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Stochastic integration with respect to additive functionals of zero quadratic variation

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We consider a Markov process X associated to a nonnecessarily symmetric Dirichlet form $\mathcal{E}$. We define a stochastic integral with respect to a class of additive functionals of zero quadratic variation and then we obtain an Itô formula for the process $u(X)$, when u is locally in the domain of $\mathcal{E}$.

Keywords: additive functional; Dirichlet form; Fukushima decomposition; Itô formula; Markov process; stochastic calculus; quadratic variation; zero energy process

1. Introduction and main results

The semimartingale theory has produced a fundamental tool based on stochastic integration and Itô's formula: the stochastic calculus. Since Markov processes are not in general semimartingales, Fukushima [17] developed another stochastic calculus in the framework of symmetric Dirichlet spaces. Let E be a Polish space, m be a Radon measure on E and $(X, \Omega, \mathcal{F}_t, \mathbf{P}_x, t \in \mathbb{R}_+, x \in E)$ be a E-valued m-symmetric Markov process with regular Dirichlet form $\mathcal{E}$. For any element u of the domain $\mathcal{F}$ of $\mathcal{E}$, the process $(u(X_t) - u(X_0), t \geq 0)$ admits the decomposition

$$u(X_t) - u(X_0) = M_t^u + N_t^u,$$

where M^u is a martingale additive functional of finite energy and N^u is a continuous additive functional of zero energy. This decomposition is called Fukushima's decomposition and it can be seen as a substitute of the Doob–Meyer decomposition of super-martingales and Itô's formula for semimartingales. The part of the class of bounded variation processes in the semimartingale theory is played by the class of continuous additive functionals of zero energy. In general, these additive functionals are not of bounded variation and therefore the Lebesgue–Stieltjes integrals can not be defined. Nevertheless, the concepts of energy and quadratic variation are closely related, see Graversen and Rao [20]. In particular, it is well known that for any function $g \in L^1(E; m)$, the process N^u has 0-quadratic variation with respect to the measure $\mathbf{P}_{g\cdot m} := \int_E \mathbf{P}_x(\cdot) m(\mathrm{d}x)$, that is, $u(X)$ is a Dirichlet process in the Föllmer sense [13]. Is shown in [13] (see also [14]) that for any function φ in $\mathcal{C}^1(\mathbb{R})$, the following limit exists $\mathbf{P}_x$-a.s. for m-a.e. $x \in E$:

$$\int_0^t \varphi\bigl(u(X_s)\bigr)\,\mathrm{d}N_s^u := \lim_{n\to\infty} \sum_{i=0}^{n-1} \varphi\bigl(u(X_{ti/n})\bigr)\bigl(N^u_{t(i+1)/n} - N^u_{ti/n}\bigr). \tag{1.1}$$

In his Itô formula expending $u(X)$ [13,14], this integral replaces the Lebesgue–Stieltjes integral in the classical Itô formula for semimartingales. In this connection, Russo and Vallois [32] and [33] have obtained a similar stochastic calculus through a so-called regularization procedure.

However, in the theory of symmetric Markov processes has been necessary to extend (1.1) to more general integrands than whose of type $\varphi \circ u$, for $\varphi \in C^1(\mathbb{R})$. For example, in order to define stochastic line integrals of 1-forms along the paths of symmetric diffusion processes on manifolds, Nakao [29] introduced an integral $\int_0^t f(X_s)\,\mathrm{d}N_s^u$ for f bounded function element of $\mathcal{F}$. His new integral is an additive functional in the sense of Fukushima *et al.* [18], and therefore it is defined $\mathbf{P}_x$-a.s. for q.e. (quasi every) $x \in E$ (i.e., outside of an exceptional set [18]). This allows to get a refinement of the Föllmer's Itô formula, for which, the expansion of $u(X)$ holds $\mathbf{P}_x$-a.s. for q.e. $x \in E$. Besides, this integral is used by Fitzsimmons and Kuwae [12], to study the lower order perturbation of diffusion processes.

We emphasize that following [13], for f and u in $\mathcal{E}$ it is possible to show the existence of the limit $\lim_{n\to\infty}\sum_{i=0}^{n-1}[f(X_{ti/n})(N^u_{t(i+1)/n} - N^u_{ti/n}) + u(X_{ti/n})(N^f_{t(i+1)/n} - N^f_{ti/n})]$. $\mathbf{P}_x$-a.e. for m-a.e. $x \in E$, however it is not possible to show the existence of the limits $\lim_{n\to\infty}\sum_{i=0}^{n-1} f(X_{ti/n})(N^u_{t(i+1)/n} - N^u_{ti/n})$ and $\lim_{n\to\infty}\sum_{i=0}^{n-1} u(X_{ti/n})(N^f_{t(i+1)/n} - N^f_{ti/n})$ and therefore to define the integrals $\int_0^t f(X_s)\,\mathrm{d}N_s^u$ and $\int_0^t u(X_s)\,\mathrm{d}N_s^f$ in separated way.

The conditions of existence of Nakao's integral being too restrictive, this notion could not be used by Chen *et al.* [5] to study the lower order perturbation of symmetric Markov processes that are not diffusions. Chen *et al.* [6], have extended Nakao's integral to a larger class of integrators as well as integrands. Using time reversal, they have defined an integral $\int_0^t f(X_s)\,\mathrm{d}C_s$ for f in $\mathcal{F}_{loc}$, the set of functions locally in $\mathcal{F}$ and C in a large class of processes containing $\{N^u\colon u \in \mathcal{F}\}$. The process C is not in general of zero energy but of zero quadratic variation and the integral is not an additive functional or a local additive functional but a local additive functional admitting null set. Kuwae [26] gives a refinement of Chen *et al.* work, redefining the stochastic integral without using time reversal.

Our aim in this paper, is to construct an integral $\int_0^t f(X_s)\,\mathrm{d}C_s$ for a Markov process X associated to a nonnecessarily symmetric regular Dirichlet form $(\mathcal{E},\mathcal{F})$ in a Hilbert space $L^2(E,m)$, f locally in $\mathcal{F}$ and C local continuous additive functional with zero quadratic variation. To do so, one can not extend the construction of Chen *et al.* neither Kuwae's construction because they both heavily rely on the symmetry of the Markov process.

On one hand, it is legitimate to solve this question since many results for symmetric Dirichlet forms hold for nonsymmetric Dirichlet forms, see, for example, [23–25,28] and [30]. In particular, Fukushima's decomposition holds for nonsymmetric regular Dirichlet forms, but also the correspondence between Markov processes and (nonnecessarily symmetric) Dirichlet forms, Revuz correspondence and other relations between probabilistic notions for X and analytic notions for $\mathcal{E}$.

In order to introduce our stochastic integral, we need the following definitions: A sequence $(\Pi_n := \{0 = t_{n,0} < t_{n,1} < \cdots < t_{p_n,n} < \infty\})_{n\in\mathbb{N}}$ of partitions of $\mathbb{R}_+$ is said to tend to the identity if $\|\Pi_n\| := \max\{t_{n,k+1} - t_{n,k}\} \to 0$ as $n \to \infty$ and $t_{p_n,n} \to \infty$. We denote by $\mathcal{N}$ the set of continuous additive functionals of finite energy and of zero quadratic variation. Denote by $\mathcal{N}_{f\text{-}loc}$ the set of process locally in $\mathcal{N}$. (See Definitions 2.4 and 2.5 below.)

Theorem 1.1. *For a function f locally in $\mathcal{F}$ and an element C of $\mathcal{N}_{f\text{-}loc}$, there exists an unique local additive functional I such that*:

For any sequence (Π_n) of partitions tending to the identity, there exists a subsequence (Π_{n_k}) such that $\mathbf{P}_x$-a.e. for m-a.e. x in E: $\sum_{i=0}^{p_{n_k}-1} f(X_{t_{n_k,i}})[C(t\wedge t_{n_k,i+1}) - C(t\wedge t_{n_k,1})]$ converges to I_t uniformly on any compact of $[0,\zeta)$. Moreover, I belongs to $\mathcal{N}_{f\text{-}loc}$.

The local additive functional obtained in the theorem below is denoted by $f * C_t$ or by $\int_0^t f(X_s)\,\mathrm{d}C_s$. Then when C is of bounded variation, $f * C$ coincides with the Lebesgue–Stieltjes integral. If there exists a local martingale M, a real function u on E and a C^1-real function φ such that $f = \varphi \circ u$ and $u(X) = M + C$, $\mathbf{P}_x$-a.e. for m-a.e. then $f * C$ coincides with the Föllmer integral.

This new integral leads to an Itô formula for $u(X)$ when u belongs to $\mathcal{F}_{loc}$. On this purpose, we need first an extension of the Fukushima decomposition of $u(X)$ for the elements u locally in $\mathcal{F}$. This extension is well-known for diffusions processes.

When X is not a diffusion we have the following substitute of the Fukushima decomposition: Denote by $\mathcal{M}_{f\text{-}loc}$ the set of local martingale additive functionals locally of finite energy.

Proposition 1.2. *For u in $\mathcal{F}_{loc}$, the process $u(X)$ admits the following decomposition $\mathbf{P}_x$-a.e. for q.e. $x \in E$*:

$$u(X_t) = u(X_0) + V_t^u + W_t^u + C_t^u, \qquad t < \zeta \ (t < \infty \text{ if } u \in \mathcal{F}),$$

where $W^u \in \mathcal{M}_{f\text{-}loc}$, $C^u \in \mathcal{N}_{c,f\text{-}loc}$ and V^u is the AF of bounded variation given by:

$$V_t^u = \sum_{s\le t}\big(u(X_s) - u(X_{s-})\big)1_{\{|u(X_s)-u(X_{s-})|>1\}} - u(X_{\zeta-})1_{\{t\ge\zeta\}}.$$

Moreover, the jumps of W^u are bounded by 1.

In particular, if $E = \mathbb{R}^d$ and we take u the coordinate function $\pi_i : x \to x_i$, $i = 1, \dots, d$, the above result can be seen as a generalization of the Itô–Lévy decomposition for Lévy processes (e.g., Sato [34]).

Using the notation of Proposition 1.2, we introduce the following extension of the Itô formula.

Proposition 1.3. *Suppose that Φ belongs to $\mathcal{C}^2(\mathbb{R}^d)$ and $u = (u_1, \dots, u_d)$ belongs to $\mathcal{F}^d_{loc}$. Then for q.e. $x \in E$, $\mathbf{P}_x$-a.s. for all $t \in [0,\zeta)$ ($[0,\infty)$ if $u \in \mathcal{F}^d$)*:

$$\begin{aligned}
&\Phi\big(u(X_t)\big) - \Phi\big(u(X_0)\big) \\
&\quad= \sum_{k=1}^d \int_0^t \frac{\partial\Phi}{\partial x_k}\big(u(X_{s-})\big)\,\mathrm{d}W_s^{u_k} + \sum_{k=1}^d \int_0^t \frac{\partial\Phi}{\partial x_k}\big(u(X_s)\big)\,\mathrm{d}C_s^{u_k} \\
&\qquad + \frac{1}{2}\sum_{k,\ell=1}^d \int_0^t \frac{\partial^2\Phi}{\partial x_k\,\partial x_\ell}\big(u(X_s)\big)\,\mathrm{d}\big\langle W^{u_k,c}, W^{u_\ell,c}\big\rangle_s
\end{aligned} \tag{1.2}$$

$$
\begin{aligned}
&+\sum_{s\le t}\Bigg[\Phi\big(u(X_s)\big)-\Phi\big(u(X_{s-})\big)\\
&\qquad -\sum_{k=1}^{d}\frac{\partial\Phi}{\partial x_k}\big(u(X_{s-})\big)\Delta u_k(X_s)1_{\{|\Delta(u_k(X_s))|<1\}}\Bigg]\\
&-\sum_{k=1}^{d}\frac{\partial\Phi}{\partial x_k}\big(u(X_{\zeta-})\big)u(X_{\zeta-})1_{\{t\ge\zeta\}}.
\end{aligned}
$$

In the case that $E=\mathbb{R}^d$, if we take $u=(\pi_1,\dots,\pi_d)$, we obtain a Itô formula for the process X and therefore the Fukushima decomposition of $\Phi(X)$ for $\Phi\in C_0^2(\mathbb{R}^d)$. Following Albeverio *et al.* [1], the Dirichlet form $\mathcal{E}(\Phi,\Psi)$ for Φ and Ψ in $\mathcal{F}$ can be approximate by $\frac{1}{t}\int_E \mathbf{E}_x(\Phi(X_0)-\Phi(X_t))\Psi(x)m(\mathrm{d}x)$, then we hope that the Itô formula can be used, for example, in order to give a probabilistic approach to the work of Hu *et al.* [21,22] concerning Beurling–Deny decomposition for nonsymmetric Dirichlet forms.

As mentioned above, following [13], it is possible to define the second term on the right-hand side of (1.2) as the limit $\mathbf{P}_x$-a.s. for m-a.e. $x\in E$:

$$
\lim_{n\to\infty}\sum_{i=0}^{n-1}\sum_{k=1}^{d}\int_0^t\frac{\partial\Phi}{\partial x_k}\big(u(X_{ti/n})\big)\big[C^{u_k}_{t(i+1)/n}-C^{u_k}_{ti/n}\big]\tag{1.3}
$$

and to get (1.2), $\mathbf{P}_x$-a.s. for m-a.e. $x\in E$. If we want to use the results of [13] to ensure the existence of the above limit and (1.2), $\mathbf{P}_x$-a.s. for q.e. $x\in E$, we need to show that the process N^u is of zero quadratic variation, and then $u(X)$ is a Dirichlet process in the Föllmer sense, $\mathbf{P}_x$-a.s. for q.e. $x\in E$. This has been demonstrated by many authors for certain class of processes and functions u. For example, this has been shown by Bouleau and Yor [4] for the case when X is a unidimensional semimartingale with discontinuous part of bounded variation and u an absolutely continuous function with bounded weak derivative. Also it has been shown for Föllmer and Protter [15] for a multidimensional Brownian Motion and a function u locally in the Sobolev space $\mathcal{W}^{1,2}$. There are even, similar results for a time dependent function, maybe the first one shown by Föllmer *et al.* [16] for the case of a unidimensional Brownian Motion and a time dependent function u with locally square integrable weak derivatives satisfying a mild condition of continuity. The results of [16] have been extended, for example, by Bardina and Jolis [2], Bardina and Rovira [3] for elliptic diffusion processes, by Ghomrasni and Peskir [19] for continuous semimartingales, by Eisenbaum [7,8], Eisenbaum and Walsh [10] for Lévy processes with Brownian component, by Eisenbaum and Kyprianou [9] and Walsh [36] for Lévy processes without Brownian component. The precedent list is not exhaustive, in fact in the references of these cited papers, we can find more examples of stochastic processes X and functions u, time-dependent or not, for which $u(X\cdot,\cdot)$ is a Dirichlet process in the Föllmer sense and therefore, Proposition 1.3 is already known.

The integrals used in (1.2) are based on the Fukushima stochastic calculus using the concept of additive functional, the additive property is then essential. In this context, is not possible to extend Proposition 1.3 to time-dependent functions u.

The paper is organized as follows. In Section 2, we present some preliminaries. In Section 3, we construct a stochastic integration with respect to N^u. To do so, we first establish a decomposition of N^u as the sum of three processes N_1^u, N_2^u and N_3^u such that N_1^u and N_2^u are respectively associated to the diffusion part and the jumping part of the symmetric part of $\mathcal{E}$, and N_3^u is of bounded variations. Next, we present respectively stochastic integration with respect to N_1^u and N_2^u. These results lead to an integral with respect to N^u which is used with an argument of localization to introduce the stochastic integral with respect to C in $\mathcal{N}_{c,f\text{-}loc}$. In Section 4, we prove Theorem 1.1, that is, the stochastic integral with respect to C can be approximated by Riemman sums. We also show that when the Dirichlet form is symmetric, the obtained stochastic integral with respect to C coincides with the integral defined by Chen *et al.* [6]. In Section 5, we establish Proposition 1.2 and the Itô formula in which this new integral takes the place of the Lebesgue–Stieltjes integral in the classical Itô formula for semimartingales.

2. Preliminaries

In this paper, we use mostly notation and vocabulary from the book of Fukushima *et al.* [18] still available in the nonnecessarily symmetric case (see Ma and Rockner [28] and Oshima [30]). This section contains existing results or some immediate consequences of existing results that will be useful for the other sections.

Throughout this paper, we assume that $X = (\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \{X_t\}_{t\geq 0}, \{\mathbf{P}_z\}_{z\in E})$ is a Hunt process on a locally compact separable metric space E, properly associated to a regular Dirichlet form $\mathcal{E}$ with domain $\mathcal{F}$ in a Hilbert space $L^2(E;m)$. We do not assume that $\mathcal{E}$ is symmetric. Set $\mathcal{E}_1(u,v) := \mathcal{E}(u,v) + (u,v)$, where $(\cdot,\cdot)$ denotes the inner product in $L^2(E,m)$. It is known that $\mathcal{F}$ is a Hilbert space with inner product $\tilde{\mathcal{E}}_1(u,v) := \frac{1}{2}(\mathcal{E}_1(u,v) + \mathcal{E}_1(v,u))$. Denote by ζ the life time of X and ∂ the extra point such that $X_t(\omega) = \partial$ for all $t \geq \zeta(\omega)$ and $\omega \in \Omega$. A real function on E is extended to a function on $E \cup \partial$ by setting $f(\partial) = 0$.

The energy of an AF (additive functional) A is defined by

$$e(A) := \lim_{t\to 0} \frac{1}{2t} \mathbf{E}_m\big[A_t^2\big]$$

when the limit exists and for two AF A, B, their mutual energy is defined by

$$e(A,B) := \tfrac{1}{2}\big[e(A+B) - e(A) - e(B)\big].$$

An AF M is called a martingale additive functional (abbreviated as MAF) if it is finite, càdlàg and for q.e. x in E: $\mathbf{E}_x[M_t^2] < \infty$ and $\mathbf{E}_x[M_t] = 0$ for any $t \geq 0$. Denote by $\mathcal{M}$ the set of MAF's of finite energy and

$$\mathcal{N}_c := \big\{N\colon N \text{ is a finite continuous AF}, e(A) = 0, \mathbf{E}_x(|N_t|) < \infty \text{ q.e. for each } t > 0\big\}.$$

For any $u \in \mathcal{F}$, M^u and N^u denote the elements of $\mathcal{M}$ and $\mathcal{N}_c$, respectively, that are present in Fukushima decomposition of $u(X_t) - u(X_0)$, $t \geq 0$, that is:

$$u(X_t) - u(X_0) = M_t^u + N_t^u \qquad \text{for } t \geq 0, \mathbf{P}_x\text{-a.e. for q.e. } x \in E.$$

In this paper, we always assume that the elements of $\mathcal{F}$ are always represented by its quasi-continuous m-versions.

For a nearly Borel set $B(\subset E)$, σ_B and τ_B represent the first hitting time to B and the first exit time from B respectively, that is:

$$\sigma_B := \inf\{t > 0\colon\ X_t \in B\},$$
$$\tau_B := \inf\{t > 0\colon\ X_t \notin B\}.$$

We denote by $\mathcal{F}_b$ the subset of $\mathcal{F}$ of bounded functions and for a nearly Borel finely open set, $\mathcal{F}_G$ the set of functions $u \in \mathcal{F}$ such that $u(x) = 0$ for q.e. $x \in E \setminus G$. The subset of $\mathcal{F}_G$ of bounded functions is denoted by $\mathcal{F}_{b,G}$. The abbreviations CAF and PCAF stand for continuous additive functionals and positive continuous additive functional, respectively. The Revuz measure of a PCAF is the measure given by the Revuz correspondence between PCAFs and smooth measures. All these definitions are found in [18].

The following theorem is a small modification of Theorem 5.4.2 of [18] established for the symmetric case, but it holds also for the nonsymmetric case. (See [30] and [35].)

Theorem 2.1. *Let u be an element of $\mathcal{F}$ and let G be a nearly Borel finely open set. Let A^1 and A^2 be two PCAFs with Revuz measure μ_1 and μ_2, respectively, such that $\mathcal{F}_{b,G} \subset L^1(E,\mu_i)$ for $i = 1, 2$. Then $\mathbf{P}_x(N_t^u = A_t^1 - A_t^2$ for $t < \tau_G) = 1$ for q.e. $x \in E$, if and only if*:

$$\mathcal{E}(u,h) = \langle \mu_2 - \mu_1, h\rangle \qquad \forall h \in \mathcal{F}_{b,G}.$$

Definition 2.2. *We define $\mathcal{N}_c^0$ as the set of CAFs C such that, there exists u in $\mathcal{F}$ and finite PCAFs A^1, A^2 with Revuz measure μ_1 and μ_2, respectively, satisfying: $\mathcal{F}_b \subset L^1(E,\mu_i)$ and $\mathbf{P}_x(C_t = N_t^u + A_t^2 - A_t^1$ for $t < \infty) = 1$ for q.e. $x \in E$. In this case, we define the linear functional $\Theta(C)$ on $\mathcal{F}_b$ by*

$$\bigl\langle \Theta(C), h\bigr\rangle := -\mathcal{E}(u,h) + \langle \mu_2 - \mu_1, h\rangle, \qquad h \in \mathcal{F}_b.$$

It follows from Theorem 2.1 that the definition of $\Theta(C)$ for $C \in \mathcal{N}_c^0$ is consistent in the sense that it does not depend of the elements which represent C.

The following lemma is an immediate consequence of Theorem 2.1.

Lemma 2.3. *Let $C^{(1)}$ and $C^{(2)}$ be elements of $\mathcal{N}_c^0$ and G a nearly Borel finely open set. Then $C^{(1)} = C^{(2)}$ on $[\![0, \sigma_{E\setminus G}[\![$ $\mathbf{P}_x$-a.e. for q.e. $x \in E$ if and only if*

$$\bigl\langle \Theta\bigl(C^{(1)}\bigr), h\bigr\rangle = \bigl\langle \Theta\bigl(C^{(2)}\bigr), h\bigr\rangle \qquad \text{for all } h \in \mathcal{F}_{b,G}.$$

We recall that an increasing sequence of nearly Borel finely open sets $(G_n)_{n\in\mathbb{N}}$ is called a nest if $\tau_{G_n} \uparrow \zeta$ $\mathbf{P}_x$-a.s. for q.e. $x \in E$.

Definition 2.4. *Let Γ be a class of local AF's. Following* [6], *we say that a $(\mathcal{F}_t)$-adapted process A is locally in Γ, and write: $A \in \Gamma_{f\text{-}loc}$, if there exists a sequence A^n in Γ and a nest of nearly Borel finely open sets $\{G_n\}$ such that $A_t = A_t^n$ for $t < \tau_{G_n}$ $\mathbf{P}_x$-a.e. for q.e. $x \in E$. In this case, A is hence a local AF.* (*See* [6], *page* 939, *for the definition of local AF.*)

Definition 2.5. *A local AF V is said to be of zero quadratic variation if for any $t > 0$: $\sum_{k=0}^{n-1}(V_{t(i+1)/n} - V_{ti/n})^2$ converges to zero as $n \to \infty$ in $\mathbf{P}_{g\cdot m}$ measure on $\{t < \zeta\}$ for some (and therefore for all) strictly positive $g \in L^1(E, m)$.*

We denote by $\mathcal{N}$ the set of CAFs of finite energy and of zero quadratic variation. In [35], we have established the following theorem of representation for the elements of $\mathcal{N}$.

Theorem 2.6. *Let C be an element of $\mathcal{N}_{f\text{-}loc}$. There exists a nest of nearly Borel finely open sets (G_n) and $(u_n) \in \mathcal{F}$ such that $\mathbf{P}_x$-a.e. for q.e. $x \in E$:*

$$C_t = N_t^{u_n} - \int_0^t u_n(X_s)\,\mathrm{d}s \qquad \text{for all } t < \tau_{G_n}.$$

3. Stochastic integration

Consider an element u of $\mathcal{F}$ and two finite smooth measure μ_1 and μ_2 such that $\mathcal{E}(u, h) = \langle \mu_1 - \mu_2, h\rangle$ for any element h of $\mathcal{F}_b$. Thanks to Theorem 2.1, we know that N^u is of bounded variation. The integral $(f * N^u)_t := \int_0^t f(X_s)\,\mathrm{d}N_s^u$ is hence well-defined as a Lebesgue–Stieltjes integral, moreover, if f belongs to $\mathcal{F}_b$, $f * N^u$ belongs to $\mathcal{N}_c^0$ (see Definition 2.2) and for any h in $\mathcal{F}_b$ we have:

$$\langle \Theta(f * N^u), h\rangle = \langle \Theta(N^u), fh\rangle. \tag{3.1}$$

Thanks to Lemma 2.3, the above equation characterizes the local CAF $f * N^u$. In order to define the integral of f with respect to a process N^u which is not necessarily of bounded variation, it is hence natural to construct a local CAF still denoted by $f * N^u$ satisfying the equation (3.1). This has been done by Nakao [29] for the symmetric case and the aim of this section is to do it for the nonnecessarily symmetric case.

The construction of $f * N^u$ is based on a decomposition of N^u in three components (see Lemma 3.3 below). The first component is associated to the diffusion part of $\tilde{\mathcal{E}}$, the symmetric component of $\mathcal{E}$. The second one is associated to the jump part of $\tilde{\mathcal{E}}$ and the third one is a local CAF of bounded variation. Once this decomposition done, the construction of $f * N^u$ will be close to Nakao's construction in the symmetric case.

Thanks to a localization argument and Theorem 2.6, we will construct the integral $f * C$ for any $f \in \mathcal{F}_{loc}$ and $A \in \mathcal{N}_{f\text{-}loc}$. We always consider $\mathcal{F}$ to be equipped with the norm $\tilde{\mathcal{E}}_1$. We will use repeatedly the following facts:

(1) If a PCAF A with Revuz measure μ satisfies $\mu(E) < \infty$ then A is finite continuous. Indeed, it is consequence of Lemma 4.3 of [23]. This is the case when $A = \langle M\rangle$ for $M \in \mathcal{M}$.
(2) If A is a PCAF A with Revuz measure μ of finite energy integral (i.e., there exists $U_1\mu$ in $\mathcal{F}$ such that $\int_E h(x)\mu(\mathrm{d}x) = \mathcal{E}_1(U_1\mu, h)$ for all $h \in \mathcal{F}$), then A is finite continuous. In fact, for any t, $\mathbf{E}_x(A_t) \le \mathrm{e}^t U_1\mu(x) < \infty$ q.e.
(3) For two CAF, A, B and a nearly Borel set G, we have for q.e. $x \in E$, $\mathbf{P}_x(A = B \text{ on } [\![0, \tau_G]\!]) = 1$ if and only if for q.e. $x \in E$, $\mathbf{P}_x(A = B \text{ on } [\![0, \sigma_{E\setminus G}]\!]) = 1$.

(4) If $J:\mathcal{F}\to\mathbb{R}$ is a continuous linear functional, there exists a unique $w\in\mathcal{F}$ such that $J(h)=\mathcal{E}_1(w,h)$ for any $h\in\mathcal{F}$. (See Theorem I.2.6. in [28].)

3.1. A decomposition of N^u

We denote by $\tilde{\mathcal{E}}$ the symmetric part of $\mathcal{E}$ and denote by $\tilde{\mathcal{E}}^{(c)}$ and $\tilde{\mathcal{E}}^{(j)}$ the diffusion part and the jumping part of $\tilde{\mathcal{E}}$, respectively, in the Beurling–Deny decomposition of $\tilde{\mathcal{E}}$. (See Section 5.3 in [18].) For u in $\mathcal{F}$, the applications $h\to\tilde{\mathcal{E}}^{(c)}(u,h)$ and $h\to\tilde{\mathcal{E}}^{(j)}(u,h)$ are continuous. This leads to the following lemma.

Lemma 3.1. *For u in $\mathcal{F}$, there exists unique elements w and v of $\mathcal{F}$ such that $\mathcal{E}_1(w,h)=\tilde{\mathcal{E}}^{(c)}(u,h)$ and $\mathcal{E}_1(v,h)=\tilde{\mathcal{E}}^{(j)}(u,h)$ for any $h\in\mathcal{F}$.*

Definition 3.2. *For any $u\in\mathcal{F}$, set: ${}^c\tilde{N}^u_t:=N^w_t-\int_0^t w(X_s)\,\mathrm{d}s$ and ${}^j\tilde{N}^u_t:=N^v_t-\int_0^t v(X_s)\,\mathrm{d}s$ where w and v are the elements of $\mathcal{F}$ given by Lemma 3.1.*

It is clear that ${}^c\tilde{N}^u$ and ${}^j\tilde{N}^u$ belongs to $\mathcal{N}^0_c$ and

$$\begin{aligned}&\langle\Theta({}^c\tilde{N}^u),h\rangle=-\tilde{\mathcal{E}}^{(c)}(u,h)\quad\text{and}\\&\langle\Theta({}^j\tilde{N}^u),h\rangle=-\tilde{\mathcal{E}}^{(j)}(u,h)\qquad\text{for all } h\in\mathcal{F}_b.\end{aligned}\tag{3.2}$$

For u in $\mathcal{F}$, the application $h\to\mathcal{E}_1(u,h)$ is continuous. Hence, there exists a unique u^* in $\mathcal{F}$ such that

$$\mathcal{E}_1(u,h)=\tilde{\mathcal{E}}_1(u^*,h),\qquad h\in\mathcal{F}.\tag{3.3}$$

Moreover, we have:

$$\mathcal{E}_1(u^*,u^*)\le K^2\mathcal{E}_1(u,u),\tag{3.4}$$

where K is a continuity constant of $\mathcal{E}$, which means that $\mathcal{E}$ satisfies the sector condition:

$$|\mathcal{E}_1(v,w)|\le K(\mathcal{E}_1(v,v))^{1/2}(\mathcal{E}_1(w,w))^{1/2}\qquad\text{for all } v,w\in\mathcal{F}.$$

Lemma 3.3. *For u in $\mathcal{F}$, let u^* be given by (3.3). Denote by $\tilde{k}$ the killing measure of $\tilde{\mathcal{E}}$ and by $\tilde{K}$ the PCAF associated to $\tilde{k}(\mathrm{d}x)$ by the Revuz correspondence. Then we have $\mathbf{P}_x$-a.e. for q.e. $x\in E$ for any $t<\infty$*

$$N^u_t={}^c\tilde{N}^{u^*}+{}^j\tilde{N}^{u^*}-\int_0^t u^*(X_s)\,\mathrm{d}\tilde{K}_s+\int_0^t(u-u^*)(X_s)\,\mathrm{d}s.\tag{3.5}$$

Proof. From the Beurling–Deny decomposition of $\tilde{\mathcal{E}}$, we have that for any $h\in\mathcal{F}$,

$$\int_E|h(x)u^*(x)|\tilde{k}(\mathrm{d}x)\le[\mathcal{E}_1(h,h)]^{1/2}[\mathcal{E}_1(u^*,u^*)]^{1/2}$$

thus $\int_0^t |u^*(X_s)|\,\mathrm{d}\tilde{K}_s$ is a finite PCAF. Then $\int_0^t u^*(X_s)\,\mathrm{d}\tilde{K}_s$ is an element of $\mathcal{N}_c^0$ and then, the right-hand side of (3.5) belongs to $\mathcal{N}_c^0$. Denote this element by C. The killing part $\tilde{\mathcal{E}}^{(k)}$ of $\tilde{\mathcal{E}}$ satisfies

$$\tilde{\mathcal{E}}^{(k)}\big(u^*, h\big) = \int_E h(x)u^*(x)\tilde{k}(\mathrm{d}x) \qquad \text{for any } h \in \mathcal{F}.$$

It follows from (3.2) that for all $h \in \mathcal{F}$:

$$\begin{aligned}\big\langle \Theta(C), h\big\rangle &= -\tilde{\mathcal{E}}\big(u^*, h\big) + \big(u - u^*, h\big)\\ &= -\mathcal{E}(u, h).\end{aligned}$$

Then (3.5) follows from Lemma 2.3. □

3.2. Stochastic integration with respect to ${}^c\tilde{N}^u$

The following lemma is Lemma 5.1.2 and Corollary 5.2.1 of [18] that we recall for reader's convenience. In [18], it is established for the symmetric case but is also valid for the nonsymmetric case. In fact, its proof is based on the inequality (5.1.1) of [18] which is proved, for example, in Lemma 4.7 of [23] for the nonsymmetric case.

Lemma 3.4. *Let (u_n) be a sequence of quasi continuous functions in $\mathcal{F}$ and $\tilde{\mathcal{E}}_1$-convergent to u. Then there exists a subsequence $\{u_{n_k}\}$ such that for q.e. $x \in E$,*

$$\mathbf{P}_x\big(u_{n_k}(X_t) \text{ converges uniformly to } u(X_t) \text{ on each compact interval of } [0, \infty)\big) = 1$$

and the same holds for $N^{u_{n_k}}$ and N^u, and for $M^{u_{n_k}}$ and M^u, replacing $u_{n_k}(X)$ and $u(X)$, respectively.

Using Lemma 4.3 of [23], we can obtain the following lemma.

Lemma 3.5. *Let A^n be a sequence of PCAFs. Suppose that $\mu_n(E)$ converges to zero as $n \to \infty$, where μ_n represents the Revuz measures of A_n. Then there exists a subsequence (n_k) satisfying the condition that for q.e. $x \in E$,*

$$\mathbf{P}_x\big(A_t^{n_k} \text{ converges to zero uniformly on any compact}\big) = 1.$$

Lemma 3.6. *For every u in $\mathcal{F}$ and f in $\mathcal{F}_b$, there exists a unique w in $\mathcal{F}$, such that:*

$$e\big(f * M^{u,c}, M^h\big) = \mathcal{E}_1(w, h) \qquad \forall h \in \mathcal{F}.$$

Proof. For $h \in \mathcal{F}$, $[e(f * M^{u,c}, M^h)]^2 \le e(f * M^{u,c})e(M^{h,c}) \le e(f * M^{u,c})\tilde{\mathcal{E}}_1(h, h)$. Since $e(f * M^{u,c}) < \infty$, the functional $h \to e(f * M^{u,c}, M^h)$ is continuous. □

Definition 3.7. *For every u in $\mathcal{F}$ and f in $\mathcal{F}_b$, the stochastic integral of f with respect to ${}^c\tilde{N}^u$ denoted by $\int_0^\cdot f(X_s)\,\mathrm{d}^c\tilde{N}_s^u$ or by $f * {}^c\tilde{N}^u$ is defined by*:

$$\int_0^t f(X_s)\,\mathrm{d}^c\tilde{N}_s^u := N_t^w - \int_0^t w(X_s)\,\mathrm{d}s - \frac{1}{2}\langle M^{f,c}, M^{u,c}\rangle_t, \qquad t \geq 0,$$

where w is the element of $\mathcal{F}$ associated to (u, f) by Lemma 3.6.

For any $u, v \in \mathcal{F}$, let $\mu^c_{\langle u,v\rangle}$ be the signed Revuz measure associated to $\langle M^{u,c}, M^{v,c}\rangle$. We have: $\frac{1}{2}\mu^c_{\langle u,v\rangle}(E) = \tilde{\mathcal{E}}^{(c)}(u, v)$. For f, h in $\mathcal{F}_b$, we have (Theorem 5.4 of [23])

$$\mathrm{d}\mu^c_{\langle u,hf\rangle} = f\,\mathrm{d}\mu^c_{\langle u,h\rangle} + h\,\mathrm{d}\mu^c_{\langle u,f\rangle}. \tag{3.6}$$

Lemma 3.8.

(i) *For u in $\mathcal{F}$ and f in $\mathcal{F}_b$, we have $f * {}^c\tilde{N}^u \in \mathcal{N}_c^0$ and*

$$\langle\Theta(f * {}^c\tilde{N}^u), h\rangle = \langle\Theta({}^c\tilde{N}^u), fh\rangle \qquad \textit{for all } h \in \mathcal{F}_b. \tag{3.7}$$

In particular, the integral is well defined in the following sense. If $u, v \in \mathcal{F}$ are such that ${}^c\tilde{N}^u = {}^c\tilde{N}^v$, then for any $f \in \mathcal{F}_b$, $f * {}^c\tilde{N}^u = f * {}^c\tilde{N}^v$.

(ii) *For (u_n) a sequence of $\mathcal{F}$ $\tilde{\mathcal{E}}_1$-converging to u, there exists a subsequence (n_k) such that for q.e. $x \in E$*:

$$\mathbf{P}_x\bigl(f * {}^c\tilde{N}^{u_{n_k}} \textit{ converges to } f * {}^c\tilde{N}^u \textit{ uniformly on any compact}\bigr) = 1.$$

Proof. (i) $f * {}^c\tilde{N}^u \in \mathcal{N}_c^0$ because $|\mu^c_{\langle u,f\rangle}|(E) < \infty$. Besides for any $h \in \mathcal{F}_b$,

$$\begin{aligned}\langle\Theta(f * {}^c\tilde{N}^u), h\rangle &= -e(f * M^{u,c}, M^h) - \frac{1}{2}\int_E h(x)\,\mathrm{d}\mu^c_{\langle f,u\rangle} \\ &= -\frac{1}{2}\int_E f(x)\,\mathrm{d}\mu^c_{\langle h,u\rangle} - \frac{1}{2}\int_E h(x)\,\mathrm{d}\mu^c_{\langle f,u\rangle}.\end{aligned}$$

Then (3.7) is consequence of (3.2) and (3.6). The second statement is consequence of Lemma 2.3.

(ii) Note that for any u, v in $\mathcal{F}$, $f * {}^c\tilde{N}^u - f * {}^c\tilde{N}^v = f * {}^c\tilde{N}^{u-v}$. Thus we need only to show that if (u_n) converges to 0 and $f \in \mathcal{F}_b$, there exists a subsequence (n_k) such that for q.e. $x \in E$:

$$\mathbf{P}_x\bigl(f * {}^c\tilde{N}^{u_{n_k}} \text{ converges to 0 uniformly on any compact}\bigr) = 1.$$

For each n, let w_n be the function associated to (f, u_n) by Lemma 3.6. Then for any $h \in \mathcal{F}$ we have: $\mathcal{E}_1(w_n, h)^2 \leq \|f^2\|_\infty \mathcal{E}_1(h, h)\mathcal{E}_1(u_n, u_n)$. In particular, choosing $h = w_n$, one obtains:

$$\mathcal{E}_1(w_n, w_n) \leq \|f^2\|_\infty \mathcal{E}_1(u_n, u_n) \to 0 \qquad \text{as } n \to \infty.$$

It follows from Lemma 3.4 that there exists a subsequence (n_k) such that $\mathbf{P}_x$-a.e. for q.e. $x \in E$, $N_t^{w_{n_k}} - \int_0^t w_{n_k}(X_s)\,\mathrm{d}s$ converges to 0 uniformly on compacts.

Besides: $\mu^c_{\langle u_n\rangle}(E)=\tilde{\mathcal{E}}^{(c)}(u_n,u_n)$, which converges to 0. Hence, by Lemma 3.5, there exists a subsequence (n_k) such that

$$\left|\left\langle M^{u_{n_k},c},M^{f,c}\right\rangle\right|\le\left\langle M^{f,c}\right\rangle^{1/2}\left\langle M^{u_{n_k},c}\right\rangle^{1/2}$$

converges to 0 on compacts $\mathbf{P}_x$-a.e. for q.e. $x\in E$. □

3.3. Stochastic integration with respect to ${}^j\tilde{N}^u$

Denote by (N,H) the Lévy system of X. Let $\hat{X}$ be the Markov process properly associated to the Dirichlet form $\hat{\mathcal{E}}(u,v):=\mathcal{E}(v,u)$, $u,v\in\mathcal{F}$ and $(\hat{N},H_{\hat{X}})$ its Lévy system. Let ν_H be the Revuz measure associated to H and let $\nu_{\hat{H}}$ be the Revuz measure associated to $H_{\hat{X}}$ and $\hat{H}$ be the PCAF of X associated to $\nu_{\hat{H}}$ by the Revuz correspondence. Let J, $\hat{J}$ and $\tilde{J}$ denote respectively the jumping measure of $\mathcal{E}$, $\hat{\mathcal{E}}$ and $\tilde{\mathcal{E}}$, that is, $J(\mathrm{d}y,\mathrm{d}x)=\frac{1}{2}N(x,\mathrm{d}y)\nu_H(\mathrm{d}x)$, $\hat{J}(\mathrm{d}y,\mathrm{d}x)=\frac{1}{2}\hat{N}(x,\mathrm{d}y)\nu_{\hat{H}}(\mathrm{d}x)$ and $\tilde{J}(\mathrm{d}x,\mathrm{d}y)=\frac{1}{2}[J(\mathrm{d}x,\mathrm{d}y)+\hat{J}(\mathrm{d}x,\mathrm{d}y)]$. It is known that $\hat{J}(\mathrm{d}y,\mathrm{d}x)=J(\mathrm{d}x,\mathrm{d}y)$.

We will use the following notations:

$$\mathbf{N}(\mathrm{d}y,\mathrm{d}s):=N(X_s,\mathrm{d}y)\,\mathrm{d}H_s,\quad\text{and}$$
$$\tilde{\mathbf{N}}(\mathrm{d}y,\mathrm{d}s):=\tfrac{1}{2}\big(N(X_s,\mathrm{d}y)\,\mathrm{d}H_s+\hat{N}(X_s,\mathrm{d}y)\,\mathrm{d}\hat{H}_s\big).$$

For any $u\in\mathcal{F}$, denote by $M^{u,j}$ the jump part of M^u (see page 213 of [18] for the definition), this is an element of $\mathring{\mathcal{M}}$ and for all $h\in\mathcal{F}$, $e(M^{u,j},M^h)=\tilde{\mathcal{E}}^{(j)}(u,h)$. With the same arguments used to show Lemma 3.6, we can obtain the following lemma.

Lemma 3.9. *For every u in $\mathcal{F}$ and f in $\mathcal{F}_b$, there exists a unique w in $\mathcal{F}$, such that*:

$$e\big(f*M^{u,j},M^h\big)=\mathcal{E}_1(w,h)\qquad\forall h\in\mathcal{F}.$$

Definition 3.10. *For every u in $\mathcal{F}$ and f in $\mathcal{F}_b$, the stochastic integral of f with respect to ${}^j\tilde{N}^u$ denoted by $\int_0^\cdot f(X_s)\,\mathrm{d}^j\tilde{N}^u_s$ or by $f*{}^j\tilde{N}^u$ is defined by*:

$$\begin{aligned}\int_0^t f(X_s)\,\mathrm{d}^j\tilde{N}^u_s:={}&N^w_t-\int_0^t w(X_s)\,\mathrm{d}s\\&-\frac{1}{2}\int_0^t\int_E\big[f(x)-f(X_s)\big]\big[u(x)-u(X_s)\big]\tilde{\mathbf{N}}(\mathrm{d}x,\mathrm{d}s),\qquad t\ge 0,\end{aligned}$$

where w is the element of $\mathcal{F}$ associated to (u,f) by Lemma 3.9.

Lemma 3.11.

(i) *For $u\in\mathcal{F}$ and $f\in\mathcal{F}_b$, $f*{}^j\tilde{N}^u$ belongs to $\mathcal{N}^0_c$ and for h in $\mathcal{F}_b$*:

$$\big\langle\Theta\big(f*{}^j\tilde{N}^u\big),h\big\rangle=\big\langle\Theta\big({}^j\tilde{N}^u\big),fh\big\rangle.\tag{3.8}$$

In particular, the integral is well defined in the following sense. If u, v in $\mathcal{F}$ are such that ${}^j\tilde{N}^u = {}^j\tilde{N}^v$, then for any f in $\mathcal{F}_b$: $f * {}^j\tilde{N}^u = f * {}^j\tilde{N}^v$.

(ii) *If (u_n) is $\tilde{\mathcal{E}}_1$-converging to u, there exists a subsequence (n_k) such that for q.e. $x \in E$:*

$$\mathbf{P}_x\big(f * {}^j\tilde{N}^{u_{n_k}} \text{ converges to } f * {}^j\tilde{N}^u \text{ uniformly on any compact}\big) = 1.$$

Proof. The proof of (ii) is similar to the proof of (ii) of Lemma 3.8. We prove (i). Clearly, $f * {}^j\tilde{N}^u$ belongs to $\mathcal{N}_c^0$ and for any $h \in \mathcal{F}_b$:

$$\begin{aligned}
&\big\langle \Theta\big(f * {}^j\tilde{N}^u, h\big)\big\rangle \\
&\quad = e\big(f * M^{u,j}, M^h\big) - \int_{E\times E\setminus\delta} h(y)\big[f(x)-f(y)\big]\big[u(x)-u(y)\big]\tilde{J}(\mathrm{d}x,\mathrm{d}y) \\
&\quad = -\int_{E\times E\setminus\delta} \big[f(y)\big\{h(x)-h(y)\big\} + h(y)\big\{f(x)-f(y)\big\}\big]\big[u(x)-u(y)\big]\tilde{J}(\mathrm{d}x,\mathrm{d}y),
\end{aligned}$$

where $\delta := \{(x,x)\colon x \in E\}$. Using the symmetry of $\tilde{J}$ and the fact that $J(\mathrm{d}x,\mathrm{d}y) + J(\mathrm{d}y,\mathrm{d}x) = 2\tilde{J}(\mathrm{d}x,\mathrm{d}y)$, one proves that the right-hand side of the above equation coincides with:

$$-\int_{E\times E\setminus\delta} \big[h(x)f(x) - h(y)f(y)\big]\big[u(x)-u(y)\big]J(\mathrm{d}x,\mathrm{d}y) = -\tilde{\mathcal{E}}^{(j)}(u, hf).$$

Then 3.8 is consequence of 3.2. The second statement can be shown in the same way that its analogous in Lemma 3.8(i). □

3.4. Stochastic integration with respect to N^u

In view of the decomposition (3.5), we can define the stochastic integral of $f(X)$ with respect to N^u for $f \in \mathcal{F}_b$ and $u \in \mathcal{F}$ as follows.

Definition 3.12. *For any $u \in \mathcal{F}$ and $f \in \mathcal{F}_b$, the stochastic integral of $f(X)$ with respect to N^u denoted by $f * N^u$ or by $\int_0^t f(X_s)\,\mathrm{d}N_s^u$ is defined by*

$$\begin{aligned}
\int_0^t f(X_s)\,\mathrm{d}N_s^u := {}& \int_0^t f(X_s)\,\mathrm{d}^c\tilde{N}_s^{u^*} + \int_0^t f(X_s)\,\mathrm{d}^j\tilde{N}_s^{u^*} \\
& - \int_0^t f(X_s)u^*(X_s)\,\mathrm{d}\tilde{K}_s + \int_0^t f(X_s)\big(u(X_s) - u^*(X_s)\big)\,\mathrm{d}s,
\end{aligned}$$

where the first two integrals are in the sense of the definitions (3.7) and (3.10), respectively, and the others integrals are Lebesgue–Stieltjes integrals.

It is clear that for any u in $\mathcal{F}$ and f in $\mathcal{F}_b$, the stochastic integral $f * N^u$ belongs to $\mathcal{N}_c^0$ and in view of (3.7) and (3.8), we have

$$\big\langle\Theta\big(f * N^u\big), h\big\rangle = \big\langle\Theta\big(N^u\big), fh\big\rangle = -\mathcal{E}(u, fh) \qquad \text{for all } h \in \mathcal{F}_b.$$

Let (u_n) be a sequence on $\mathcal{F}$ $\tilde{\mathcal{E}}_1$-converging to $u \in \mathcal{F}$, it follows from (3.4) that (u_n^*) $\tilde{\mathcal{E}}_1$-converges to u^* then thanks to Lemma 3.4, Lemma 3.8(ii) and Lemma 3.11(ii) we have the following lemma.

Lemma 3.13. *Let f be a function in $\mathcal{F}_b$ and (u_n) a sequence on $\mathcal{F}$ $\tilde{\mathcal{E}}_1$-converging to $u \in \mathcal{F}$. Then there exists a subsequence (n_k) such that for q.e. $x \in E$:*

$$\mathbf{P}_x\big(f * N^{u_{n_k}} \textit{ converges uniformly on any compact to } f * N^u\big) = 1.$$

Let A be the CAF defined by $A_t := N_t^u - \int_0^t u(X_s)\,\mathrm{d}s$, for an element u of $\mathcal{F}$ and let f be a function in $\mathcal{F}_b$, the stochastic integral of $f(X)$ with respect to A is defined by:

$$f * A_t = \int_0^t f(X_s)\,\mathrm{d}A_s := \int_0^t f(X_s)\,\mathrm{d}N_s^u - \int_0^t f(X_s)u(X_s)\,\mathrm{d}s.$$

Lemma 3.14. *Let u and v be elements of $\mathcal{F}$, f and g elements of $\mathcal{F}_b$ and G a nearly Borel finely open set. Set $A_t := N_t^u - \int_0^t u(X_s)\,\mathrm{d}s$ and $B_t := N_t^v - \int_0^t v(X_s)\,\mathrm{d}s$. Suppose that $f(x) = g(x)$ for q.e. $x \in G$ and $\mathbf{P}_x(A_t = B_t$, for any $t < \sigma_{E\setminus G}) = 1$ for q.e. $x \in E$. Then $\mathbf{P}_x(f * A_t = g * B_t$, for any $t < \sigma_{E\setminus G}) = 1$ for q.e. $x \in E$.*

Proof. It follows from Lemma 2.3 that for any $h \in \mathcal{F}_G$ $\langle \Theta(A), h\rangle = \langle \Theta(B), h\rangle$ then $\langle \Theta(f * A), h\rangle = \langle \Theta(A), fh\rangle = \langle \Theta(B), gh\rangle = \langle \Theta(g * B), h\rangle$. We conclude thanks to Lemma 2.3. □

A function f belongs to $\mathcal{F}_{loc}$ if there exists a sequence (f_n) of $\mathcal{F}$ and a nest of nearly Borel finely open sets (G_n) such that $f(x) = f_n(x)$ for q.e. $x \in G_n$. In fact the sequence (f_n) can be taken in $\mathcal{F}_b$ (see Lemma 3.1 in [6]).

With the above lemma and Theorem 2.6, we can define the stochastic integral of $f(X_s)$ with respect to C for any $f \in \mathcal{F}_{loc}$ and C in $\mathcal{N}_{f\text{-}loc}$.

Definition 3.15. *Let C be an element of $\mathcal{N}_{f\text{-}loc}$ and f in $\mathcal{F}_{loc}$. Let (G_n) and (u_n) be the sequences of the conclusion of Theorem 2.6 and $(f_n) \subset \mathcal{F}_b$ such that $f(x) = f_n(x)$ for q.e. $x \in G_n$. Set $C_t^n := N_t^{u_n} - \int_0^t u_n(X_s)\,\mathrm{d}s$. Then if $\sigma := \lim_{n\to\infty} \sigma_{E\setminus G_n}$, we define the stochastic integral of f with respect to A and denoted by $f * C_t, t \geq 0$ or by $\int_0^t f(X_s)\,\mathrm{d}C_s, t \geq 0$ as the following local CAF:*

$$f * C_t := \begin{cases} f_n * C_t^n & \textit{for } t < \sigma_{E\setminus G_n}, \\ 0 & \textit{for } t \geq \sigma. \end{cases}$$

Remark 3.16.

(i) It follows from Lemma 3.14 that the above definition makes sense and not depend of the sequences C^n, (f_n) nor (G_n).

(ii) Any PCAF belongs to $\mathcal{N}_{f\text{-}loc}$, then with the notation of the above definition, $f * C$ belongs to $(\mathcal{N}_{f\text{-}loc})_{f\text{-}loc} = \mathcal{N}_{f\text{-}loc}$.

(iii) Let $\varphi:\mathbb{R}\to\mathbb{R}$ be a function admitting a continuous derivative. For any $n\in\mathbb{N}$ let φ_n be a function admitting a bounded continuous derivative such that $\varphi_n(x)=\varphi(x)$ if $|x|<n$. We know that $\varphi_n(u)-\varphi(0)$ belongs to $\mathcal{F}$ for any $u\in\mathcal{F}$ and if we set $G_n:=\{x\colon |u(x)|<n\}$, (G_n) is a nest of finely open sets. Since u is quasi continuous in the strict sense (i.e., $u(X_t)\to u(X_{\zeta-})\in\mathbb{R}$ as $t\uparrow\zeta$), $\sigma_{E\backslash G_n}\uparrow\infty$ $\mathbf{P}_x$-a.s. for q.e. $x\in E$. Therefore for any $v\in\mathcal{F}$ the stochastic integral $\varphi(u)*N^v=[\varphi(u)-\varphi(0)]*N^v+\varphi(0)N^v$ is a finite CAF. This hold also for $\varphi(u_1,\dots,u_k)*N^v$ for any $u_1,\dots,u_k$ and v in $\mathcal{F}$ and $\varphi\in C^1(\mathbb{R}^k)$.

4. Proof of Theorem 1.1

In this section, we show that for $f\in\mathcal{F}_{loc}$ and $C\in\mathcal{N}_{f\text{-}loc}$, the additive functional $f*C$ built in the precedent section satisfies the conclusion of Theorem 1.1.

Without loss of generality, we take in this section Ω to be the canonical path space $D([0,\infty)\to\infty)\to E_\partial$ of càdlàg functions from $[0,\infty)$ to E_∂ for which $w(t)=\partial$ for all $t\ge\zeta(\omega):=\inf\{s\ge 0\colon w(s)=\partial\}$.

Given $\omega\in\{\omega\in\Omega\colon t<\zeta(\omega)\}$, the operator r_t is defined by:

$$r_t(\omega)(s):=\begin{cases}\omega\big((t-s)-\big) & \text{if } 0\le s<t,\\ \omega(0) & \text{if } s\ge t.\end{cases}$$

We denote by $\{\hat{P}_x,x\in E\}$ the law of $\hat{X}$, the dual process of X. The following lemma can be established using the same arguments as Lemma 5.7.1 in [18].

Lemma 4.1. *For positive t and every $\mathcal{F}_t$-measurable set Γ,*

$$P_m\big(r_t^{-1}\Gamma;t<\zeta\big)=\hat{P}_m(\Gamma,t<\zeta).$$

Lemma 4.2. *For any u in $\mathcal{F}$, there exists a unique $\hat{u}\in\mathcal{F}$ such that $\mathcal{E}_1(\hat{u},h)=\mathcal{E}_1(h,u)$ for any h in $\mathcal{F}$. If we set*:

$$\hat{N}_t^u:=N_t^{\hat{u}}+\int_0^t\big(u(X_s)-\hat{u}(X_s)\big)\,\mathrm{d}s,\qquad t\ge 0,$$
$$\hat{M}_t^u:=u(X_t)-u(X_0)-\hat{N}_t^u,$$

then under $(\hat{P}_x,x\in E)$, $\hat{N}^u$ and $\hat{M}^u$ are, respectively, the CAF of zero energy and the MAF of finite energy of the Fukushima decomposition for $u(X_t)-u(X_0)$, $t\ge 0$.

Proof. For any $n\in\mathbb{N}$ set $\hat{f}_n:=n(u-n\hat{R}_{n+1}u)$. The constant K was introduced in (3.4). For any h in $\mathcal{F}$ and n,m in $\mathbb{N}$:

$$\begin{aligned}\mathcal{E}_1\big(R_1(\hat{f}_n-\hat{f}_m),h\big)&=\mathcal{E}_1\big(h,\hat{R}_1(\hat{f}_n-\hat{f}_m)\big)\\ &\le K\big(\mathcal{E}_1(h,h)\big)^{1/2}\big(\mathcal{E}_1\big(\hat{R}_1(\hat{f}_n-\hat{f}_m),\hat{R}_1(\hat{f}_n-\hat{f}_m)\big)\big)^{1/2}.\end{aligned}$$

In particular, if $h = R_1(\hat{f}_n - \hat{f}_m)$ we obtain:

$$\mathcal{E}_1\big(R_1(\hat{f}_n - \hat{f}_m), R_1(\hat{f}_n - \hat{f}_m)\big) \le K^2 \mathcal{E}_1\big(\hat{R}_1(\hat{f}_n - \hat{f}_m), \hat{R}_1(\hat{f}_n - \hat{f}_m)\big).$$

It is known that the right-hand side of the above equation tends to 0 as n, m tends to infinity (see Theorem I.2.13 in [28]) then there exists $\hat{u}$ in $\mathcal{F}$ such that $R_1 \hat{f}_n$ converges to $\hat{u}$ with respect to the $\tilde{\mathcal{E}}_1$-norm. Besides, for any h in $\mathcal{F}$: $\mathcal{E}_1(\hat{u}, h) = \lim \mathcal{E}_1(R_1 \hat{f}_n, h) = \lim \mathcal{E}_1(h, \hat{R}_1 \hat{f}_n) = \mathcal{E}_1(h, u)$.

Let A_t be the CAF of zero energy of the Fukusmima decomposition of $u(X_t) - u(X_0)$ with respect to $\hat{P}_x, x \in E$. By taking a subsequence if necessary, we have $\hat{P}_x$-a.e. for q.e. $x \in E$: For all $t \ge 0$

$$\begin{aligned} A_t &= \lim_{n\to\infty} \int_0^t \big[u(X_s) - \hat{f}_n(X_s)\big]\,\mathrm{d}s \\ &= \lim_{n\to\infty} \int_0^t \big[\hat{u}(X_s) - \hat{f}_n(X_s)\big]\,\mathrm{d}s + \int_0^t \big[u(X_s) - \hat{u}(X_s)\big]\,\mathrm{d}s \\ &= \hat{N}_t^u. \end{aligned}$$ □

Clearly, $\hat{N}^u$ belongs to $\mathcal{N}_c^0$ and:

$$\big\langle \Theta\big(\hat{N}^u\big), h\big\rangle = -\mathcal{E}(h, u), \qquad h \in \mathcal{F}. \tag{4.1}$$

Lemma 4.3. *Let $\hat{A}$ be a PCAF with respect to $(\hat{P}_x, x \in E)$ and with Revuz measure μ. Then under $(P_x, x \in E)$, $\hat{A}$ is the PCAF with Revuz measure μ.*

Proof. We suppose without loss of generality that $\mu \in S_0$. Let $u = \hat{U}_1\mu$ and $v := U_1\mu$ be the 1-potentials of μ with respect to $\mathcal{E}$ and $\hat{\mathcal{E}}$, respectively. Let A be the PCAF with respect to $(P_x, x \in E)$ and with Revuz measure μ. For any h in $\mathcal{F}$, $\mathcal{E}_1(h, u) = \mathcal{E}_1(v, h)$, then with the notation of Lemma 4.1, $v = \hat{u}$. It follows from Theorem 2.1 that P_x-a.e. for q.e. $x \in E$:

$$\begin{aligned} A_t &= -N_t^v + \int_0^t v(X_s)\,\mathrm{d}s = -N_t^v - \int_0^t \big[u(X_s) - v(X_s)\big]\,\mathrm{d}s + \int u(X_s)\,\mathrm{d}s \\ &= -\hat{N}^u + \int_0^t u(X_s)\,\mathrm{d}s = \hat{A}_t. \end{aligned}$$ □

The following lemma can be found in [11], Lemma 3.21, for symmetric diffusions.

Lemma 4.4. *Let u be in $\mathcal{F}$. For any $t \le T$ we have P_m-a.e. on $\{T < \zeta\}$:*

$$\begin{aligned} &\hat{N}_t^u \circ r_T = \hat{N}_T^u - \hat{N}_{T-t}^u \qquad \textit{and} \\ &\hat{M}_t^u \circ r_T = \hat{M}_T^u \circ r_T - \hat{M}_{T-t}^u \circ r_{T-t}. \end{aligned}$$

Proof. Define $\hat{u}$ and $(\hat{f}_n)$ as in Lemma 4.2. $\hat{P}_m$-a.e. and by taking subsequences if necessary we have: $\hat{N}_t^u = \lim_{n\to\infty} \int_0^t [u(X_s) - \hat{f}_n(X_s)]\,\mathrm{d}s$, thus in view of Lemma 4.1, we have P_m-a.e. on $\{T < \zeta\}$:

$$\begin{aligned}
\hat{N}_t^u \circ r_T &= \lim_{n\to\infty} \int_0^t \big[u(X_s) - \hat{f}_n(X_s)\big]\,\mathrm{d}s \circ r_T \\
&= \lim_{n\to\infty} \int_0^T \big[u(X_s) - \hat{f}_n(X_s)\big]\,\mathrm{d}s - \lim_{n\to\infty} \int_0^{T-t} \big[u(X_s) - \hat{f}_n(X_s)\big]\,\mathrm{d}s \\
&= \hat{N}_T^u - \hat{N}_{T-t}^u.
\end{aligned}$$

The second equality can be shown with easy computations using the first one. □

Remark 4.5. The first equality in Lema 4.4 is in fact true for N^u and therefore for the elements in $\mathcal{N}_{f\text{-}loc}$, in particular for any PCAF.

Similarly to [6], the proof of Theorem 1.1 is based in an extension of the Lyons and Zheng decomposition [27], that is, in a representation of N^u using forward and backward MAF. We recall that for u in $\mathcal{F}$, u^* was defined as the unique element of $\mathcal{F}$ satisfying (3.3).

Lemma 4.6. *Let u be in $\mathcal{F}$ and T in $\mathbb{R}_+$. Set $v := u^*$. Then we have $\mathbf{P}_m$-a.e. on $\{T < \zeta\}$:*

$$\begin{aligned}
N_t^u = &-\frac{1}{2}\big(M_t^v + \hat{M}_t^v \circ r_t\big) + \frac{1}{2}\big(v(X_t) - v(X_{t-})\big) \\
&+ \int_0^t \big[u(X_s) - v(X_s)\big]\,\mathrm{d}s, \qquad t \le T.
\end{aligned} \tag{4.2}$$

Proof. In view of Lemma 4.4, the right-hand side of (4.2) coincides $\mathbf{P}_m$-a.e. on $\{T < \zeta\}$ with A, where for all $t \le T$, $A_t := \frac{1}{2}(N_t^v + \hat{N}_t^v) + \int_0^t [u(X_s) - v(X_s)]\,\mathrm{d}s$. It follows from (4.1) that $\langle \Theta(A), h\rangle = -\tilde{\mathcal{E}}(v,h) + (u,h) - (v,h) = -\mathcal{E}(u,h)$, for all $h \in \mathcal{F}$. Now, 4.2 is consequence of Lemma 2.3. □

Lemma 4.7. *Let $(N^\ell)_{\ell\in\mathbb{N}}$ be a sequence of elements of $\in \mathcal{N}_{c,f\text{-}loc}$ and let (Π_n) be a sequence of partitions tending to the identity. Then there exists a subsequence (Π_{n_j}) of (Π_n) such that $\mathbf{P}_x$-a.s. for m-a.e. x in E we have: For all $\ell \in \mathbb{N}$,*

$$\sum_{k=0}^{p_{n_j}-1} \big[N^\ell(t \wedge t_{n_j,k+1}) - N(t \wedge t_{n_j,k})\big]^2$$

converges to zero as $n \to \infty$, uniformly in any compact of $[0,\infty)$.

Proof. Let g be a function belongs to $L^1(E,m)$ such that $0 < g(x) \le 1$ for all $x \in E$. For any $t \in \mathbb{R}_+$ set $a_n(t) := \sup\{t_{n,k}\colon\ t_{n,k} < t\}$. For all $n, \ell, K \in \mathbb{N}$, $\eta > 0$:

$$\begin{aligned}
&\mathbf{P}_{g\cdot m}\left(\sup_{t\le K}\sum_{k=0}^{p_n-1}\left[N^\ell(t\wedge t_{n,k+1}) - N^\ell(t\wedge t_{n,k})\right]^2 > \eta\right)\\
&\quad \le \frac{2}{\eta}\mathbf{E}_m\left(\sum_{k=0}^{p_n-1}\left[N^\ell(K\wedge t_{n,k+1}) - N^\ell(K\wedge t_{n,k})\right]^2\right)\\
&\qquad + \mathbf{P}_{g\cdot m}\left(\sup_{t\le K}\left[N^\ell(t) - N^\ell\big(a_n(t)\big)\right]^2 > \frac{\eta}{2}\right).
\end{aligned} \tag{4.3}$$

Since $N^\ell \in \mathcal{N}_c$, the last term in the above equation converges to zero as $n \to \infty$. (See (5.2.14) in [18].) For all $n, \ell, K \in \mathbb{N}$ and $\eta > 0$ let $\alpha(n,\ell,K,\eta)$ be the left-hand side of (4.3). Then for all $\eta > 0$:

$$\beta(n,\eta) := \sum_{\ell,K\in\mathbb{N}} \frac{1}{\ell^2K^2}\alpha(n,\ell,K,\eta) \to 0 \qquad \text{as } n\to\infty.$$

For any $j \in \mathbb{N}$ take n_j such that $\beta(n_j, j^{-1}) \le j^{-2}$. Then for any $j,\ell,K \in \mathbb{N}$, $\alpha(n_j,\ell,K,j^{-1}) \le \ell^2K^2j^{-2}$ therefore, it follows from Borel–Cantelli that for all ℓ, K, $\sum_{k=0}^{p_{n_j}-1}[N^\ell(t\wedge t_{n_j,k+1}) - N^\ell(t\wedge t_{n_j,k})]^2$ converges to zero as $n\to\infty$ uniformly on $[0,K]$ $\mathbf{P}_{g\cdot m}$-a.s. □

Proof of Theorem 1.1. As usually, the uniqueness in the theorem is the following sense: two local AF A, B are equivalent if $\mathbf{P}_x(A_t = B_t, t < \zeta) = 1$ for q.e. $x \in E$. Evidently if I^1 and I^2 are two local AF satisfying the conclusion of the theorem then, $\mathbf{P}_x(I^1_t = I^2_t, t < \zeta) = 1$ for m-a.e. $x \in E$. We can show that this hold for q.e. $x \in E$ using an argument of the proof of Proposition 4.6 in [6].

Now we shall proof that the stochastic integral $\int_0^t f(X_s)\,\mathrm{d}C_s$ of the precedent section satisfies the conclusion of theorem. Let $\{u_n\}$, $\{f_n\}$ and $\{G_n\}$ be a sequence of $\mathcal{F}$, $\mathcal{F}_b$ and Ξ, respectively, such that $C_t = C^n_t := N^{u_n}_t - \int_0^t u_n(X_s)\,\mathrm{d}s$ on $[\![0,\tau_{G_n}[\![$ $\mathbf{P}_m$-a.e. and $f = f_n$ q.e. on G_n (Theorem 2.6). For each n set $v_n := u_n^*$. In order to simplify the notation let M^n be M^{v_n} and in the same way define $\hat{N}^n$ and $\hat{M}^n$. For all $t \le T$ set $a_n(t) = \sup\{t_{n,k}\colon\ t_{n,k} < t\}$ and set:

$$X^n_t := \sum_{k=0}^{p_n-1} X(t_{n,k})1_{\{t_{n,k}<t\le t_{n,k+1}\}}, \qquad Y^n_t := \sum_{k=0}^{p_n-1} X(t_{n,k+1})1_{\{t_{n,k}\le t<t_{n,k+1}\}} \quad \text{and}$$

$$Z^n_T(t) := Y^n_{T-t}\circ r_T = \sum_{k=0}^{p_n-1} X(T-t_{n,k+1})1_{\{T-t_{n,k+1}<t\le T-t_{n,k}\}}.$$

In view of (4.2) and Lemma 4.4, we have for any $\ell \in \mathbb{N}$:

$$
\begin{aligned}
&\sum_{k=0}^{p_n-1} f_\ell\big(X(t_{n,k})\big)\big[C^\ell(t_{n,k+1}\wedge t) - C^\ell(t_{n,k}\wedge t)\big] \\
&\quad = -\frac{1}{2}\int_0^t f_\ell\big(X_s^n\big)\,\mathrm{d}M_s^\ell - \frac{1}{2}\int_t^T f_\ell\big(Z_T^n(s)\big)\,\mathrm{d}\hat{M}_s^\ell \circ r_T - \int_0^t f_\ell\big(X_s^n\big)v_\ell(X_s)\,\mathrm{d}s \\
&\qquad - \frac{1}{2}\sum_{k=0}^{p_n-1}\big[f_\ell\big(X(t_{n,k+1})\big) - f_\ell\big(X(t_{n,k})\big)\big]\big[v_\ell\big(X(t_{n,k+1}\wedge t)\big) - v_\ell\big(X(t_{n,k}\wedge t)\big)\big] \qquad (4.4) \\
&\qquad + \frac{1}{2} f_\ell\big(X\big(a_n(t)\big)\big)\big(v_\ell(X_t) - v_\ell(X_{t-})\big) \\
&\qquad - \frac{1}{2}\sum_{k=0}^{p_n-1}\big[f_\ell\big(X(t_{n,k+1})\big) - f_\ell\big(X(t_{n,k})\big)\big]\big[\hat{N}^\ell\big(X(t_{n,k+1}\wedge t)\big) - \hat{N}^\ell\big(X(t_{n,k}\wedge t)\big)\big].
\end{aligned}
$$

In view of Lemmas 4.1 and 4.7, the right-hand side of (4.4) converges in $\mathbf{P}_{g\cdot m}$-measure on $\{T < \zeta\}$ to:

$$
\begin{aligned}
I_T^\ell(t) := &-\frac{1}{2}\int_0^t f_\ell(X_s)\,\mathrm{d}M_s^\ell - \frac{1}{2}\int_t^T f_\ell(X_s)\,\mathrm{d}\hat{M}_s^\ell \circ r_T - \int_0^t f_\ell(X_s)v_\ell(X_s)\,\mathrm{d}s \\
&- \frac{1}{2}\big[M^{f_\ell}, M^\ell\big] + \frac{1}{2} f_\ell(t-)\big(v_\ell(X_t) - v_\ell(t-)\big).
\end{aligned} \tag{4.5}
$$

Besides, if $u_\ell = R_1 h$ for some h in $L^2(E, m)$, the left-hand side of (4.4) converges in $\mathbf{P}_{g\cdot m}$-measure to $\int_0^t f_\ell(X_s)\,\mathrm{d}C_s^\ell$. Therefore, for the general case, by approximating u_ℓ for a suite $(R_1 h_n)$ with respect to $\tilde{\mathcal{E}}_1$, it follows thanks to Lemma 3.4 and Lemma 3.13 that $I_T^\ell(t)$ coincides with $\int_0^t f_\ell(X_s)\,\mathrm{d}C_s^\ell$ $\mathbf{P}_{g\cdot m}$-a.e. on $\{T < \zeta\}$.

In order to prove the theorem with need to show that there exists a subsequence of (Π_n) such that $\mathbf{P}_{g\cdot m}$-a.e. we have: For any $\ell \in \mathbb{N}$, the first five terms in the right-hand side of (4.4) converge to the corresponding terms of the right-hand side of (4.5) uniformly on any compact of $[0, \zeta)$ and the last term in the right-hand side of (4.4) converges to zero uniformly on any compact of $[0, \zeta)$.

We must show only the existence of such subsequence of (Π_n) for the second term in the right-hand side of (4.4). The existence of such subsequence for the other terms can be shown using standard results in the semimartingale theory and the arguments used to show Lemma 4.7. (See, e.g., Chapter II in [31].)

For any $n, \ell \in \mathbb{N}$ and $\eta, T > 0$ set:

$$
\begin{aligned}
\alpha(n, \ell, T, \eta) &:= \mathbf{P}_m\left(\sup_{t\le T}\left|\int_t^T \big(f_\ell\big(Z_T^n(s)\big) - f_\ell(X_s)\big)\,\mathrm{d}\hat{M}_s^\ell \circ r_T\right| > \eta;\, T < \zeta\right) \\
&= \hat{\mathbf{P}}_m\left(\sup_{t\le T}\left|\int_t^T \big(f_\ell\big(Z_T^n(s)\big) - f_\ell(X_s)\big)\,\mathrm{d}\hat{M}_s^\ell\right| > \eta;\, T < \zeta\right).
\end{aligned}
$$

Using the Doob inequalities, we have:

$$\begin{aligned}\alpha(n,\ell,T,\eta) &\le \hat{\mathbf{P}}_m\left(\left|\int_0^T \big(f_\ell\big(Z_T^n(s)\big)-f_\ell(X_s)\big)\,\mathrm{d}\hat{M}_s^\ell\right| > \frac{\eta}{2};\, T<\zeta\right)\\ &\quad + \hat{\mathbf{P}}_m\left(\sup_{t\le T}\left|\int_0^t \big(f_\ell\big(Z_T^n(s)\big)-f_\ell(X_s)\big)\,\mathrm{d}\hat{M}_s^\ell\right| > \frac{\eta}{2};\, T<\zeta\right)\\ &\le \frac{4}{\eta}\hat{\mathbf{E}}_m\left(\int_0^T \big(f_\ell\big(Z_T^n(s)\big)-f_\ell(X_s)\big)^2\,\mathrm{d}\big\langle\hat{M}^\ell\big\rangle_s;\, T<\zeta\right)\\ &\le \frac{4}{\eta}\mathbf{E}_m\left(\int_0^T \big(f_\ell\big(Z_T^n(s)\big)-f_\ell(X_s)\big)^2\,\mathrm{d}\big\langle\hat{M}^\ell\big\rangle_s\circ r_T;\, T<\zeta\right).\end{aligned}$$

In view of Remark 4.5, $\mathbf{P}_m$-a.e. on $\{T<\zeta\}$:

$$\begin{aligned}\int_0^T \big(f_\ell\big(Z_T^n(s)\big)-f_\ell(X_s)\big)^2\,\mathrm{d}\big\langle\hat{M}^\ell\big\rangle_s\circ r_T &= -\int_0^T \big(f_\ell\big(Y_{T-s}^n\big)-f_\ell(X_{T-s})\big)^2\,\mathrm{d}\big\langle\hat{M}^\ell\big\rangle_{T-s}\\ &= \int_0^T \big(f_\ell\big(Y_s^n\big)-f_\ell(X_s)\big)^2\,\mathrm{d}\big\langle\hat{M}^\ell\big\rangle_s\\ &\le \mathrm{e}^T\int_0^\infty \mathrm{e}^{-s}\big(f_\ell\big(Y_s^n\big)-f_\ell(X_s)\big)^2\,\mathrm{d}\big\langle\hat{M}^\ell\big\rangle_s.\end{aligned}$$

Let μ be the Revuz measure of $\langle\hat{M}^\ell\rangle$, it follows from Lemma 4.3 that $\mu(E)=2\hat{e}(\hat{M}^\ell)<\infty$ where $\hat{e}$ denote the energy with respect to $(\hat{\mathbf{P}}_x, x\in E)$. Therefore, we have:

$$\begin{aligned}\mathbf{E}_m\left[\int_0^\infty \mathrm{e}^{-s}\,\mathrm{d}\big\langle\hat{M}^\ell\big\rangle_s\right] &= \lim_{x\to\infty}\mathbf{E}_m\left[\mathrm{e}^{-x}\big\langle\hat{M}^\ell\big\rangle_x+\int_0^x \mathrm{e}^{-s}\big\langle\hat{M}^\ell\big\rangle_s\,\mathrm{d}s\right]\\ &\le \lim_{x\to\infty}\mathrm{e}^{-x}x\mu(E)+\lim_{x\to\infty}\int_0^x \mathrm{e}^{-s}s\,\mathrm{d}s\,\mu(E)\\ &= \mu(E)<\infty.\end{aligned}$$

Since f_ℓ is quasi-continuous in the strict sense, $f_\ell(Y_t^n)$ converges to $f_\ell(X_t)$ uniformly on $\mathbb{R}_+$, $\mathbf{P}_m$-a.e. Therefore by dominated convergence, we have:

$$\beta(n,\ell):=\mathbf{E}_m\left[\int_0^\infty \mathrm{e}^{-s}\big(f_\ell\big(Y_s^n\big)-f_\ell(X_s)\big)^2\,\mathrm{d}\big\langle\hat{M}^\ell\big\rangle_s\right]\to 0 \qquad \text{as } n\to\infty.$$

For any $j\in\mathbb{N}$ let n_j such that:

$$\sum_{\ell=1}^\infty \frac{1}{\ell^2}\beta\big(n_j, j^{-1}\big)\le\frac{1}{j^3}\qquad \forall n\in\mathbb{N}.$$

Since for all $\eta, T > 0$: $\alpha(n, l, T, \eta) \leq \frac{4}{\eta} e^T \beta(n, \ell)$, we have that $\alpha(n_j, \ell, T, j^{-1}) \leq \frac{4}{j^2} e^T$ $\forall T > 0$. It follows from Borel–Cantelli lemma that for any T, ℓ: $\mathbf{P}_m(\Omega \setminus \Omega_{T,\ell}) = 0$ where:

$$\Omega_{T,\ell} := \{\zeta \leq T\}$$
$$\cup \left\{ \int_0^t \big(f_\ell\big(Z_T^{n_j}(s)\big) - f_\ell(X_s)\big) \, d\hat{M}_s^\ell \to 0 \text{ uniformly on any compact of } [0, T]; T < \zeta \right\},$$

then $\mathbf{P}_m(\Omega \setminus \Omega^*) = 0$ where $\Omega^* = \bigcap_{T \in Q_+, \ell \in \mathbb{N}} \Omega_{T,\ell}$. It is easy to show that for $\omega \in \Omega^*$, $\int_0^t (f_\ell(Z_T^{n_j}(s)) - f_\ell(X_s)) \, d\hat{M}_s^\ell$ converges to zero uniformly on any compact of $[0, \zeta(\omega))$. $\square$

Example 4.8. In this example, we show that the stochastic integral constructed by Chen *et al.* [6] for symmetric Dirichlet forms can be defined in the sense of Definition 3.15. Moreover, both definitions coincide $\mathbf{P}_m$-a.e. $[\![0, \zeta[\![$. We use the notations and definitions of [6], thus Λ is a linear operator that maps some class of local MAF's into even local CAF's admitting m-null set. Let M be a locally square-integrable MAF on $[\![0, \zeta[\![$ that belongs to the domain of Λ. We see from the proof of [6], Theorem 3.7 and Lemma 3.2, that there exists a nest $\{F_k\}$ of closed sets such that $\mathbf{P}_m$-a.e. on $[\![0, \tau_{F_k}[\![$:

$$\Lambda(M) = \Lambda\big(M^k\big) + A_t^k + L_t^k, \tag{4.6}$$

where $M^k \in \mathcal{M}$, A is a CAF of bounded variation and $L^k \in (\mathcal{M}_{loc})^{[\![0,\zeta[\![}$. With a refinement argument used in the proof of [6], Lemma 4.6, one checks that $\Lambda(M)$ is a local CAF of X. Denote by $\mathfrak{C}$ the set of CAF of X of finite energy. In view of [6], Proposition 2.8, the right-hand side of (4.6) belongs to $\mathfrak{C}_{f\text{-}loc}$, hence $\Lambda(M)$ belongs to $(\mathfrak{C}_{f\text{-}loc})_{f\text{-}loc} = \mathfrak{C}_{f\text{-}loc}$.

By [6], Theorem 3.7, $\Lambda(M)$ is of zero quadratic variation in the sense of Definition 2.5. Then $\Lambda(M)$ belongs to $\mathcal{N}_{f\text{-}loc}$ and therefore the integral $f * \Lambda(M)$ is well defined for any $f \in \mathcal{F}_{loc}$.

The stochastic integral defined in [6] can be approximate in some sense by Riemann sums. (See [6], Theorem 4.4.) Consequently, thanks to Theorem 1.1 the integrals $f * \Lambda(M)$ given by [6] and Definition 3.15 both coincide $\mathbf{P}_m$-a.e. on $[\![0, \zeta[\![$ for any $f \in \mathcal{F}_b$ and therefore for any $f \in \mathcal{F}_{loc}$.

5. Proof of Propositions 1.2 and 1.3

Proof of Proposition 1.2. From the proof of Lemma 1.1 in [37], there exist sequences (u_n), (g_n) in $\mathcal{F}$ and nest of nearly Borel finely open sets (G_n) and $(\mathcal{G}_n)$ such that for any n: $u(x) = u_n(x)$ q.e. on G_n, $G_n \subset \mathcal{G}_n$, $g_n(x) = 1$ q.e. on G_n, $g_n(x) = 0$ q.e. on $E \setminus \mathcal{G}_n$ and $\|g_n\|_\infty \leq 1$. Moreover, there exists a sequence of positive numbers (ε_n) converging to 0 that the following limit define and element in $\mathcal{M}_{f\text{-}loc}$, where the convergence is uniformly on any compact of $[0, \zeta)$ $\mathbf{P}_x$-a.e. for q.e. $x \in E$.

$$M_t^{bj} := \lim_{n \to \infty} \sum_{s \leq t} \big[u(X_s) - u(X_{s-})\big] 1_{\{\varepsilon_n < |\Delta u(X_s)| < 1\}}$$
$$- \int_0^t \int_E 1_{\{\varepsilon_n < |u(y) - u(X_s)| < 1\}} \big[u(y) - u(X_s)\big] N(X_s, dy) \, dH_s, \qquad t < \zeta.$$

Besides, it is known that $M_t^c := M_t^{u_n,c}$ if $t < \tau_{\mathcal{G}_n}$ define a local CAF in $\mathcal{M}_{f\text{-}loc}$. Then set $W^u = M^{bj} - M^c$. We shall proof that $C^u \in \mathcal{N}_{c,f\text{-}loc}$, where:

$$C_t^u := u(X_t) - u(X_0) - \sum_{s \le t} \big[u(X_s) - u(X_{s-})\big] 1_{\{|\Delta u(X_s)| \ge 1\}} - W^u, \qquad t < \zeta.$$

For any $\ell \in \mathbb{N}$, by taking a subsequence of (ε_n), if necessary, we have that $\mathbf{P}_x$-a.e. for q.e. $x \in E$. For all $t < \tau_{G_\ell}$,

$$\begin{aligned}
&C_t^u - N_t^{u_\ell} - \int_0^t u_\ell(X_s) N(X_s, \partial)\,\mathrm{d}H_s \\
&\quad + \int_0^t \int_E 1_{\{|u_\ell(X_s) - u_\ell(y)| \ge 1\}} \big[u_\ell(y) - u_\ell(X_s)\big] N(X_s, \mathrm{d}y)\,\mathrm{d}H_s \\
&= \lim_{n\to\infty} \bigg(\int_0^t \int_E 1_{\{\varepsilon_n < |u(y) - u(X_s)| < 1\}} \big[u(y) - u(X_s)\big] N(X_s, \mathrm{d}y)\,\mathrm{d}H_s \\
&\qquad - \int_0^t \int_E 1_{\{\varepsilon_n < |u_\ell(y) - u_\ell(X_s)| < 1\}} \big[u_\ell(y) - u_\ell(X_s)\big] N(X_s, \mathrm{d}y)\,\mathrm{d}H_s \bigg) \\
&= \lim_{n\to\infty} \bigg(\int_0^t \int_E g_\ell(X_s) 1_{\{\varepsilon_n < |u(y) - u(X_s)| < 1\}} \big[u(y) - u(X_s)\big] N(X_s, \mathrm{d}y)\,\mathrm{d}H_s \\
&\qquad - \int_0^t \int_E g_\ell(X_s) 1_{\{\varepsilon_n < |u_\ell(y) - u_\ell(X_s)| < 1\}} \big[u_\ell(y) - u_\ell(X_s)\big] N(X_s, \mathrm{d}y)\,\mathrm{d}H_s \bigg) \\
&= - \int_0^t \int_E 1_{\{|u(y) - u(X_s)| < 1\}} \big[g_\ell(y) - g_\ell(X_s)\big]\big[u(y) - u(X_s)\big] N(X_s, \mathrm{d}y)\,\mathrm{d}H_s \\
&\quad + \int_0^t \int_E 1_{\{|u_\ell(y) - u_\ell(X_s)| < 1\}} \big[g_\ell(y) - g_\ell(X_s)\big]\big[u_\ell(y) - u_\ell(X_s)\big] N(X_s, \mathrm{d}y)\,\mathrm{d}H_s
\end{aligned}$$

and the last term belongs to $\mathcal{N}_{f\text{-}loc}$, in fact, for $t < \tau_{G_\ell}$:

$$\begin{aligned}
&\int_0^t \int_E 1_{\{|u(y) - u(X_s)| < 1\}} \big|g_\ell(y) - g_\ell(X_s)\big|\big|u(y) - u(X_s)\big| N(X_s, \mathrm{d}y)\,\mathrm{d}H_s \\
&\quad = \int_0^t \int_E 1_{\{|u(y) - u(X_s)| < 1\}} g_\ell(X_s) \big|g_\ell(y) - g_\ell(X_s)\big|\big|u(y) - u(X_s)\big| N(X_s, \mathrm{d}y)\,\mathrm{d}H_s \\
&\quad \le \int_0^t \int_E \big[g_\ell(X_s) - g_\ell(y)\big]^2 N(X_s, \mathrm{d}y)\,\mathrm{d}H_s \\
&\qquad + \int_0^t \int_E g_\ell(y) \big|g_\ell(y) - g_\ell(X_s)\big|\big|u_\ell(y) - u_\ell(X_s)\big| N(X_s, \mathrm{d}y)\,\mathrm{d}H_s \\
&\quad < \infty.
\end{aligned}$$

Therefore, $C^u \in (\mathcal{N}_{f\text{-}loc})_{f\text{-}loc} = \mathcal{N}_{f\text{-}loc}$. $\square$

Proof of Proposition 1.3. Thanks to Theorem 1.1, the Itô formula can be proved up to ζ with the same argument used to prove the generalized Itô formula of [6]. (Theorem 4.7. of [6].) When $u \in \mathcal{F}$, all terms in the decomposition of Theorem 1.2 are finite AF. Moreover, $C_t^u = N_t^u - \int_0^t \int_E 1_{\{|u(X_s)-u(y)|\geq 1\}}[u(y) - u(X_s)]N(X_s, \mathrm{d}y)\,\mathrm{d}H_s$, then it follows from Remark 3.16(iii) that the stochastic integrals $\int_0^t \frac{\partial \Phi}{\partial x_i}(u(X_s))\,\mathrm{d}C_s^u$ are finite AF. Besides W^u belongs to $\mathcal{M}$ and therefore the integrals $\int_0^t \frac{\partial \Phi}{\partial x_i}(u(X_s))\,\mathrm{d}W_s^u$ are also finite AF. Therefore, when $u \in \mathcal{F}$ all terms present in the Itô formula are finite AF, then the Itô formula can be extended from $[0, \zeta)$ to $[0, \infty)$. $\square$

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