Sample path properties of bifractional Brownian motion

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Let $B^{H,K} = \{B^{H,K}(t), t \in \mathbb{R}_+\}$ be a bifractional Brownian motion in $\mathbb{R}^d$. We prove that $B^{H,K}$ is strongly locally non-deterministic. Applying this property and a stochastic integral representation of $B^{H,K}$, we establish Chung's law of the iterated logarithm for $B^{H,K}$, as well as sharp Hölder conditions and tail probability estimates for the local times of $B^{H,K}$.

We also consider the existence and regularity of the local times of the multiparameter bifractional Brownian motion $B^{\overline{H,K}} = \{B^{\overline{H,K}}(t), t \in \mathbb{R}_+^N\}$ in $\mathbb{R}^d$ using the Wiener–Itô chaos expansion.

Keywords: bifractional Brownian motion; chaos expansion; Chung's law of the iterated logarithm; Hausdorff dimension; level set; local times; multiple Wiener–Itô stochastic integrals; self-similar Gaussian processes; small ball probability

1. Introduction

In recent years, there has been considerable interest in studying fractional Brownian motion due to its applications in various scientific areas including telecommunications, turbulence, image processing and finance. Many authors have also proposed using more general self-similar Gaussian processes and random fields as stochastic models (see, e.g., Addie *et al.* [1], Anh *et al.* [3], Benassi *et al.* [7], Mannersalo and Norros [30], Bonami and Estrade [12], Cheridito [13] and Benson *et al.* [8]). Such applications have raised many interesting theoretical questions about self-similar Gaussian processes and fields in general. However, in contrast to the extensive studies on fractional Brownian motion, there has been little systematic investigation on other self-similar Gaussian processes. The main reasons for this, in our opinion, are the complexity of dependence structures and the lack of availability of convenient stochastic integral representations for self-similar Gaussian processes that do not have stationary increments.

The objective of this paper is to fill this gap by developing systematic ways to study sample path properties of self-similar Gaussian processes. Our main tools are the Lamperti transformation (which provides a powerful connection between self-similar processes and stationary processes; see Lamperti [26]) and the strong local non-determinism of Gaussian processes (see Xiao [45]). In particular, for any self-similar Gaussian process $X = \{X(t), t \in \mathbb{R}\}$, the Lamperti transformation leads to a stochastic integral representation for X. We will show the usefulness of such a representation in studying sample path properties of X.

To illustrate our methods, we only consider a rather special class of self-similar Gaussian processes, namely, the bifractional Brownian motions introduced by Houdré and Villa [21]. Given constants $H \in (0, 1)$ and $K \in (0, 1]$, the bifractional Brownian motion (bi-fBm, in short) in $\mathbb{R}$ is a centered Gaussian process $B_0^{H,K} = \{B_0^{H,K}(t), t \in \mathbb{R}_+\}$ with covariance function

$$R^{H,K}(s,t) := R(s,t) = \frac{1}{2^K}[(t^{2H} + s^{2H})^K - |t - s|^{2HK}] \tag{1.1}$$

and $B_0^{H,K}(0) = 0$.

Let $B_1^{H,K}, \ldots, B_d^{H,K}$ be independent copies of $B_0^{H,K}$. We define the Gaussian process $B^{H,K} = \{B^{H,K}(t), t \in \mathbb{R}_+\}$ with values in $\mathbb{R}^d$ by

$$B^{H,K}(t) = (B_1^{H,K}(t), \ldots, B_d^{H,K}(t)) \qquad \forall t \in \mathbb{R}_+. \tag{1.2}$$

By (1.1), we can easily verify that $B^{H,K}$ is a self-similar process with index HK, that is, for every constant $a > 0$,

$$\{B^{H,K}(at), t \in \mathbb{R}_+\} \overset{d}{=} \{a^{HK} B^{H,K}(t), t \in \mathbb{R}_+\}, \tag{1.3}$$

where $X \overset{d}{=} Y$ means that the two processes have the same finite-dimensional distributions. Note that when $K = 1$, $B^{H,K}$ is the ordinary fractional Brownian motion in $\mathbb{R}^d$. However, if $K \neq 1$, $B^{H,K}$ does not have stationary increments. In fact, fractional Brownian motion is the only Gaussian self-similar process with stationary increments (see Samorodnitsky and Taqqu [38]).

Russo and Tudor [37] established some properties concerning the strong variations, local times and stochastic calculus of real-valued bifractional Brownian motion. An interesting property that deserves to be recalled is the fact that when $HK = \frac{1}{2}$, the quadratic variation of this family of stochastic processes (which includes the standard Brownian motion when $K = 1$ and $HK = \frac{1}{2}$) on $[0, t]$ is equal to a constant times t. This is really remarkable since, as far as we know, these are the only Gaussian self-similar processes with this quadratic variation and besides the well-known case of Brownian motion, the other members of this family are not semimartingale. Taking into account this property, it is natural to ask if the bifractional Brownian motion $B^{H,K}$ with $KH = \frac{1}{2}$ shares other properties with Brownian motion (from the sample path regularity point of view). As can be seen from the rest of the paper, the answer is often positive: for example, the bi-fBm with $HK = \frac{1}{2}$ and Brownian motion satisfy the same forms of Chung's laws of the iterated logarithm and the Hölder conditions for their local times.

The rest of this paper is organized as follows. In Section 2, we apply the Lamperti transformation to prove the strong local non-determinism of $B_0^{H,K}$. This property plays an essential role in the proofs of most of our results. In Section 3, we derive small ball probability estimates and a stochastic integral representation for $B_0^{H,K}$. Applying these results, we prove a version of Chung's law of the iterated logarithm for bifractional Brownian motion.

Section 4 is devoted to the study of local times of one-parameter bifractional Brownian motion and the corresponding N-parameter fields. In general, there are mainly two methods for studying local times of Gaussian processes: the Fourier analysis approach introduced by Berman and the Malliavin calculus approach. It is known that the Fourier analysis approach, combined

with various properties of *local non-determinism*, yields strong regularity properties such as the joint continuity and sharp Hölder conditions for the local times (see Berman [10], Pitt [36], Geman and Horowitz [20] and Xiao [44,45]), while the Malliavin calculus approach requires fewer conditions on the process and establishes regularity of the local times in the sense of Sobolev–Watanabe spaces (see Watanabe [42], Imkeller *et al.* [23], Coutin *et al.* [16], Hu and Oksendal [22] and Eddahbi *et al.* [18]). In this paper, we make use of both approaches to obtain more comprehensive results on local times of bifractional Brownian motion and fields.

Throughout this paper, an unspecified positive and finite constant is denoted by c, which may not be the same in each occurrence. More specific constants in Section i are labeled as $c_{i,1}, c_{i,2}, \ldots$.

2. Strong local non-determinism

The following proposition is essential in this paper. From its proof, we see that the same conclusion holds for quite general self-similar Gaussian processes.

Proposition 2.1. *For all constants $0 < a < b$, $B_0^{H,K}$ is strongly locally φ-non-deterministic on $I = [a, b]$ with $\varphi(r) = r^{2HK}$. That is, there exist positive constants $c_{2,1}$ and r_0 such that for all $t \in I$ and all $0 < r \leq \min\{t, r_0\}$,*

$$\mathrm{Var}\big(B_0^{H,K}(t)|B_0^{H,K}(s) : s \in I, r \leq |s - t| \leq r_0\big) \geq c_{2,1}\varphi(r). \tag{2.1}$$

Proof. We consider the centered stationary Gaussian process $Y_0 = \{Y_0(t), t \in \mathbb{R}\}$ defined through Lamperti's transformation (Lamperti [26]),

$$Y_0(t) = \mathrm{e}^{-HKt} B_0^{H,K}(\mathrm{e}^t) \qquad \text{for every } t \in \mathbb{R}. \tag{2.2}$$

The covariance function $r(t) := \mathbb{E}(Y_0(0)Y_0(t))$ is given by

$$\begin{aligned} r(t) &= \frac{1}{2^K} \mathrm{e}^{-HKt}[(\mathrm{e}^{2Ht} + 1)^K - |\mathrm{e}^t - 1|^{2HK}] \\ &= \frac{1}{2^K} \mathrm{e}^{HKt}[(1 + \mathrm{e}^{-2Ht})^K - |1 - \mathrm{e}^{-t}|^{2HK}]. \end{aligned} \tag{2.3}$$

Hence, $r(t)$ is an even function and, by (2.3) and the Taylor expansion, we verify that $r(t) = O(\mathrm{e}^{-\beta t})$ as $t \to \infty$, where $\beta = \min\{H(2 - K), 1 - HK\}$. It follows that $r(\cdot) \in L^1(\mathbb{R})$. Also, by again using (2.3) and the Taylor expansion we have

$$r(t) \sim 1 - \frac{1}{2^K}|t|^{2HK} \qquad \text{as } t \to 0. \tag{2.4}$$

The stationary Gaussian process Y_0 is sometimes called the Ornstein–Uhlenbeck process associated with $B_0^{H,K}$ (note that it does not coincide with the solution of the fractional Langevin

equation; see Cheridito *et al.* [14] for a proof in the case $K = 1$). By Bochner's theorem, Y_0 has the stochastic integral representation

$$Y_0(t) = \int_{\mathbb{R}} e^{i\lambda t} W(d\lambda) \qquad \forall t \in \mathbb{R}, \tag{2.5}$$

where W is a complex Gaussian measure with control measure Δ, whose Fourier transform is $r(\cdot)$. The measure Δ is called the *spectral measure* of Y.

Since $r(\cdot) \in L^1(\mathbb{R})$, the spectral measure Δ of Y has a continuous density function $f(\lambda)$ which can be represented as the inverse Fourier transform of $r(\cdot)$:

$$f(\lambda) = \frac{1}{\pi} \int_0^\infty r(t) \cos(t\lambda) \, dt. \tag{2.6}$$

We wish to prove that f has the asymptotic property

$$f(\lambda) \sim c_{2,2} |\lambda|^{-(1+2HK)} \qquad \text{as } \lambda \to \infty, \tag{2.7}$$

where $c_{2,2} > 0$ is an explicit constant depending only on HK. Note that (2.4) and the Tauberian theorem due to Pitman ([35], Theorem 5) only imply that $\int_\lambda^\infty f(x) \, dx \sim c|\lambda|^{-2HK}$ as $\lambda \to \infty$. Some extra Tauberian condition on f is usually needed if we wish to obtain (2.7) by using the Tauberian theorem; see Bingham *et al.* [11].

In the following, we give a direct proof of (2.7) by using (2.6) and an Abelian argument similar to that used in the proof of Theorem 1 of Pitman [35]. Without loss of generality, we assume from now on that $\lambda > 0$. Applying integration by parts to (2.6), we get

$$f(\lambda) = -\frac{1}{\pi\lambda} \int_0^\infty r'(t) \sin(t\lambda) \, dt \tag{2.8}$$

with

$$r'(t) = \frac{HK}{2^K} e^{HKt} [(1 + e^{-2Ht})^{K-1}(1 - e^{-2Ht}) - (1 + e^{-t})(1 - e^{-t})^{2HK-1}]. \tag{2.9}$$

We need to distinguish three cases: $2HK < 1$, $2HK = 1$ and $2HK > 1$. In the first case, it can be verified from (2.9) that $r'(t) = O(e^{-\beta t})$ as $t \to \infty$, hence $r'(\cdot) \in L^1(\mathbb{R})$ and

$$r'(t) \sim -2^{1-K} HK |t|^{2HK-1} \qquad \text{as } t \to 0. \tag{2.10}$$

We will also make use of the properties of higher-order derivatives of $r(t)$. It is elementary to compute $r''(t)$ and verify that, when $2HK < 1$, we have

$$r''(t) \sim -2^{1-K} HK (2HK - 1) |t|^{2HK-2} \qquad \text{as } t \to 0 \tag{2.11}$$

and $r''(t) = O(e^{-\beta t})$ as $t \to \infty$, which implies that $r''(\cdot) \in L^1(\mathbb{R})$. Moreover, we can show that $r''(t) > 0$ for all sufficiently large t and that $r''(t)$ is eventually monotone decreasing.

The behavior of the derivatives of $r(t)$ is slightly different when $2HK = 1$. (2.9) becomes

$$r'(t) = \frac{1}{2^{K+1}} e^{t/2} [(1 + e^{-2Ht})^{K-1}(1 - e^{-2Ht}) - 1 - e^{-t}] \qquad (2.12)$$

and

$$r''(t) = \frac{1}{2^{K+2}} e^{t/2} [(1 + e^{-2Ht})^{K-2}(1 + 2(4H - 1)e^{-2Ht} + e^{-4Ht}) - 1 + e^{-t}]. \qquad (2.13)$$

Hence, we have $r'(0) = -2^{-K}$, $r''(0) = H/2$ and both $r'(\cdot)$ and $r''(\cdot)$ are in $L^1(\mathbb{R})$.

When $2HK > 1$, it can be shown that (2.11) still holds, $r''(t) = O(e^{-\beta t})$ as $t \to \infty$, $r''(t) > 0$ for all sufficiently large t and $r''(t)$ is eventually monotone decreasing. We omit the details.

We now proceed to prove (2.7). First, we consider the case when $0 < 2HK < 1$. By a change of variable, we can write

$$f(\lambda) = -\frac{1}{\pi \lambda^2} \int_0^\infty r'\left(\frac{t}{\lambda}\right) \sin t \, dt. \qquad (2.14)$$

Hence,

$$\frac{f(\lambda)}{-(\pi \lambda^2)^{-1} r'(1/\lambda)} = \int_0^\infty \frac{r'(t/\lambda)}{r'(1/\lambda)} \sin t \, dt. \qquad (2.15)$$

Let $p \in (0, \infty)$ be a fixed constant such that $r''(t) > 0$ on $[p, \infty)$. It follows from (2.10) and the dominated convergence theorem that

$$\lim_{\lambda \to \infty} \int_0^p \frac{r'(t/\lambda)}{r'(1/\lambda)} \sin t \, dt = \int_0^p t^{2HK-1} \sin t \, dt. \qquad (2.16)$$

On the other hand, integration by parts yields

$$\int_p^\infty r'(t/\lambda) \sin t \, dt = r'(p/\lambda) \cos p + \frac{1}{\lambda} \int_p^\infty r''(t/\lambda) \cos t \, dt. \qquad (2.17)$$

Using the fact that $r''(t) > 0$ on $[p, \infty)$ and the Riemann–Lebesgue lemma, we derive

$$\left| \int_p^\infty r'(t/\lambda) \sin t \, dt \right| \leq |r'(p/\lambda) \cos p| + \left| \frac{1}{\lambda} \int_p^\infty r''(t/\lambda) \cos t \, dt \right| \leq 2|r'(p/\lambda)|. \qquad (2.18)$$

Hence, we have

$$\limsup_{\lambda \to \infty} \left| \int_p^\infty \frac{r'(t/\lambda)}{r'(1/\lambda)} \sin t \, dt \right| \leq 2p^{2HK-1}. \qquad (2.19)$$

Combining (2.15), (2.16), (2.19) and letting $p \to \infty$, we see that when $0 < 2HK < 1$, (2.7) holds with $c_{2,2} = 2^{1-K} HK \pi^{-1} \int_0^\infty t^{2HK-1} \sin t \, dt$.

Second, we consider the case $2HK = 1$. Since $r'(t)$ is continuous and $r'(0) = -2^{-K}$, (2.16) becomes

$$\lim_{\lambda \to \infty} \int_0^p r'(t/\lambda) \sin t \, dt = r'(0) \int_0^p \sin t \, dt = r'(0)(1 - \cos p). \qquad (2.20)$$

Using (2.17) and integration by parts again, we derive

$$\int_p^\infty r'(t/\lambda) \sin t \, \mathrm{d}t = r'(p/\lambda) \cos p + \frac{1}{\lambda} \int_p^\infty r''(t/\lambda) \cos t \, \mathrm{d}t. \tag{2.21}$$

It follows from (2.21), (2.13) and the Riemann–Lebesgue lemma that

$$\lim_{\lambda \to \infty} \int_p^\infty r'(t/\lambda) \sin t \, \mathrm{d}t = r'(0) \cos p. \tag{2.22}$$

We see from the above and (2.14) that

$$f(\lambda) \sim \frac{1}{2^K \pi} |\lambda|^{-2} \qquad \text{as } \lambda \to \infty. \tag{2.23}$$

This verifies that (4.10) holds when $2HK = 1$.

Finally, we consider the case $1 < 2HK < 2$. Note that (2.16) and (2.19) are no longer useful and we must modify the above argument. By applying integration by parts to (2.8), we obtain

$$f(\lambda) = -\frac{1}{\pi\lambda^2} \int_0^\infty r''(t) \cos(t\lambda) \, \mathrm{d}t. \tag{2.24}$$

Note that we have $-1 < 2HK - 2 < 0$. Hence, $r''(t)$ is integrable in the neighborhood of $t = 0$. Consequently, the proof for this case is very similar to that of the case $0 < 2HK < 1$. From (2.24) and (2.11), we can verify that (2.7) also holds and the constant $c_{2,2}$ is explicitly determined by H and K. Hence, we have proved (2.7) in general.

It follows from (2.7) and Lemma 1 of Cuzick and DuPreez [17] (see also Xiao [45] for more general results) that $Y_0 = \{Y_0(t), t \in \mathbb{R}\}$ is strongly locally φ-non-deterministic on any interval $J = [-T, T]$ with $\varphi(r) = r^{2HK}$ in the following sense. There exist positive constants δ and $c_{2,3}$ such that for all $t \in [-T, T]$ and all $r \in (0, |t| \wedge \delta)$,

$$\mathrm{Var}\big(Y_0(t)|Y_0(s) : s \in J, r \le |s - t| \le \delta\big) \ge c_{2,3}\varphi(r). \tag{2.25}$$

We now prove the strong local non-determinism of $B_0^{H,K}$ on I. To this end, note that $B_0^{H,K}(t) = t^{HK} Y_0(\log t)$ for all $t > 0$. We choose $r_0 = a\delta$. For all $s, t \in I$ with $r \le |s - t| \le r_0$, we then have

$$\frac{r}{b} \le |\log s - \log t| \le \delta. \tag{2.26}$$

Hence, it follows from (2.25) and (2.26) that for all $t \in [a, b]$ and $r < r_0$,

$$\begin{aligned}
\mathrm{Var}\big(&B_0^{H,K}(t)|B_0^{H,K}(s) : s \in I, r \le |s - t| \le r_0\big) \\
&= \mathrm{Var}\big(t^{HK} Y_0(\log t)|s^{HK} Y_0(\log s) : s \in I, r \le |s - t| \le r_0\big) \\
&\ge t^{2HK} \mathrm{Var}\big(Y_0(\log t)|Y_0(\log s) : s \in I, r \le |s - t| \le r_0\big) \\
&\ge a^{2HK} \mathrm{Var}\big(Y_0(\log t)|Y_0(\log s) : s \in I, r/b \le |\log s - \log t| \le \delta\big) \\
&\ge c_{2,3}\varphi(r).
\end{aligned} \tag{2.27}$$

This proves Proposition 2.1. $\qquad\square$

For use in next section, we now state two properties of the spectral density $f(\lambda)$ of Y. They follow from (2.7) or, more generally, from (2.4) and the truncation inequalities in Loéve [28], page 209; see also Monrad and Rootzén [33].

Lemma 2.2. *There exist positive constants $c_{2,4}$ and $c_{2,5}$ such that for $u > 1$,*

$$\int_{|\lambda|<u} \lambda^2 f(\lambda)\,\mathrm{d}\lambda \le c_{2,4} u^{2(1-HK)} \tag{2.28}$$

and

$$\int_{|\lambda|\ge u} f(\lambda)\,\mathrm{d}\lambda \le c_{2,5} u^{-2HK}. \tag{2.29}$$

We will also need the following lemma from Houdré and Villa [21].

Lemma 2.3. *There exist positive constants $c_{2,6}$ and $c_{2,7}$ such that for all $s, t \in \mathbb{R}_+$, we have*

$$c_{2,6}|t-s|^{2HK} \le \mathbb{E}\big[\big(B_0^{H,K}(t) - B_0^{H,K}(s)\big)^2\big] \le c_{2,7}|t-s|^{2HK}. \tag{2.30}$$

3. Chung's law of the iterated logarithm

As applications of small ball probability estimates, Monrad and Rootzén [33], Xiao [44] and Li and Shao [27] established Chung-type laws of the iterated logarithm for fractional Brownian motion and other strongly locally non-deterministic Gaussian processes with stationary increments. However, there have been no results on Chung's LIL for self-similar Gaussian processes that do not have stationary increments (recall that the class of self-similar Gaussian processes is large and fBm is the only such process with stationary increments).

In this section, we prove the following Chung's law of the iterated logarithm for bifractional Brownian motion in $\mathbb{R}$. It will be clear that our argument is applicable to a large class of self-similar Gaussian processes.

Theorem 3.1. *Let $B_0^{H,K} = \{B_0^{H,K}(t), t \in \mathbb{R}_+\}$ be a bifractional Brownian motion in $\mathbb{R}$. There then exists a positive and finite constant $c_{3,1}$ such that*

$$\liminf_{r\to 0} \frac{\max_{t\in[0,r]}|B_0^{H,K}(t)|}{r^{HK}/(\log\log(1/r))^{HK}} = c_{3,1} \qquad a.s. \tag{3.1}$$

In order to prove Theorem 3.1, we need several preliminary results. Lemma 3.2 gives estimates on the small ball probability of $B_0^{H,K}$.

Lemma 3.2. *There exist positive constants $c_{3.2}$ and $c_{3.3}$ such that for all $t_0 \in [0, 1]$ and $x \in (0, 1)$,*

$$\exp\left(-\frac{c_{3.2}}{x^{1/(HK)}}\right) \leq \mathbb{P}\left\{\max_{t \in [0,1]} |B_0^{H,K}(t) - B_0^{H,K}(t_0)| \leq x\right\} \leq \exp\left(-\frac{c_{3.3}}{x^{1/(HK)}}\right). \tag{3.2}$$

Proof. By Proposition 2.1 and Lemma 2.3, we see that $B_0^{H,K}$ satisfies conditions (C1) and (C2) of Xiao [45]. Hence, this lemma follows from Theorem 3.1 of Xiao [45]. $\qquad\square$

Proposition 3.3 provides a zero–one law for ergodic self-similar processes, which complements the results of Takashima [40]. In order to state it, we need to recall some definitions.

Let $X = \{X(t), t \in \mathbb{R}\}$ be a separable, self-similar process with index κ. For any constant $a > 0$, the scaling transformation $S_{\kappa,a}$ of X is defined by

$$(S_{\kappa,a}X)(t) = a^{-\kappa}X(at) \qquad \forall t \in \mathbb{R}. \tag{3.3}$$

Note that saying X is κ-self-similar is equivalent to saying that for every $a > 0$, the process $\{(S_{\kappa,a}X)(t), t \in \mathbb{R}\}$ has the same finite-dimensional distributions as those of X. That is, for a κ-self-similar process X, a scaling transformation $S_{\kappa,a}$ preserves the distribution of X, so the notions of ergodicity and mixing of $S_{\kappa,a}$ can be defined in the usual way (cf. Cornfeld *et al.* [15]). Following Takashima [40], we say that a κ-self-similar process $X = \{X(t), t \in \mathbb{R}\}$ is *ergodic* (resp. *strong mixing*) if for every $a > 0$, $a \neq 1$, the scaling transformation $S_{\kappa,a}$ is ergodic (resp. *strong mixing*). This, in turn, is equivalent to saying that the shift transformations for the corresponding stationary process $Y = \{Y(t), t \in \mathbb{R}\}$ defined by $Y(t) = \mathrm{e}^{-\kappa t}X(\mathrm{e}^t)$ are ergodic (resp. strong mixing).

Proposition 3.3. *Let $X = \{X(t), t \in \mathbb{R}\}$ be a separable, self-similar process with index κ. We assume that $X(0) = 0$ and that X is ergodic. Then, for any increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, we have $\mathbb{P}(E_{\kappa,\psi}) = 0$ or 1, where*

$$E_{\kappa,\psi} = \left\{\omega : \textit{there exists } \delta > 0 \textit{ such that } \sup_{0 \leq s \leq t} |X(s)| \geq t^\kappa \psi(t) \textit{ for all } 0 < t \leq \delta\right\}. \tag{3.4}$$

Proof. We will prove that for every $a > 0$, the event $E_{\kappa,\psi}$ is invariant with respect to the transformation $S_{\kappa,a}$. The conclusion then follows from the ergodicity of X.

Fix a constant $a > 0$ and $a \neq 1$. We consider two cases: (i) $a > 1$ and (ii) $a < 1$. In the first case, since ψ is increasing, we have $\psi(au) \geq \psi(u)$ for all $u > 0$. Assuming that a.s. there is a $\delta > 0$ such that

$$\sup_{0 \leq s \leq t} |X(s)| \geq t^\kappa \psi(t) \qquad \text{for all } 0 < t \leq \delta, \tag{3.5}$$

then

$$\sup_{0 \leq s \leq t} |a^{-\kappa}X(as)| = a^{-\kappa} \sup_{0 \leq s \leq at} |X(s)| \geq t^\kappa \psi(t) \qquad \text{for all } 0 < t \leq \delta/a. \tag{3.6}$$

This implies that $E_{\kappa,\psi} \subset S_{\kappa,a}^{-1}(E_{\kappa,\psi})$. By the self-similarity of X, these two events have the same probability, so it follows that $\mathbb{P}\{E_{\kappa,\psi} \Delta S_{\kappa,a}^{-1}(E_{\kappa,\psi})\} = 0$. This proves that $E_{\kappa,\psi}$ is $S_{\kappa,a}$-invariant and, hence, has probability 0 or 1.

In case (ii), we have $\psi(au) \leq \psi(u)$ for all $u > 0$ and the proof is similar to the above. If $S_{\kappa,a} X \in E_{\kappa,\psi}$, then we have $X \in E_{\kappa,\psi}$. This implies that $S_{\kappa,a}^{-1}(E_{\kappa,\psi}) \subset E_{\kappa,\psi}$ and, again, $E_{\kappa,\psi}$ is $S_{\kappa,a}$-invariant. This completes the proof. $\qquad\square$

By a result of Maruyama [31] on ergodicity and mixing properties of stationary Gaussian processes, we see that $B_0^{H,K}$ is mixing. Hence, we have the following corollary of Proposition 3.3.

Corollary 3.4. *There exists a constant $c_{3,4} \in [0, \infty]$ such that*

$$\liminf_{t \to 0+} \frac{(\log\log 1/t)^{HK}}{t^{HK}} \max_{0 \leq s \leq t} |B_0^{H,K}(s)| = c_{3,4} \qquad a.s. \tag{3.7}$$

Proof. We take $\psi_c(t) = c(\log\log 1/t)^{-HK}$ and define $c_{3,4} = \sup\{c \geq 0 : \mathbb{P}\{E_{\kappa,\psi_c}\} = 1\}$. It can be verified that (3.7) follows from Proposition 3.3. $\qquad\square$

It follows from Corollary 3.4 that Theorem 3.1 will be established if we show $c_{3,4} \in (0, \infty)$. This is where Lemma 3.2 and the following lemma from Talagrand [41] are needed.

Lemma 3.5. *Let $X = \{X(t), t \in \mathbb{R}\}$ be a centered Gaussian process in $\mathbb{R}$ and let $S \subset \mathbb{R}$ be a closed set equipped with the canonical metric defined by*

$$d(s, t) = \left[\mathbb{E}\big(X(s) - X(t)\big)^2\right]^{1/2}.$$

There then exists a positive constant $c_{3,5}$ such that for all $u > 0$,

$$\mathbb{P}\left\{\sup_{s,t \in S} |X(s) - X(t)| \geq c_{3,5}\left(u + \int_0^D \sqrt{\log N_d(S, \varepsilon)}\, d\varepsilon\right)\right\} \leq \exp\left(-\frac{u^2}{D^2}\right), \tag{3.8}$$

where $N_d(S, \varepsilon)$ denotes the smallest number of open d-balls of radius ε needed to cover S and where $D = \sup\{d(s, t) : s, t \in S\}$ is the diameter of S.

We now proceed to prove Theorem 3.1.

Proof of Theorem 3.1. We prove the lower bound first. For any integer $n \geq 1$, let $r_n = e^{-n}$. Let $0 < \gamma < c_{3,3}$ be a constant and consider the event

$$A_n = \left\{\max_{0 \leq s \leq r_n} |B_0^{H,K}(s)| \leq \gamma^{HK} r_n^{HK} / (\log\log 1/r_n)^{HK}\right\}.$$

The self-similarity of $B_0^{H,K}$ and Lemma 3.2 then imply that

$$P\{A_n\} \leq \exp\left(-\frac{c_{3,3}}{\gamma}\log n\right) = n^{-c_{3,3}/\gamma}. \tag{3.9}$$

Since $\sum_{n=1}^{\infty} P\{A_n\} < \infty$, the Borel–Cantelli lemma implies that

$$\liminf_{n\to\infty} \frac{\max_{s\in[0,r_n]}|B_0^{H,K}(s)|}{r_n^{HK}/(\log\log(1/r_n))^{HK}} \geq c_{3,3} \qquad \text{a.s.} \tag{3.10}$$

It follows from (3.10) and a standard monotonicity argument that

$$\liminf_{r\to 0} \frac{\max_{t\in[0,r]}|B_0^{H,K}(t)|}{r^{HK}/(\log\log(1/r))^{HK}} \geq c_{3,6} \qquad \text{a.s.} \tag{3.11}$$

The upper bound is a little more difficult to prove, due to the dependence structure of $B_0^{H,K}$. In order to create independence, we will make use of the following stochastic integral representation of $B_0^{H,K}$. For every $t > 0$,

$$B_0^{H,K}(t) = t^{HK} \int_{\mathbb{R}} e^{i\lambda \log t} W(d\lambda). \tag{3.12}$$

This follows from the spectral representation (2.5) of Y and its connection with $B_0^{H,K}$.

For every integer $n \geq 1$, we take

$$t_n = n^{-n} \quad \text{and} \quad d_n = n^{\beta}, \tag{3.13}$$

where $\beta > 0$ is a constant whose value will be determined later. It is sufficient to prove that there exists a finite constant $c_{3,7}$ such that

$$\liminf_{n\to\infty} \frac{\max_{s\in[0,t_n]}|B_0^{H,K}(s)|}{t_n^{HK}/(\log\log(1/t_n))^{HK}} \leq c_{3,7} \qquad \text{a.s.} \tag{3.14}$$

Let us define two Gaussian processes, X_n and $\widetilde{X}_n$, by

$$X_n(t) = t^{HK} \int_{|\lambda|\in(d_{n-1},d_n]} e^{i\lambda \log t} W(d\lambda) \tag{3.15}$$

and

$$\widetilde{X}_n(t) = t^{HK} \int_{|\lambda|\notin(d_{n-1},d_n]} e^{i\lambda \log t} W(d\lambda), \tag{3.16}$$

respectively. Clearly, $B_0^{H,K}(t) = X_n(t) + \widetilde{X}_n(t)$ for all $t \geq 0$. It is important to note that the Gaussian processes X_n ($n = 1, 2, \ldots$) are independent and, moreover, for every $n \geq 1$, the processes X_n and $\widetilde{X}_n$ are also independent.

Let $h(r) = r^{HK}(\log\log 1/r)^{-HK}$. We make the following two claims:

(i) there is a constant $\gamma > 0$ such that

$$\sum_{n=1}^{\infty} \mathbb{P}\left\{ \max_{s \in [0,t_n]} |X_n(s)| \leq \gamma^{HK} h(t_n) \right\} = \infty; \tag{3.17}$$

(ii) for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}\left\{ \max_{s \in [0,t_n]} |\widetilde{X}_n(s)| > \varepsilon h(t_n) \right\} < \infty. \tag{3.18}$$

Since the events in (3.17) are independent, we see that (3.14) follows from (3.17), (3.18) and a standard Borel–Cantelli argument.

It remains to verify the claims (i) and (ii) above. By Lemma 3.2 and Anderson's inequality (see Anderson [2]), we have

$$\mathbb{P}\left\{ \max_{s \in [0,t_n]} |X_n(s)| \leq \gamma^{HK} h(t_n) \right\} \geq \mathbb{P}\left\{ \max_{s \in [0,t_n]} |B_0^{H,K}(s)| \leq \gamma^{HK} h(t_n) \right\}$$

$$\geq \exp\left(-\frac{c_{3,2}}{\gamma} \log(n \log n) \right)$$

$$= (n \log n)^{-c_{3,2}/\gamma}. \tag{3.19}$$

Hence, (i) holds for $\gamma \geq c_{3,2}$.

In order to prove (ii), we divide $[0, t_n]$ into $p_n + 1$ non-overlapping subintervals $J_{n,j} = [a_{n,j-1}, a_{n,j}]$, $j = 0, 1, \ldots, p_n$, and then apply Lemma 3.5 to $\widetilde{X}_n$ on each of $J_{n,j}$. Let $\beta > 0$ be the constant in (3.13) and take $J_{n,0} = [0, t_n n^{-\beta}]$. After $J_{n,j}$ has been defined, we take $a_{n,j+1} = a_{n,j}(1 + n^{-\beta})$. It can be verified that the number of such subintervals of $[0, t_n]$ satisfies the following bound:

$$p_n + 1 \leq c n^{\beta} \log n. \tag{3.20}$$

Moreover, for every $j \geq 1$, if $s, t \in J_{n,j}$ and $s < t$, then we have $t/s - 1 \leq n^{-\beta}$ and this yields

$$t - s \leq s n^{-\beta} \quad \text{and} \quad \log\left(\frac{t}{s}\right) \leq n^{-\beta}. \tag{3.21}$$

Lemma 2.3 implies that the canonical metric d for the process $\widetilde{X}_n$ satisfies

$$d(s, t) \leq c|s - t|^{HK} \qquad \text{for all } s, t > 0 \tag{3.22}$$

and $d(0, s) \leq c t_n^{HK} n^{-\beta HK}$ for every $s \in J_{n,0}$. It follows that $D_0 := \sup\{d(s, t); s, t \in J_{n,0}\} \leq c t_n^{HK} n^{-\beta HK}$ and

$$N_d(J_{n,0}, \varepsilon) \leq c \frac{t_n n^{-\beta}}{\varepsilon^{1/(HK)}}. \tag{3.23}$$

Some simple calculation yields

$$
\begin{aligned}
\int_0^{D_0} \sqrt{\log N_d(J_{n,0}, \varepsilon)}\, d\varepsilon &\leq \int_0^{t_n^{HK} n^{-\beta HK}} \sqrt{\log\left(\frac{t_n n^{-\beta}}{\varepsilon^{1/(HK)}}\right)}\, d\varepsilon \\
&= t_n^{HK} n^{-\beta HK} \int_0^1 \sqrt{\log\left(\frac{1}{u}\right)}\, du \\
&= c_{3,8} t_n^{HK} n^{-\beta HK}.
\end{aligned} \tag{3.24}
$$

It follows from Lemma 3.5 and (3.24) that

$$
\mathbb{P}\left\{\max_{s \in J_{n,0}} |\widetilde{X}_n(s)| > \varepsilon h(t_n)\right\} \leq \exp\left(-c\frac{n^{2\beta HK}}{(\log(n \log n))^{2HK}}\right). \tag{3.25}
$$

For every $1 \leq j \leq p_n$, we estimate the d-diameter of $J_{n,j}$. It follows from (3.16) that for any $s, t \in J_{n,j}$ with $s < t$,

$$
\begin{aligned}
\mathbb{E}\big(\widetilde{X}_n(s) - \widetilde{X}_n(t)\big)^2 &= \int_{|\lambda| \leq d_{n-1}} |t^{HK} e^{i\lambda \log t} - s^{HK} e^{i\lambda \log s}|^2 f(\lambda)\, d\lambda \\
&\quad + \int_{|\lambda| > d_n} |t^{HK} e^{i\lambda \log t} - s^{HK} e^{i\lambda \log s}|^2 f(\lambda)\, d\lambda \\
&:= \mathfrak{I}_1 + \mathfrak{I}_2.
\end{aligned} \tag{3.26}
$$

The second term is easy to estimate: for all $s, t \in J_{n,j}$,

$$
\mathfrak{I}_2 \leq 4 t_n^{2HK} \int_{|\lambda| > d_n} f(\lambda)\, d\lambda \leq c_{3,9} t_n^{2HK} n^{-2\beta HK}, \tag{3.27}
$$

where the last inequality follows from (2.29).

For the first term $\mathfrak{I}_1$, we use the elementary inequality $1 - \cos x \leq x^2$ to derive that for all $s, t \in J_{n,j}$ with $s < t$,

$$
\begin{aligned}
\mathfrak{I}_1 &= \int_{|\lambda| \leq d_{n-1}} \left[(t^{HK} - s^{HK})^2 + 2 t^{HK} s^{HK}\left(1 - \cos\left(\lambda \log \frac{t}{s}\right)\right)\right] f(\lambda)\, d\lambda \\
&\leq s^{2HK}\left(\frac{t}{s} - 1\right)^{2HK} \int_{\mathbb{R}} f(\lambda)\, d\lambda + 2 t^{2HK} \log^2\left(\frac{t}{s}\right) \int_{|\lambda| \leq d_{n-1}} \lambda^2 f(\lambda)\, d\lambda \\
&\leq c_{3,10} t_n^{2HK} n^{-2\beta HK},
\end{aligned} \tag{3.28}
$$

where, in deriving the last inequality, we have used (3.21) and (2.28), respectively.

It follows from (3.26), (3.27) and (3.28) that the d-diameter of $J_{n,j}$ satisfies

$$
D_j \leq c_{3,11} t_n^{HK} n^{-\beta HK}. \tag{3.29}
$$

Hence, similar to (3.25), we use Lemma 3.5 and (3.29) to derive

$$\mathbb{P}\left\{\max_{s\in J_{n,j}}|\widetilde{X}_n(s)| > \varepsilon h(t_n)\right\} \leq \exp\left(-c\frac{n^{2\beta HK}}{(\log(n\log n))^{2HK}}\right). \tag{3.30}$$

By combining (3.20), (3.25) and (3.30), we derive that for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty}\mathbb{P}\left\{\max_{s\in[0,t_n]}|\widetilde{X}_n(s)| > \varepsilon h(t_n)\right\} \leq \sum_{n=1}^{\infty}\sum_{j=0}^{p_n}\mathbb{P}\left\{\max_{s\in J_{n,j}}|\widetilde{X}_n(s)| > \varepsilon h(t_n)\right\}$$

$$\leq c\sum_{n=1}^{\infty}n^{\beta}\log n\exp\left(-c\frac{n^{2\beta HK}}{(\log(n\log n))^{2HK}}\right)$$

$$< \infty. \tag{3.31}$$

This proves (3.18) and hence the theorem. □

Remark 3.6. Let $t_0 \in [0,1]$ be fixed and consider the process $X = \{X(t), t \in \mathbb{R}_+\}$ defined by $X(t) = B_0^{H,K}(t + t_0) - B_0^{H,K}(t_0)$. By applying Lemma 3.2 and modifying the proof of Theorem 3.1, we can show that

$$c_{3,12}^{-1} \leq \liminf_{r\to 0}\frac{\max_{t\in[0,r]}|B_0^{H,K}(t+t_0) - B_0^{H,K}(t_0)|}{r^{HK}/(\log\log(1/r))^{HK}} \leq c_{3,12} \qquad \text{a.s.,} \tag{3.32}$$

where $c_{3,12} > 1$ is a constant depending only on HK.

Corresponding to Lemma 3.2, we can also consider the small ball probability of $B_0^{H,K}$ under the Hölder-type norm. For $\alpha \in (0,1)$ and any function $y \in C_0([0,1])$, we consider the α-Hölder norm of y defined by

$$\|y\|_{\alpha} = \sup_{s,t\in[0,1],s\neq t}\frac{|y(s) - y(t)|}{|s - t|^{\alpha}}. \tag{3.33}$$

The following proposition extends the results of Stolz [39] and Theorem 2.1 of Kuelbs, Li and Shao [25] to bifractional Brownian motion.

Proposition 3.7. *Let $B_0^{H,K}$ be a bifractional Brownian motion in $\mathbb{R}$ and $\alpha \in (0, HK)$. There exist positive constants $c_{3,13}$ and $c_{3,14}$ such that for all $\varepsilon \in (0,1)$,*

$$\exp\left(-c_{3,13}\varepsilon^{-1/(HK-\alpha)}\right) \leq \mathbb{P}\{\|B_0^{H,K}\|_{\alpha} \leq \varepsilon\} \leq \exp\left(-c_{3,14}\varepsilon^{-1/(HK-\alpha)}\right). \tag{3.34}$$

Proof. The result follows from Theorem 3.4 of Xiao [45]. □

4. Local times of bifractional Brownian motion

This section is devoted to the study of the local times of the bi-fBm, both in the one-parameter and multiparameter cases. As was pointed out in the Introduction, there are essentially two ways to prove the existence and regularity properties of local times for Gaussian processes: the first is related to Fourier analysis and the local nondeterminism property; the second is based on Malliavin calculus and the Wiener–Itô chaos expansion. We will apply the Fourier analysis approach to the one-parameter case and the Malliavin calculus approach to the multiparameter case.

4.1. The one-parameter case

Let $B^{H,K} = \{B^{H,K}(t), t \in \mathbb{R}_+\}$ be a bifractional Brownian motion with indices H and K in $\mathbb{R}^d$. For any closed interval $I \subset \mathbb{R}_+$ and for any $x \in \mathbb{R}^d$, the local time $L(x, I)$ of $B^{H,K}$ is defined as the density of the occupation measure μ_I defined by

$$\mu_I(A) = \int_I \mathbb{1}_A(B^{H,K}(s))\,\mathrm{d}s, \qquad A \in \mathcal{B}(\mathbb{R}^d).$$

It can be shown (cf. Geman and Horowitz [20], Theorem 6.4) that the following *occupation density formula* holds. For every Borel function $g(t, x) \geq 0$ on $I \times \mathbb{R}^d$,

$$\int_I g(t, B^{H,K}(t))\,\mathrm{d}t = \int_{\mathbb{R}^d} \int_I g(t, x) L(x, \mathrm{d}t)\,\mathrm{d}x. \tag{4.1}$$

Lemma 2.3 and Theorem 21.9 of Geman and Horowitz [20] together imply that if $1/(HK) > d$, then $B^{H,K}$ has a local time $L(x, t) := L(x, [0, t])$, where $(x, t) \in \mathbb{R}^d \times [0, \infty)$. In fact, more regularity properties of $L(x, t)$ can be derived from Theorem 3.14 of Xiao [45], which we summarize in the following theorem. Besides being of interest in their own right, such results are also useful in studying the fractal properties of the sample paths of $B^{H,K}$.

Theorem 4.1. *Let* $B^{H,K} = \{B^{H,K}(t), t \in \mathbb{R}\}$ *be a bifractional Brownian motion with indices H and K in $\mathbb{R}^d$. If $1/(HK) > d$, then the following properties hold*:

   (i) $B^{H,K}$ *has a local time $L(x, t)$ that is jointly continuous in (x, t) almost surely.*

   (ii) [Local Hölder condition] *For every $B \in \mathcal{B}(\mathbb{R})$, let $L^*(B) = \sup_{x \in \mathbb{R}^d} L(x, B)$ be the maximum local time. There then exists a positive constant $c_{4,1}$ such that for all $t_0 \in \mathbb{R}_+$,*

$$\limsup_{r \to 0} \frac{L^*(B(t_0, r))}{\varphi_1(r)} \leq c_{4,1} \qquad a.s. \tag{4.2}$$

Here and in the sequel, $B(t, r) = (t - r, t + r)$ and $\varphi_1(r) = r^{1 - HKd}(\log \log 1/r)^{HKd}$.

   (iii) [Uniform Hölder condition] *For every finite interval $I \subseteq \mathbb{R}$, there exists a positive finite constant $c_{4,2}$ such that*

$$\limsup_{r \to 0} \sup_{t_0 \in I} \frac{L^*(B(t_0, r))}{\varphi_2(r)} \leq c_{4,2} \qquad a.s., \tag{4.3}$$

where $\varphi_2(r) = r^{1-HKd}(\log 1/r)^{HKd}$.

Proof. By Proposition 2.1 and Lemma 2.3, we see that the conditions of Theorem 3.14 of Xiao [45] are satisfied. Hence, the results follow. □

The following states that the local Hölder condition for the maximum local time is sharp.

Remark 4.2. By the definition of local times, we have that for every interval $Q \subseteq \mathbb{R}_+$,

$$|Q| = \int_{\overline{B^{H,K}(Q)}} L(x, Q) \, dx \le L^*(Q) \cdot \left(\max_{s,t \in Q} |B^{H,K}(s) - B^{H,K}(t)| \right)^d. \tag{4.4}$$

By taking $Q = B(t_0, r)$ in (4.4) and using (3.32) in Remark 3.6, we derive the lower bound in the following

$$c_{4,3} \le \limsup_{r \to 0} \frac{L^*(B(t_0, r))}{\varphi_1(r)} \le c_{4,4} \qquad \text{a.s.,} \tag{4.5}$$

where $c_{4,3} > 0$ is a constant independent of t_0 and the upper bound is given by (4.2). A similar lower bound for (4.3) could also be established by using (4.4), if we were to prove that for every interval $I \subseteq \mathbb{R}_+$,

$$\liminf_{r \to 0} \inf_{t \in I} \max_{s \in B(t,r)} \frac{|B^{H,K}(s) - B^{H,K}(t)|}{r^{Hk}/(\log 1/r)^{HK}} \le c_{4,5} \qquad \text{a. s.} \tag{4.6}$$

Theorem 4.1 can be applied to determine the Hausdorff dimension and Hausdorff measure of the level set $Z_x = \{t \in \mathbb{R}_+ : B^{H,K}(t) = x\}$, where $x \in \mathbb{R}^d$; see Berman [9], Monrad and Pitt [32] and Xiao [44,45]. In the following theorem, we prove a uniform Hausdorff dimension result for the level sets of $B^{H,K}$.

Theorem 4.3. *If* $1/(HK) > d$, *then, with probability one,*

$$\dim_H Z_x = 1 - HKd \qquad \text{for all } x \in \mathbb{R}^d, \tag{4.7}$$

where $\dim_H$ *denotes Hausdorff dimension.*

Proof. It follows from Theorem 3.19 of Xiao [45] that, with probability one,

$$\dim_H Z_x = 1 - HKd \qquad \text{for all } x \in \mathcal{O}, \tag{4.8}$$

where $\mathcal{O}$ is the random open set defined by

$$\mathcal{O} = \bigcup_{s,t \in \mathbb{Q}; s < t} \{x \in \mathbb{R}^d : L(x, [s, t]) > 0\}.$$

Hence, it only remains to show that $\mathcal{O} = \mathbb{R}^d$ a.s. For this purpose, we consider the stationary Gaussian process $Y = \{Y(t), t \in \mathbb{R}\}$ defined by $Y(t) = e^{-HKt} B^{H,K}(e^t)$, using the Lamperti transformation.

Note that the component processes of Y are independent and, as shown in the proof of Proposition 2.1, they are strongly locally φ-non-deterministic, with $\varphi(r) = r^{2HK}$. It follows from Theorem 3.14 of Xiao [45] that Y has a jointly continuous local time $L_Y(x, t)$, where $(x, t) \in \mathbb{R}^d \times \mathbb{R}$. From the proof of Proposition 2.1, it can be verified that Y satisfies the conditions of Theorem 2 of Monrad and Pitt [32] and it follows that, almost surely for every $y \in \mathbb{R}^d$, there exists a finite interval $J \subset \mathbb{R}$ such that $L_Y(y, J) > 0$.

On the other hand, by using the occupation density formula (4.1), we can verify that the local times of $B^{H,K}$ and Y are related by the following equation. For all $x \in \mathbb{R}^d$ and any finite interval $I = [a, b] \subset [0, \infty)$,

$$L(x, I) = \int_{[\log a, \log b]} e^{(1-HK)s} L_Y(e^{-HKs} x, \mathrm{d}s). \tag{4.9}$$

Hence, there exists a.s. a finite interval I such that $L(0, I) > 0$. The continuity of $L(x, I)$ implies the a.s. existence of $\delta > 0$ such that $L(y, I) > 0$ for all $y \in \mathbb{R}^d$ with $|y| \leq \delta$. Observe that the scaling property of $B^{H,K}$ implies that for all constants $c > 0$, the scaled local time $c^{-(1-HKd)} L(x, ct)$ is a version of $L(c^{-HK} x, t)$. It follows that, a.s. for every $x \in \mathbb{R}^d$, $L(x, J) > 0$ for some finite interval $J \subset [0, \infty)$. $\qquad \square$

Since there is little knowledge on the explicit distribution of $L(0, 1)$, it is of interest to estimate the tail probability $\mathbb{P}\{L(0, 1) > x\}$ as $x \to \infty$. This problem has been considered by Kasahara *et al.* [24] for certain fractional Brownian motions and by Xiao [45] for a large class of Gaussian processes. Our next result is a consequence of Theorem 3.20 of Xiao [45].

Theorem 4.4. *Let $B^{H,K} = \{B^{H,K}(t), t \in \mathbb{R}\}$ be a bifractional Brownian motion in $\mathbb{R}^d$ with indices H and K. If $1/(HK) > d$, then for $x > 0$ sufficiently large,*

$$-\log \mathbb{P}\{L(0, 1) > x\} \asymp x^{HK}, \tag{4.10}$$

where $a(x) \asymp b(x)$ means that $a(x)/b(x)$ is bounded from below and above by positive and finite constants for all sufficiently large x.

Proof. By Proposition 2.1 and Lemma 2.3, we see that the conditions of Theorem 3.20 in Xiao [45] are satisfied. This proves (4.10). $\qquad \square$

Let us also note that the existence of the jointly continuous version of the local time and the self-similarity allow us to prove the following renormalization result. The case $d = 1$ has been proven in Russo and Tudor [37].

Proposition 4.5. *If $1/(HK) > d$, then for any integrable function $F : \mathbb{R}^d \to \mathbb{R}$,*

$$t^{HKd-1} \int_{[0,t]} F(B^{H,K}(u)) \, \mathrm{d}u \xrightarrow{(d)} \widetilde{F} L(0, 1) \qquad as \ t \to \infty, \tag{4.11}$$

where $\widetilde{F} = \int_{\mathbb{R}^d} F(x) \, dx$.

Proof. It holds that

$$\int_{[0,t]} F(B^{H,K}(u)) \, du = t \int_{[0,1]} F(B^{H,K}(tv)) \, dv \stackrel{d}{=} t \int_{[0,1]} F(t^{HK} B^{H,K}(v)) \, dv. \qquad (4.12)$$

By using the occupation density formula, we derive

$$\int_{[0,t]} F(B^{H,K}(u)) \, du = t \int_{\mathbb{R}^d} F(t^{HK} x) L(x, 1) \, dx$$

$$= t^{1-HKd} \int_{\mathbb{R}^d} F(y) L(y t^{-HK}, 1) \, dy. \qquad (4.13)$$

Since the function $y \mapsto L(y, 1)$ is almost surely continuous and bounded, the dominated convergence theorem implies that, as $t \to \infty$, the last integral in (4.13) tends to $\widetilde{F} L(0, 1)$ almost surely. This and (4.12) together yield (4.11). □

4.2. Oscillation of bifractional Brownian motion

The oscillations of certain classes of stochastic processes, especially Gaussian processes, in the measure space $([0, 1], \lambda_1)$, where λ_1 is the Lebesgue measure on $\mathbb{R}$, have been studied by, among others, Wschebor [43] and Azaïs and Wschebor [6]. The following is an analogous result for bifractional Brownian motion.

Proposition 4.6. *Let $B^{H,K}$ be a bi-fBm in $\mathbb{R}$ with indices $H \in (0, 1)$ and $K \in (0, 1]$. For every $t \in [0, 1]$, let*

$$Z_\varepsilon(t) = \frac{B^{H,K}(t + \varepsilon) - B^{H,K}(t)}{\varepsilon^{HK}}.$$

Then the following statements hold:

    (i) *for every integer $k \geq 1$, almost surely,*

$$\int_0^1 (Z_\varepsilon(t))^k \, dt \to \mathbb{E}(\rho^k) \qquad \text{as } \varepsilon \to 0,$$

where ρ is a centered normal random variable with variance $\sigma^2 = 2^{1-K}$;
    (ii) *for every interval $J \subset [0, 1]$, almost surely, for every $x \in \mathbb{R}$*

$$\lambda_1\{t \in J : Z_\varepsilon(t) \leq x\} \to \lambda_1(J) \mathbb{P}(\rho \leq x) \qquad \text{as } \varepsilon \to 0.$$

Proof. Let us define

$$Y^{\varepsilon,k} = \int_0^1 (Z_\varepsilon(t))^k \, dt.$$

It is sufficient to prove that

$$\text{Var}(Y^{\varepsilon,k}) \leq c(k)\varepsilon^{\beta} \qquad \text{for some } c(k) \text{ and } \beta > 0. \tag{4.14}$$

The conclusions (i) and (ii) will then follow, as in Azaïs and Wschebor [6], by means of a Borel–Cantelli argument.

Note that

$$\text{Var}(Y^{\varepsilon,k}) = \int_0^1 \int_0^1 \text{Cov}(Z_\varepsilon(u)^k, Z_\varepsilon(u)^k) \, du \, dv.$$

We will make use of the fact that for a centered Gaussian vector (U, V),

$$\text{Cov}(U^k, V^k) = \sum_{1 \leq p \leq k} c(p,k)[\text{Cov}(U, V)]^p [\text{Var}(U)\,\text{Var}(V)]^{k-p}.$$

Since the random variable Z_ε has clearly bounded variance (cf. Lemma 2.3), it suffices to show that for every $1 \leq p \leq k$,

$$\int_0^1 \int_0^1 [\mathbb{E}(Z_\varepsilon(u)Z_\varepsilon(v))]^p \, du \, dv \leq c_{4,6}\varepsilon^{\beta}. \tag{4.15}$$

We can write

$$\int_0^1 \int_0^1 [\mathbb{E}(Z_\varepsilon(u)Z_\varepsilon(v))]^p \, dv \, du = 2\int_0^1 \int_0^u \mathbb{1}_{(u-v<\varepsilon)}[\mathbb{E}(Z_\varepsilon(u)Z_\varepsilon(v))]^p \, dv \, du$$

$$+ 2\int_0^1 \int_0^u \mathbb{1}_{(u-v\geq\varepsilon)}[\mathbb{E}(Z_\varepsilon(u)Z_\varepsilon(v))]^p \, dv \, du$$

$$:= A + B.$$

Clearly, $A \leq c\varepsilon$, hence it suffices to bound the term B. Note that

$$\mathbb{E}(Z_\varepsilon(u)Z_\varepsilon(v)) = \frac{1}{\varepsilon^{2HK}} \int_{u-\varepsilon}^u \int_{v-\varepsilon}^v \frac{\partial^2 R}{\partial a\,\partial b} \, db \, da.$$

Since

$$\frac{\partial^2 R}{\partial a\,\partial b}(a,b) = \frac{2HK}{2^K}[(a^{2H} + b^{2H})^{K-2}a^{2H-1}b^{2H-1} - (2HK-1)|a-b|^{2HK-2}],$$

we have

$$B \leq c(p,H,K)\int_0^1 \int_0^{u-\varepsilon}\left[\frac{1}{\varepsilon^{2HK}}\int_{u-\varepsilon}^u \int_{v-\varepsilon}^v (a^{2H} + b^{2H})^{K-2}a^{2H-1}b^{2H-1} \, db \, da\right]^p dv \, du$$

$$+ c(p,H,K)\int_0^1 \int_0^{u-\varepsilon}\left[\frac{1}{\varepsilon^{2HK}}\int_{u-\varepsilon}^u \int_{v-\varepsilon}^v |a-b|^{2HK-2} \, db \, da\right]^p dv \, du$$

$$:= B_1 + B_2.$$

The term B_2 can be treated as in the fBm case (see Azaïs and Wschebor [6], Proposition 2.1) and we get $B_2 \le c\varepsilon^\beta$ for some constant $\beta > 0$. Finally, since $a^{2HK} + b^{2HK} \ge a^{HK}b^{HK}$, we can write

$$B_1 \le c(p, H, K) \int_0^1 \int_0^{u-\varepsilon} \left[\frac{1}{\varepsilon^{2HK}} \int_{u-\varepsilon}^u \int_{v-\varepsilon}^v a^{HK-1} b^{HK-1} \, db \, da \right]^p dv \, du$$

$$= c(p, H, K) \int_0^1 \int_0^{u-\varepsilon} \left(\frac{u^{HK} - (u-\varepsilon)^{HK}}{\varepsilon^{HK}} \right)^p \left(\frac{v^{HK} - (v-\varepsilon)^{HK}}{\varepsilon^{HK}} \right)^p dv \, du$$

$$\le c \left[\int_0^1 \left(\frac{u^{HK} - (u-\varepsilon)^{HK}}{\varepsilon^{HK}} \right)^p dv \, du \right]^2.$$

A change of variable shows that $B_1 \le c\varepsilon^{2(1-HK)}$. Combining the above yields (4.15). Therefore, we have proven (4.14) and thereby the proposition is proved. $\qquad\square$

The above result can be extended to obtain the almost sure weak approximation of the occupation measure of the bi-fBm $B^{H,K}$ by means of the normalized number of crossing of $B_\varepsilon^{H,K}$, where $B_\varepsilon^{H,K}$ represents the convolution of $B^{H,K}$ with an approximation of the identity $\Phi_\varepsilon(t) = \frac{1}{\varepsilon}\Phi(\frac{t}{\varepsilon})$ with $\Phi = \mathbb{1}_{[-1,0]}$. If g is a real-valued function defined on an interval I, then the number of crossing of level u is

$$N_u(g, I) = \#\{t \in I : g(t) = u\},$$

where $\#E$ denotes the cardinality of E.

Proposition 4.7. *Almost surely for every continuous function f and for every bounded interval $I \subset \mathbb{R}_+$,*

$$\left(\frac{\pi}{2} \right)^{1/2} \varepsilon^{1-HK} \int_{-\infty}^\infty f(u) N_u(B_\varepsilon^{H,K}, I) \, du \to \int_{-\infty}^\infty f(u) L(u, I) \, du \qquad \text{as } \varepsilon \to 0.$$

Proof. The arguments in Azaïs and Wschebor [6], Section 5, apply. Details are left to the reader. $\qquad\square$

4.3. The multiparameter case

For any given vectors $H = (H_1, \ldots, H_N) \in (0, 1)^N$ and $K = (K_1, \ldots, K_N) \in (0, 1]^N$, an (N, d)-bifractional Brownian sheet $B^{H,K} = \{B^{H,K}(t), t \in \mathbb{R}_+^N\}$ is a centered Gaussian random field in $\mathbb{R}^d$ with i.i.d. components whose covariance functions are given by

$$\mathbb{E}(B_1^{H,K}(s) B_1^{H,K}(t)) = \prod_{j=1}^N \frac{1}{2^{K_j}} [(s_j^{2H_j} + t_j^{2H_j})^{K_j} - |t_j - s_j|^{2H_j K_j}]. \tag{4.16}$$

It follows from (4.16) that, similarly to an (N, d)-fractional Brownian sheet (cf. Xiao and Zhang [46] and Ayache and Xiao [5]), $B^{H,K}$ is operator-self-similar. However, it does not have convenient stochastic integral representations, which have played essential roles in the study of fractional Brownian sheets. Nevertheless, we will prove that the sample path properties of $B^{H,K}$ are very similar to those of fractional Brownian sheets and we can describe the anisotropic properties of $B^{H,K}$ in terms of the vectors H and K.

We start with the following useful lemma.

Lemma 4.8. *For any $\varepsilon > 0$, there exist positive and finite constants $c_{4,7}$ and $c_{4,8}$ such that for all $s, t \in [\varepsilon, 1]^N$,*

$$c_{4,7} \sum_{j=1}^{N} |s_j - t_j|^{2H_j K_j} \leq \mathbb{E}\big[\big(B_1^{H,K}(s) - B_1^{H,K}(t)\big)^2\big] \leq c_{4,8} \sum_{j=1}^{N} |s_j - t_j|^{2H_j K_j} \tag{4.17}$$

and

$$c_{4,7} \sum_{j=1}^{N} |s_j - t_j|^{2H_j K_j} \leq \det \mathrm{Cov}(B_1^{H,K}(s), B_1^{H,K}(t)) \leq c_{4,8} \sum_{j=1}^{N} |s_j - t_j|^{2H_j K_j}. \tag{4.18}$$

Here and in the sequel, $\det \mathrm{Cov}$ *denotes determinant of the covariance matrix.*

Proof. We will make use of the following, easily verifiable, fact. For any Gaussian random vector (Z_1, Z_2),

$$\det \mathrm{Cov}(Z_1, Z_2) = \mathrm{Var}(Z_1)\, \mathrm{Var}(Z_2|Z_1), \tag{4.19}$$

where $\mathrm{Var}(Z_1)$ and $\mathrm{Var}(Z_2|Z_1)$ denote the variance of Z_1 and the conditional variance of Z_2, given Z_1, respectively.

By (4.19), we see that for all $s, t \in [\varepsilon, 1]^N$,

$$\det \mathrm{Cov}(B_1^{H,K}(s), B_1^{H,K}(t)) = \mathbb{E}[B_1^{H,K}(s)^2]\, \mathrm{Var}(B_1^{H,K}(t)|B_1^{H,K}(s))$$

$$\leq \mathbb{E}[B_1^{H,K}(s)^2]\mathbb{E}\big[\big(B_1^{H,K}(s) - B_1^{H,K}(t)\big)^2\big]. \tag{4.20}$$

Since $\mathrm{Var}(B_1^{H,K}(s))$ is bounded from above and below by positive and finite constants, it is sufficient to prove the upper bound in (4.17) and the lower bound in (4.18).

When $N = 1$, Lemma 2.3, Proposition 2.1 and (4.19) collectively imply that both (4.17) and (4.18) hold. Next, we show that if the lemma holds for any $B^{H,K}$ with at most n parameters, then it holds for $B^{H,K}$ with $n + 1$ parameters.

We first verify the upper bound in (4.17). For any $s, t \in [\varepsilon, 1]^{n+1}$, let $s' = (s_1, \ldots, s_n, t_{n+1})$. We then have

$$\mathbb{E}\big[\big(B_1^{H,K}(s) - B_1^{H,K}(t)\big)^2\big] \leq 2\mathbb{E}\big[\big(B_1^{H,K}(s) - B_1^{H,K}(s')\big)^2\big]$$

$$+ 2\mathbb{E}\big[\big(B_1^{H,K}(s') - B_1^{H,K}(t)\big)^2\big]. \tag{4.21}$$

For the first term, we note that whenever $s_1, \ldots, s_n \in [\varepsilon, 1]$ are fixed, $B^{H,K}$ is a (rescaled) bifractional Brownian motion in s_{n+1}. Hence, Lemma 2.3 implies that the first term in the right-hand side of (4.21) is bounded by $c|t_n - s_n|^{2H_{n+1}K_{n+1}}$, where the constant c is independent of $s_1, \ldots, s_n \in [\varepsilon, 1]$. On the other hand, when $t_{n+1} \in [\varepsilon, 1]$ is fixed, $B^{H,K}$ is a (rescaled) (N, d)-bifractional Brownian sheet. Hence, the induction hypothesis implies that the second term in the right-hand side of (4.21) is bounded by $c \sum_{j=1}^{n} |t_j - s_j|^{2H_j K_j}$. This and (4.21) together prove the upper bound in (4.17).

Suppose that the lower bound in (4.18) holds for any $B^{H,K}$ with at most n parameters. For $N = n + 1$, we write $\det \operatorname{Cov}(B_1^{H,K}(s), B_1^{H,K}(t))$ as

$$\prod_{j=1}^{n+1} t_j^{2H_j K_j} s_j^{2H_j K_j} - \prod_{j=1}^{n+1} \frac{1}{2^{2K_j}} [(t_j^{2H_j} + s_j^{2H_j})^{K_j} - |t_j - s_j|^{2H_j K_j}]^2$$

$$= \prod_{j=2}^{n+1} t_j^{2H_j K_j} s_j^{2H_j K_j} \left\{ s_1^{2H_1 K_1} t_1^{2H_1 K_1} - \frac{1}{2^{2K_1}} [(t_1^{2H_1} + s_1^{2H_1})^{K_1} - |t_1 - s_1|^{2H_1 K_1}]^2 \right\}$$

$$+ \frac{1}{2^{2K_1}} [(t_1^{2H_1} + s_1^{2H_1})^{K_1} - |t_1 - s_1|^{2H_1 K_1}]^2$$

$$\times \left\{ \prod_{j=2}^{n+1} t_j^{2H_j K_j} s_j^{2H_j K_j} - \prod_{j=2}^{n+1} \frac{1}{2^{2K_j}} [(t_j^{2H_j} + s_j^{2H_j})^{K_j} - |t_j - s_j|^{2H_j K_j}]^2 \right\}$$

$$\geq c \sum_{j=1}^{n+1} |s_j - t_j|^{2H_j K_j}, \tag{4.22}$$

where the last inequality follows from the induction hypothesis. This proves the lower bound in (4.18). □

Applying Lemma 4.8, we can prove that many results in Xiao and Zhang [46] and Ayache and Xiao [5] on sample path properties of fractional Brownian sheet also hold for $B^{H,K}$. Theorem 4.9 is concerned with the existence of local times of $B^{H,K}$.

Theorem 4.9. *Let $B^{H,K} = \{B^{H,K}(t), t \in \mathbb{R}_+^N\}$ be an (N, d)-bifractional Brownian sheet with parameters $H \in (0, 1)^N$ and $K \in (0, 1]^N$. If $d < \sum_{j=1}^{N} \frac{1}{H_j K_j}$, then for any N-dimensional closed interval $I \subset (0, \infty)^N$, $B^{H,K}$ has a local time $L(x, I)$, $x \in \mathbb{R}^d$. Moreover, the local time admits the following L^2-representation:*

$$L(x, I) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\langle y, x \rangle} \int_I e^{i\langle y, B^{H,K}(s) \rangle} \, ds \, dy, \qquad x \in \mathbb{R}^d. \tag{4.23}$$

Remark 4.10. Although the existence of local times can also be proven by using the Malliavin calculus (see Proposition 4.15 below), we prefer to provide a Fourier analytic proof because

(1) we can compare the two methods in this way and (2) the above theorem also gives the representation (4.23).

Proof of Theorem 4.9. Without loss of generality, we may assume that $I = [\varepsilon, 1]^N$, where $\varepsilon > 0$. Let λ_N be the Lebesgue measure on I. We denote by μ the image measure of λ_N under the mapping $t \mapsto B^{H,K}(t)$. The Fourier transform of μ is then

$$\widehat{\mu}(\xi) = \int_I e^{i\langle \xi, B^{H,K}(t)\rangle} \, dt. \tag{4.24}$$

It follows from Fubini's theorem and (4.17) that

$$
\begin{aligned}
\mathbb{E} \int_{\mathbb{R}^d} |\widehat{\mu}(\xi)|^2 \, d\xi &= \int_I \int_I \int_{\mathbb{R}^d} \mathbb{E}\left(e^{i\langle \xi, B^{H,K}(s) - B^{H,K}(t)\rangle}\right) d\xi \, ds \, dt \\
&= c \int_I \int_I \frac{1}{[\mathbb{E}(B_1^{H,K}(s) - B_1^{H,K}(t))^2]^{d/2}} \, ds \, dt \\
&\leq c \int_I \int_I \frac{1}{[\sum_{j=1}^N |s_j - t_j|^{2H_j K_j}]^{d/2}} \, ds \, dt.
\end{aligned}
\tag{4.25}
$$

The same argument on page 214 of Xiao and Zhang [46] shows that the last integral is finite whenever $d < \sum_{j=1}^N \frac{1}{H_j K_j}$. Hence, in this case, $\widehat{\mu} \in L^2(\mathbb{R}^d)$ a.s. and Theorem 4.9 follows from the Plancherel theorem. $\qquad \square$

Remark 4.11. Recently, Ayache, Wu and Xiao [4] have shown that fractional Brownian sheets have jointly continuous local times based on "sectorial local non-determinism." It would be interesting to prove that $B^{H,K}$ is sectorially locally non-deterministic and to establish joint continuity and sharp Hölder conditions for the local times of $B^{H,K}$.

We now consider the Hausdorff and packing dimensions of the image, graph and level set of $B^{H,K}$. In order to state our theorems conveniently, we assume that

$$0 < H_1 K_1 \leq \cdots \leq H_N K_N < 1. \tag{4.26}$$

We denote packing dimension by $\dim_{\mathrm{P}}$; see Falconer [19] for its definition and properties. The following theorems can be proven by using Lemma 4.8 and the same arguments as in Section 3 of Ayache and Xiao [5]. We leave the details to the interested reader.

Theorem 4.12. *With probability* 1,

$$\dim_{\mathrm{H}} B^{H,K}([0,1]^N) = \dim_{\mathrm{P}} B^{H,K}([0,1]^N) = \min\left\{ d; \sum_{j=1}^N \frac{1}{H_j K_j} \right\} \tag{4.27}$$

and

$$
\dim_{\mathrm{H}} \mathrm{Gr}\, B^{H,K}([0,1]^N)
$$
$$
= \dim_{\mathrm{P}} \mathrm{Gr}\, B^{H,K}([0,1]^N) \tag{4.28}
$$
$$
= \begin{cases} \displaystyle\sum_{j=1}^{N} \frac{1}{H_j K_j}, & \text{if } \displaystyle\sum_{j=1}^{N} \frac{1}{H_j K_j} \le d, \\[2.5ex] \displaystyle\sum_{j=1}^{k} \frac{H_k K_k}{H_j K_j} + N - k + (1 - H_k K_k)d, & \text{if } \displaystyle\sum_{j=1}^{k-1} \frac{1}{H_j K_j} \le d < \sum_{j=1}^{k} \frac{1}{H_j K_j}, \end{cases}
$$

where $\sum_{j=1}^{0} \frac{1}{H_j K_j} := 0$.

Theorem 4.13. *Let* $L_x = \{t \in (0,\infty)^N : B^{H,K}(t) = x\}$ *be the level set of* $B^{H,K}$. *The following statements hold*:

   (i) *if* $\sum_{j=1}^{N} \frac{1}{H_j} < d$, *then for every* $x \in \mathbb{R}^d$, *we have* $L_x = \varnothing$ *a.s.*;

   (ii) *if* $\sum_{j=1}^{N} \frac{1}{H_j} > d$, *then for every* $x \in \mathbb{R}^d$ *and* $0 < \varepsilon < 1$, *with positive probability,*

$$
\dim_{\mathrm{H}}(L_x \cap [\varepsilon,1]^N) = \dim_{\mathrm{P}}(L_x \cap [\varepsilon,1]^N)
$$
$$
= \min\left\{ \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k - H_k d, 1 \le k \le N \right\}
$$
$$
= \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k - H_k d, \qquad \text{if } \sum_{j=1}^{k-1} \frac{1}{H_j} \le d < \sum_{j=1}^{k} \frac{1}{H_j}. \tag{4.29}
$$

4.4. A Malliavin calculus approach

Using the Malliavin calculus approach, we can study the local times of more general bifractional Brownian sheets. Consider the $(N \times d)$-matrices

$$
\overline{H} = (\overline{H}_1, \dots, \overline{H}_d) \quad \text{and} \quad \overline{K} = (\overline{K}_1, \dots, \overline{K}_d),
$$

where for any $i = 1, \dots, d$,

$$
\overline{H}_i = (H_{i,1}, \dots, H_{i,N}) \quad \text{and} \quad \overline{K}_i = (K_{i,1}, \dots, K_{i,N}),
$$

with $H_{i,j} \in (0,1)$ and $K_{i,j} \in (0,1]$ for every $i = 1, \dots, d$ and $j = 1, \dots, N$.

   We will say that the Gaussian field $B^{\overline{H},\overline{K}}$ is an (N,d)-*bifractional Brownian sheet* with indices $\overline{H}$ and $\overline{K}$ if

$$
B^{\overline{H},\overline{K}}(t) = (B^{\overline{H}_1}(t), \dots, B^{\overline{H}_d}(t)), \qquad t \in [0,\infty)^N,
$$

and for every $i = 1, \ldots, d$, the random field $\{B^{\overline{H_i}}(t), t \in \mathbb{R}_+^N\}$ is centered and has covariance function

$$\mathbb{E}(B^{\overline{H_i}, \overline{K_i}}(t) B^{\overline{H_i}, \overline{K_i}}(s)) = R^{\overline{H_i}, \overline{K_i}}(s, t) = \prod_{j=1}^N R^{H_{i,j}, K_{i,j}}(s_j, t_j).$$

As in Section 4.1, the local time $L(x, t)$ ($t \in \mathbb{R}_+^N$ and $x \in \mathbb{R}^d$) of $B^{\overline{H}, \overline{K}}$ is defined as the density of the occupation measure μ_t, defined by

$$\mu_t(A) = \int_{[0,t]} \mathbb{1}_A(B^{\overline{H}, \overline{K}}(s)) \, ds, \qquad A \in \mathcal{B}(\mathbb{R}^d).$$

Formally, we can write

$$L(x, t) = \int_{[0,t]} \delta_x(B^{\overline{H}, \overline{K}}(s)) \, ds,$$

where δ_x denotes the Dirac function and $\delta_x(B_s^{\overline{H}, \overline{K}})$ is therefore a distribution in the Watanabe sense (see Watanabe [42]).

We require some notation. For $x \in \mathbb{R}$, let $p_\sigma(x)$ be the centered Gaussian kernel with variance $\sigma > 0$. Also, consider the Gaussian kernel on $\mathbb{R}^d$ given by

$$p_\sigma^d(x) = \prod_{i=1}^d p_\sigma(x_i), \qquad x = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

Denote by $\mathbf{H}_n(x)$ the nth Hermite polynomial defined by $\mathbf{H}_0(x) = 1$ and, for $n \geq 1$,

$$\mathbf{H}_n(x) = \frac{(-1)^n}{n!} \exp\left(\frac{x^2}{2}\right) \frac{d^n}{dx^n} \exp\left(-\frac{x^2}{2}\right), \qquad x \in \mathbb{R}.$$

We will make use of the following technical lemma.

Lemma 4.14. *For any $H \in (0, 1)$ and $K \in (0, 1]$, let us define the function*

$$Q_{H,K}(z) = \frac{R^{H,K}(1, z)}{z^{HK}}, \qquad z \in (0, 1],$$

and $Q_{H,K}(0) = 0$. The function $Q_{H,K}$ then takes values in $[0, 1]$, $Q_{H,K}(1) = 1$ and it is strictly increasing. Moreover, there exists a constant $\delta > 0$ such that for all $z \in (1 - \delta, 1)$,

$$(Q_{H,K}(z))^n \leq \exp\left(-c(\delta, H, K) n (1 - z)^{2HK}\right). \tag{4.30}$$

Proof. Clearly, the Cauchy–Schwarz inequality implies that $0 \leq Q_{H,K}(z) \leq 1$. Let us prove that the function $Q_{H,K}$ is strictly increasing. By computing the derivative $Q'_{H,K}(z)$ and multiplying this by z^{HK+1}, we observe that it is sufficient to show that

$$(1 - z)^{2HK-1}(1 + z) - (1 + z^{2H})^{K-1}(1 - z^{2H}) > 0 \qquad \text{for all } z \in (0, 1). \tag{4.31}$$

If $HK \leq \frac{1}{2}$, since $(1 + z^{2H})^{K-1} \leq 1 + z$, the left-hand side of (4.31) can be minorized by $(1 + z^{2H})^K ((1 - z)^{2HK-1} - 1 + z^{2H})$ and this is positive since $(1 - z)^{2HK-1} \geq 1$.

If $HK > \frac{1}{2}$, we note that

$$
(1 - z)^{2HK-1}(1 + z) + (1 + z^{2H})^{K-1} z^{2H}
$$
$$
\geq (1 - z)(1 + z) + (1 + z^{2H})^{K-1} z^2
$$
$$
\geq (1 + z^{2H})^{K-1}(1 - z^2) + (1 + z^{2H})^{K-1} z^2
$$
$$
\geq (1 + z^{2H})^{K-1}
$$

and this implies (4.31). Concerning inequality (4.30), we note that

$$
Q_{H,K}(z)^n = \exp(n \log Q_{H,K}(z)) \geq \exp\big(-n\big(1 - Q_{H,K}(z)\big)\big).
$$

Now, by Taylor's formula,

$$
(1 + z^{2H})^K z^{-HK} \leq 2^K + c(H, K, \delta)(1 - z)^2
$$

and therefore

$$
Q_{H,K}(z) \leq 1 + c(H, K, \delta)(1 - z)^2 - \frac{1}{2^K}(1 - z)^{2HK}
$$
$$
\leq 1 + c(H, K, \delta)(1 - z)^{2HK} \delta^{2-2HK} - \frac{1}{2^K}(1 - z)^{2HK}.
$$

The conclusion follows as in the proof of Lemma 2 of Eddahbi *et al.* [18] since

$$
1 - Q_{H,K}(z) \geq \frac{1}{2^K}(1 - z)^{2HK}\big(1 - c(H, K, \delta)\big)
$$

for any $z \in (1 - \delta, 1)$ with δ close to zero and with $c(H, K, \delta)$ tending to zero as $\delta \to 0$. $\qquad\square$

The following proposition gives a chaotic expansion of the local time of the (N, d)-bifractional Brownian sheet. The stochastic integral $I_n(h)$ which appears below is the multiple Wiener–Itô integral of order n of the function h of nN variables with respect to an $(N, 1)$-bifractional Brownian motion with parameters $H = (H_1, \ldots, H_N)$ and $K = (K_1, \ldots, K_N)$. Recall that such integrals can be constructed in general on a Gaussian space; see, for example, Major [29] or Nualart [34]. We will only need the following isometry formula:

$$
\mathbb{E}\big(I_n\big(\mathbb{1}_{[0,t]}^{\otimes n}\big) I_m\big(\mathbb{1}_{[0,s]}^{\otimes m}\big)\big) = n! R^{H,K}(t, s)^n \mathbb{1}_{(n=m)}
$$
$$
= n! \prod_{j=1}^N (R^{H_j, K_j}(t_j, s_j))^n \mathbb{1}_{(n=m)}
$$

(4.32)

for all $s, t \in \mathbb{R}_+^N$.

Proposition 4.15. *For any $x \in \mathbb{R}^d$ and $t \in (0, \infty)^N$, the local time $L(x, t)$ admits the following chaotic expansion*

$$L(x, t) = \sum_{n_1, \ldots, n_d \geq 0} \int_{[0,t]} \prod_{i=1}^{d} \frac{p_{\underline{s}^{2\overline{H}_i \overline{K}_i}}(x_i)}{\underline{s}^{n_i \overline{H}_i \overline{K}_i}} \mathbf{H}_{n_i}\left(\frac{x_i}{\underline{s}^{\overline{H}_i}}\right) I_{n_i}^i\left(\mathbb{1}_{[0,s]}(\cdot)^{\otimes n_i}\right) \mathrm{d}s, \qquad (4.33)$$

where $\underline{s} = s_1 \cdots s_N$ and $\underline{s}^{\overline{H}_i \overline{K}_i} = \prod_{j=1}^{N} s_j^{H_{i,j} K_{i,j}}$. The $I_{n_i}^i$ denote the multiple Wiener–Itô stochastic integrals with respect to the independent N-parameter bifractional Brownian motion $B^{\overline{H}_i, \overline{K}_i}$.
   Moreover, if $\sum_{j=1}^{N} \frac{1}{H_j^* K_j^*} > d$, where $H_j^* = \max\{H_{i,j} : i = 1, \ldots, d\}$ and $K_j^* = \max\{K_{i,j} : i = 1, \ldots, d\}$, then $L(x, t)$ is a random variable in $L^2(\Omega)$.

Proof. The chaotic expression (4.33) can be obtained similarly as in Eddahbi *et al.* [18] or Russo and Tudor [37]. It is based on the approximation of the Dirac delta function by Gaussian kernels with variance converging to zero. Let us evaluate the $L^2(\Omega)$ norm of $L(x, t)$. By the independence of components and the isometry of multiple stochastic integrals, we obtain

$$\|L(x, t)\|_2^2 = \sum_{m \geq 0} \sum_{n_1 + \cdots + n_d = m} \int_{[0,t]} \mathrm{d}u \int_{[0,t]} \mathrm{d}v \prod_{i=1}^{d} \beta_{n_i}(u) \beta_{n_i}(v) R^{\overline{H}_i, \overline{K}_i}(u, v)^{n_i}, \qquad (4.34)$$

where

$$\beta_{n_i}(u) = \frac{p_{\underline{s}^{2\overline{H}_i \overline{K}_i}}(x_i)}{\underline{s}^{n_i \overline{H}_i \overline{K}_i}} \mathbf{H}_{n_i}\left(\frac{x_i}{\underline{s}^{\overline{H}_i \overline{K}_i}}\right).$$

By Propositions 3 and 6 of Imkeller *et al.* [23] (see also Lemma 11 of Eddahbi *et al.* [18]), we have the bound

$$\beta_{n_i}(u) \beta_{n_i}(v) \leq c_{4,9} \frac{1}{(n_i \vee 1)^{(8\beta-1)/6}} \frac{1}{\underline{u}^{n_i \overline{H}_i \overline{K}_i} \underline{v}^{n_i \overline{H}_i \overline{K}_i}} \qquad (4.35)$$

for any $\beta \in [\frac{1}{4}, \frac{1}{2})$. Using inequality (4.35), we derive from (4.34) that $\|L(x, t)\|_2^2$ is at most

$$c \sum_{m \geq 0} \sum_{n_1 + \cdots + n_d = m} \left(\prod_{i=1}^{d} \frac{1}{(n_i \vee 1)^{(8\beta-1)/6}}\right) \int_{[0,t]} \mathrm{d}u \int_{[0,u]} \mathrm{d}v \prod_{i=1}^{d} \prod_{j=1}^{N} \frac{R^{H_{i,j}, K_{i,j}}(u_j, v_j)^{n_i}}{(u_j v_j)^{n_i H_{i,j} K_{i,j}}}$$

$$= c \sum_{m \geq 0} \sum_{n_1 + \cdots + n_d = m} \left(\prod_{i=1}^{d} \frac{1}{(n_i \vee 1)^{(8\beta-1)/6}}\right) \prod_{j=1}^{N} \int_{0}^{t_j} u_j \, \mathrm{d}u_j \int_{0}^{1} \left(\prod_{i=1}^{d} Q_{H_{i,j}, K_{i,j}}(z)^{n_i}\right) \mathrm{d}z$$

$$= c_{4,10} \underline{t}^2 \sum_{m \geq 0} \sum_{n_1 + \cdots + n_d = m} \left(\prod_{i=1}^{d} \frac{1}{(n_i \vee 1)^{(8\beta-1)/6}}\right) \prod_{j=1}^{N} \int_{0}^{1} \left(\prod_{i=1}^{d} Q_{H_{i,j}, K_{i,j}}(z)^{n_i}\right) \mathrm{d}z, \qquad (4.36)$$

where we have used the changes of variable $u_j = u_j$ and $v_j = z_j u_j$. Using the above lemma and following along the lines of the proof of Lemma 2 of Eddahbi *et al.* [18], we can prove the bound

$$\int_0^1 \left(\prod_{i=1}^d Q_{H_{i,j}, K_{i,j}}(z)^{n_i} \right) dz \le c_{4,11} m^{-1/(2H_j^* K_j^*)}. \tag{4.37}$$

Here, $c_{4,11} = c_{4,11}(\overline{H}, \overline{K})$ depends on $\overline{H}$ and $\overline{K}$. Finally, (4.37) implies that

$$\|L(x,t)\|_2^2 \le c_{4,12} \sum_{m \ge 1} \left(\prod_{j=1}^N m^{-1/(2H_j^* K_j^*)} \right) \sum_{n_1 + \cdots + n_d = m} \left(\prod_{i=1}^d \frac{1}{(n_i \vee 1)^{(8\beta - 1)/6}} \right)$$

$$\le c_{4,13} \sum_{m \ge 1} m^{-\sum_{j=1}^N 1/(2H_j^* K_j^*) + d(1 - (8\beta - 1)/6) - 1}, \tag{4.38}$$

where $c_{4,12}$ and $c_{4,13}$ depend only on $\overline{H}, \overline{K}$ and t. The last series in (4.38) converges if

$$\sum_{j=1}^N \frac{1}{2H_j^* K_j^*} > d \left(1 - \frac{8\beta - 1}{6} \right). \tag{4.39}$$

To conclude, observe that by choosing β close to $\frac{1}{2}$, $\sum_{j=1}^N \frac{1}{H_j^* K_j^*} > d$ implies the required condition (4.39). □

We recall that a random variable $F = \sum_n I_n(f_n)$ belongs to the Watanabe space $\mathbb{D}^{\alpha,2}$ if

$$\|F\|_{\alpha,2}^2 := \sum_{n \ge 0} (1 + m)^\alpha \|I_n(f_n)\|_2^2 < \infty.$$

Corollary 4.16. *For every $t \in (0, \infty)^N$ and $x \in \mathbb{R}^d$, the local time $L(x,t)$ of the (N,d)-bifractional Brownian sheet $B^{H,K}$ belongs to the Watanabe space $\mathbb{D}^{\alpha,2}$ for every $0 < \alpha < \sum_{j=1}^N \frac{1}{2H_j^* K_j^*} - \frac{d}{2}$.*

Proof. This is a consequence of the proof of Proposition 4.15. Using the computation contained there, we obtain, for any $\beta \in [\frac{1}{4}, \frac{1}{2})$,

$$\|L(x,t)\|_{\alpha,2}^2 \le c_{4,14}(\overline{H}, \overline{K}, d, t) \sum_{m \ge 1} (1 + m)^\alpha m^{d(1 - (8\beta - 1)/6) - 1 - \sum_{j=1}^N 1/(2H_j^* K_j^*)},$$

which is convergent if $\alpha < \sum_{j=1}^N \frac{1}{2H_j^* K_j^*} - d(1 - \frac{8\beta - 1}{6}) - 1 - \sum_{j=1}^N \frac{1}{2H_j^* K_j^*}$. Choosing β close to $\frac{1}{2}$, we obtain the conclusion. □

Acknowledgements

References

[1] Addie, R., Mannersalo, P. and Norros, I. (2002). Performance formulae for queues with Gaussian input. *European Trans. Telecommunications* **13** 183–196.

[2] Anderson, T.W. (1955). The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities. *Proc. Amer. Math. Soc.* **6** 170–176. MR0069229

[3] Anh, V.V., Angulo, J.M. and Ruiz-Medina, M.D. (1999). Possible long-range dependence in fractional random fields. *J. Statist. Plann. Inference* **80** 95–110. MR1713795

[4] Ayache, A., Wu, D. and Xiao, Y. (2007). Joint continuity of the local times of fractional Brownian sheets. *Ann. Inst. H. Poincaré Probab. Statist.* To appear.

[5] Ayache, A. and Xiao, Y. (2005). Asymptotic properties and Hausdorff dimensions of fractional Brownian sheets. *J. Fourier Anal. Appl.* **11** 407–439. MR2169474

[6] Azaïs, J.M. and Wschebor, M. (1996). Almost sure oscillation of certain random processes. *Bernoulli* **2** 257–270. MR1416866

[7] Benassi, A., Bertrand, P. and Istas, J. (2000). Identification of the Hurst exponent of a step multifractional Brownian motion. *Statist. Inference Stoch. Process.* **13** 101–111. MR1819289

[8] Benson, D.A., Meerschaert, M.M. and Baeumer, B. (2006). Aquifer operator-scaling and the effect on solute mixing and dispersion. *Water Resour. Res.* **42** W01415.

[9] Berman, S.M. (1972). Gaussian sample function: Uniform dimension and Hölder conditions nowhere. *Nagoya Math. J.* **46** 63–86. MR0307320

[10] Berman, S.M. (1973). Local nondeterminism and local times of Gaussian processes. *Indiana Univ. Math. J.* **23** 69–94. MR0317397

[11] Bingham, N.H., Goldie, C.M. and Teugels, J.L. (1987). *Regular Variation*. Cambridge: Cambridge Univ. Press. MR0898871

[12] Bonami, A. and Estrade, A. (2003). Anisotropic analysis of some Gaussian models. *J. Fourier Anal. Appl.* **9** 215–236. MR1988750

[13] Cheridito, P. (2004). Gaussian moving averages, semimartingales and option pricing. *Stochastic Process. Appl.* **109** 47–68. MR2024843

[14] Cheridito, P., Kawaguchi, H. and Maejima, M. (2003). Fractional Ornstein–Uhlenbeck processes. *Electron. J. Probab.* **8** 1–14. MR1961165

[15] Cornfeld, I.P., Fomin, S.V. and Sinai, Ya.G. (1982). *Ergodic Theory*. New York: Springer. MR0832433

[16] Coutin, L., Nualart, D. and Tudor, C.A. (2001). Tanaka formula for the fractional Brownian motion. *Stochastic Process. Appl.* **94** 301–315. MR1840834

[17] Cuzick, J. and DuPreez, J. (1982). Joint continuity of Gaussian local times. *Ann. Probab.* **10** 810–817. MR0659550

[18] Eddahbi, M., Lacayo, R., Solé, J.L., Tudor, C.A. and Vives, J. (2005). Regularity of the local time for the d-dimensional fractional Brownian motion with N-parameters. *Stoch. Anal. Appl.* **23** 383–400. MR2130355

[19] Falconer, K.J. (1990). *Fractal Geometry – Mathematical Foundations and Applications*. Chichester: Wiley. MR1102677

[20] Geman, D. and Horowitz, J. (1980). Occupation densities. *Ann. Probab.* **8** 1–67. MR0556414

[21] Houdré, C. and Villa, J. (2003). An example of infinite dimensional quasi-helix. In *Stochastic Models (Mexico City, 2002)* 195–201. *Contemp. Math.* **336**. Providence, RI: Amer. Math. Soc. MR2037165

[22] Hu, Y.Z. and Oksendal, B. (2002). Chaos expansion of local time of fractional Brownian motions. *Stochastic Anal. Appl.* **20** 815–837. MR1921068

[23] Imkeller, P., Perez-Abreu, V. and Vives, J. (1995). Chaos expansion of double intersection local time of Brownian motion in $\mathbb{R}^d$ and renormalization. *Stoch. Process. Appl.* **56** 1–34. MR1324319

[24] Kasahara, Y., Kôno, N. and Ogawa, T. (1999). On tail probability of local times of Gaussian processes. *Stochastic Process. Appl.* **82** 15–21. MR1695067

[25] Kuelbs, J., Li, W.V. and Shao, Q.-M. (1995). Small ball probabilities for Gaussian processes with stationary increments under Hölder norms. *J. Theoret. Probab.* **8** 361–386. MR1325856

[26] Lamperti, J. (1962). Semi-stable stochastic processes. *Trans. Amer. Math. Soc.* **104** 64–78. MR0138128

[27] Li, W.V. and Shao, Q.-M. (2001). Gaussian processes: Inequalities, small ball probabilities and applications. In *Stochastic Processes: Theory and Methods. Handbook of Statistics* **19** (C.R. Rao and D. Shanbhag, eds.) 533–597. North-Holland. MR1861734

[28] Loéve, L. (1977). *Probability Theory I*. New York: Springer. MR0651017

[29] Major, P. (1981). *Multiple Wiener–Itô Integrals. Lecture Notes in Math.* **849**. Berlin: Springer. MR0611334

[30] Mannersalo, P. and Norros, I. (2002). A most probable path approach to queueing systems with general Gaussian input. *Comp. Networks* **40** 399–412.

[31] Maruyama, G. (1949). The harmonic analysis of stationary stochastic processes. *Mem. Fac. Sci. Kyushu Univ. A* **4** 45–106. MR0032127

[32] Monrad, D. and Pitt, L.D. (1987). Local nondeterminism and Hausdorff dimension. In *Progress in Probability and Statistics. Seminar on Stochastic Processes 1986* (E. Cinlar, K.L. Chung and R.K. Getoor, eds.) 163–189. Boston: Birkhäuser. MR0902433

[33] Monrad, D. and Rootzén, H. (1995). Small values of Gaussian processes and functional laws of the iterated logarithm. *Probab. Theory Related Fields* **101** 173–192. MR1318191

[34] Nualart, D. (1995). *Malliavin Calculus and Related Topics*. New York: Springer. MR1344217

[35] Pitman, E.J.G. (1968). On the behavior of the characteristic function of a probability distribution in the neighbourhood of the origin. *J. Australian Math. Soc. Series A* **8** 422–443. MR0231423

[36] Pitt, L.D. (1978). Local times for Gaussian vector fields. *Indiana Univ. Math. J.* **27** 309–330. MR0471055

[37] Russo, F. and Tudor, C.A. (2006). On the bifractional Brownian motion. *Stoch. Process. Appl.* **5** 830–856. MR2218338

[38] Samorodnitsky, G. and Taqqu, M.S. (1994). *Stable non-Gaussian Random Processes: Stochastic Models with Infinite Variance*. New York: Chapman and Hall. MR1280932

[39] Stolz, W. (1996). Some small ball probabilities for Gaussian processes under nonuniform norms. *J. Theoret. Probab.* **9** 613–630. MR1400590

[40] Takashima, K. (1989). Sample path properties of ergodic self-similar processes. *Osaka Math. J.* **26** 159–189. MR0991287

[41] Talagrand, M. (1995). Hausdorff measure of trajectories of multiparameter fractional Brownian motion. *Ann. Probab.* **23** 767–775. MR1334170

[42] Watanabe, S. (1984). *Lectures on Stochastic Differential Equations and Malliavin Calculus*. Berlin: Springer. MR0742628

[43] Wschebor, M. (1992). Sur les accroissements du processus de Wiener. *C. R. Acad. Sci. Paris* **315** 1293–1296. MR1194538

[44] Xiao, Y. (1997). Hölder conditions for the local times and the Hausdorff measure of the level sets of Gaussian random fields. *Probab. Theory Related Fields* **109** 129–157. MR1469923

[45] Xiao, Y. (2007). Strong local nondeterminism and the sample path properties of Gaussian random fields. In *Asymptotic Theory in Probability and Statistics with Applications* (T.L. Lai, Q. Shao and L. Qian, eds.) 136–176. Higher Education Press, Beijing.

[46] Xiao, Y. and Zhang, T. (2002). Local times of fractional Brownian sheets. *Probab. Theory Related Fields* **124** 204–226. MR1936017