -\frac{1}{q} \sum_{j=1}^{q} (1 - q\delta_{j, \pi_k(r)}) \mathbf{I}_{jj}(\omega_k).$$

3.1. Asymptotic results for T_n^*

In the following, we will analyze in which situations our randomization method leads to asymptotically valid tests, that is, whether some suitable centered T_n^* converges to the same distribution as $T_n - h^{-1/2}\mu_0$. Therefore, we have to exploit the limiting behavior of the randomization statistic T_n^*. It will turn out that we have to center T_n^* at

$$\widehat{\mu}^* := A_K \int_{-\pi}^{\pi} \left(\frac{1}{q} \sum_{j_1, j_2=1}^{q} (-1 + q\delta_{j_1 j_2}) \right.$$
$$\left. \times \big\{ \big| \mathrm{tr}\big(\widehat{\mathbf{F}}_{j_1 j_2}(\omega)\big) \big|^2 + \mathrm{tr}\big(\widehat{\mathbf{F}}_{j_1 j_1}(\omega) \overline{\widehat{\mathbf{F}}_{j_2 j_2}(\omega)}^T\big) \big\} \right) d\omega \tag{3.2}$$

to gain asymptotic normality.

Theorem 3.1 (Asymptotic distribution of T_n^*). *Suppose that the mixing condition* (2.1) *holds for all $k \leq 32$. Under the Assumption* 2.2, *we have (conditioned on $\underline{X}_1, \ldots, \underline{X}_n$) convergence in distribution*

$$T_n^* - \frac{\widehat{\mu}^*}{\sqrt{h}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \tau^{*2}) \tag{3.3}$$

in probability as $n \to \infty$, where $\widehat{\mu}^*$ is as in (3.2) *and τ^{*2} is defined by*

$$
\tau^{*2} := B_K \int_{-\pi}^{\pi} \frac{1}{q^2} \sum_{j_1, j_2, j_3, j_4 = 1}^{q} \left(-1 + q\delta_{j_1 j_3}\delta_{j_2 j_4} + \frac{q}{q-1}(1 - \delta_{j_1 j_3})(1 - \delta_{j_2 j_4}) \right)
$$
$$
\times \left\{ \left| \text{tr}\left(\mathbf{F}_{j_1 j_3}(\omega)\overline{\mathbf{F}_{j_2 j_4}(\omega)}^T \right) \right|^2 \right. \tag{3.4}
$$
$$
\left. + \text{tr}\left(\mathbf{F}_{j_1 j_1}(\omega)\overline{\mathbf{F}_{j_2 j_2}(\omega)}^T \right) \text{tr}\left(\overline{\mathbf{F}_{j_3 j_3}(\omega)}\mathbf{F}_{j_4 j_4}(\omega)^T \right) \right\} d\omega.
$$

Note, that the conditions of the above theorem are weaker than the assumptions of Theorem 2.1. Moreover, remark that the centering term $\widehat{\mu}^*$ defined in (3.2) fulfills $\widehat{\mu}^* = \mu^* + o_P(\sqrt{h})$, where

$$
\mu^* := A_K \int_{-\pi}^{\pi} \left(\frac{1}{q} \sum_{j_1, j_2 = 1}^{q} (-1 + q\delta_{j_1 j_2}) \right.
$$
$$
\left. \times \left\{ \left| \text{tr}\left(\mathbf{F}_{j_1 j_2}(\omega) \right) \right|^2 + \text{tr}\left(\mathbf{F}_{j_1 j_1}(\omega)\overline{\mathbf{F}_{j_2 j_2}(\omega)}^T \right) \right\} \right) d\omega. \tag{3.5}
$$

Hence, to compare the above results with Theorem 2.1 we have to analyze μ^* and τ^{*2} under H_0 which leads to μ_0^* and τ_0^{*2} in the following remark.

Remark 3.1. Under H_0 the constants μ^* and τ^{*2} of Theorem 3.1, reduce to

$$
\mu_0^* = A_K \int_{-\pi}^{\pi} \left(\frac{1}{q} \sum_{j_1, j_2 = 1}^{q} (-1 + q\delta_{j_1 j_2}) \left\{ \left| \text{tr}\left(\mathbf{F}_{j_1 j_2}(\omega) \right) \right|^2 \right\} \right) d\omega = \mu_0 \tag{3.6}
$$

and

$$
\tau_0^{*2} = B_K \int_{-\pi}^{\pi} \frac{1}{q^2} \sum_{j_1, j_2, j_3, j_4 = 1}^{q} \left(-1 + q\delta_{j_1 j_3}\delta_{j_2 j_4} + \frac{q}{q-1}(1 - \delta_{j_1 j_3})(1 - \delta_{j_2 j_4}) \right)
$$
$$
\times \left| \text{tr}\left(\mathbf{F}_{j_1 j_3}(\omega)\overline{\mathbf{F}_{j_2 j_4}(\omega)}^T \right) \right|^2 d\omega, \tag{3.7}
$$

where $\mathbf{F}_{jj}(\omega) = \mathbf{F}_{11}(\omega)$ holds for all ω and $j = 1, \ldots, q$.

It is interesting to note that the unconditional and conditional centering parts μ_0 in (2.4) and μ_0^* in (3.6) coincide. Moreover, $T_n - h^{-1/2}\mu_0$ as well as $T_n^* - h^{-1/2}\widehat{\mu}^*$ posses a Gaussian limit distribution under the null. However, in order to construct an asymptotically exact randomization test based on $T_n^* - h^{-1/2}\widehat{\mu}^*$ we have to be sure that the limit variances τ_0^2 in (2.5) and τ_0^{*2} in (3.7) under H_0 are equal as well. This will be discussed in more detail in the next subsection. Here we just state an example (corresponding to Example 2.1) where this requirement is fulfilled.

Example 3.1 (The case $p = 1$ and $q = 2$). For $p = 1$ and $q = 2$ the quantities μ^* and τ^{*2} defined in Theorem 3.1 become

$$\mu^* = A_K \int_{-\pi}^{\pi} \frac{1}{2}\big\{\big(f_{11}(\omega) - f_{22}(\omega)\big)^2 + f_{11}^2(\omega) + f_{22}^2(\omega) - 2\big|f_{12}(\omega)\big|^2\big\}\,\mathrm{d}\omega$$

and

$$\tau^{*2} = B_K \int_{-\pi}^{\pi} \frac{1}{4}\big\{\big(f_{11}(\omega) - f_{22}(\omega)\big)^4 + \big(f_{11}^2(\omega) + f_{22}^2(\omega) - 2\big|f_{12}(\omega)\big|^2\big)^2\big\}\,\mathrm{d}\omega.$$

Under H_0, we have

$$\mu_0^* = A_K \int_{-\pi}^{\pi} f_{11}^2(\omega)\big(1 - C_{12}(\omega)\big)\,\mathrm{d}\omega = \mu_0$$

and

$$\tau_0^{*2} = B_K \int_{-\pi}^{\pi} f_{11}^4(\omega)\big(1 - C_{12}(\omega)\big)^2\,\mathrm{d}\omega = \tau_0^2.$$

3.2. The randomization test procedures

Based on the conditional CLTs above we can now define different randomization tests. The first natural approach is to use the test $\varphi_{n,\mathrm{cent}}^* := \mathbf{1}_{(c_{n,\mathrm{cent}}^*(\alpha),\infty)}(T_n - h^{-1/2}\widehat{\mu})$, where $c_{n,\mathrm{cent}}^*(\alpha)$ is the data-dependent $(1 - \alpha)$-quantile of the conditional distribution of $T_n^* - h^{-1/2}\widehat{\mu}^*$ given the data. As typical for resampling methods, note that we still use the same test statistic as for the unconditional case and only apply the randomization statistic to calculate the critical value. By Theorems 2.1 and 3.1, this test will be asymptotically exact, i.e. $E(\varphi_{n,\mathrm{cent}}^*) \to \alpha$ holds under H_0, if the asymptotic variances τ_0^2 and τ_0^{*2} of the test statistic $T_n - h^{-1/2}\widehat{\mu}$ and its randomization version $T_n^* - h^{-1/2}\widehat{\mu}^*$ posses the same limit under the null. Although $T_n^* - h^{-1/2}\widehat{\mu}^*$ does in general not mimic the null distribution under the alternative, the following corollary shows that $\varphi_{n,\mathrm{cent}}^*$ will also be asymptotically consistent in these situations. Before we state these properties, we like to introduce a computational less demanding version of the above test. Therefore note that Remark 3.1 implies that the difference $h^{-1/2}(\widehat{\mu}^* - \widehat{\mu})$ converges to zero in probability under H_0. Hence, it may be convenient to use the test

$$\varphi_n^* := \mathbf{1}_{(c_n^*(\alpha),\infty)}(T_n) \tag{3.8}$$

without centering part, where $c_n^*(\alpha)$ is the data-dependent $(1 - \alpha)$-quantile of the conditional distribution of T_n^*. For completeness, we shortly explain the numerical algorithm for the implementation of φ_n^*. The algorithm for $\varphi_{n,\text{cent}}^*$ is analogue.

Step 1: Compute the test statistic T_n as given in (1.5) based on data $\underline{X}_1, \ldots, \underline{X}_n$.

Step 2: Generate independent random permutations $\pi_0, \ldots, \pi_{\lfloor n/2 \rfloor}$ as in (3.1) that are uniformly distributed on the symmetric group $\mathcal{S}_q$.

Step 3: Calculate the randomization statistic T_n^* (given $\underline{X}_1, \ldots, \underline{X}_n$).

Step 4: Repeat the Steps 2 and 3 B-times, where B is large, which leads to $T_n^{*(1)}, \ldots, T_n^{*(B)}$.

Step 5: Reject the null hypothesis H_0 if $B^{-1} \sum_{b=1}^{B} \mathbf{1}\{T_n > T_n^{*(b)}\} > \alpha$.

In the sequel, we analyze the asymptotic properties of both tests φ_n^* and $\varphi_{n,\text{cent}}^*$ and compare it with the unconditional benchmark test φ_n.

Corollary 3.1 (Exactness and consistency of φ_n^* and $\varphi_{n,\text{cent}}^*$). *Suppose the assumptions of Theorem* 2.1 *are satisfied.*

(a) *If H_0 is true, the following assertions are equivalent*:

(i) *The randomization test φ_n^* is asymptotically exact and equivalent to φ_n, that is,*

$$E\left(\left|\varphi_n - \varphi_n^*\right|\right) \to 0 \qquad \text{as } n \to \infty. \tag{3.9}$$

(ii) *It holds*

$$
\begin{aligned}
0 = {} & \sum_{\substack{j_1, j_3 = 1 \\ j_1 \neq j_3}}^{q} \left((q-1)^3 - 1\right) \left|\text{tr}\left(\mathbf{F}_{j_1 j_3}(\omega)\overline{\mathbf{F}_{j_1 j_3}(\omega)}^T\right)\right|^2 \\
& - 2 \sum_{\substack{j_1, j_3, j_4 = 1 \\ \text{all} \neq}}^{q} \left((q-1)^2 + 1\right) \left|\text{tr}\left(\mathbf{F}_{j_1 j_3}(\omega)\overline{\mathbf{F}_{j_1 j_4}(\omega)}^T\right)\right|^2 \\
& + \sum_{\substack{j_1, j_2, j_3, j_4 = 1 \\ j_1 \neq j_3, j_2 \neq j_4, j_1 \neq j_2, j_3 \neq j_4}}^{q} (q-2) \left|\text{tr}\left(\mathbf{F}_{j_1 j_3}(\omega)\overline{\mathbf{F}_{j_2 j_4}(\omega)}^T\right)\right|^2.
\end{aligned} \tag{3.10}
$$

Moreover, if (3.10) *holds, φ_n^* will also be asymptotically consistent, that is, $E(\varphi_n^*) \to 1$ under H_1 as $n \to \infty$.*

(b) *The above statement* (a) *also holds true for $\varphi_{n,\text{cent}}^*$ instead of φ_n^*.*

Note that the asymptotic equivalence in (a) implies that both tests posses the same power for contiguous alternatives. Moreover, remark that the above stated consistency of the tests is caused by the fact that $T_n - h^{-1/2}\widehat{\mu}$ converges to $+\infty$ in probability for fixed alternatives, see the proof section for more details, and the non-degenerated limit law of its randomization version. In the following, we will give some necessary and sufficient conditions for (ii) above.

Corollary 3.2 (Necessary and sufficient conditions for exactness of φ_n^* and $\varphi_{n,\text{cent}}^*$). *Suppose that the assumptions of Theorem* 2.1 *hold.*

  (i) *In the case $q = 2$, that is, we are testing for the equality of two $(p \times p)$ spectral density matrices, condition* (ii) *of Corollary* 3.1 *is fulfilled for all $p \in \mathbb{N}$.*
 (ii) *For any $q, p \in \mathbb{N}, q \geq 2$, a sufficient condition for condition* (ii) *of Corollary* 3.1 *is*

$$\mathbf{F}_{ij}(\omega) = \mathbf{F}_{12}(\omega) \tag{3.11}$$

for all $\omega \in [-\pi, \pi]$ and all $i, j \in \{1, \ldots, q\}$ with $i \neq j$. This means that all $(p \times p)$ block matrices on all secondary diagonals have to be equal and Hermitian everywhere.
 (iii) *For $p = 1$ and $q \geq 2$, a sufficient condition weaker than* (ii) *above is*

$$\left| f_{ij}(\omega) \right|^2 = \left| f_{12}(\omega) \right|^2$$

for all $\omega \in [-\pi, \pi]$ and all $i, j \in \{1, \ldots, q\}$ with $i \neq j$.
 (iv) *For $p = 1$ and $q = 3$, the condition*

$$\left| f_{ij}(\omega) \right|^2 = \left| f_{12}(\omega) \right|^2$$

for all $\omega \in [-\pi, \pi]$ and all $i, j \in \{1, 2, 3\}$ with $i \neq j$ is not only sufficient, but also necessary for condition (ii) *of Corollary* 3.1.

Note that the randomization tests φ_n^* and $\varphi_{n,\text{cent}}^*$ are asymptotically exact particularly in the case of uncorrelated p-variate (sub-)time series $\underline{X}_1, \ldots, \underline{X}_q$ due to (ii) above. Intuitively, this makes sense because permuting the block diagonal matrices distorts the correlation structure between these time series, if there is any. This explains also why both randomization tests are asymptotically exact in the more general case of equal covariance structure between the time series $\underline{X}_1, \ldots, \underline{X}_q$ as shown in (ii) and (iii) of Corollary 3.2 above, which is often denoted by equicorrelatedness.

Observe that due to the sophisticated condition in Corollary 3.1, a result for general $q \geq 3$ equivalent to (iv) in Corollary 3.2 does not seem to be achievable.

Nevertheless, there is an obvious possibility for constructing another randomization test that works even if the presumption (3.10) of Corollary 3.1 is not fulfilled. We just have to impose an estimator for the conditional limit variance τ^{*2} of T_n^* as introduced in Theorem 3.1. An appropriate candidate is given by

$$\widehat{\tau}^{*2} := B_K \int_{-\pi}^{\pi} \frac{1}{q^2} \sum_{j_1, j_2, j_3, j_4 = 1}^{q} \left(-1 + q \delta_{j_1 j_3} \delta_{j_2 j_4} + \frac{q}{q-1}(1 - \delta_{j_1 j_3})(1 - \delta_{j_2 j_4}) \right)$$
$$\times \left\{ \text{tr}\left(\widehat{\mathbf{F}}_{j_1 j_1}(\omega) \overline{\widehat{\mathbf{F}}_{j_2 j_2}(\omega)}^T \right) \text{tr}\left(\overline{\widehat{\mathbf{F}}_{j_3 j_3}(\omega)} \widehat{\mathbf{F}}_{j_4 j_4}(\omega)^T \right) \tag{3.12}$$
$$+ \left| \text{tr}\left(\widehat{\mathbf{F}}_{j_1 j_3}(\omega) \overline{\widehat{\mathbf{F}}_{j_2 j_4}(\omega)}^T \right) \right|^2 \right\} d\omega$$

and the corresponding randomization test is $\varphi_{n,\text{stud}}^* := \mathbf{1}_{(c_{n,\text{stud}}^*(\alpha),\infty)}((T_n - h^{-1/2}\widehat{\mu})/\widehat{\tau})$, where $c_{n,\text{stud}}^*(\alpha)$ denotes the data-dependent $(1 - \alpha)$-quantile of the conditional distribution of $(T_n^* - h^{-1/2}\widehat{\mu}^*)/\widehat{\tau}^*$ given the data. Let us shortly state its properties.

Corollary 3.3. *Suppose that the assumptions of Theorem* 2.1 *hold. Then the test* $\varphi_{n,\text{stud}}^*$ *is not only of asymptotic level* α *under* H_0 *but also consistent for testing* H_0 *versus* H_1, *that is,* $E(\varphi_{n,\text{stud}}^*) \to 1$ *under* H_1 *as* $n \to \infty$.

Finally, we like to note that some of the assumptions can be weakened for linear processes. In particular, by combining Theorem 3.1 above with results of Dette and Paparoditis [9] we gain the following remark.

Remark 3.2 (Exactness and consistency of the tests for linear processes). If the process posses a linear structure, that is, $\underline{X}_t = \sum_{j=-\infty}^{\infty} \boldsymbol{\Psi}_j \underline{e}_{t-j}$ for a d-dimensional i.i.d. white noise $(\underline{e}_t, t \in \mathbb{Z})$ and a sequence of $(d \times d)$-matrices $(\boldsymbol{\Psi}_j)_j = ((\psi_j(r,s))_{r,s})_j$, Condition (2.2) is not needed and the mixing Assumption 2.1 in the above Corollaries 3.1–3.3 can be substituted by the summable condition $\sum_j |j|^{1/2} |\psi_j(r,s)| < \infty$ and the moment condition $E(\|e_t\|^{32}) < \infty$.

For the two sample testing problem as described in the introductory part our randomization test has a nice reading as a symmetry test. This is part of the following remark.

Remark 3.3 (The case $q = 2$: Interpretation as a conditional symmetry test). Remark that we can rewrite (in distributional equality) the summands of our randomization statistic for $q = 2$ as

$$\mathbf{I}_{\pi_k(r),\pi_k(r)}(\omega_k) - \widetilde{\mathbf{I}}(\omega_k) \overset{\mathcal{D}}{=} \frac{1}{2} e_k (\mathbf{I}_{11}(\omega_k) - \mathbf{I}_{22}(\omega_k)),$$

where $(e_k)_{k\geq 0}$ are i.i.d. signs, that is, independent and on $\{+1, -1\}$ uniformly distributed r.v.s, and we set $e_{-k} := e_k$ as well as $e_{k+sn} := e_k$ for $s \in \mathbb{Z}$. Hence, the above defined randomization tests can be interpreted as some general kind of conditional symmetry tests in this situation.

Remark 3.4 (Choice and influence of the bandwidth). As already mentioned in the Introduction, our randomization approach has the nice advantage that the bandwidth is the only tuning parameter that has to be assessed. This feature becomes even better as we will see in our extensive simulation study, see Section 4 below, where our approach does not react very sensitive to variations of the bandwidth h. However, it is of course desirable to have a detached principle for selecting the bandwidth. Therefore, we use a data driven cross validation method for choosing h for the kernel spectral density estimation. This method is due to Beltrão and Bloomfield [2] and Robinson [28] and has also been applied by Paparoditis [21–23].

Finally, we also like to discuss the local power of all randomization tests.

Corollary 3.4. *Suppose that the assumptions of Theorem* 2.3 *hold.*

(i) *The test $\varphi^*_{n,\text{stud}}$ has the same local power as the asymptotic test φ_n, i.e. we have $E(\varphi^*_{n,\text{stud}}) \to 1 - \Phi(u_{1-\alpha} - v/\tau_0)$ under local alternatives (2.7) with $\alpha_n = h^{-1/4}n^{-1/2}$, where v is given in Theorem 2.3.*

(ii) *In addition, suppose that $\mathbf{f}$ in (2.7) satisfies (3.10). Then φ^*_n and $\varphi^*_{n,\text{cent}}$ also posses the same power under local alternatives (2.7) with $\alpha_n = h^{-1/4}n^{-1/2}$.*

4. Simulation studies

In this section, we illustrate the performance of the randomization procedure in comparison with the asymptotic (unconditional) benchmark test φ_n as described in the previous sections.

For better lucidity, we thereby only analyze the finite sample behavior of the computational least-demanding randomization test φ^*_n as proposed in (3.8). The other two randomization procedures from Section 3 behave similar to or slightly worse than φ^*_n. For more details, we refer the reader to our supplementary material (cf. Jentsch and Pauly [17]).

4.1. The setup

Suppose we observe bivariate time series data $(\underline{X}_t = (X_{t,1}, X_{t,2})^T, t = 1, \ldots, n)$ and we want to test the null hypothesis H_0 of equality of both corresponding one-dimensional spectral densities $f_1(\omega)$ and $f_2(\omega)$. In the setup of Section 1, this means $q = 2$, $p = 1$ and $f_j(\omega) = \mathbf{F}_{jj}(\omega)$, $j = 1, 2$ and we test

$$H_0: \left\{ f_1(\omega) = f_2(\omega) \text{ for all } \omega \in [-\pi, \pi] \right\}$$

against

$$H_1: \left\{ \exists A \subset \mathcal{B}([-\pi, \pi]) \text{ with } \lambda(A) > 0 : f_1(\omega) \neq f_2(\omega) \text{ for all } \omega \in A \right\}.$$

In the following, we consider data from several well-established time series models. In particular, our analysis includes Gaussian and non-Gaussian linear time series as well as non-linear time series models. The linear models under consideration cover moving average (MA) models and autoregressive (AR) models with innovations following Gaussian, logistic and double-exponential distributions, respectively. GARCH models, threshold AR (TAR) and random coefficient autoregressive (RCA) models are investigated to cover important classes of non-linear time series.

Although GARCH processes are known to have power law tails (see Basrak *et al.* [1], Section 4) and, consequently, only moments up to some finite order exists, we include GARCH models to investigate the general performance of the randomization approach for processes that go beyond our Assumption 2.1. For the same reasons, we consider also RCA models in our simulation study.

The performance of the randomization test φ^*_n in comparison to the unconditional benchmark test φ_n is investigated under the null and under the alternative.

For all models under consideration, we have generated $T = 400$ time series. For evaluation of the test statistic, the bandwidth has been chosen by cross validation as proposed in Remark 3.4 and is denoted by h_{CV}. Further, we use the Bartlett–Priestley kernel (see Priestley [27], page 448) for which the constants in (2.3) become $A_K = \frac{6}{5}$ and $B_K = \frac{2672\pi}{385}$ for this particular kernel function. For each time series, the test φ_n has been executed with critical values from normal approximation as discussed in Section 2 and the randomization test φ_n^* as discussed in Section 3, where $B = 300$ randomization replications have been used.

4.2. Analysis of the size

To investigate the behavior of the tests under the null, we consider realizations from vector autoregressive models (VAR), vector moving-average models (VMA) to cover linear time series and from GARCH, TAR and RCA models to cover non-linear cases. We consider data from the bivariate vector AR(1) model

$$\text{AR(1):} \quad \underline{X}_t = \mathbf{A}\underline{X}_{t-1} + \underline{e}_t, \qquad t \in \mathbb{Z}, \tag{4.1}$$

where $\underline{e}_t \sim (0, \mathbf{\Sigma}_1)$ is an independent and identically distributed (i.i.d.) bivariate white noise with covariance matrix $\mathbf{\Sigma}_1$, $\mathbf{A}$ is chosen from

$$\mathbf{A}_1 = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}, \qquad \mathbf{A}_2 = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}, \qquad \mathbf{A}_3 = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.9 \end{pmatrix}$$

and $\mathbf{\Sigma}_1 = \mathbf{Id}$ is the unit matrix. The VAR model corresponding to coefficient matrix $\mathbf{A}_i$ is denoted by AR_i. All models have been investigated for i.i.d. Gaussian innovations $(\underline{e}_t, t \in \mathbb{Z})$, whereas the most critical model AR_3 has also been analyzed for logistic (with c.d.f. $F(x) = (1 + \exp(-x))^{-1}$) and double-exponential distributions (with p.d.f. $f(x) = \exp(-|x|)/2$) of the i.i.d. innovations $(\underline{e}_t, t \in \mathbb{Z})$, respectively. Observe that due to the diagonal shape of all involved matrices $\mathbf{\Sigma}_1$ and $\mathbf{A}_i$, we are dealing with two *independent* univariate time series here. Furthermore, we consider data from the bivariate vector MA(1) model

$$\text{MA(1):} \quad \underline{X}_t = \mathbf{B}\underline{e}_{t-1} + \underline{e}_t, \qquad t \in \mathbb{Z}, \tag{4.2}$$

where $\underline{e}_t \sim (0, \mathbf{\Sigma}_2)$ is an i.i.d. bivariate white noise with covariance matrix $\mathbf{\Sigma}_2$, $\mathbf{B}$ is chosen from

$$\mathbf{B}_1 = \begin{pmatrix} 0.1 & 0.5 \\ 0.5 & 0.1 \end{pmatrix}, \qquad \mathbf{B}_2 = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}, \qquad \mathbf{B}_3 = \begin{pmatrix} 0.9 & 0.5 \\ 0.5 & 0.9 \end{pmatrix},$$

and

$$\mathbf{\Sigma}_2 = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}. \tag{4.3}$$

The VMA model corresponding to $\mathbf{B}_i$ is denoted by MA_i. Again all models have been investigated for i.i.d. Gaussian innovations $(\underline{e}_t, t \in \mathbb{Z})$, whereas the model MA_3 has also been analyzed

for logistic and double-exponential distributions of the innovations, respectively, as discussed above, but with covariance matrix $\mathbf{\Sigma}_2$. In this setting, we are dealing with two *dependent* time series whose marginal spectral densities are equal due to the symmetric shape of $\mathbf{\Sigma}_2$ and $\mathbf{B}_i$. Note that the application of the randomization technique to the case of two dependent time series is justified by Corollary 3.2(i). Also, we investigate three different non-linear time series models. First, we consider bivariate data $(\underline{X}_t = (X_{t,1}, X_{t,2})^T, t = 1, \ldots, n)$ from two independent, but identically distributed univariate GARCH(1, 1) processes $\{X_{t,i}, t \in \mathbb{Z}\}$, $i = 1, 2$, with

$$\text{GARCH}(1, 1): \quad X_{t,i} = \sigma_{t,i} e_{t,i}, \qquad \sigma_{t,i}^2 = \omega + a X_{t-1,i}^2 + b \sigma_{t-1,i}^2, \qquad t \in \mathbb{Z}, \quad (4.4)$$

where $\omega = 0.01$, $a = 0.1$ and the coefficient b is chosen from

$$b_1 = 0.2, \qquad b_2 = 0.3, \qquad b_3 = 0.4.$$

The corresponding models are denoted by $GARCH_i$, $i = 1, 2, 3$. Further, two (centered) independent, but identically distributed univariate TAR(1) models $\{X_{t,i}, t \in \mathbb{Z}\}$, $i = 1, 2$, that follow the model equation

$$\text{TAR}(1): \quad X_{t,i} = \begin{cases} a(1)X_{t-1,i} + e_{t,i}, & X_{t-1,i} < 0, \\ a(2)X_{t-1,i} + e_{t,i}, & X_{t-1,i} \geq 0, \end{cases} \qquad t \in \mathbb{Z} \qquad (4.5)$$

with coefficients $\underline{a} = (a(1), a(2))'$ chosen from

$$\underline{a}_1 = (-0.2, 0.1)^T, \qquad \underline{a}_2 = (-0.3, 0.2)^T, \qquad \underline{a}_3 = (-0.4, 0.3)^T$$

are studied and denoted by TAR_i, $i = 1, 2, 3$. Also two independent, but identically distributed univariate RCA(1) models $\{X_{t,i}, t \in \mathbb{Z}\}$, $i = 1, 2$, that follow the model equation

$$\text{RCA}(1): \quad X_{t,i} = a_t X_{t-1} + e_{t,i}, \qquad t \in \mathbb{Z}, \qquad (4.6)$$

where $a_t \sim \mathcal{N}(0, \sigma^2)$ are i.i.d. centered normally distributed random variables with standard deviation σ chosen from

$$\sigma_1 = 0.1, \qquad \sigma_2 = 0.2, \qquad \sigma_3 = 0.3$$

are considered and denoted by RCA_i, $i = 1, 2, 3$. For all non-linear models above, we have used (independent) standard normal i.i.d. white noise processes $\{e_{t,i}\}$.

For nominal sizes $\alpha \in \{1\%, 5\%, 10\%\}$ and sample sizes $n \in \{50, 100, 200\}$, the corresponding results for all combinations are displayed in Tables 1–3. To check how sensitive the tests react on the bandwidth choice, we report the simulation results for bandwidths $c \cdot h_{\text{CV}}$ and $c \in \{0.5, 1, 1.5\}$ to cover under-smoothing and over-smoothing with respect to the bandwidth h_{CV} chosen via cross validation.

To illustrate the slow convergence of the actual size of the unconditional test φ_n for the different models under the null and to emphasize the need of resampling techniques to resolve this issue, we report its performance also for larger sample sizes in Table 4.

Table 1. Actual size of φ_n and φ_n^* for nominal size $\alpha \in \{1\%, 5\%, 10\%\}$, sample size $n \in \{50, 100, 200\}$, bandwidth $h = c \cdot h_{\mathrm{CV}}$ for $c \in \{0.5, 1, 1.5\}$, autoregressive models AR_1–AR_3 and moving-average models MA_1–MA_3, all with Gaussian innovations

| | | AR_1 | | | | | | AR_2 | | | | | | AR_3 | | | | | |
| | | 50 | | 100 | | 200 | | 50 | | 100 | | 200 | | 50 | | 100 | | 200 | |
α	c	φ_n	φ_n^*	φ_n	φ_n^*	φ_n	φ_n^*	φ_n	φ_n^*	φ_n	φ_n^*	φ_n	φ_n^*	φ_n	φ_n^*	φ_n	φ_n^*	φ_n	φ_n^*
1	0.5	5.5	1.5	8.5	2.0	6.5	1.0	10.0	1.5	7.5	1.3	7.3	1.8	17.8	2.8	13.0	1.3	11.3	1.0
	1	11.3	1.3	10.5	1.0	11.3	0.8	13.8	1.5	11.3	2.3	12.0	1.3	27.0	2.0	20.0	1.5	21.8	1.8
	1.5	9.8	1.5	17.0	1.8	10.5	1.8	16.8	0.5	13.3	1.0	11.5	1.5	36.3	2.0	26.5	1.3	24.0	1.0
5	0.5	10.5	5.0	14.5	5.8	12.3	4.8	14.3	6.0	14.8	4.5	12.3	6.8	27.3	7.5	20.8	5.8	17.5	4.8
	1	17.0	8.3	17.3	6.3	17.3	4.5	20.5	5.5	17.8	6.3	18.5	7.0	33.5	6.5	27.3	5.3	27.3	6.3
	1.5	18.0	5.3	21.8	8.3	17.5	5.3	23.5	4.8	18.5	4.8	17.5	4.8	42.5	7.3	33.8	4.3	30.8	5.8
10	0.5	15.0	8.3	20.3	11.8	18.3	10.8	20.0	11.0	19.0	10.0	16.3	11.0	33.0	12.8	28.8	11.5	22.5	10.5
	1	19.8	12.3	22.0	12.0	23.8	11.3	24.8	11.5	23.5	11.5	21.3	12.8	38.0	13.3	33.5	9.8	32.8	11.5
	1.5	22.3	8.0	27.3	13.3	21.8	11.8	27.3	11.3	24.3	9.0	23.8	9.3	48.0	13.8	41.0	8.5	34.3	10.0

| | | MA_1 | | | | | | MA_2 | | | | | | MA_3 | | | | | |
| | | 50 | | 100 | | 200 | | 50 | | 100 | | 200 | | 50 | | 100 | | 200 | |
α	c	φ_n	φ_n^*	φ_n	φ_n^*	φ_n	φ_n^*	φ_n	φ_n^*	φ_n	φ_n^*	φ_n	φ_n^*	φ_n	φ_n^*	φ_n	φ_n^*	φ_n	φ_n^*
1	0.5	4.0	2.0	4.0	1.5	4.8	1.8	6.0	1.8	5.8	0.5	4.0	1.3	6.5	1.3	5.3	1.8	5.3	1.3
	1	5.0	1.5	4.3	2.0	5.5	1.5	7.5	1.8	4.3	0.5	5.5	1.8	8.3	2.0	6.5	1.0	5.8	1.8
	1.5	3.3	0.5	4.0	1.5	3.3	0.5	9.5	1.3	6.8	1.3	7.0	1.0	11.5	1.0	8.5	2.0	11.0	2.5
5	0.5	7.8	5.8	8.5	5.0	8.5	6.8	11.0	5.5	10.0	5.3	11.3	5.3	11.0	5.8	10.5	6.3	8.8	5.5
	1	8.8	5.0	9.5	4.8	10.0	6.5	13.5	7.3	8.0	4.8	11.3	6.0	13.5	7.3	12.0	6.5	11.8	5.5
	1.5	8.3	4.8	7.5	4.3	9.5	6.3	14.3	6.5	11.0	5.3	11.5	5.3	17.8	5.0	13.5	5.5	17.0	7.5
10	0.5	13.0	8.3	13.5	10.3	12.3	11.5	13.8	9.5	16.0	10.3	15.8	10.3	16.0	9.8	16.5	10.8	12.8	11.3
	1	12.8	11.3	14.0	11.5	12.0	11.5	18.5	12.8	13.0	11.0	16.8	12.3	17.8	11.0	15.8	11.5	16.3	9.5
	1.5	11.5	10.5	12.8	10.3	13.3	13.0	18.0	13.5	14.5	11.0	15.3	10.8	23.0	8.8	18.0	9.3	20.8	12.8

Table 2. Actual size of φ_n and φ_n^* for nominal size $\alpha \in \{1\%, 5\%, 10\%\}$, sample size $n \in \{50, 100, 200\}$, bandwidth $h = c \cdot h_{CV}$ for $c \in \{0.5, 1, 1.5\}$, autoregressive model AR_3 and moving-average model MA_3, both with logistic and double-exponential distribution of the innovations

| | | | AR_3 | | | | | | MA_3 | | | | | |
| | | n: | 50 | | 100 | | 200 | | 50 | | 100 | | 200 | |
$\{e_t\}$	α	c	φ_n	φ_n^*	φ_n	φ_n^*	φ_n	φ_n^*	φ_n	φ_n^*	φ_n	φ_n^*	φ_n	φ_n^*
Logistic	1	0.5	16.3	3.3	15.8	2.5	14.3	2.0	6.8	2.8	6.3	1.5	8.0	2.0
		1	33.8	4.0	23.3	0.8	21.0	1.8	9.8	1.8	9.0	2.0	10.0	2.8
		1.5	39.3	3.3	36.3	2.0	29.8	2.5	16.5	4.8	14.5	1.8	10.5	2.5
	5	0.5	26.8	6.5	23.8	7.3	21.0	7.0	12.8	9.3	12.0	6.8	13.5	8.3
		1	40.8	8.0	30.5	7.3	27.5	4.8	17.0	8.5	16.5	8.3	17.0	9.3
		1.5	46.3	7.5	45.3	7.5	34.8	8.8	24.0	11.0	20.0	7.8	17.5	6.5
	10	0.5	33.3	13.5	27.8	11.8	24.5	12.8	17.5	13.3	16.5	13.5	17.0	15.3
		1	45.5	12.3	35.0	11.5	33.0	10.3	19.8	13.5	21.5	14.0	21.8	16.3
		1.5	51.5	12.5	49.0	15.3	39.5	13.5	27.3	17.5	25.0	15.5	21.8	11.5
Double -exp.	1	0.5	18.0	2.3	11.3	0.5	11.0	0.8	11.5	3.3	10.5	3.8	10.3	4.0
		1	27.8	2.8	27.8	2.0	20.0	2.0	16.0	4.0	17.0	4.8	14.3	4.0
		1.5	34.0	2.5	31.8	3.8	30.8	1.3	17.0	1.5	18.3	2.8	17.5	4.0
	5	0.5	29.0	6.5	20.0	5.5	19.0	4.8	17.8	10.3	17.3	10.8	16.8	8.5
		1	35.5	7.3	36.5	7.3	25.5	4.3	20.8	12.0	23.5	13.5	20.3	14.8
		1.5	42.3	9.0	38.8	7.5	37.3	4.8	23.8	8.0	25.0	9.5	23.0	12.8
	10	0.5	36.0	11.0	26.0	11.8	22.5	8.8	23.0	16.5	22.3	15.5	19.8	15.8
		1	43.8	13.5	40.5	13.3	29.8	10.3	26.0	17.8	28.3	23.0	23.5	20.0
		1.5	46.5	14.3	42.5	14.5	40.5	12.3	29.0	15.3	29.5	16.5	27.5	20.5

4.3. Analysis of the power

As can be seen in Tables 1–3, the asymptotic test tends to overreject the null hypothesis systematically in most situations and should not be applied, at least for the simulated sample sizes. In particular, for the autoregressive models where the actually achieved size is far too large compared to nominal size (see Tables 1–2), the unconditional benchmark test φ_n cannot be judged. Moreover, even for larger sample sizes up to $n = 2000$ it does not keep the prescribed level satisfactorily, see Table 4. Thereofree, we present here only the power behavior of the randomization test φ_n^*. However, we present a small power simulation study of φ_n as well as all other randomization tests in the supplementary material (cf. Jentsch and Pauly [17]). It can be seen that there is actually no big difference in the power behavior (measured as achieved power) between the asymptotic test φ_n and all randomization tests.

To illustrate the behavior of φ_n^* under the alternative, that is for inequality of both spectral densities, we consider several models belonging to the same model classes that have already been considered above under the null. First, we consider realizations from the autoregressive

Table 3. Actual size of φ_n and φ_n^* for nominal size $\alpha \in \{1\%, 5\%, 10\%\}$, sample size $n \in \{50, 100, 200\}$, bandwidth $h = c \cdot h_{CV}$ for $c \in \{0.5, 1, 1.5\}$ and non-linear models $GARCH_i$, TAR_i and RCA_i, respectively

| | | | i: 1 | | | | | | i: 2 | | | | | | i: 3 | | | | | |
| | | | n: 50 | | 100 | | 200 | | 50 | | 100 | | 200 | | 50 | | 100 | | 200 | |
Model	α	c	φ_n	φ_n^*	φ_n	φ_n^*	φ_n	φ_n^*	φ_n	φ_n^*	φ_n	φ_n^*	φ_n	φ_n^*	φ_n	φ_n^*	φ_n	φ_n^*	φ_n	φ_n^*
$GARCH_i$	1	0.5	13.0	3.5	11.8	3.8	14.0	3.3	11.0	3.5	15.0	2.3	14.5	2.5	12.8	2.5	14.8	4.5	16.8	6.5
		1	15.0	2.8	14.5	2.8	14.3	3.5	12.5	2.5	21.8	4.5	17.3	3.5	16.3	4.0	16.0	3.8	17.8	4.3
		1.5	17.3	3.0	16.0	3.5	21.5	3.0	22.3	3.8	18.8	4.0	22.0	3.8	22.3	3.8	21.5	4.0	20.5	3.5
	5	0.5	20.5	11.0	17.5	9.8	22.8	10.8	19.5	12.0	24.0	11.8	22.3	9.8	20.0	11.5	23.0	13.0	23.5	14.0
		1	21.3	12.0	20.3	10.3	21.5	9.0	18.8	8.3	28.0	14.3	23.3	12.5	22.8	9.5	22.5	11.0	25.8	11.3
		1.5	25.8	7.3	21.3	8.8	26.5	11.3	29.8	11.0	24.8	12.0	26.8	13.0	25.8	12.0	28.0	14.0	27.0	9.8
	10	0.5	24.0	16.8	22.5	15.3	27.3	19.0	26.5	18.0	28.3	17.8	28.8	19.5	27.5	20.8	28.0	19.8	29.3	21.5
		1	28.3	18.5	27.0	16.5	28.8	17.3	25.8	14.3	32.5	22.5	27.5	18.3	29.3	17.5	29.0	18.5	31.3	19.0
		1.5	29.8	13.5	25.5	13.8	32.3	17.5	33.8	17.8	29.8	16.8	30.5	19.5	30.3	18.8	33.5	20.5	31.8	17.5
TAR_i	1	0.5	7.0	1.3	8.3	1.8	6.0	1.3	8.5	1.5	8.3	0.8	8.3	1.0	10.5	1.8	10.0	1.8	11.0	3.8
		1	11.5	1.5	10.3	1.0	11.0	1.5	11.8	1.0	11.0	1.8	11.8	1.3	13.3	1.3	12.0	2.0	13.0	2.3
		1.5	15.0	1.0	12.0	1.5	12.0	0.5	16.0	2.8	13.5	2.3	12.3	2.3	11.8	0.8	13.8	2.0	13.0	2.5
	5	0.5	13.0	5.3	14.0	5.3	11.5	5.0	15.0	6.0	12.3	6.0	14.0	5.3	15.8	6.3	15.8	8.5	19.3	10.8
		1	16.3	5.3	15.8	6.3	15.8	6.3	17.0	4.8	17.5	7.0	17.3	7.3	21.3	7.0	17.5	6.8	21.0	9.5
		1.5	20.8	6.3	17.5	4.5	18.0	5.3	23.5	8.5	21.5	6.8	17.5	5.3	17.8	5.0	19.5	5.8	22.5	6.8
	10	0.5	16.5	8.8	19.3	13.3	15.8	10.3	19.5	13.3	19.0	11.5	18.0	12.0	22.3	10.8	21.3	14.8	23.8	15.3
		1	19.8	11.5	19.8	12.8	20.5	11.5	21.3	10.3	21.5	12.5	21.8	14.0	27.8	14.5	22.5	12.8	26.8	16.3
		1.5	23.8	10.8	20.5	8.8	22.8	9.5	29.0	14.0	26.3	12.0	24.0	11.8	22.0	9.3	24.0	10.3	27.5	11.0
RCA_i	1	0.5	6.5	2.5	7.0	1.3	8.3	0.8	9.0	1.8	11.3	2.3	11.8	2.5	10.3	3.0	12.3	4.0	11.8	2.8
		1	12.3	1.0	11.0	2.0	12.0	2.5	10.8	1.8	13.5	2.5	13.0	0.8	15.0	3.8	12.8	0.8	12.3	2.0
		1.5	12.5	2.5	13.3	1.8	14.3	1.5	15.5	4.0	12.0	1.5	15.8	2.0	15.8	0.8	16.5	1.8	17.3	2.3
	5	0.5	13.0	6.5	13.8	7.5	14.8	5.5	14.3	6.5	18.8	9.3	18.8	8.3	17.8	9.0	18.5	11.3	19.8	10.8
		1	18.0	7.0	17.3	5.0	19.0	5.8	16.3	7.3	21.0	8.5	20.0	7.3	20.3	8.5	19.0	6.8	18.8	8.3
		1.5	18.8	6.8	17.8	4.8	21.5	6.3	23.8	8.0	18.3	4.0	22.0	7.8	20.8	6.8	25.5	8.3	24.5	7.3
	10	0.5	20.5	12.3	21.0	13.3	17.5	11.0	20.8	14.0	24.0	15.8	23.3	15.0	22.8	14.3	25.5	18.8	24.5	16.3
		1	22.3	11.8	21.5	10.3	23.0	12.8	21.8	12.3	26.0	15.0	25.0	15.0	25.8	13.8	23.8	14.3	24.3	14.8
		1.5	23.5	12.3	20.8	8.3	27.0	13.3	28.3	14.5	21.0	9.8	25.5	13.8	26.0	12.8	31.8	14.8	29.5	15.5

Table 4. Actual size of φ_n for nominal size $\alpha \in \{1\%, 5\%, 10\%\}$, sample size $n \in \{50, 100, 200, 500, 1000, 2000\}$, bandwidth h_{CV} and models AR_3, MA_3, $GARCH_3$, TAR_3 and RCA_3, respectively

Model	α	n 50	100	200	500	1000	2000
AR_3	1	27.0	20.0	21.8	19.3	18.5	18.3
	5	33.5	27.3	27.3	25.8	23.5	24.0
	10	38.0	33.5	32.8	30.0	28.5	27.3
MA_3	1	8.3	6.5	5.8	6.3	5.8	4.8
	5	13.5	12.0	11.8	11.0	10.0	9.3
	10	17.8	15.8	16.3	17.0	14.0	13.3
$GARCH_3$	1	16.3	16.0	17.8	16.8	13.3	13.5
	5	22.8	22.5	25.8	26.3	19.5	20.8
	10	29.3	29.0	31.3	30.0	27.3	26.0
TAR_3	1	13.3	12.0	13.0	10.0	8.5	7.8
	5	21.3	17.5	21.0	15.5	15.3	14.8
	10	27.8	22.5	26.8	20.3	21.3	17.8
RCA_3	1	15.0	12.8	12.3	13.5	11.8	11.5
	5	20.3	19.0	18.8	19.8	18.8	17.8
	10	25.8	23.8	24.3	25.3	24.3	22.0

model in (4.1). Here, **A** is chosen from

$$\mathbf{A}_4 = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.8 \end{pmatrix}, \qquad \mathbf{A}_5 = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.7 \end{pmatrix}, \qquad \mathbf{A}_6 = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.6 \end{pmatrix}$$

and $\mathbf{\Sigma} = \mathbf{Id}$. The corresponding models are now denoted by AR_i, $i = 4, 5, 6$ and due to the diagonal shape, we are dealing with two independent time series. In the second case, we generate realizations from the moving average model in (4.2) with **B** chosen from

$$\mathbf{B}_4 = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.7 \end{pmatrix}, \qquad \mathbf{B}_5 = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.8 \end{pmatrix}, \qquad \mathbf{B}_6 = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.9 \end{pmatrix}$$

and $\mathbf{\Sigma} = \mathbf{\Sigma}_2$ defined in (4.3) denoted by MA_i, $i = 4, 5, 6$. Due to non-diagonal shape, we are dealing with two dependent time series in this case. To investigate the power performance of the tests, we consider the same non-linear time series models as above. First, we use independent GARCH(1, 1) models, where the first process $\{X_{t,1}, t \in \mathbb{Z}\}$ follows GARCH(1, 1) equation in (4.4) with $\omega = 0.01$, $a = 0.1$, $b = 0.2$ and the second process $\{X_{t,2}, t \in \mathbb{Z}\}$ is generated by the same model, where $\omega = 0.01$, $a = 0.1$, but b is chosen from

$$b_4 = 0.3, \qquad b_5 = 0.4, \qquad b_6 = 0.5$$

referred to as model $GARCH_i$, $i = 4, 5, 6$. Two (centered) independent threshold AR models under the alternative are generated by equation (4.5), where $\{X_{t,1}, t \in \mathbb{Z}\}$ corresponds to coefficients

$(a(1), a(2)) = (-0.2, 0.1)$ and those for $\{X_{t,2}, t \in \mathbb{Z}\}$ are chosen from

$$\underline{a}_4 = (-0.3, 0.2)^T, \qquad \underline{a}_5 = (-0.4, 0.3)^T, \qquad \underline{a}_6 = (-0.5, 0.4)^T.$$

Remark in this context, that for the asymptotic test φ_n we would observe a larger absolute power since this test is very liberal in the considered situations. In the supplementary material (cf. Jentsch and Pauly [17]), we therefore present its power performance in comparison to all three randomization tests by a comparison of actually achieved size obtained from the simulations in Section 4.2 with rejection rates in Tables 5–6 instead of using the nominal size. There it can be seen that the power behavior of all three tests (measured as power in comparison to actual size) appear to be quite similar for all models under investigation.

4.4. Discussion

From Tables 1–4, it can be seen that the asymptotic test φ_n has difficulties in keeping the prescribed level and tends to over rejects the null systematically for all small ($n = 50$) and moderate ($n = 100$) sample sizes. Its performance is not even desirable for larger sample sizes ($n \geq 200$). This is the case for all linear and non-linear as well as all three innovation distributions under consideration. In Table 1, especially for the most critical autoregressive models AR_3, where the AR coefficient is near to unity and the corresponding spectral densities have a non-flat shape, the null approximation of φ_n is extremely poor and the performance is unacceptable (see Table 1, right panel). Nevertheless, this generally poor performance is not surprising since the slow convergence speed of L_2-type statistics is already known, see for instance Paparoditis [21], and Table 4 in this paper.

In comparison to that, the randomization test φ_n^* performs better than φ_n. For all models under the null and innovation distributions under investigation, the randomization test holds the prescribed level more satisfactorily than φ_n.

Especially for Gaussian innovations and linear time series, the usage of φ_n^* can be recommended for all dependent and independent settings and sample sizes. In these cases, a close inspection of Tables 1–3 shows also that the bandwidth choice seems to have only a slight effect on the behavior of the randomization test, where this choice appears to be more crucial for φ_n. To demonstrate this, compare for instance in Tables 1 and 2 the performance of model AR_3 for $n = 200$ and $\alpha = 10\%$, where the range of the actual size is from 22.5% to 40.5% for φ_n and from 8.8% to 12.3% for φ_n^* over the bandwidths $0.5h_{CV}$, h_{CV} and $1.5h_{CV}$.

For linear time series with non-Gaussian innovations the performance of φ_n is still worse than that of φ_n^*. However, here the randomization test is slightly more effected by the choice of the bandwidth and its accuracy under H_0 often needs larger sample sizes ($n \geq 100$ or even $n \geq 200$).

For non-linear time series, see Table 3, a similar observation can be made. The performance of φ_n is poor for all sample sizes from 50 to 200 and φ_n^* again keeps the prescribed level much better. In particular, for the *TAR* models, the control of the nominal size is quite accurate. For the RCA and GARCH models under consideration, which are not covered by our Assumption 2.1, the performance of φ_n^* is still quite good and, in particular, improved in comparison to φ_n. However, the finite sample performance seems to be more effected by the bandwidth choice for RCA models and φ_n^* still tends to overreject the null for all GARCH models.

Table 5. Power of φ_n^* for nominal size $\alpha \in \{1\%, 5\%, 10\%\}$, sample size $n \in \{50, 100, 200\}$, bandwidth $h = c \cdot h_{CV}$ for $c \in \{0.5, 1, 1.5\}$ and autoregressive models AR_4, AR_5 and AR_6 with Gaussian, logistic and double-exponential distribution of the innovations

			AR_4			AR_5			AR_6		
		n:	50	100	200	50	100	200	50	100	200
$\{e_t\}$	α	c									
Gauss	1	0.5	1.5	3.3	10.8	5.3	9.0	35.0	4.8	19.3	64.3
		1	3.0	5.3	10.0	4.5	11.8	37.0	6.8	19.0	67.0
		1.5	3.8	2.5	11.5	3.8	10.0	36.8	5.3	20.5	67.5
	5	0.5	7.5	14.3	32.3	16.5	29.0	64.8	19.0	49.5	89.0
		1	8.5	15.3	25.8	14.5	33.8	70.3	23.3	51.5	91.0
		1.5	12.5	12.3	33.0	12.8	29.5	71.0	20.3	48.3	89.3
	10	0.5	16.5	24.0	46.3	28.5	46.8	80.8	31.5	72.5	95.0
		1	17.0	26.0	39.0	26.0	47.5	83.8	39.5	71.8	96.3
		1.5	18.8	25.0	46.5	23.3	43.3	84.5	36.0	64.3	97.3
Logistic	1	0.5	3.3	5.0	10.8	6.3	10.0	36.8	8.0	24.0	64.5
		1	3.8	4.8	11.0	6.5	12.5	31.5	7.3	16.3	70.0
		1.5	2.8	5.8	10.5	6.0	12.3	34.3	10.3	22.3	66.5
	5	0.5	10.0	17.0	29.3	16.5	31.5	65.5	22.8	55.5	88.3
		1	11.3	16.5	26.5	20.3	33.3	64.5	21.8	53.0	90.0
		1.5	9.0	18.3	26.8	16.3	30.0	64.5	24.5	52.5	91.0
	10	0.5	18.0	26.5	45.3	25.8	48.5	77.3	37.8	69.8	94.8
		1	18.5	26.0	44.0	29.5	50.5	81.0	35.5	71.5	96.0
		1.5	18.3	26.5	39.0	26.5	44.0	81.5	39.5	70.5	98.3
Double -exp.	1	0.5	4.0	5.0	9.8	6.3	10.3	39.8	10.3	24.0	59.3
		1	6.0	6.0	8.5	5.8	13.8	38.3	8.8	23.5	65.5
		1.5	4.5	6.8	12.5	8.0	15.0	35.3	9.3	23.5	69.0
	5	0.5	13.3	16.5	27.0	18.5	31.5	67.3	27.3	54.8	86.8
		1	14.5	16.8	24.0	17.3	34.3	66.8	23.8	55.3	91.0
		1.5	12.0	17.0	32.8	19.0	38.3	66.5	24.0	50.3	89.0
	10	0.5	22.3	25.5	43.0	28.8	48.3	82.0	41.0	75.0	93.8
		1	23.3	28.5	40.5	31.0	49.3	82.5	39.0	71.8	95.8
		1.5	21.0	23.3	43.8	31.0	51.8	79.5	39.0	66.5	97.0

Tables 5–7 show the power behavior of the randomization test φ_n^*, where we compare its power to its nominal size. When studying the panels with increasing sample sizes (from left to right) the consistency results of Theorem 2.2 and Corollary 3.1 under the alternative can be confirmed by the simulations. In particular, for the non-linear RCA and GARCH models a typical consistency behavior can also be observed. Similar to the situation under the null, for Gaussian innovations and linear time series the bandwidth choice does not seem to be crucial for φ_n^*, but for other innovations and non-linear time series it has a considerable effect on its power behavior. To this

Table 6. Power of φ_n^* for nominal size $\alpha \in \{1\%, 5\%, 10\%\}$, sample size $n \in \{50, 100, 200\}$, bandwidth $h = c \cdot h_{CV}$ for $c \in \{0.5, 1, 1.5\}$ and moving-average models MA_4, MA_5 and MA_6 with Gaussian, logistic and double-exponential distribution of the innovations

			MA_4			MA_5			MA_6		
		n:	50	100	200	50	100	200	50	100	200
$\{e_t\}$	α	c									
Gauss	1	0.5	3.3	7.0	14.3	4.5	12.3	34.8	12.8	29.8	65.5
		1	4.8	11.0	17.8	7.8	19.5	38.0	12.3	38.3	69.0
		1.5	3.3	9.5	20.8	9.5	19.0	44.3	14.0	34.8	76.5
	5	0.5	10.0	17.0	31.0	13.8	29.3	54.0	31.5	50.0	84.8
		1	12.8	21.5	34.3	21.8	40.3	62.8	29.8	62.0	87.0
		1.5	12.5	20.8	39.0	23.3	39.0	66.3	29.3	62.0	90.0
	10	0.5	19.8	26.8	41.3	22.8	41.0	64.5	41.5	63.8	91.5
		1	20.0	34.0	47.3	31.8	53.0	74.5	43.0	72.0	94.3
		1.5	23.3	30.5	50.0	35.3	52.8	78.8	43.3	74.5	95.5
Logistic	1	0.5	2.5	3.8	8.3	3.0	10.8	16.0	9.3	14.8	30.0
		1	2.5	7.0	12.3	5.3	11.8	14.5	10.8	20.0	36.8
		1.5	2.5	5.0	9.0	5.0	11.0	21.0	10.3	14.8	40.0
	5	0.5	13.5	13.3	19.0	13.5	23.5	29.5	22.5	29.0	49.0
		1	11.8	13.3	22.8	14.8	26.0	34.3	24.3	39.3	59.5
		1.5	10.8	12.3	22.0	15.3	24.8	39.3	21.0	33.5	61.8
	10	0.5	19.0	21.3	28.3	21.3	34.3	41.8	31.3	39.3	59.8
		1	19.8	22.5	33.0	24.0	36.8	47.3	32.3	51.0	71.3
		1.5	18.5	19.8	31.0	21.5	37.0	51.5	34.5	45.5	72.0
Double -exp.	1	0.5	3.5	4.8	9.0	6.3	11.8	18.5	11.3	19.8	35.0
		1	6.5	10.5	11.3	8.0	13.3	23.0	10.3	20.5	37.3
		1.5	5.3	9.8	13.5	7.0	12.0	22.3	12.3	23.5	43.5
	5	0.5	12.5	14.8	20.5	19.5	24.5	32.8	26.8	34.8	51.8
		1	13.3	19.8	25.3	19.5	26.0	39.3	23.8	39.3	55.5
		1.5	18.0	21.5	24.8	18.3	25.8	40.8	25.0	40.5	64.5
	10	0.5	21.5	23.8	28.3	29.8	36.3	42.3	35.8	45.0	60.5
		1	23.0	27.3	36.3	29.3	35.8	48.5	33.0	50.3	64.5
		1.5	26.5	34.0	36.5	26.8	36.5	49.3	34.0	50.3	73.8

end, we think that the typically applied cross-validation selector leads to quite adequate finite sample performances.

Our simulation experience may be summarized as follows:

- The randomization technique makes sure that φ_n^* keeps the prescribed level well for very small sample sizes (especially for most linear time series and *TAR* models under considera-

Table 7. Power of φ_n^* for nominal size $\alpha \in \{1\%, 5\%, 10\%\}$, sample size $n \in \{50, 100, 200\}$, bandwidth $h = c \cdot h_{\text{CV}}$ for $c \in \{0.5, 1, 1.5\}$ and non-linear models $GARCH_i$, TAR_i and RCA_i for $i = 4, 5, 6$, respectively

		i:	4			5			6		
		n:	50	100	200	50	100	200	50	100	200
Model	α	c									
$GARCH_i$	1	0.5	5.3	6.5	8.5	12.3	22.8	42.0	21.0	53.5	86.3
		1	4.3	7.5	11.8	12.3	23.8	46.5	26.5	56.0	83.0
		1.5	4.0	5.3	11.3	12.8	24.0	46.8	25.5	55.3	83.5
	5	0.5	12.8	16.3	20.5	26.3	35.8	61.3	41.5	71.3	95.0
		1	10.3	17.8	25.3	24.5	38.5	66.8	47.3	72.3	92.8
		1.5	12.8	13.8	21.0	28.3	40.5	66.3	49.5	73.5	94.3
	10	0.5	19.8	24.5	27.3	37.0	48.8	70.3	56.5	82.5	96.3
		1	16.8	25.3	34.8	32.8	50.0	77.3	58.5	81.0	95.8
		1.5	21.3	23.0	33.0	37.3	51.5	74.8	59.0	81.3	96.8
TAR_i	1	0.5	2.5	2.3	3.5	2.0	1.3	2.8	2.0	3.8	6.8
		1	1.5	1.3	1.8	1.3	0.8	3.8	2.0	3.5	5.8
		1.5	1.5	1.5	2.8	2.0	1.3	1.0	1.8	1.5	3.3
	5	0.5	6.3	8.3	8.0	8.0	6.0	12.8	6.5	11.3	21.3
		1	6.5	5.5	4.5	6.3	4.8	9.8	6.3	10.5	18.5
		1.5	5.8	5.5	7.8	6.3	5.5	5.8	5.8	6.5	12.5
	10	0.5	13.3	12.5	13.3	12.8	10.3	22.3	14.0	22.5	30.8
		1	12.8	10.8	10.5	11.5	9.8	15.8	11.8	17.5	27.5
		1.5	11.5	9.3	12.5	11.8	11.0	9.0	13.8	14.0	19.3
RCA_i	1	0.5	1.8	1.3	2.3	2.5	3.0	2.8	2.3	6.0	10.5
		1	1.3	2.5	2.3	1.5	3.8	2.5	2.8	5.8	12.0
		1.5	2.3	0.8	1.3	1.5	1.5	2.5	3.8	5.8	12.0
	5	0.5	6.0	7.3	7.0	6.8	11.3	10.8	8.5	15.0	20.3
		1	5.8	7.0	8.3	6.5	11.3	9.3	13.0	15.3	27.3
		1.5	6.8	7.5	5.5	6.3	8.8	9.8	10.3	16.0	24.5
	10	0.5	14.8	12.5	12.5	12.3	14.8	16.5	17.3	23.3	30.3
		1	10.0	14.3	12.3	12.5	16.5	15.0	23.3	23.8	39.3
		1.5	14.0	13.0	13.5	11.0	14.8	17.0	19.8	22.8	33.0

tion) as shown in Tables 1–3. Its performance is *in general* significantly better than those of the asymptotic benchmark test φ_n.

- Even for larger sample sizes up to $n = 2000$ the asymptotic test φ_n cannot be recommended due to its very slow convergence as emphasized in Table 4.
- In comparison to other resampling methods applied in time series analysis, the randomization technique used here has the big advantage that their performance does not depend on the choice of any tuning parameter in addition to the bandwidth. The choice of the band-

- width can be done automatically by standard methods and does not seem to be as crucial as for φ_n.
- The performance of φ_n^* becomes even more excellent if we compare its behavior for the very small sample size of $n = 50$ with the poor performance of the unconditional test.
- In comparison to that the only other known and mathematically analyzed resampling test for H_0, the bootstrap test of Dette and Paparoditis [9], needs sample sizes of $n \geq 512$ for gaining comparable results (see Table 1 in their paper). Remark that they have also modeled a VAR(1) model under the null with Gaussian innovations. When comparing their results with ours (see Table 1 in this paper) note that $\rho = 0$ in their paper corresponds to our model AR_3, but with the (slightly less critical) choice

$$\mathbf{A} = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.8 \end{pmatrix}.$$

- Furthermore, the power performance of φ_n^* improves with increasing sample size (as usual under consistency). The simulations in the supplementary material even show that the power behavior (measured as power in comparison to actual size) is similar to that of the benchmark test φ_n and all other randomization tests.
- Finally, to sum up, the randomization procedure helps to hold the prescribed level under the null more satisfactorily and does not forfeit power under the alternative in comparison to the unconditional case.

5. Final remarks and outlook

In this paper, we have introduced novel randomization-type tests for comparing spectral density matrices. Their theoretical properties have been analyzed in detail and we have also studied their finite sample performance in extensive simulation studies. The asymptotic behavior under the null as well as for fixed and local alternatives have been developed for non-linear time series under a joint cumulant condition (2.1), whereas for linear processes finite 32nd moments are required. Although these conditions seem to be rather strong, we like to mention that only a few mathematical results on this topic can be found in the current literature, where most of them are only developed for the one-dimensional case. Note, that such, or stronger linearity or even Gaussianity conditions are typical assumptions in recent time series publications, see, for example, Eichler [11], Dette and Paparoditis [9], Dette and Hildebrandt [8], Jentsch [15], Jentsch and Pauly [16], Preuss and Hildebrandt [25] or Preuss *et al.* [26]. Moreover, one of the main contributions of the current paper is the theoretical justification of the randomization approach, which has yet not been achieved in the time series context.

In future research, we aim to get rid of the last tuning parameter (i.e., the bandwidth) by studying a different test statistic that is based on integrated periodograms rather than kernel spectral density estimators. In doing so, we also plan to relax the conditions on the process in the non-linear case and to investigate to what extend our theory can still be established. As pointed out by one referee, a promising approach may be given by substituting our joint cumulant condition by physical-dependence-type-conditions introduced in Wu [32] and further studied in for example, Liu and Wu [19] or Xiao and Wu [33]. However, to our knowledge the theory is

currently not available in full strength for our setting, that is, for multivariate k-sample problems with dependencies and triangular arrays. The latter is required to derive the local power behavior of our test statistic.

6. Proofs

In the following and for a better lucidity of the proofs, we will use the abbreviate notation $I_{w,m_1,m_2;k} = I_{pw-m_1,pw-m_2}(\omega_k)$ and $f_{w_1,w_2,m_1,m_2;k} = f_{pw_1-m_1,pw_2-m_2}(\omega_k)$.

6.1. Proofs of Section 2

Proof of Theorem 2.1. Under the Assumptions 2.1 and 2.2, which accord with Assumptions 3.1 and 3.3 (with taper function $h(t) \equiv 1$) in Eichler [11], Theorem 3.5 in his paper is applicable to the test statistic T_n, where the also required Assumption 3.2 is obviously satisfied due to the quite simple shape of T_n. Also, by direct computation, we get

$$E(T_n) = \frac{h^{1/2}}{n} \int_{-\pi}^{\pi} \sum_{m_1,m_2=0}^{p-1} \sum_{k_1,k_2} K_h(\omega - \omega_{k_1}) K_h(\omega - \omega_{k_2})$$

$$\times \frac{1}{q^2} \sum_{j_1,j_2=1}^{q} \left(\sum_{r=1}^{q} (1 - q\delta_{j_1 r})(1 - q\delta_{j_2 r}) \right)$$

$$\times E(I_{j_1,m_1,m_2;k_1} \overline{I_{j_2,m_1,m_2;k_2}}) \, d\omega$$

$$= \frac{1}{nh^{3/2}} \int_{-\pi}^{\pi} \sum_{m_1,m_2=0}^{p-1} \sum_{k} K^2\left(\frac{\omega - \omega_k}{h}\right)$$

$$\times \frac{1}{q} \sum_{j_1,j_2=1}^{q} (-1 + q\delta_{j_1 j_2}) f_{j_1,j_2,m_1,m_1;k} \overline{f_{j_1,j_2,m_2,m_2;k}} \, d\omega + o(1),$$

which is asymptotically equivalent to

$$\frac{1}{h^{1/2}} \mu_0 = \frac{1}{h^{1/2}} \frac{1}{2\pi} \int_{-\pi}^{\pi} K^2(v) \, dv \int_{-\pi}^{\pi} \left(\frac{1}{q} \sum_{j_1,j_2=1}^{q} (-1 + q\delta_{j_1 j_2}) |\text{tr}(\mathbf{F}_{j_1 j_2}(\omega))|^2 \right) d\omega.$$

For the variance, we have

$$\text{Var}(T_n)$$

$$= \frac{h}{n^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \sum_{m_1,m_2,m_3,m_4=0}^{p-1} \sum_{k_1,k_2,k_3,k_4} K_h(\omega - \omega_{k_1}) K_h(\omega - \omega_{k_2}) K_h(\lambda - \omega_{k_3}) K_h(\lambda - \omega_{k_4})$$

$$\times \frac{1}{q^4}\left(\sum_{j_1,j_2,j_3,j_4=1}^{q}\left(\sum_{r=1}^{q}(1-q\delta_{j_1r_1})(1-q\delta_{j_2r_1})\right)\right.$$

$$\left.\times\left(\sum_{r_2=1}^{q}(1-q\delta_{j_3r_2})(1-q\delta_{j_4r_2})\right)\right)$$

$$\times\left\{E(I_{j_1,m_1,m_2;k_1}\,\overline{I_{j_2,m_1,m_2;k_2}}\,I_{j_3,m_3,m_4;k_3}\,\overline{I_{j_4,m_3,m_4;k_4}})\right.$$

$$-E(I_{j_1,m_1,m_2;k_1}\,\overline{I_{j_2,m_1,m_2;k_2}})$$

$$\left.\times E(I_{j_3,m_3,m_4;k_3}\,\overline{I_{j_4,m_3,m_4;k_4}})\right\}\,d\omega\,d\lambda.$$

In evaluation of the difference of the expectations above, the only asymptotically non-vanishing cases are $\omega_{k_1}=\omega_{k_3}\neq\omega_{k_2}=\omega_{k_4}$, $\omega_{k_1}=\omega_{k_4}\neq\omega_{k_2}=\omega_{k_3}$, $\omega_{k_1}=-\omega_{k_3}\neq\omega_{k_2}=-\omega_{k_4}$ and $\omega_{k_1}=-\omega_{k_4}\neq\omega_{k_2}=-\omega_{k_3}$, where all of them make the same contribution, which enables us to consider the first case only equipped with a factor 4. In this situation, we have

$$\left\{E(\cdots)-E(\cdots)E(\cdots)\right\}=f_{j_1,j_3,m_1,m_4;k_1}\,\overline{f_{j_1,j_3,m_2,m_3;k_1}}\,f_{j_2,j_4,m_1,m_4;k_2}\,\overline{f_{j_2,j_4,m_2,m_3;k_2}}\,\delta_{k_1k_3}\delta_{k_2k_4},$$

which results in

$$\mathrm{Var}(T_n)$$

$$=\frac{4h}{n^2}\sum_{k_1,k_2}\left(\int_{-\pi}^{\pi}K_h(\omega-\omega_{k_1})K_h(\omega-\omega_{k_2})\,d\omega\right)^2\frac{1}{q^2}$$

$$\times\sum_{j_1,j_2,j_3,j_4=1}^{q}(-1+q\delta_{j_1j_2})(-1+q\delta_{j_3j_4})$$

$$\times\left(\sum_{m_1,m_4=0}^{p-1}f_{j_1,j_3,m_1,m_4;k_1}\,\overline{f_{j_2,j_4,m_1,m_4;k_2}}\right)$$

$$\times\left(\sum_{m_2,m_3=0}^{p-1}\overline{f_{j_1,j_3,m_2,m_3;k_1}}\,f_{j_2,j_4,m_2,m_3;k_2}\right)+o(1).$$

The term on the right-hand side is asymptotically equivalent to

$$\tau_0^2=B_K\int_{-\pi}^{\pi}\left(\frac{1}{q^2}\sum_{j_1,j_2,j_3,j_4=1}^{q}(-1+q\delta_{j_1j_2})(-1+q\delta_{j_3j_4})\left|\mathrm{tr}\left(\mathbf{F}_{j_1j_3}(\omega)\overline{\mathbf{F}_{j_2j_4}(\omega)}^T\right)\right|^2\right)d\omega,$$

which concludes this proof. □

Proof of Theorem 2.2. Note first that the $\sqrt{nh}$-consistency of the kernel density estimators, see, for example, Equation (5) in Eichler [11], implies $\widehat{\mu}-\mu_0=o_P(h^{1/2})$. Thus Slutsky's lemma

together with Theorem 2.1 show that under H_0

$$\frac{T_n - h^{-1/2}\widehat{\mu}}{\tau_0} \xrightarrow{\mathcal{D}} N(0, 1)$$

which corresponds to the limit result for Q_T on page 976 in Eichler [11]. Together with the consistency of $\widehat{\tau}^2$ this proves the asymptotic exactness of φ_n under the null hypothesis. In contrast to that, similar to Eichler's Theorem 5.1, it can be verified by direct computations and Assumption 2.2 that $h^{1/2}T_n$ converges in probability to $+\infty$ under the alternative H_1. Thus, the same hold true for $T_n - h^{-1/2}\widehat{\mu}$, which implicates consistency of the test. $\qquad\square$

Proof of Theorem 2.3. Remark that the imposed uniform mixing conditions still imply the $\sqrt{nh}$-consistency of the kernel density estimators for the processes. Hence, the estimators for mean and variance can be replaced by their limit values and we can apply Theorem 5.4 in Eichler [11] with the special function

$$\Psi(\mathbf{f}) = \left(\mathrm{vec}\left(\mathbf{F}_{rr} - \frac{1}{q}\sum_{j=1}^{q}\mathbf{F}_{jj}\right)\right)_{1 \le r \le q}.$$

Noting that

$$\left.\frac{\partial\Psi(\mathbf{Z})}{\partial Z_{i,j}}\right|_{\mathbf{Z}=\mathbf{f}} = \mathrm{vec}\left((\mathbf{1}\{(r, s) = (i, j)\})_{(r,s)\in\Xi}\right) - \frac{1}{q}\mathrm{vec}\left((\mathbf{1}\{(r, s) \in \Xi_{(i,j)}\})_{(r,s)\in\Xi}\right)$$

for

$$\Xi_{(i,j)} = \begin{cases} \{(r, s) \in \Xi: \exists c \in \mathbb{Z}_{\neq 0} \text{ with } (r, s) = (cp + i, cp + j)\}, & \text{for } (i, j) \in \Xi, \\ \varnothing, & \text{elsewise} \end{cases}$$

and with $\Xi = \bigcup_{c=1}^{q}\{(r, s): 1 + (c - 1)p \le r, s \le cp\}$ that theorem implies, that our standardized test statistic converges in distribution to a normally distributed random variable with mean ν and variance τ_0^2. Hence, the results follows. $\qquad\square$

6.2. Proofs of Section 3

Throughout this section, $E^*(\cdot) := E(\cdot|\underline{X}_1, \ldots, \underline{X}_n)$, $\mathrm{Var}^*(\cdot) := \mathrm{Var}(\cdot|\underline{X}_1, \ldots, \underline{X}_n)$ and $\mathrm{Cov}^*(\cdot, \cdot) := \mathrm{Cov}(\cdot, \cdot|\underline{X}_1, \ldots, \underline{X}_n)$ denote mean, variance and covariance conditioned on the data, respectively.

Lemma 6.1 (Brillinger [3], Theorem 4.3.2). *Suppose that the mixing condition* (2.1) *is satisfied for all* $k \le s \in \mathbb{N}$. *Then for any* $k \in \{1, \ldots, s\}$ *and* $\omega_1, \ldots, \omega_k \in [0, 2\pi]$, *we have*

$$\mathrm{cum}\left(J_{a_1}(\omega_1), \ldots, J_{a_k}(\omega_k)\right)$$
$$= \frac{(2\pi)^{(k/2)-1}}{n^{k/2}}f_k^{(a_1,\ldots,a_k)}(\omega_1, \ldots, \omega_{k-1})\sum_{t=1}^{n}e^{\mathrm{i}t\sum_{j=1}^{k}\omega_j} + \mathrm{O}\left(\frac{1}{n^{k/2}}\right),$$
(6.1)

where $J_a(\omega)$ denotes the ath component of the d-variate discrete Fourier transform $\underline{J}(\omega)$, see (1.4), $f_k^{(a_1,\ldots,a_k)}$ is defined as

$$f_k^{(a_1,\ldots,a_k)}(\omega_1,\ldots,\omega_{k-1})$$

$$= \frac{1}{(2\pi)^{k-1}} \sum_{t_1,\ldots,t_{k-1}=-\infty}^{\infty} c_{a_1,\ldots,a_k}(t_1,\ldots,t_{k-1}) \exp\big(\mathrm{i}(t_1\omega_1 + \cdots + t_{k-1}\omega_{k-1})\big)$$

the kth order cumulant spectral density and f_k is the $(d \times d \times \cdots \times d)$ (n times) kth order cumulant spectra of the d-variate time series $(\underline{X}_t, t \in \mathbb{Z})$.

Lemma 6.2. *It holds*

(i) $\displaystyle\sum_{r=1}^{q}(1 - q\delta_{j_1,\pi_{k_1}(r)})(1 - q\delta_{j_2,\pi_{k_2}(r)}) = -q + q^2 \sum_{r=1}^{q} \delta_{j_1,\pi_{k_1}(r)}\delta_{j_2,\pi_{k_2}(r)},$

(ii) $\displaystyle\sum_{r=1}^{q} E^*\big((1 - q\delta_{j_1,\pi_{k_1}(r)})(1 - q\delta_{j_2,\pi_{k_2}(r)})\big) = \big(-q + q^2\delta_{j_1 j_2}\big)\delta_{|k_1|,|k_2|},$

(iii) $\displaystyle\sum_{r_1,r_2=1}^{q} E^*\big((1 - q\delta_{j_1,\pi_{k_1}(r_1)})(1 - q\delta_{j_2,\pi_{k_2}(r_1)})(1 - q\delta_{j_3,\pi_{k_3}(r_2)})(1 - q\delta_{j_4,\pi_{k_4}(r_2)})\big)$

$$= q^2(-1 + q\delta_{j_1 j_2})(-1 + q\delta_{j_3 j_4})\delta_{|k_1|,|k_2|}\delta_{|k_3|,|k_4|}(1 - \delta_{|k_1|,|k_3|})$$

$$+ q^2\left(-1 + q\delta_{j_1 j_3}\delta_{j_2 j_4} + \frac{q}{q-1}(1 - \delta_{j_1 j_3})(1 - \delta_{j_2 j_4})\right)$$

$$\times \delta_{|k_1|,|k_3|}\delta_{|k_2|,|k_4|}(1 - \delta_{|k_1|,|k_2|})$$

$$+ q^2\left(-1 + q\delta_{j_1 j_4}\delta_{j_2 j_3} + \frac{q}{q-1}(1 - \delta_{j_1 j_4})(1 - \delta_{j_2 j_3})\right)$$

$$\times \delta_{|k_1|,|k_4|}\delta_{|k_2|,|k_3|}(1 - \delta_{|k_1|,|k_2|})$$

$$+ C\delta_{|k_1|,|k_2|,|k_3|,|k_4|},$$

for some constant $C < \infty$, where $\delta_{|k_1|,|k_2|,|k_3|,|k_4|} = 1$ if $|k_1| = |k_2| = |k_3| = |k_4|$ and 0 otherwise.

Proof. The first assertion (i) follows from $\sum_{r=1}^{q} \delta_{j,\pi_k(r)} = 1$. Due to $\widetilde{\mathbb{P}}(\pi_k(r) = j) = \frac{1}{q}$ for all k, r and j, we get

$$E^*(\delta_{j_1,\pi_{k_1}(r_1)}\delta_{j_2,\pi_{k_2}(r_2)}) = \begin{cases} E^*(\delta_{j_1,\pi_k(r)}\delta_{j_2,\pi_k(r)}), & |k_1| = |k_2|, r_1 = r_2, \\ E^*(\delta_{j_1,\pi_k(r_1)}\delta_{j_2,\pi_k(r_2)}), & |k_1| = |k_2|, r_1 \neq r_2, \\ E^*(\delta_{j_1,\pi_{k_1}(r)})E^*(\delta_{j_2,\pi_{k_2}(r)}), & |k_1| \neq |k_2| \end{cases} \quad (6.2)$$

$$= \begin{cases} \dfrac{1}{q}\delta_{j_1 j_2}, & |k_1| = |k_2|, r_1 = r_2, \\[2mm] \dfrac{1}{q(q-1)}(1 - \delta_{j_1 j_2}), & |k_1| = |k_2|, r_1 \neq r_2, \\[2mm] \dfrac{1}{q^2}, & |k_1| \neq |k_2| \end{cases}$$

which yields (ii). Now consider (iii). First, set $Z_{j,k,r} := 1 - q\delta_{j,\pi_k(r)}$ and note that $E^*(Z_{j,k,r}) = 0$ for all j, k and r. Therefore, we have to consider the following non-vanishing cases (a) $|k_1| = |k_2| \neq |k_3| = |k_4|$, (b) $|k_1| = |k_3| \neq |k_2| = |k_4|$, (c) $|k_1| = |k_4| \neq |k_2| = |k_3|$ and (d) $|k_1| = |k_2| = |k_3| = |k_4|$. The first case (a) gives

$$q^2(-1 + q\delta_{j_1 j_2})(-1 + q\delta_{j_3 j_4})\delta_{|k_1|,|k_2|}\delta_{|k_3|,|k_4|}(1 - \delta_{|k_1|,|k_3|})$$

which follows immediately from (ii). Case (b) becomes

$$\sum_{r_1,r_2=1}^{q} E^*(Z_{j_1,k_1,r_1}Z_{j_2,k_2,r_1}Z_{j_3,k_3,r_2}Z_{j_4,k_4,r_2}) = \sum_{r_1,r_2=1}^{q} E^*(Z_{j_1,k_1,r_1}Z_{j_3,k_1,r_2})E^*(Z_{j_2,k_2,r_1}Z_{j_4,k_2,r_2})$$

and together with (6.2), we get

$$E^*(Z_{j_1,k_1,r_1}Z_{j_3,k_1,r_2}) = -1 + q\delta_{r_1 r_2}\delta_{j_1 j_3} + \frac{q}{q-1}(1 - \delta_{r_1 r_2})(1 - \delta_{j_1 j_3})$$

and an analogue result holds for $E^*(Z_{j_2,k_2,r_1}Z_{j_4,k_2,r_2})$. By multiplication and summing up, the contribution of case (b) becomes

$$q^2\left(-1 + q\delta_{j_1 j_3}\delta_{j_2 j_4} + \frac{q}{q-1}(1 - \delta_{j_1 j_3})(1 - \delta_{j_2 j_4})\right)$$

and case (c) contributes analogously.   □

With these results, we can analyze the conditional expectation and variance of T_n^*.

Theorem 6.1 (Conditional mean and variance of T_n^*). *Under the assumptions of Theorem* 3.1, *it holds*

$$E^*(T_n^*) = h^{-1/2}\mu_n^* + o_P(1),$$

where

$$\mu_n^* := \frac{1}{n}\int_{-\pi}^{\pi}\sum_{m_1,m_2=0}^{p-1}\sum_{k}K_h^2(\omega - \omega_k)\frac{1}{q}\sum_{j_1,j_2=1}^{q}(-1 + q\delta_{j_1 j_2})I_{pj_1-m_1,pj_1-m_2}(\omega_k)$$
$$\times \overline{I_{pj_2-m_1,pj_2-m_2}(\omega_k)}\,d\omega. \tag{6.3}$$

Moreover, we have

$$
\mathrm{Var}^*(T_n^*) = \frac{4h}{n^2} \sum_{\substack{k_1,k_2 \\ |k_1|\neq|k_2|}} \left(\int_{-\pi}^{\pi} K_h(\omega - \omega_{k_1}) K_h(\omega - \omega_{k_2}) \, d\omega \right)^2
$$

$$
\times \sum_{m_1,m_2,m_3,m_4=0}^{p-1} \frac{1}{q^2} \sum_{j_1,j_2,j_3,j_4=1}^{q} \left(-1 + q\delta_{j_1 j_3}\delta_{j_2 j_4} \right.
$$

$$
\left. + \frac{q}{q-1}(1 - \delta_{j_1 j_3})(1 - \delta_{j_2 j_4}) \right)
$$

$$
\times I_{j_1,m_1,m_2;k_1} \overline{I_{j_2,m_1,m_2;k_2}}
$$

$$
\times I_{j_3,m_3,m_4;k_1} \overline{I_{j_4,m_3,m_4;k_2}} + o_P(1)
$$

$$
=: \tau_n^{*2} + o_P(1).
$$

Proof. By using Lemma 6.2(ii), we get

$$
E^*(T_n^*) = \frac{h^{1/2}}{n} \int_{-\pi}^{\pi} \sum_{m_1,m_2=0}^{p-1} \sum_{k_1,k_2} K_h(\omega - \omega_{k_1}) K_h(\omega - \omega_{k_2}) \frac{1}{q^2}
$$

$$
\times \sum_{j_1,j_2=1}^{q} E^* \left(\sum_{r=1}^{q} (1 - q\delta_{j_1,\pi_{k_1}(r)})(1 - q\delta_{j_2,\pi_{k_2}(r)}) \right)
$$

$$
\times I_{j_1,m_1,m_2;k_1} \overline{I_{j_2,m_1,m_2;k_2}} \, d\omega
$$

$$
= \frac{h^{1/2}}{n} \int_{-\pi}^{\pi} \sum_{m_1,m_2=0}^{p-1} \sum_{k_1,k_2} K_h(\omega - \omega_{k_1}) K_h(\omega - \omega_{k_2}) \frac{1}{q}
$$

$$
\times \sum_{j_1,j_2=1}^{q} (-1 + q\delta_{j_1 j_2})\delta_{|k_1|,|k_2|}
$$

$$
\times I_{j_1,m_1,m_2;k_1} \overline{I_{j_2,m_1,m_2;k_2}} \, d\omega.
$$

Furthermore, the case $k_1 = -k_2$ is asymptotically negligible, which yields the first assertion of Theorem 6.1. Considering the conditional variance of T_n^*, we get

$$
\mathrm{Var}^*(T_n^{*2})
$$

$$
= \frac{h}{n^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \sum_{m_1,m_2,m_3,m_4=0}^{p-1} \sum_{k_1,k_2,k_3,k_4} K_h(\omega - \omega_{k_1}) K_h(\omega - \omega_{k_2}) K_h(\lambda - \omega_{k_3}) K_h(\lambda - \omega_{k_4})
$$

$$\times \frac{1}{q^4}$$

$$\times \sum_{j_1,j_2,j_3,j_4=1}^{q} \sum_{r_1,r_2=1}^{q} \mathrm{Cov}^*(Z_{j_1,k_1,r_1} Z_{j_2,k_2,r_1},$$

$$Z_{j_3,k_3,r_2} Z_{j_4,k_4,r_2})$$

$$\times I_{j_1,m_1,m_2;k_1} \overline{I_{j_2,m_1,m_2;k_2}}$$

$$\times I_{j_3,m_3,m_4;k_3} \overline{I_{j_4,m_3,m_4;k_4}} \, d\omega \, d\lambda.$$

By using Lemma 6.2(ii) and (iii), only the cases $|k_1| = |k_3| \neq |k_2| = |k_4|$ and $|k_1| = |k_4| \neq |k_2| = |k_3|$ play a role asymptotically. More precisely, only the four cases $k_1 = k_3 \neq k_2 = k_4$ and $k_1 = -k_3 \neq k_2 = -k_4$ and $k_1 = k_4 \neq k_2 = k_3$ and $k_1 = -k_4 \neq k_2 = -k_3$ do not vanish, but all of them make the same contribution asymptotically. This gives a factor 4 and we get up to an $o_P(1)$-term

$$\mathrm{Var}^*\big(T_n^{*2}\big)$$

$$= \frac{4h}{n^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \sum_{m_1,m_2,m_3,m_4=0}^{p-1} \sum_{\substack{k_1,k_2 \\ |k_1|\neq|k_2|}} K_h(\omega - \omega_{k_1}) K_h(\omega - \omega_{k_2}) K_h(\lambda - \omega_{k_1}) K_h(\lambda - \omega_{k_2})$$

$$\times \frac{1}{q^2} \sum_{j_1,j_2,j_3,j_4=1}^{q} \left(-1 + q \delta_{j_1 j_3} \delta_{j_2 j_4} \right.$$

$$\left. + \frac{q}{q-1}(1 - \delta_{j_1 j_3})(1 - \delta_{j_2 j_4}) \right)$$

$$\times I_{j_1,m_1,m_2;k_1} \overline{I_{j_2,m_1,m_2;k_2}}$$

$$\times I_{j_3,m_3,m_4;k_1} \overline{I_{j_4,m_3,m_4;k_2}} \, d\omega \, d\lambda,$$

which concludes this proof.   □

Theorem 6.2 (On $E(\mu_n^*)$, $\mathrm{Var}(\mu_n^*)$, $E(\tau_n^{*2})$ and $\mathrm{Var}(\tau_n^{*2})$). *Under the assumptions of Theorem* 3.1, *it holds*

$$E\big(\mu_n^*\big) = \mu^* + \mathrm{o}\big(h^{1/2}\big) \quad and \quad \mathrm{Var}\big(\mu_n^*\big) = \mathrm{o}(h),$$

$$E\big(\tau_n^{*2}\big) = \tau^{*2} + \mathrm{o}(1) \quad and \quad \mathrm{Var}\big(\tau_n^{*2}\big) = \mathrm{o}(1) \tag{6.4}$$

as $n \to \infty$.

Proof. Because arguments are completely analogue, we prove only the more complicated part that deals with τ_n^{*2}. First, by introducing the notation $J_{j,m;k} = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} X_{t,pj-m} e^{-it\omega_k}$, we get

$$
E(\tau_n^{*2}) = \frac{4h}{n^2} \sum_{\substack{k_1,k_2 \\ |k_1| \neq |k_2|}} \left(\int_{-\pi}^{\pi} K_h(\omega - \omega_{k_1}) K_h(\omega - \omega_{k_2}) \, d\omega \right)^2
$$

$$
\times \sum_{m_1,m_2,m_3,m_4=0}^{p-1} \frac{1}{q^2} \sum_{j_1,j_2,j_3,j_4=1}^{q} \left(-1 + q\delta_{j_1 j_3}\delta_{j_2 j_4} \right.
$$

$$
\left. + \frac{q}{q-1}(1 - \delta_{j_1 j_3})(1 - \delta_{j_2 j_4}) \right)
$$

$$
\times E(J_{j_1,m_1;k_1} \overline{J_{j_1,m_2;k_1}} J_{j_3,m_3;k_1}
$$

$$
\times \overline{J_{j_3,m_4;k_1}} J_{j_2,m_1;k_2}
$$

$$
\times J_{j_2,m_2;k_2} \overline{J_{j_4,m_3;k_2}} J_{j_4,m_4;k_2}).
$$

(6.5)

The last expectation above can be expressed in terms of cumulants and we have

$$
E(J_1 \cdots J_8) = \sum_{\sigma} \prod_{B \in \sigma} \mathrm{cum}(J_i : i \in B),
$$

(6.6)

where we identify the DFTs with $J_1, \ldots, J_8$ for notational convenience, $\mathrm{cum}(J_i : i \in B)$ is the joint cumulant of $(J_i : i \in B)$, σ runs through the list of all partitions of $\{1, \ldots, 8\}$ and B runs through the list of all blocks of the partition σ. Note that the largest contribution in (6.6) is made by products of four cumulants of second order. Moreover, all of those combinations where cumulants $\mathrm{cum}(J_{\cdot,\cdot;k_i}, J_{\cdot,\cdot;k_j})$ or $\mathrm{cum}(J_{\cdot,\cdot;k_i}, \overline{J_{\cdot,\cdot;k_j}})$ for $i \neq j$ occur, contain a factor $\delta_{k_1 k_2}$ or $\delta_{k_1-k_2}$, respectively, but these cases are excluded in the summation in (6.5). The cases with $\mathrm{cum}(J_{\cdot,\cdot;k_i}, J_{\cdot,\cdot;k_i})$ or $\mathrm{cum}(\overline{J_{\cdot,\cdot;k_i}}, \overline{J_{\cdot,\cdot;k_i}})$ contain a factor $\delta_{k_i 0}$, which causes the sum over k_i to collapse and, therefore, they are of lower order and asymptotically negligible. The only two remaining products consisting of four cumulants of second order are by (6.1)

$$
\mathrm{cum}(J_{j_1,m_1;k_1}, \overline{J_{j_1,m_2;k_1}}) \, \mathrm{cum}(J_{j_3,m_3;k_1}, \overline{J_{j_3,m_4;k_1}})
$$

$$
\times \mathrm{cum}(\overline{J_{j_2,m_1;k_2}}, J_{j_2,m_2;k_2}) \, \mathrm{cum}(\overline{J_{j_4,m_3;k_2}}, J_{j_4,m_4;k_2})
$$

$$
= f_{j_1,j_1,m_1,m_2;k_1} f_{j_3,j_3,m_3,m_4;k_1} \overline{f_{j_2,j_2,m_1,m_2;k_2}} f_{j_4,j_4,m_3,m_4;k_2} + \mathrm{O}\left(\frac{1}{n}\right)
$$

and

$$
\mathrm{cum}(J_{j_1,m_1;k_1}, \overline{J_{j_3,m_4;k_1}}) \, \mathrm{cum}(\overline{J_{j_1,m_2;k_1}}, J_{j_3,m_3;k_1})
$$

$$
\times \mathrm{cum}(\overline{J_{j_2,m_1;k_2}}, J_{j_4,m_4;k_2}) \, \mathrm{cum}(J_{j_2,m_2;k_2}, \overline{J_{j_4,m_3;k_2}})
$$

$$
= f_{j_1,j_3,m_1,m_4;k_1} \overline{f_{j_1,j_3,m_2,m_3;k_1}} f_{j_2,j_4,m_1,m_4;k_2} f_{j_2,j_4,m_2,m_3;k_2} + \mathrm{O}\left(\frac{1}{n}\right).
$$

Now, by taking the sums over m_1, m_2, m_3, m_4 of both expressions, we get

$$\text{tr}\big(\mathbf{F}_{j_1 j_1}(\omega_{k_1})\overline{\mathbf{F}_{j_2 j_2}(\omega_{k_2})}^T\big)\text{tr}\big(\overline{\mathbf{F}_{j_3 j_3}(\omega_{k_1})}\mathbf{F}_{j_4 j_4}(\omega_{k_2})^T\big) + \text{O}\Big(\frac{1}{n}\Big) \tag{6.7}$$

and

$$\text{tr}\big(\mathbf{F}_{j_1 j_3}(\omega_{k_1})\overline{\mathbf{F}_{j_2 j_4}(\omega_{k_2})}^T\big)\text{tr}\big(\overline{\mathbf{F}_{j_1 j_3}(\omega_{k_1})}\mathbf{F}_{j_2 j_4}(\omega_{k_2})^T\big) + \text{O}\Big(\frac{1}{n}\Big),$$

respectively. Asymptotically equivalent to (6.5), this results in

$$B_K \int_{-\pi}^{\pi} \frac{1}{q^2} \sum_{j_1, j_2, j_3, j_4 = 1}^{q} \left(-1 + q\delta_{j_1 j_3}\delta_{j_2 j_4} + \frac{q}{q-1}(1 - \delta_{j_1 j_3})(1 - \delta_{j_2 j_4}) \right)$$
$$\times \Big\{ \text{tr}\big(\mathbf{F}_{j_1 j_1}(\omega_{k_1})\overline{\mathbf{F}_{j_2 j_2}(\omega_{k_2})}^T\big)\text{tr}\big(\mathbf{F}_{j_3 j_3}(\omega_{k_1})\overline{\mathbf{F}_{j_4 j_4}(\omega_{k_2})}^T\big)$$
$$+ \big|\text{tr}\big(\mathbf{F}_{j_1 j_3}(\omega)\overline{\mathbf{F}_{j_2 j_4}(\omega)}\big)\big|^2 \Big\} \, d\omega,$$

which shows the first assertion of (6.4). For its second part, we have

$$\text{Var}\big(\tau_n^{*2}\big) = \frac{16h^2}{n^4} \sum_{\substack{k_1, k_2 \\ |k_1| \neq |k_2|}} \sum_{\substack{k_3, k_4 \\ |k_3| \neq |k_4|}} \left(\int_{-\pi}^{\pi} K_h(\omega - \omega_{k_1})K_h(\omega - \omega_{k_2}) \, d\omega \right)^2$$
$$\times \left(\int_{-\pi}^{\pi} K_h(\lambda - \omega_{k_3})K_h(\lambda - \omega_{k_4}) \, d\omega \right)^2$$
$$\times \sum_{m_1, \ldots, m_8 = 0}^{p-1} \frac{1}{q^4} \sum_{j_1, \ldots, j_8 = 1}^{q} \left(-1 + q\delta_{j_1 j_3}\delta_{j_2 j_4} \right.$$
$$\left. + \frac{q}{q-1}(1 - \delta_{j_1 j_3})(1 - \delta_{j_2 j_4}) \right)$$
$$\times \left(-1 + q\delta_{j_5 j_7}\delta_{j_6 j_8} \right.$$
$$\left. + \frac{q}{q-1}(1 - \delta_{j_5 j_7})(1 - \delta_{j_6 j_8}) \right)$$
$$\times \Big\{ E(J_{j_1, m_1; k_1} \overline{J_{j_1, m_2; k_1}} J_{j_3, m_3; k_1}$$
$$\times \overline{J_{j_3, m_4; k_1}} J_{j_2, m_1; k_2} J_{j_2, m_2; k_2}$$
$$\times \overline{J_{j_4, m_3; k_2}} J_{j_4, m_4; k_2} J_{j_5, m_5; k_3}$$
$$\times \overline{J_{j_5, m_6; k_3}} J_{j_7, m_7; k_3} \tag{6.8}$$

$$\times \overline{J_{j_7,m_8;k_3}} \, \overline{J_{j_6,m_5;k_4}} \, J_{j_6,m_6;k_4}$$
$$\times \overline{J_{j_8,m_7;k_4}} \, J_{j_8,m_8;k_4})$$
$$- E(J_{j_1,m_1;k_1} \, \overline{J_{j_1,m_2;k_1}} \, J_{j_3,m_3;k_1}$$
$$\times \overline{J_{j_3,m_4;k_1}} \, J_{j_2,m_1;k_2}$$
$$\times J_{j_2,m_2;k_2} \, \overline{J_{j_4,m_3;k_2}}$$
$$\times J_{j_4,m_4;k_2})$$
$$\times E(J_{j_5,m_5;k_3} \, \overline{J_{j_5,m_6;k_3}}$$
$$\times J_{j_7,m_7;k_3} \, \overline{J_{j_7,m_8;k_3}}$$
$$\times \overline{J_{j_6,m_5;k_4}} \, J_{j_6,m_6;k_4}$$
$$\times \overline{J_{j_8,m_7;k_4}} \, J_{j_8,m_8;k_4})\}$$

and similar to the computations for $E(\tau_n^{*2})$, we are able to express the 16th and both 8th moments with cumulants and the difference above becomes

$$\{E(J_1 \cdots J_{16}) - E(J_1 \cdots J_8)E(J_9 \cdots J_{16})\} = \sum_{\widetilde{\sigma}} \prod_{B \in \widetilde{\sigma}} \mathrm{cum}(J_i : i \in B), \qquad (6.9)$$

where we identify the DFTs $J_{l,m;k}$ from above with $J_1, \ldots, J_{16}$ for notational convenience and now $\widetilde{\sigma}$ runs through the list of all partitions of $\{1, \ldots, 16\}$ that can not be written as cumulants depending on subsets of $\{J_1, \ldots, J_8\}$ multiplied with cumulants depending on subsets of $\{J_9, \ldots, J_{16}\}$ and B runs through the list of all blocks of the partition $\widetilde{\sigma}$. Now, we can use Lemma 6.1 and insert (6.1) in (6.9). Again the largest contribution comes from products of eight cumulants of second order and due to $\sum_{t=1}^{n} e^{it \sum_{j=1}^{k} \omega_j} = n$ if $\sum_{j=1}^{k} \omega_j = 0$ and $\sum_{t=1}^{n} e^{it \sum_{j=1}^{k} \omega_j} = 0$ otherwise, all these combinations that occur in (6.9) contain a factor $\delta_{|k_i|,|k_j|}$ for $i \neq j$. This is either excluded in the summation in (6.8) or results in at least one collapsing sum, which eventually implies $\mathrm{Var}(\tau_n^{*2}) = o(1)$. $\qquad \square$

Proof of Remark 3.1. We only treat the variance. Under H_0, the second summand in (3.4) does not depend on $j_1, \ldots, j_4$, so that this case vanishes thanks to

$$\sum_{j_1,j_2,j_3,j_4=1}^{q} \left(-1 + q \delta_{j_1 j_3} \delta_{j_2 j_4} + \frac{q}{q-1}(1 - \delta_{j_1 j_3})(1 - \delta_{j_2 j_4}) \right) = 0,$$

which yields the desired assertion. $\qquad \square$

Proof of Theorem 3.1. Note first that by Theorem 6.2 and the $\sqrt{nh}$-consistency of the spectral density estimators, see the proof of Theorem 2.2, we have $\mu_n^* - \widehat{\mu}^* = o_P(h^{1/2})$. Hence, it is sufficient to prove a central limit theorem for $T_n^* - h^{-1/2}\mu_n^*$.

By using Lemma 6.2(i), we have

$$T_n^* = \frac{h^{1/2}}{n} \sum_{k_1,k_2} \int_{-\pi}^{\pi} K_h(\omega - \omega_{k_1}) K_h(\omega - \omega_{k_2}) \, d\omega$$

$$\times \sum_{m_1,m_2=0}^{p-1} \sum_{j_1,j_2=1}^{q} \left(-\frac{1}{q} + \sum_{r=1}^{q} \delta_{j_1,\pi_{k_1}(r)} \delta_{j_2,\pi_{k_2}(r)} \right) I_{j_1,m_1,m_2;k_1} \overline{I_{j_2,m_1,m_2;k_2}}$$

$$=: \sum_{k_1,k_2} w_{n,k_1,k_2}^*(\pi_{k_1}, \pi_{k_2})$$

with an obvious notation for $w_{n,k_1,k_2}^*(\pi_{k_1}, \pi_{k_2}) =: w_{k_1,k_2}^*(\pi_{k_1}, \pi_{k_2})$. Because of

$$E^* \left(\sum_{r=1}^{q} \delta_{\pi_{k_1}(r),j_1} \delta_{\pi_{k_2}(r),j_2} \Big| \pi_{k_1} \right) = \begin{cases} q^{-1}, & |k_1| \neq |k_2|, \\ \delta_{j_1 j_2}, & |k_1| = |k_2|, \end{cases} \tag{6.10}$$

it follows that

$$T_n^* = \sum_{|k_1| \neq |k_2|} w_{k_1,k_2}^*(\pi_{k_1}, \pi_{k_2}) + \sum_{|k_1| = |k_2|} w_{k_1,k_2}^*(\pi_{k_1}, \pi_{k_2})$$

$$=: W_n^* + \sum_{|k_1| = |k_2|} w_{k_1,k_2}^*(\pi_{k_1}, \pi_{k_2}) = W_n^* + h^{-1/2} \mu_n^* + o_{P^*}(1),$$

see Theorem 6.1 above. Setting $\widetilde{W}_{k_1,k_2}^* := w_{k_1,k_2}^*(\pi_{k_1}, \pi_{k_2}) + w_{k_2,k_1}^*(\pi_{k_2}, \pi_{k_1})$, we obtain up to a negligible term

$$W_n^* = \sum_{0 \leq k_1 < k_2 \leq \lfloor n/2 \rfloor} \widetilde{W}_{k_1,k_2}^* + \widetilde{W}_{-k_1,k_2}^* + \widetilde{W}_{k_1,-k_2}^* + \widetilde{W}_{-k_1,-k_2}^*$$

$$=: \sum_{0 \leq k_1 < k_2 \leq \lfloor n/2 \rfloor} W_{k_1,k_2}^*.$$

Since the random variables W_{k_1,k_2}^* are clean by (6.10) in the sense of Definition 2.1 of De Jong [7], that is, $E^*(W_{k_1,k_2}^* | \pi_{k_1}) = 0$ holds a.s., we are in the situation to apply Proposition 3.2. of his paper. Hence, for obtaining convergence in distribution (conditioned on the data)

$$\frac{1}{\tau_n^*} W_n^* \xrightarrow{\mathcal{D}} N(0,1) \tag{6.11}$$

in probability, it remains to prove that

$$G_1 := \sum_{0 \leq k_1 < k_2 \leq \lfloor n/2 \rfloor} E^*\left(W_{k_1,k_2}^{*4} \right), \tag{6.12}$$

$$G_2 := \sum_{0 \leq k_1 < k_2 < k_3 \leq \lfloor n/2 \rfloor} E^*\left(W_{k_1,k_2}^{*2} W_{k_1,k_3}^{*2} + W_{k_2,k_1}^{*2} W_{k_2,k_3}^{*2} + W_{k_3,k_1}^{*2} W_{k_3,k_2}^{*2} \right) \tag{6.13}$$

and

$$G_4 := \sum_{0 \le k_1 < k_2 < k_3 < k_4 \le \lfloor n/2 \rfloor} E^* \big(W^*_{k_1,k_2} W^*_{k_1,k_3} W^*_{k_4,k_2} W^*_{k_4,k_3} + W^*_{k_1,k_2} W^*_{k_1,k_4} W^*_{k_3,k_2} W^*_{k_3,k_4}$$
$$+ W^*_{k_1,k_3} W^*_{k_4,k_1} W^*_{k_2,k_3} W^*_{k_2,k_4} \big) \tag{6.14}$$

are all of lower order (in probability) than τ_n^{*2}, that is, are all negligible. We prove the requested result only for the most contributing case (6.14). Because of the inherited symmetries and the constant number of periodogram factors with frequencies $\omega_{k_1}, \omega_{k_2}, \omega_{k_3}$ and ω_{k_4}, it suffices to consider the representative

$$\widetilde{G}_4 := \sum_{0 \le k_1 < k_2 < k_3 < k_4 \le \lfloor n/2 \rfloor} E^* \big(w^*_{k_1,k_2}(\pi_{k_1}, \pi_{k_2}) w^*_{k_1,k_3}(\pi_{k_1}, \pi_{k_3}) w^*_{k_4,k_2}(\pi_{k_4}, \pi_{k_2}) w^*_{k_4,k_3}(\pi_{k_4}, \pi_{k_3}) \big)$$

$$= \sum_{0 \le k_1 < k_2 < k_3 < k_4 \le \lfloor n/2 \rfloor} \frac{h^2}{n^4} \sum_{m_1,\dots,m_8=0}^{p-1} \sum_{j_1,\dots,j_8=1}^{q} \int_{-\pi}^{\pi} K_h(\lambda_1 - \omega_{k_1}) K_h(\lambda_1 - \omega_{k_2}) \, d\lambda_1$$

$$\times \int_{-\pi}^{\pi} K_h(\lambda_2 - \omega_{k_1}) K_h(\lambda_2 - \omega_{k_3}) \, d\lambda_2$$

$$\times \int_{-\pi}^{\pi} K_h(\lambda_3 - \omega_{k_4}) K_h(\lambda_3 - \omega_{k_2}) \, d\lambda_3$$

$$\times \int_{-\pi}^{\pi} K_h(\lambda_4 - \omega_{k_4}) K_h(\lambda_4 - \omega_{k_3}) \, d\lambda_4$$

$$\times I_{j_1,m_1,m_2;k_1} \overline{I_{j_2,m_1,m_2;k_2}} I_{j_3,m_3,m_4;k_1}$$
$$\times \overline{I_{j_4,m_3,m_4;k_3}} I_{j_5,m_5,m_6;k_4} \overline{I_{j_6,m_5,m_6;k_2}}$$
$$\times I_{j_7,m_7,m_8;k_4} \overline{I_{j_8,m_7,m_8;k_3}}$$

$$\times E^* \bigg(\sum_{r_1,\dots,r_4=1}^{q} \big(\delta_{\pi_{k_1}(r_1), j_1} \delta_{\pi_{k_2}(r_1), j_2} - q^{-1} \big) \cdots$$

$$\times \big(\delta_{\pi_{k_4}(r_4), j_7} \delta_{\pi_{k_3}(r_4), j_8} - q^{-1} \big) \bigg).$$

Since the last conditional expectation above is bounded by 0 from below and by q^4 from above, taking the expectation above and applying (6.1) for $k = 16$ results in

$$E(\widetilde{G}_4) = \bigg(\sum_{0 \le k_1 < k_2 < k_3 < k_4 \le \lfloor n/2 \rfloor} \frac{h^2}{n^4} \sum_{m_1,\dots,m_8=0}^{p-1} \sum_{j_1,\dots,j_8=1}^{q} \int_{-\pi}^{\pi} K_h(\lambda_1 - \omega_{k_1}) K_h(\lambda_1 - \omega_{k_2}) \, d\lambda_1$$

$$\times \int_{-\pi}^{\pi} K_h(\lambda_2 - \omega_{k_1}) K_h(\lambda_2 - \omega_{k_3}) \, d\lambda_2$$

$$\times \int_{-\pi}^{\pi} K_h(\lambda_3 - \omega_{k_4}) K_h(\lambda_3 - \omega_{k_2}) \, \mathrm{d}\lambda_3$$

$$\times \int_{-\pi}^{\pi} K_h(\lambda_4 - \omega_{k_4}) K_h(\lambda_4 - \omega_{k_3}) \, \mathrm{d}\lambda_4 \Bigg)$$

$$\times \mathrm{O}(1).$$

Approximating all Riemann sums by their corresponding integrals and by using standard substitutions, the expression above becomes an $\mathrm{O}(h)$ term. Similar arguments and using (6.1) for $k = 32$ yield $\mathrm{Var}(\widetilde{G}_4) = \mathrm{O}(h^2)$. This completes the proof. □

Proof of Corollary 3.1. By comparing the results in Theorem 2.1 and Theorem 3.1, we have asymptotic exactness of $\varphi_{n,\mathrm{cent}}^*$ if and only if $\tau_0^2 = \tau_0^{*2}$ holds under H_0. Rearrangements of the summations over j_1, j_2, j_3, j_4 in the integrands of (2.5) and (3.7) yield to the condition that

$$(q-1)\big|\mathrm{tr}\big(\mathbf{F}_{11}(\omega)\overline{\mathbf{F}_{11}(\omega)}^T\big)\big|^2 - \frac{2}{q} \sum_{\substack{j_2,j_4=1 \\ j_2 \neq j_4}}^{q} \big|\mathrm{tr}\big(\mathbf{F}_{11}(\omega)\overline{\mathbf{F}_{j_2 j_4}(\omega)}^T\big)\big|^2$$

$$+ \sum_{\substack{j_1,j_2,j_3,j_4=1 \\ j_1 \neq j_3, j_2 \neq j_4}}^{q} \big(-q+q^2\delta_{j_1 j_2}\big)\big(-q+q^2\delta_{j_3 j_4}\big)\big|\mathrm{tr}\big(\mathbf{F}_{j_1 j_3}(\omega)\overline{\mathbf{F}_{j_2 j_4}(\omega)}^T\big)\big|^2 \tag{6.15}$$

and

$$(q-1)\big|\mathrm{tr}\big(\mathbf{F}_{11}(\omega)\overline{\mathbf{F}_{11}(\omega)}^T\big)\big|^2 - \frac{2}{q} \sum_{\substack{j_2,j_4=1 \\ j_2 \neq j_4}}^{q} \big|\mathrm{tr}\big(\mathbf{F}_{11}(\omega)\overline{\mathbf{F}_{j_2 j_4}(\omega)}^T\big)\big|^2$$

$$+ \sum_{\substack{j_1,j_2,j_3,j_4=1 \\ j_1 \neq j_3, j_2 \neq j_4}}^{q} \frac{1}{q^2(q-1)} \big|\mathrm{tr}\big(\mathbf{F}_{j_1 j_3}(\omega)\overline{\mathbf{F}_{j_2 j_4}(\omega)}^T\big)\big|^2, \tag{6.16}$$

have to be equal. Equalizing both quantities and another rearrangement of both last sums in (6.15) and in (6.16) gives the desired result. By Lemma 1 in Janssen and Pauls [14], this even shows the asymptotic equivalence of the tests in (3.9). Since we have $\widehat{\mu} - \mu_n^* = \mathrm{o}_P(h^{1/2})$ under H_0, the same holds true for the other version φ_n^*.

Consistency for $\varphi_{n,\mathrm{cent}}^*$ follows as in the proof of Theorem 2.1 since $T_n - h^{-1/2}\widehat{\mu}$ converges in probability to $+\infty$ under H_1 but the critical value still converges in probability to a quantile of a normal distribution. For the other test, we can rewrite φ_n^* asymptotically equivalent as $\mathbf{1}_{(c_n^*(\alpha),\infty)}(T_n - h^{-1/2}\mu_n^*)$ and consistency follows as above since $T_n - h^{-1/2}\mu_n^*$ also converges in probability to $+\infty$ under H_1. □

Proof of Corollary 3.2. For $q = 2$ the ratios occurring in the first and in the third sum in (3.10) are zero and the second sum does not occur at all. In the cases (ii) and (iii), we can treat $|\operatorname{tr}(\mathbf{F}_{j_1 j_3}(\omega)\overline{\mathbf{F}_{j_2 j_4}(\omega)}^T)|^2$ as a constant, which causes the three sums in (3.10) to cumulate to zero. For (iv) the right-hand side of Corollary 3.1(ii) becomes

$$\frac{8}{9}\left(|f_{12}(\omega)|^4 + |f_{13}(\omega)|^4 + |f_{23}(\omega)|^4\right)$$

$$- \frac{8}{9}\left(|f_{12}(\omega)|^2|f_{13}(\omega)|^2 + |f_{12}(\omega)|^2|f_{23}(\omega)|^2 + |f_{13}(\omega)|^2|f_{23}(\omega)|^2\right)$$

$$= \frac{4}{9}\left\{\left(|f_{12}(\omega)|^2 - |f_{13}(\omega)|^2\right)^2 + \left(|f_{12}(\omega)|^2 - |f_{23}(\omega)|^2\right)^2 + \left(|f_{13}(\omega)|^2 - |f_{23}(\omega)|^2\right)^2\right\},$$

which gives the desired result. $\square$

Proof of Corollary 3.3. Since $\widehat{\tau}^{*2}$ converges in $\mathbb{P} \otimes \widetilde{\mathbb{P}}$ probability to τ^{*2}, the critical value $c^*_{n,\text{stud}}(\alpha)$ always converges in probability to $u_{1-\alpha}$. Hence, we can close the proof as in Theorem 2.1. $\square$

Proof of Remark 3.2. Note that the assumptions guarantee that the conditional CLT, see Theorem 3.1, holds true as above. Moreover, the test statistic also satisfies the CLT stated in Theorem 2.1 under the null, see Equations (2.4)–(2.6) in Dette and Paparoditis [9], which shows the asymptotic exactness under the null. Finally, consistency of the tests follows from Theorem 2 of their paper. $\square$

Proof of Corollary 3.4. Checking through the proof of Theorem 3.1 (where the central limit theorem (3.3) holds under the null as well as for fixed alternatives) we see that all results even remain valid for the processes $\underline{X}^n_1, \ldots, \underline{X}^n_n$ with spectral density given by $\mathbf{f}^n$, that is, we also have convergence in distribution given the data

$$(\widehat{\tau}^*)^{-1}\left(T^*_n - \frac{\widehat{\mu}^*}{\sqrt{h}}\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1)$$

in probability in this case. Since the test statistic is asymptotically normally distributed with mean ν and variance τ_0^2, see the proof of Theorem 2.3, the result (i) follows as in the proof of Corollary 3.3.

For proving (ii), recall that our mixing conditions imply the $\sqrt{nh}$-consistency of the kernel density estimators for the processes. This shows that $\widehat{\mu} - \widehat{\mu}^* = \mathrm{o}_P(h^{1/2})$ even holds under the given local alternatives (2.7) with $\alpha_n = h^{-1/4}n^{-1/2}$. Since we also have $\widehat{\tau} - \widehat{\tau}^* = \mathrm{o}_P(h^{1/2})$, the result follows from Slutsky's lemma as in the proof of Corollary 3.1. $\square$

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Supplementary Material

Supplement to "Testing equality of spectral densities using randomization techniques" (DOI: 10.3150/13-BEJ584SUPP; .pdf). In the supplement to the current paper (cf. Jentsch and Pauly [17]), we provide additional supporting simulations for the asymptotic test and all three randomization tests under consideration in a variety of examples.

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