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ON THE NUMBER OF CUSPS OF PERTURBATIONS OF COMPLEX POLYNOMIALS

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Abstract

Let f be a 1-variable complex polynomial such that f has an isolated singularity at the origin. In the present paper, we show that there exists a perturbation f_t of f which has only fold singularities and cusps as singularities of a real polynomial map from $\mathbf{R}^2$ to $\mathbf{R}^2$. We then calculate the number of cusps of f_t in a sufficiently small neighborhood of the origin and estimate the number of cusps of f_t in $\mathbf{R}^2$.

1. Introduction

Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be a smooth map which has fold singularities and cusps as singularities. We call such a map an *excellent map*. In [10], Whitney showed that the set of excellent maps is dense in $C^\infty(\mathbf{R}^2, \mathbf{R}^2)$.

It's known that there is a relation between the topology of surfaces and the topology of the critical locus of a map. Quine [8] and Fukuda-Ishikawa [2] studied the number of cusps of stable maps between oriented 2-manifolds. The degree of cusps of a stable map is determined by the topological degree of a stable map and the Euler characteristics of surfaces. Fukuda and Ishikawa also studied the number of cusps of stable perturbations of generic map germs [2]. They showed the number of cusps modulo 2 is a topological invariant of generic map germs. Moreover, the number of cusps modulo 2 depends only on the topology of surfaces. Krzyżanowska and Szafraniec gave a criterion to determine if a polynomial map is an excellent map or not [5]. They also gave an algorithm to compute the number of cusps of generic polynomial maps. In [9], Szafraniec considered bifurcations of cusps of families of plane-to-plane maps and presented an algebraic method for computing the number of cusps of analytic families. In holomorphic case, Gaffney and Mond gave an algebraic formula to count the number of cusps and nodes of a generic perturbation of finitely determined holomorphic map germs from $(\mathbf{C}^2, \mathbf{o})$ to $(\mathbf{C}^2, \mathbf{o})$, where $\mathbf{o}$ is the origin of $\mathbf{C}^2$ [3]. Farnik, Jelonek and Ruas described the number of cusps and nodes of generic complex polynomial maps [1].

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Let $f(z)$ be a complex polynomial such that $f(0) = 0$. Then there exist a positive integer k and a complex polynomial $g(z)$ such that $f(z) = z^k g(z)$ and $g(0) \neq 0$. We call k the *multiplicity of f at the origin.* In this paper, we study certain perturbations of complex polynomials and calculate explicitly the number of cusps of perturbations by using multiplicities of singularities of complex polynomials. We identify $\mathbf{C}$ with $\mathbf{R}^2$. Then $f(z)$ defines a real polynomial map

$$f : \mathbf{R}^2 \to \mathbf{R}^2, \quad (x, y) \mapsto (\Re f(x, y), \Im f(x, y)),$$

where $z = x + \sqrt{-1}y$. Assume that the origin 0 of $\mathbf{C}$ is a singularity of f. We define a *linear perturbation f_t of f* as follows:

$$f_t(z) := f(z) + t(a + ib)\bar{z},$$

where $a, b, t \in \mathbf{R}$, $i = \sqrt{-1}$ and $0 < |t| \ll 1$. Note that a linear perturbation f_t of f is not a complex polynomial, but is a 1-variable mixed polynomial in the sense of Oka [7]. We now regard a mixed polynomial map $f_t : \mathbf{C} \to \mathbf{C}$ as a real polynomial map $(\Re f_t, \Im f_t) : \mathbf{R}^2 \to \mathbf{R}^2$. If $f(z) = z^n$, Fukuda and Ishikawa showed that the number of cusps of a linear perturbation of f is congruent to $n + 1$ modulo 2, see [2, Example 2.3]. In [4], the author, Ishikawa, Kawashima and Nguyen showed that there exist linear perturbations of 2-variable Brieskorn polynomials which are excellent maps and we estimated the number of cusps of linear perturbations.

As we will show in Lemma 2, f_t is an excellent map for $0 < |t| \ll 1$ if a and b lie outside the union of zero sets of analytic functions determined by a, b and f. In particular, such a and b are generic. The main theorem is the following.

THEOREM 1. *Let $f(z)$ be a complex polynomial and k be the multiplicity of f at the origin. Suppose that $k \geq 2$. If a linear perturbation f_t of f is an excellent map for $0 < |t| \ll 1$, then the number of cusps of $f_t|_U$ is equal to $k + 1$, where U is a sufficiently small neighborhood of the origin.*

We estimate the number of cusps of f_t in $\mathbf{R}^2$.

COROLLARY 1. *Let f_t be a liner perturbation of a complex polynomial f in Theorem 1 and $n = \deg f$. Assume that $n \geq 2$. Then the number of cusps of f_t belongs to $[n + 1, 3n - 3]$. In particular, the number of cusps of f_t is at least three.*

This paper is organized as follows. In Section 2 we give the definition of excellent maps, a criterion to study generic polynomial mappings and introduce the notation of the multiplicity of roots of mixed polynomials. In Section 3 we show the existence of linear perturbations which are excellent maps. In Section 4 we prove Theorem 1 and give an example of a perturbation of a complex polynomial which has $(n + 1)$-cusps and also an example which has $(3n - 3)$-cusps.

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2. Preliminaries

2.1. Excellent maps. Let X and Y be 2-dimensional smooth manifolds. A smooth map $f: X \to Y$ is called an *excellent map* if for any $p \in X$, there exist local coordinates (x, y) centered at p and local coordinates centered at $f(p)$ such that f is locally described in one of the following forms:

(1) $(x, y) \mapsto (x, y)$,

(2) $(x, y) \mapsto (x, y^2)$,

(3) $(x, y) \mapsto (x, y^3 + xy)$.

A point p in the case (1) is a *regular point.* In the cases (2) (resp. (3)), a point p is called a *fold* (resp. a *cusp*).

We introduce the bundle $J^r(X, Y)$ of r-jets and its submanifolds $S_k(X, Y)$ and $S_1^2(X, Y)$ for $k = 1, 2$. For a smooth map $f: X \to Y$, a point p and a positive integer r, let $j^r f(p)$ be the r-jet of f at p. Set

$$J^r(X, Y) := \bigcup_{(p,q) \in X \times Y} J^r(X, Y, p, q),$$

where $J^r(X, Y, p, q) = \{j^r f(p) \mid f(p) = q\}$. The set $J^r(X, Y)$ is called *the bundle of r-jets of maps from X into Y*. The *r-extension* $j^r f: X \to J^r(X, Y)$ of f is defined by $p \mapsto j^r f(p)$, where $p \in X$. It is known that $J^r(X, Y)$ is a smooth manifold and the r-extension $j^r f$ of f is a smooth map. We define a submanifold of $J^1(X, Y)$ for $k = 1, 2$ as follows:

$$S_k(X, Y) = \{j^1 f(p) \in J^1(X, Y) \mid \operatorname{rank} df_p = 2 - k\}.$$

A smooth map $f: X \to Y$ is an excellent map if and only if

(1) $j^1 f$ is transversal to $S_1(X, Y)$ and $S_2(X, Y)$,

(2) $j^2 f$ is transversal to $S_1^2(X, Y)$,

where $S_1^2(X, Y)$ is defined as follows:

$$S_1^2(X, Y) = \left\{ j^2 f(p) \in J^2(X, Y) \;\middle|\; \begin{array}{l} j^1 f(p) \in S_1(X, Y), \\ j^1 f \text{ is transversal to } S_1(X, Y) \text{ at } p, \\ \operatorname{rank} d(f \mid S_1(f))(p) = 0 \end{array} \right\}.$$

Denote by $C^\infty(X, Y)$ the set of all smooth maps $X \to Y$ equipped with the C^∞-topology. It is known that the subset of smooth maps from X to Y which are excellent maps is open and dense in $C^\infty(X, Y)$ topologized with the C^∞-topology [6, 10].

2.2. Singularities of polynomial maps. Let $g = (g_1, g_2): U \to \mathbf{R}^2$ be a polynomial map, where U is an open set. Set $J = \dfrac{\partial(g_1, g_2)}{\partial(x, y)}$, $G_i = \dfrac{\partial(g_i, J)}{\partial(x, y)}$ for $i = 1, 2$.

We define the algebraic set G' as follows:

$$G' := \left\{ (x, y) \in U \mid J(x, y) = G_1(x, y) = G_2(x, y) = \frac{\partial(G_1, J)}{\partial(x, y)} = \frac{\partial(G_2, J)}{\partial(x, y)} = 0 \right\}.$$

In [5, Proposition 2] and [9, Proposition 2.2], Krzyżanowska and Szafraniec showed the following proposition:

Proposition 1. *The algebraic set G' is empty if and only if the set of singularities of g consists of either fold singularities or cusps. Moreover, the number of cusps of g is equal to the number of $\{(x, y) \in U \mid J(x, y) = G_1(x, y) = G_2(x, y) = 0\}$.*

2.3. Multiplicity with sign. Set $z = x + iy$. Then a pair of real polynomials (g_1, g_2) defines a mixed polynomial $g(z, \bar{z})$ as follows:

$$\begin{aligned} g(z, \bar{z}) &= g_1(x, y) + ig_2(x, y) \\ &= g_1\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) + ig_2\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right). \end{aligned}$$

Then $\frac{\partial g}{\partial z}$ and $\frac{\partial g}{\partial \bar{z}}$ satisfy the following equations:

$$\begin{aligned} \frac{\partial g}{\partial z} &= \frac{1}{2}\left(\frac{\partial g}{\partial x} - i\frac{\partial g}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}\right), \\ \frac{\partial g}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial g}{\partial x} + i\frac{\partial g}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial g_1}{\partial x} - \frac{\partial g_2}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial g_2}{\partial x} + \frac{\partial g_1}{\partial y}\right). \end{aligned}$$

Suppose that w is a mixed singularity of a mixed polynomial g, i.e., the gradient vectors of g_1 and g_2 at w are linearly dependent. Then we have

$$\left|\frac{\partial g}{\partial z}(w)\right| = \left|\frac{\partial g}{\partial \bar{z}}(w)\right|,$$

see [7]. Let $\alpha \in \mathbf{C}$ be an isolated root of $g(z, \bar{z}) = 0$. Put

$$S^1_\varepsilon(\alpha) := \{z \in \mathbf{C} \mid |z - \alpha| = \varepsilon\},$$

where ε is a sufficiently small positive real number. We define the *multiplicity with the sign of the root* α by the mapping degree of the normalized function

$$\frac{g}{|g|} : S^1_\varepsilon(\alpha) \to S^1.$$

We denote the multiplicity with the sign of the root α by $m_s(g, \alpha)$.

We say that α is a *positive simple root* if α satisfies

$$\left|\frac{\partial g}{\partial z}(\alpha)\right| > \left|\frac{\partial g}{\partial \bar{z}}(\alpha)\right|.$$

Similarly, α is a *negative simple root* if α satisfies

$$\left|\frac{\partial g}{\partial z}(\alpha)\right| < \left|\frac{\partial g}{\partial \bar{z}}(\alpha)\right|.$$

In [7, Proposition 15], α is a positive (resp. negative) simple root if and only if $m_s(g,\alpha) = 1$ (resp. $m_s(g,\alpha) = -1$).

Consider a bifurcation family $g_t(z,\bar{z}) = 0$ for $g_0 = g$ and $t \in \mathbf{R}$, i.e., g_t is a family of mixed polynomials which satisfies $g_0 = g$. Let $\{P_1(t),\dots,P_\nu(t)\}$ be the roots of $g_t(z,\bar{z}) = 0$ which are bifurcating from $z = \alpha$. Then we have

$$\sum_{j=1}^{\nu} m_s(g_t, P_j(t)) = m_s(g,\alpha),$$

see [7, Proposition 16].

3. The existence of linear perturbations which are excellent maps

Let $f(z)$ be a complex polynomial. Assume that $f(0) = 0$ and the origin of $\mathbf{C}$ is a singularity of f. Set $f_1 = \Re f$ and $f_2 = \Im f$. We take $a, b \in \mathbf{R}$. Then a linear perturbation f_t of f is defined by $f_t(z) = f(z) + t(a+ib)\bar{z}$, where $0 < |t| \ll 1$. Note that f_t is equal to

$$\begin{aligned} f_t(z) &= f(z) + t(a+ib)\bar{z} \\ &= f_1(z) + t(ax+by) + i\{f_2(z) + t(bx-ay)\}. \end{aligned}$$

Then f_t defines a real polynomial map from $\mathbf{R}^2$ to $\mathbf{R}^2$ as follows:

$$f_t : \mathbf{R}^2 \to \mathbf{R}^2, \quad (x,y) \mapsto (f_1(x,y) + t(ax+by), f_2(x,y) + t(bx-ay)).$$

We calculate J, G_1 and G_2 of f_t. By the Cauchy–Riemann equations $\dfrac{\partial f_2}{\partial x} = -\dfrac{\partial f_1}{\partial y}$ and $\dfrac{\partial f_2}{\partial y} = \dfrac{\partial f_1}{\partial x}$, J is modified as

$$\begin{aligned} J &= \det\begin{pmatrix} \dfrac{\partial f_1}{\partial x} + ta & \dfrac{\partial f_1}{\partial y} + tb \\ \dfrac{\partial f_2}{\partial x} + tb & \dfrac{\partial f_2}{\partial y} - ta \end{pmatrix} \\ &= \det\begin{pmatrix} \dfrac{\partial f_1}{\partial x} + ta & \dfrac{\partial f_1}{\partial y} + tb \\ -\dfrac{\partial f_1}{\partial y} + tb & \dfrac{\partial f_1}{\partial x} - ta \end{pmatrix} \\ &= \left(\frac{\partial f_1}{\partial x}\right)^2 + \left(\frac{\partial f_1}{\partial y}\right)^2 - t^2(a^2+b^2) \\ &= \left|\frac{\partial f}{\partial z}\right|^2 - t^2(a^2+b^2). \end{aligned}$$

Since f is a harmonic function, i.e., $\dfrac{\partial^2 f}{\partial z \partial \bar{z}} = 0$, $\dfrac{\partial^2 f_1}{\partial x \partial x} = -\dfrac{\partial^2 f_1}{\partial y \partial y}$ by the Cauchy–Riemann equations. Then we have

$$\begin{aligned}
G_1 &= \det \begin{pmatrix} \dfrac{\partial f_1}{\partial x} + ta & \dfrac{\partial f_1}{\partial y} + tb \\ \dfrac{\partial J}{\partial x} & \dfrac{\partial J}{\partial y} \end{pmatrix} \\
&= 2\left(\frac{\partial f_1}{\partial x} + ta\right)\left(\frac{\partial f_1}{\partial x}\frac{\partial^2 f_1}{\partial x \partial y} + \frac{\partial f_1}{\partial y}\frac{\partial^2 f_1}{\partial y \partial y}\right) - 2\left(\frac{\partial f_1}{\partial y} + tb\right)\left(-\frac{\partial f_1}{\partial x}\frac{\partial^2 f_1}{\partial y \partial y} + \frac{\partial f_1}{\partial y}\frac{\partial^2 f_1}{\partial x \partial y}\right) \\
&= 2\left(\left(\frac{\partial f_1}{\partial x}\right)^2 - \left(\frac{\partial f_1}{\partial y}\right)^2\right)\frac{\partial^2 f_1}{\partial x \partial y} + 4\frac{\partial f_1}{\partial x}\frac{\partial f_1}{\partial y}\frac{\partial^2 f_1}{\partial y \partial y} \\
&\quad + 2t\left\{a\left(\frac{\partial f_1}{\partial x}\frac{\partial^2 f_1}{\partial x \partial y} + \frac{\partial f_1}{\partial y}\frac{\partial^2 f_1}{\partial y \partial y}\right) - b\left(-\frac{\partial f_1}{\partial x}\frac{\partial^2 f_1}{\partial y \partial y} + \frac{\partial f_1}{\partial y}\frac{\partial^2 f_1}{\partial x \partial y}\right)\right\}.
\end{aligned}$$

By the same argument, G_2 is equal to

$$\begin{aligned}
G_2 &= \det \begin{pmatrix} -\dfrac{\partial f_1}{\partial y} + tb & \dfrac{\partial f_1}{\partial x} - ta \\ \dfrac{\partial J}{\partial x} & \dfrac{\partial J}{\partial y} \end{pmatrix} \\
&= 2\left(-\frac{\partial f_1}{\partial y} + tb\right)\left(\frac{\partial f_1}{\partial x}\frac{\partial^2 f_1}{\partial x \partial y} + \frac{\partial f_1}{\partial y}\frac{\partial^2 f_1}{\partial y \partial y}\right) \\
&\quad - 2\left(\frac{\partial f_1}{\partial x} - ta\right)\left(-\frac{\partial f_1}{\partial x}\frac{\partial^2 f_1}{\partial y \partial y} + \frac{\partial f_1}{\partial y}\frac{\partial^2 f_1}{\partial x \partial y}\right) \\
&= 2\left(\left(\frac{\partial f_1}{\partial x}\right)^2 - \left(\frac{\partial f_1}{\partial y}\right)^2\right)\frac{\partial^2 f_1}{\partial y \partial y} - 4\frac{\partial f_1}{\partial x}\frac{\partial f_1}{\partial y}\frac{\partial^2 f_1}{\partial x \partial y} \\
&\quad + 2t\left\{a\left(-\frac{\partial f_1}{\partial x}\frac{\partial^2 f_1}{\partial y \partial y} + \frac{\partial f_1}{\partial y}\frac{\partial^2 f_1}{\partial x \partial y}\right) + b\left(\frac{\partial f_1}{\partial x}\frac{\partial^2 f_1}{\partial x \partial y} + \frac{\partial f_1}{\partial y}\frac{\partial^2 f_1}{\partial y \partial y}\right)\right\}.
\end{aligned}$$

If G_1 and G_2 are equal to 0 at (x, y), then (x, y) satisfies the following equation:

$$\begin{aligned}
&\left\{\left(\left(\frac{\partial f_1}{\partial x}\right)^2 - \left(\frac{\partial f_1}{\partial y}\right)^2\right)\frac{\partial^2 f_1}{\partial x \partial y} + 2\frac{\partial f_1}{\partial x}\frac{\partial f_1}{\partial y}\frac{\partial^2 f_1}{\partial y \partial y}\right\} \\
&\qquad \times \left\{a\left(-\frac{\partial f_1}{\partial x}\frac{\partial^2 f_1}{\partial y \partial y} + \frac{\partial f_1}{\partial y}\frac{\partial^2 f_1}{\partial x \partial y}\right) + b\left(\frac{\partial f_1}{\partial x}\frac{\partial^2 f_1}{\partial x \partial y} + \frac{\partial f_1}{\partial y}\frac{\partial^2 f_1}{\partial y \partial y}\right)\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \left\{ \left(\left(\frac{\partial f_1}{\partial x} \right)^2 - \left(\frac{\partial f_1}{\partial y} \right)^2 \right) \frac{\partial^2 f_1}{\partial y \partial y} - 2 \frac{\partial f_1}{\partial x} \frac{\partial f_1}{\partial y} \frac{\partial^2 f_1}{\partial x \partial y} \right\} \\
&\quad \times \left\{ a \left(\frac{\partial f_1}{\partial x} \frac{\partial^2 f_1}{\partial x \partial y} + \frac{\partial f_1}{\partial y} \frac{\partial^2 f_1}{\partial y \partial y} \right) - b \left(- \frac{\partial f_1}{\partial x} \frac{\partial^2 f_1}{\partial y \partial y} + \frac{\partial f_1}{\partial y} \frac{\partial^2 f_1}{\partial x \partial y} \right) \right\}.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
(1) \qquad & a \Bigg[\left(-3 \left(\frac{\partial f_1}{\partial x} \right)^2 + \left(\frac{\partial f_1}{\partial y} \right)^2 \right) \frac{\partial f_1}{\partial y} \left\{ \left(\frac{\partial^2 f_1}{\partial y \partial y} \right)^2 - \left(\frac{\partial^2 f_1}{\partial x \partial y} \right)^2 \right\} \\
& \quad + 2 \left(- \left(\frac{\partial f_1}{\partial x} \right)^2 + 3 \left(\frac{\partial f_1}{\partial y} \right)^2 \right) \frac{\partial f_1}{\partial x} \frac{\partial^2 f_1}{\partial x \partial y} \frac{\partial^2 f_1}{\partial y \partial y} \Bigg] \\
& \quad + b \Bigg[\left(- \left(\frac{\partial f_1}{\partial x} \right)^2 + 3 \left(\frac{\partial f_1}{\partial y} \right)^2 \right) \frac{\partial f_1}{\partial x} \left\{ \left(\frac{\partial^2 f_1}{\partial y \partial y} \right)^2 - \left(\frac{\partial^2 f_1}{\partial x \partial y} \right)^2 \right\} \\
& \quad - 2 \left(-3 \left(\frac{\partial f_1}{\partial x} \right)^2 + \left(\frac{\partial f_1}{\partial y} \right)^2 \right) \frac{\partial f_1}{\partial y} \frac{\partial^2 f_1}{\partial x \partial y} \frac{\partial^2 f_1}{\partial y \partial y} \Bigg] \\
& = 0.
\end{aligned}
$$

Set real polynomials ϕ_1, ϕ_2 and Φ as follows:

$$
\begin{aligned}
\phi_1 &:= \left(-3 \left(\frac{\partial f_1}{\partial x} \right)^2 + \left(\frac{\partial f_1}{\partial y} \right)^2 \right) \frac{\partial f_1}{\partial y} \left\{ \left(\frac{\partial^2 f_1}{\partial y \partial y} \right)^2 - \left(\frac{\partial^2 f_1}{\partial x \partial y} \right)^2 \right\} \\
&\quad + 2 \left(- \left(\frac{\partial f_1}{\partial x} \right)^2 + 3 \left(\frac{\partial f_1}{\partial y} \right)^2 \right) \frac{\partial f_1}{\partial x} \frac{\partial^2 f_1}{\partial x \partial y} \frac{\partial^2 f_1}{\partial y \partial y}, \\
\phi_2 &:= \left(- \left(\frac{\partial f_1}{\partial x} \right)^2 + 3 \left(\frac{\partial f_1}{\partial y} \right)^2 \right) \frac{\partial f_1}{\partial x} \left\{ \left(\frac{\partial^2 f_1}{\partial y \partial y} \right)^2 - \left(\frac{\partial^2 f_1}{\partial x \partial y} \right)^2 \right\} \\
&\quad - 2 \left(-3 \left(\frac{\partial f_1}{\partial x} \right)^2 + \left(\frac{\partial f_1}{\partial y} \right)^2 \right) \frac{\partial f_1}{\partial y} \frac{\partial^2 f_1}{\partial x \partial y} \frac{\partial^2 f_1}{\partial y \partial y}, \\
\Phi &:= a\phi_1 + b\phi_2.
\end{aligned}
$$

Suppose that G_1 and G_2 are equal to 0 at (x, y). By the equation (1) and the definitions of ϕ_1, ϕ_2 and Φ, $\Phi(x, y)$ is also equal to 0. To show the existence of linear perturbations which are excellent maps, we consider the intersection of $\phi_1^{-1}(0)$ and $\phi_2^{-1}(0)$.

LEMMA 1. *Let U be a sufficiently small neighborhood of the origin* 0 *of* $\mathbf{C}$. *Assume that U satisfies* $\left\{ w \in U \,\middle|\, \frac{\partial f}{\partial z}(w) = 0 \right\} = \{0\}$ *and* $\left\{ w \in U \,\middle|\, \frac{\partial^2 f}{\partial z \partial z}(w) = 0 \right\} \subset \{0\}$. *Then the intersection of* $\phi_1^{-1}(0)$, $\phi_2^{-1}(0)$ *and U is equal to* $\{0\}$.

Proof. Let (x, y) be a point of $\phi_1^{-1}(0) \cap \phi_2^{-1}(0) \cap U$. Assume that (x, y) satisfies $\frac{\partial f_1}{\partial x}(x, y) = 0$ or $\left(\frac{\partial f_1}{\partial x}(x, y)\right)^2 - 3\left(\frac{\partial f_1}{\partial y}(x, y)\right)^2 = 0$. Then we have

$$\left(\frac{\partial f_1}{\partial y}(x, y)\right)^3 \left\{ \left(\frac{\partial^2 f_1}{\partial y \partial y}(x, y)\right)^2 - \left(\frac{\partial^2 f_1}{\partial x \partial y}(x, y)\right)^2 \right\} = 0,$$

$$\left(\frac{\partial f_1}{\partial y}(x, y)\right)^3 \frac{\partial^2 f_1}{\partial x \partial y}(x, y) \frac{\partial^2 f_1}{\partial y \partial y}(x, y) = 0.$$

By the above equations, (x, y) satisfies $\frac{\partial f_1}{\partial y}(x, y) = 0$ or $\frac{\partial^2 f_1}{\partial x \partial y}(x, y) = \frac{\partial^2 f_1}{\partial y \partial y}(x, y) = 0$. So (x, y) belongs to $\left\{ (x, y) \in U \,\middle|\, \frac{\partial f_1}{\partial x}(x, y) = \frac{\partial f_1}{\partial y}(x, y) = 0 \right\}$ or $\left\{ (x, y) \in U \,\middle|\, \frac{\partial^2 f_1}{\partial x \partial y}(x, y) = \frac{\partial^2 f_1}{\partial y \partial y}(x, y) = 0 \right\}$. By the assumption of U, (x, y) is equal to 0. Suppose that (x, y) satisfies $\frac{\partial f_1}{\partial y}(x, y) = 0$ or $-3\left(\frac{\partial f_1}{\partial x}(x, y)\right)^2 + \left(\frac{\partial f_1}{\partial y}(x, y)\right)^2 = 0$. By the same argument, we can check that (x, y) is equal to 0.

We assume that (x, y) satisfies $\frac{\partial f_1}{\partial x} \frac{\partial f_1}{\partial y} \neq 0$ and $\left\{ \left(\frac{\partial f_1}{\partial x}\right)^2 - 3\left(\frac{\partial f_1}{\partial y}\right)^2 \right\} \cdot \left\{ 3\left(\frac{\partial f_1}{\partial x}\right)^2 - \left(\frac{\partial f_1}{\partial y}\right)^2 \right\} \neq 0$ on U. Then $(x, y) \in \phi_1^{-1}(0) \cap \phi_2^{-1}(0) \cap U$ satisfies

$$2 \frac{\partial^2 f_1}{\partial x \partial y} \frac{\partial^2 f_1}{\partial y \partial y} = \frac{\left(-3\left(\frac{\partial f_1}{\partial x}\right)^2 + \left(\frac{\partial f_1}{\partial x}\right)^2\right) \frac{\partial f_1}{\partial y} \left\{ \left(\frac{\partial^2 f_1}{\partial y \partial y}\right)^2 - \left(\frac{\partial^2 f_1}{\partial x \partial y}\right)^2 \right\}}{-\left(-\left(\frac{\partial f_1}{\partial x}\right)^2 + 3\left(\frac{\partial f_1}{\partial y}\right)^2\right) \frac{\partial f_1}{\partial x}},$$

(2)

$$2 \frac{\partial^2 f_1}{\partial x \partial y} \frac{\partial^2 f_1}{\partial y \partial y} = \frac{\left(-\left(\frac{\partial f_1}{\partial x}\right)^2 + 3\left(\frac{\partial f_1}{\partial y}\right)^2\right) \frac{\partial f_1}{\partial x} \left\{ \left(\frac{\partial^2 f_1}{\partial y \partial y}\right)^2 - \left(\frac{\partial^2 f_1}{\partial x \partial y}\right)^2 \right\}}{\left(-3\left(\frac{\partial f_1}{\partial x}\right)^2 + \left(\frac{\partial f_1}{\partial y}\right)^2\right) \frac{\partial f_1}{\partial y}}.$$

By the above equations, (x, y) satisfies the following equation:

$$\left\{\left(-3\left(\frac{\partial f_1}{\partial x}\right)^2+\left(\frac{\partial f_1}{\partial y}\right)^2\right)^2\left(\frac{\partial f_1}{\partial y}\right)^2+\left(-\left(\frac{\partial f_1}{\partial x}\right)^2+3\left(\frac{\partial f_1}{\partial y}\right)^2\right)^2\left(\frac{\partial f_1}{\partial x}\right)^2\right\}$$
$$\times\left\{\left(\frac{\partial^2 f_1}{\partial y\partial y}\right)^2-\left(\frac{\partial^2 f_1}{\partial x\partial y}\right)^2\right\}=0.$$

Thus $\left(\frac{\partial^2 f_1}{\partial y\partial y}(x,y)\right)^2-\left(\frac{\partial^2 f_1}{\partial x\partial y}(x,y)\right)^2$ is equal to 0. By the equation (2), the second differentials $\frac{\partial^2 f_1}{\partial y\partial y}(x,y)$ and $\frac{\partial^2 f_1}{\partial x\partial y}(x,y)$ of f_1 are equal to 0. By the assumption of U, the intersection $\phi_1^{-1}(0)\cap\phi_2^{-1}(0)\cap U$ is equal to $\{0\}$. □

To study singularities of f_t, we define the mixed polynomial G_t as follows:

$$\begin{aligned}
G_t &:= G_1+iG_2\\
&= \det\begin{pmatrix}\frac{\partial f_1}{\partial x}+ta & \frac{\partial f_1}{\partial y}+tb\\ \frac{\partial J}{\partial x} & \frac{\partial J}{\partial y}\end{pmatrix}+i\det\begin{pmatrix}-\frac{\partial f_1}{\partial y}+tb & \frac{\partial f_1}{\partial x}-ta\\ \frac{\partial J}{\partial x} & \frac{\partial J}{\partial y}\end{pmatrix}\\
&= \left(\frac{\partial f_1}{\partial x}+ta+i\left(-\frac{\partial f_1}{\partial y}+tb\right)\right)\frac{\partial J}{\partial y}-\left(\frac{\partial f_1}{\partial y}+tb+i\left(\frac{\partial f_1}{\partial x}-ta\right)\right)\frac{\partial J}{\partial x}\\
&= \left(\frac{\partial f}{\partial z}+t(a+ib)\right)\frac{\partial J}{\partial y}-i\left(\frac{\partial f}{\partial z}-t(a+ib)\right)\frac{\partial J}{\partial x}.
\end{aligned}$$

Since $\frac{\partial J}{\partial z}$ is equal to $\frac{1}{2}\left(\frac{\partial J}{\partial x}-i\frac{\partial J}{\partial y}\right)$, $\frac{\partial J}{\partial x}$ and $\frac{\partial J}{\partial y}$ are equal to

$$\begin{aligned}
\frac{\partial J}{\partial x} &= 2\Re\frac{\partial J}{\partial z}=2\Re\frac{\partial^2 f}{\partial z\partial z}\overline{\frac{\partial f}{\partial z}}=\frac{\partial^2 f}{\partial z\partial z}\overline{\frac{\partial f}{\partial z}}+\overline{\frac{\partial^2 f}{\partial z\partial z}}\frac{\partial f}{\partial z},\\
\frac{\partial J}{\partial y} &= -2\Im\frac{\partial J}{\partial z}=-2\Im\frac{\partial^2 f}{\partial z\partial z}\overline{\frac{\partial f}{\partial z}}=i\left(\frac{\partial^2 f}{\partial z\partial z}\overline{\frac{\partial f}{\partial z}}-\overline{\frac{\partial^2 f}{\partial z\partial z}}\frac{\partial f}{\partial z}\right),
\end{aligned}$$

where $z=x+iy$. Thus G_t is equal to

$$\begin{aligned}
&i\left(\frac{\partial f}{\partial z}+t(a+ib)\right)\left(\frac{\partial^2 f}{\partial z\partial z}\overline{\frac{\partial f}{\partial z}}-\overline{\frac{\partial^2 f}{\partial z\partial z}}\frac{\partial f}{\partial z}\right)-i\left(\frac{\partial f}{\partial z}-t(a+ib)\right)\left(\frac{\partial^2 f}{\partial z\partial z}\overline{\frac{\partial f}{\partial z}}+\overline{\frac{\partial^2 f}{\partial z\partial z}}\frac{\partial f}{\partial z}\right)\\
&\quad= -2i\left(\frac{\partial f}{\partial z}\right)^2\overline{\frac{\partial^2 f}{\partial z\partial z}}+2ti(a+ib)\frac{\partial^2 f}{\partial z\partial z}\overline{\frac{\partial f}{\partial z}}.
\end{aligned}$$

Suppose that z satisfies $G_t(z) = 0$ and $\frac{\partial f}{\partial z}(z)\frac{\partial^2 f}{\partial z \partial z}(z) \neq 0$. By using the above equation, we have

$$\left|\frac{\partial f}{\partial z}(z)\right|^2 \left|\frac{\partial^2 f}{\partial z \partial z}(z)\right| = |t(a+ib)| \left|\frac{\partial^2 f}{\partial z \partial z}(z)\right| \left|\frac{\partial f}{\partial z}(z)\right|,$$
$$\left|\frac{\partial f}{\partial z}(z)\right| = |t(a+ib)|.$$

Thus z satisfies $J(z) = 0$. Since the multiplicity k of f at the origin is greater than 1, $G_t(0) = 0$ and $\frac{\partial f}{\partial z}(0)\frac{\partial^2 f}{\partial z \partial z}(0) = 0$. Thus we have

$$\left\{ z \in U \mid G_t(z) = 0, \frac{\partial f}{\partial z}(z) \neq 0, \frac{\partial^2 f}{\partial z \partial z}(z) \neq 0 \right\}$$
$$= \{ z \in U \backslash \{0\} \mid G_t(z) = 0 \} \subset J^{-1}(0).$$

Similarly, we define the following mixed polynomial:

$$H_t := \det \begin{pmatrix} \frac{\partial G_1}{\partial x} & \frac{\partial G_1}{\partial y} \\ \frac{\partial J}{\partial x} & \frac{\partial J}{\partial y} \end{pmatrix} + i \det \begin{pmatrix} \frac{\partial G_2}{\partial x} & \frac{\partial G_2}{\partial y} \\ \frac{\partial J}{\partial x} & \frac{\partial J}{\partial y} \end{pmatrix}$$
$$= \left(\frac{\partial G_1}{\partial x} + i\frac{\partial G_2}{\partial x} \right) \frac{\partial J}{\partial y} - \left(\frac{\partial G_1}{\partial y} + i\frac{\partial G_2}{\partial y} \right) \frac{\partial J}{\partial x}.$$

The differentials of G_t satisfy the following equations:

$$\frac{\partial G_t}{\partial z} = \frac{1}{2}\left(\frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y} \right) + \frac{i}{2}\left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} \right),$$
$$\frac{\partial G_t}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial G_1}{\partial x} - \frac{\partial G_2}{\partial y} \right) + \frac{i}{2}\left(\frac{\partial G_2}{\partial x} + \frac{\partial G_1}{\partial y} \right).$$

Then we have

$$H_t = \left(\frac{\partial G_t}{\partial z} + \frac{\partial G_t}{\partial \bar{z}} \right) \frac{\partial J}{\partial y} - i\left(\frac{\partial G_t}{\partial z} - \frac{\partial G_t}{\partial \bar{z}} \right) \frac{\partial J}{\partial x}$$
$$= \frac{\partial G_t}{\partial z}\left(\frac{\partial J}{\partial y} - i\frac{\partial J}{\partial x} \right) + \frac{\partial G_t}{\partial \bar{z}}\left(\frac{\partial J}{\partial y} + i\frac{\partial J}{\partial x} \right).$$

Since $\frac{\partial J}{\partial y} - i\frac{\partial J}{\partial x} = -2i\frac{\partial J}{\partial \bar{z}}$ and $\frac{\partial J}{\partial y} + i\frac{\partial J}{\partial x} = 2i\frac{\partial J}{\partial z}$, H_t is equal to

$$\begin{aligned}
H_t &= -2i\frac{\partial G_t}{\partial z}\frac{\partial J}{\partial \bar{z}} + 2i\frac{\partial G_t}{\partial \bar{z}}\frac{\partial J}{\partial z} \\
&= -2i\left\{-4i\frac{\partial f}{\partial z}\left|\frac{\partial^2 f}{\partial z\partial z}\right|^2 + 2ti(a+ib)\frac{\partial^3 f}{\partial z\partial z\partial z}\overline{\frac{\partial f}{\partial z}}\right\}\frac{\partial f}{\partial z}\overline{\frac{\partial^2 f}{\partial z\partial z}} \\
&\quad + 2i\left\{-2i\left(\frac{\partial f}{\partial z}\right)^2\overline{\frac{\partial^3 f}{\partial z\partial z\partial z}} + 2ti(a+ib)\left|\frac{\partial^2 f}{\partial z\partial z}\right|^2\right\}\frac{\partial^2 f}{\partial z\partial z}\overline{\frac{\partial f}{\partial z}} \\
&= -4\left(\frac{\partial f}{\partial z}\right)^2\frac{\partial^2 f}{\partial z\partial z}\left\{\overline{2\left(\frac{\partial^2 f}{\partial z\partial z}\right)^2 - \frac{\partial f}{\partial z}\frac{\partial^3 f}{\partial z\partial z\partial z}}\right\} \\
&\quad + 4t(a+ib)\overline{\frac{\partial f}{\partial z}\frac{\partial^2 f}{\partial z\partial z}}\left\{-\left(\frac{\partial^2 f}{\partial z\partial z}\right)^2 + \frac{\partial f}{\partial z}\frac{\partial^3 f}{\partial z\partial z\partial z}\right\}.
\end{aligned}$$

Note that $J(0) = \left|\frac{\partial f}{\partial z}(0)\right|^2 - t^2(a^2+b^2) \neq 0$ for $t \neq 0$ and $(a,b) \neq (0,0)$. By the definitions of G_t and H_t, we have

$$\{z \in U\backslash\{0\} \mid G_t(z) = H_t(z) = 0\} = \left\{z \in U \mid J(z) = G_1(z) = G_2(z) = \frac{\partial(G_1, J)}{\partial(x,y)}(z) = \frac{\partial(G_2, J)}{\partial(x,y)}(z) = 0\right\}.$$

We show the existence of a linear perturbation f_t of f which is an excellent map for generic (a,b).

LEMMA 2. *For a generic choice of (a,b), $f_t|_U$ is an excellent map.*

Proof. We will show that there exists a perturbation f_t of f such that $\{z \in U\backslash\{0\} \mid G_t(z) = H_t(z) = 0\}$ is empty. If $z \in U\backslash\{0\}$ satisfies $G_t(z) = 0$, we have

$$t(a+ib) = \frac{\left(\frac{\partial f}{\partial z}(z)\right)^2\overline{\frac{\partial^2 f}{\partial z\partial z}(z)}}{\frac{\partial^2 f}{\partial z\partial z}(z)\overline{\frac{\partial f}{\partial z}(z)}}.$$

Assume that $z \in U\backslash\{0\}$ satisfies $G_t(z) = H_t(z) = 0$. By using the above equation, we have

$$
\begin{aligned}
&\left(\frac{\partial f}{\partial z}(z)\right)^2 \left(\frac{\partial^2 f}{\partial z\partial z}(z)\right)^2 \overline{\frac{\partial f}{\partial z}(z)}\, \overline{\left\{2\left(\frac{\partial^2 f}{\partial z\partial z}(z)\right)^2 - \frac{\partial f}{\partial z}(z)\frac{\partial^3 f}{\partial z\partial z\partial z}(z)\right\}} \\
&\quad - \left(\frac{\partial f}{\partial z}(z)\right)^2 \overline{\frac{\partial f}{\partial z}(z)\left(\frac{\partial^2 f}{\partial z\partial z}(z)\right)^2} \left\{-\left(\frac{\partial^2 f}{\partial z\partial z}(z)\right)^2 + \frac{\partial f}{\partial z}(z)\frac{\partial^3 f}{\partial z\partial z\partial z}(z)\right\} \\
&= \left(\frac{\partial f}{\partial z}(z)\right)^2 \overline{\frac{\partial f}{\partial z}(z)}\psi(z,\bar{z}) \\
&= 0,
\end{aligned}
$$

where $\psi(z,\bar{z}) = 3\left|\frac{\partial^2 f}{\partial z\partial z}\right|^4 - \left(\frac{\partial^2 f}{\partial z\partial z}\right)^2 \overline{\frac{\partial f}{\partial z}\frac{\partial^3 f}{\partial z\partial z\partial z}} - \left(\overline{\frac{\partial^2 f}{\partial z\partial z}}\right)^2 \frac{\partial f}{\partial z}\frac{\partial^3 f}{\partial z\partial z\partial z}$. Note that ψ is a real-valued polynomial function. Set $f(z) = cz^k + (\text{higher terms})$, where k is the multiplicity of f at the origin. Then ψ has the following form:

$$\psi = (k+1)k^4(k-1)^3|c|^4|z|^{4k-8} + (\text{higher terms}).$$

If $k = 2$, then $\psi(0,0)$ is not equal to 0 and $\left(\frac{\partial f}{\partial z}(z)\right)^2 \overline{\frac{\partial f}{\partial z}(z)}\psi(z,\bar{z})$ is equal to

$$
\begin{aligned}
\left(\frac{\partial f}{\partial z}(z)\right)^2 \overline{\frac{\partial f}{\partial z}(z)}\psi(z,\bar{z}) &= (2cz + (\text{higher terms}))^2(2\bar{c}\bar{z} + (\text{higher terms})) \\
&\quad \times (\psi(0,0) + (\text{higher terms})) \\
&= z^2\bar{z}(2c + (\text{higher terms}))^2(2\bar{c} + (\text{higher terms})) \\
&\quad \times (\psi(0,0) + (\text{higher terms})).
\end{aligned}
$$

Since U is sufficiently small, $\left\{z \in U\backslash\{0\} \,\middle|\, \left(\frac{\partial f}{\partial z}(z)\right)^2 \overline{\frac{\partial f}{\partial z}(z)}\psi(z,\bar{z}) = 0\right\}$ is the empty set. So $\{z \in U\backslash\{0\} \mid G_t(z) = H_t(z) = 0\}$ is empty.

We assume that $k \geq 3$. Then $\psi(z,\bar{z}) \not\equiv 0$ and $\psi(0,0) = 0$. Thus $\psi^{-1}(0)$ is a 1-dimensional algebraic set. Since $\psi^{-1}(0)$ is a 1-dimensional algebraic set, $\psi^{-1}(0)$ has finitely many branches which depend only on $f(z)$. On U, each branch of $\psi^{-1}(0)$ is given by a convergent power series

$$\xi_m(u) = \left(\sum_{\ell} c_{1,\ell}u^{\ell}, \sum_{\ell} c_{2,\ell}u^{\ell}\right),$$

where $0 \leq u \ll 1$ for $m = 1,\dots,d$. By Lemma 1, the set $\{u \neq 0 \mid \phi_1(\xi_m(u)) = \phi_2(\xi_m(u)) = 0\}$ is empty for $m = 1,\dots,d$. Since $\psi^{-1}(0) \cap U$ has finitely many branches, we can choose coefficients a and b of Φ such that

$$\Phi(\xi_m(u)) = a\phi_1(\xi_m(u)) + b\phi_2(\xi_m(u)) \neq 0$$

for $0 < u \ll 1$ and $m = 1, \dots, d$. Thus the intersection of $\{z \in U\backslash\{0\} \mid G_1(z) = G_2(z) = 0\}$ and $\left\{z \in U\backslash\{0\} \,\middle|\, \frac{\partial(G_1, J)}{\partial(x, y)}(z) = \frac{\partial(G_2, J)}{\partial(x, y)}(z) = 0\right\}$ is empty. By Proposition 1, the set of singularities of f_t consists of either fold singularities or cusps. Therefore, $f_t|_U$ is an excellent map when (a, b) satisfies $\Phi(\xi_m(u)) \neq 0$ for $0 < u \ll 1$ and $m = 1, \dots, d$. □

Let w be a singularity of f and U_w be a sufficiently small neighborhood of w. By changing coordinates of U_w and $f(U_w)$, we may assume that $w = 0$ and $f(w) = 0$. So we can apply Lemma 2 to any singularity of f. Thus we can check that f_t is an excellent map for $0 < |t| \ll 1$ if a and b are generic.

4. Proof of Theorem 1

To calculate the number of cusps of f_t, we study zero points of G_t.

LEMMA 3. *The set $\{z \in U \mid G_t(z) = 0,\ z \neq 0\}$ is the set of positive simple roots of G_t for $(a, b) \neq (0, 0)$ and $0 < |t| \ll 1$.*

Proof. If z is a singularity of G_t, z satisfies $\left|\frac{\partial G_t}{\partial z}(z)\right| = \left|\frac{\partial G_t}{\partial \bar{z}}(z)\right|$, see [7, Proposition 15]. Hence we have

$$\left|-4i\frac{\partial f}{\partial z}\left|\frac{\partial^2 f}{\partial z \partial z}\right|^2 + 2ti(a+ib)\frac{\partial^3 f}{\partial z \partial z \partial z}\,\overline{\frac{\partial f}{\partial z}}\right| = \left|-2i\left(\frac{\partial f}{\partial z}\right)^2\overline{\frac{\partial^3 f}{\partial z \partial z \partial z}} + 2ti(a+ib)\left|\frac{\partial^2 f}{\partial z \partial z}\right|^2\right|.$$

Assume that z belongs to $G_t^{-1}(0)$. By the definition of G_t and the above equation, we have

$$\left|2\frac{\partial^2 f}{\partial z \partial z}\left|\frac{\partial f}{\partial z}\right|^2\left|\frac{\partial^2 f}{\partial z \partial z}\right|^2 - \frac{\partial f}{\partial z}\frac{\partial^3 f}{\partial z \partial z \partial z}\left|\frac{\partial f}{\partial z}\right|^2\overline{\frac{\partial^2 f}{\partial z \partial z}}\right| \tag{3}$$
$$= \left|\frac{\partial f}{\partial z}\frac{\partial^2 f}{\partial z \partial z}\left|\frac{\partial f}{\partial z}\right|^2\overline{\frac{\partial^3 f}{\partial z \partial z \partial z}} - \left(\frac{\partial f}{\partial z}\right)^2\left|\frac{\partial^2 f}{\partial z \partial z}\right|^2\overline{\frac{\partial^2 f}{\partial z \partial z}}\right|.$$

Let k be the multiplicity of f at the origin. Then $f(z)$ has the following form:

$$f(z) = cz^k + (\text{higher terms}).$$

By the equation (3), we have

$$|z^{5k-8}|\,|k^6(k-1)^2c|c|^4 + (\text{higher terms})|$$
$$= |z^{5k-8}|\,|-k^5(k-1)^2c|c|^4 + (\text{higher terms})|.$$

Note that k is greater than 1. The positive integer k satisfies $k^6(k-1)^2c|c|^4 > k^5(k-1)^2c|c|^4$. Since U is sufficiently small, the above equation does not hold in $U\backslash\{0\}$. So we can show that

$$\left|\frac{\partial G_t}{\partial z}(z)\right| > \left|\frac{\partial G_t}{\partial \bar{z}}(z)\right|$$

for any $z \in (U\backslash\{0\}) \cap G_t^{-1}(0)$. Thus zero points of G_t except for the origin are positive simple. □

Assume that f_t is an excellent map for $0 < |t| \ll 1$. We prove Theorem 1.

Proof of Theorem 1. By Proposition 1, the number of cusps of $f_t|_U$ is equal to the number of $\{z \in U \mid G_t(z) = 0, z \neq 0\}$. Set $\{z \in U \mid G_t(z) = 0, z \neq 0\} = \{w_1, \dots, w_v\}$. We denote the multiplicity of sign by $m_s(G_t, w_j)$ for $j = 1, \dots, v$. By [7, Proposition 16] and Lemma 3, we have

$$\left(\sum_{j=1}^{v} m_s(G_t, w_j)\right) + m_s(G_t, 0) = v + m_s(G_t, 0) = m_s(G_0, 0).$$

The multiplicity $m_s(G_0, 0)$ is equal to

$$\begin{aligned}\deg&\left(-2i\left(\frac{\partial f}{\partial z}\right)^2\overline{\frac{\partial^2 f}{\partial z \partial z}}\Big/\left|-2i\left(\frac{\partial f}{\partial z}\right)^2\overline{\frac{\partial^2 f}{\partial z \partial z}}\right| : S_\varepsilon^1(0) \to S^1\right)\\ &= 2(k-1) - (k-2) = k,\end{aligned}$$

where $S_\varepsilon^1(0) = \{z \in U \mid |z| = \varepsilon\}$ and $0 < \varepsilon \ll 1$. By the definition of G_t, for any $t \neq 0$, $m_s(G_t, 0)$ is equal to

$$\begin{aligned}\deg&\left(2ti(a+ib)\frac{\partial^2 f}{\partial z \partial z}\overline{\frac{\partial f}{\partial z}}\Big/\left|2ti(a+ib)\frac{\partial^2 f}{\partial z \partial z}\overline{\frac{\partial f}{\partial z}}\right| : S_{\varepsilon_t}^1(0) \to S^1\right)\\ &= k-2-(k-1) = -1,\end{aligned}$$

where $0 < \varepsilon_t \ll \varepsilon$. Thus the number of cusps of $f_t|_U$ is equal to $k+1$. □

Proof of Corollary 1. Set $\dfrac{\partial f}{\partial z} = n\prod_{j=1}^{\ell}(z - w_j)^{m_j}$. Let U_j be a sufficiently small neighborhood of w_j. By the same argument as in the proof of Theorem 1, the number of cusps of $f_t|_{U_j}$ is equal to $m_j + 2$. Note that $\sum_{j=1}^{\ell} m_j = n-1$. Then the number of cusps of f_t is equal to

$$\begin{aligned}\sum_{j=1}^{\ell}(m_j+2) &= \left(\sum_{j=1}^{\ell} m_j\right) + 2\ell\\ &= n - 1 + 2\ell.\end{aligned}$$

Since the number ℓ of singularities of f belongs to $[1, n-1]$, the number of cusps of f_t belongs to $[n+1, 3n-3]$.

By the change of coordinates, we may assume that the origin 0 is a singularity of f and $f(0) = 0$. Then the multiplicity of f at 0 is greater than 1. By Theorem 1, the number of cusps of $f_t|_U$ is at least three, where U is a sufficiently small neighborhood of 0. □

We construct a perturbation of a complex polynomial which has $(n+1)$-cusps and also a perturbation which has $(3n-3)$-cusps.

Example 1. Let $f(z) = z^n$ and $f_t(z) = z^n + t(a+ib)\bar{z}$ be a perturbation of f which is an excellent map. Then $G_t(z)$ is equal to

$$\begin{aligned} G_t(z) &= -2in^3(n-1)z^{2n-2}\bar{z}^{n-2} + 2tn^2(n-1)(a+ib)z^{n-2}\bar{z}^{n-1} \\ &= -2in^2(n-1)|z|^{2n-4}\{nz^n - t(a+ib)\bar{z}\}. \end{aligned}$$

Set $z = re^{i\theta}$ and $a + ib = \tau e^{i\iota}$, where $\tau > 0$. Then we have

$$-2in^2(n-1)r^{2n-4}\{nr^n e^{ni\theta} - t\tau r e^{i(\iota-\theta)}\}.$$

Assume that $z \neq 0$ and $G_t(z) = 0$. Then z satisfies

$$r = \left(\frac{t\tau}{n}\right)^{1/(n-1)}, \quad \theta = \frac{\iota + 2j\pi}{n+1},$$

for $j = 0, \dots, n$. Thus the number of cusps of f_t is equal to $n+1$.

Example 2. Let $f(z) = z^n + z$. Then the number of singularities of f is equal to $n-1$ and the multiplicity at each singularity of f is equal to 2. Let $f_t(z) = z^n + z + t(a+ib)\bar{z}$ be a perturbation of f which is an excellent map. By the same argument as in the proof of Corollary 1, the number of cusps of f_t is equal to $3n-3$.

4.1. Perturbations of f_t. Let f_t be a linear perturbation of f which is an excellent map. We fix a, b and t. Let $g(z, \bar{z})$ be a mixed polynomial which satisfies $\dfrac{\partial g}{\partial z}(0) = \dfrac{\partial g}{\partial \bar{z}}(0) = 0$. In this subsection, we study a perturbation of f_t:

$$f_{t,s}(z) := f(z) + t(a+ib)\bar{z} + sg(z, \bar{z}),$$

where $0 < |s| \ll |t| \ll 1$.

THEOREM 2. *The set of singularities of $f_{t,s}$ consists of either fold singularities or cusps and the number of cusps of $f_{t,s}$ is constant for $0 \leq |s| \ll |t| \ll 1$.*

Proof. Let w be a singularity of f and U_w be a sufficiently small neighborhood of w. Set $J_s = \dfrac{\partial(\Re f_{t,s}, \Im f_{t,s})}{\partial(x,y)}$, $G_{1,s} = \dfrac{\partial(\Re f_{t,s}, J_s)}{\partial(x,y)}$ and $G_{2,s} = \dfrac{\partial(\Im f_{t,s}, J_s)}{\partial(x,y)}$.

We define mixed polynomials $G_{t,s}$ and $H_{t,s}$ as follows:

$$G_{t,s} := G_{1,s} + iG_{2,s}, \quad H_{t,s} := \frac{\partial(G_{1,s}, J_s)}{\partial(x, y)} + i\frac{\partial(G_{2,s}, J_s)}{\partial(x, y)}.$$

Since the set of singularities of $f_{t,0}$ consists of either fold singularities or cusps, we have

$$\{z \in U_w \mid J_0(z) = G_{t,0} = H_{t,0} = 0\} = \emptyset.$$

Then there exists a positive real number s_0 such that

$$\{z \in U_w \mid J_s(z) = G_{t,s} = H_{t,s} = 0\} = \emptyset,$$

for any $0 \le |s| \le s_0$. Thus any singularity of $f_{t,s}$ is a fold singularity or a cusp. By the definition of $J_{t,s}$, the origin 0 is a regular point of $f_{t,s}$. Since the set $\{z \in \mathbf{C} \mid G_{t,0}(z) = 0, z \neq 0\}$ is the set of positive simple roots of $f_{t,0}$, $\{z \in \mathbf{C} \mid G_{t,s}(z) = 0, z \neq 0\}$ is also the set of positive simple roots of $f_{t,s}$ for $0 \le |s| \le s_0$. By [7, Proposition 16], the number of cusps of $f_{t,s}$ is constant for $0 \le |s| \le s_0$. $\square$

4.2. Lower bounds of the numbers of cusps of non-linear perturbations. Let $h(z, \bar{z})$ be a mixed polynomial which satisfies $h(0) = 0$ and $\left|\frac{\partial h}{\partial z}(0)\right| \neq \left|\frac{\partial h}{\partial \bar{z}}(0)\right|$. We define a perturbation $f_{t,h}$ of a complex polynomial f as follows:

$$f_{t,h}(z) := f(z) + th(z, \bar{z}),$$

where $0 < |t| \ll 1$. Set $h_1 = \Re h$, $h_2 = \Im h$ and

$$J_{t,h} = \det\begin{pmatrix} \frac{\partial f_1}{\partial x} + t\frac{\partial h_1}{\partial x} & \frac{\partial f_1}{\partial y} + t\frac{\partial h_1}{\partial y} \\ -\frac{\partial f_1}{\partial y} + t\frac{\partial h_2}{\partial x} & \frac{\partial f_1}{\partial x} + t\frac{\partial h_2}{\partial y} \end{pmatrix}.$$

Then any singularity of $f_{t,h}$ belongs to $J_{t,h}^{-1}(0)$. Assume that $f_{t,h}$ satisfies the following conditions:

(i) $f_{t,h}$ is an excellent map for $0 < |t| \ll 1$,

(ii) any cusp of $f_{t,h}$ is a simple root of $G_{t,h}$, where

$$G_{t,h} := \det\begin{pmatrix} \frac{\partial f_1}{\partial x} + t\frac{\partial h_1}{\partial x} & \frac{\partial f_1}{\partial y} + t\frac{\partial h_1}{\partial y} \\ \frac{\partial J_{t,h}}{\partial x} & \frac{\partial J_{t,h}}{\partial y} \end{pmatrix} + i\det\begin{pmatrix} -\frac{\partial f_1}{\partial y} + t\frac{\partial h_2}{\partial x} & \frac{\partial f_1}{\partial x} + t\frac{\partial h_2}{\partial y} \\ \frac{\partial J_{t,h}}{\partial x} & \frac{\partial J_{t,h}}{\partial y} \end{pmatrix}$$

$$= -2i\left(\frac{\partial f}{\partial z} + t\frac{\partial h}{\partial z}\right)\overline{\frac{\partial J_{t,h}}{\partial z}} + 2ti\frac{\partial h}{\partial \bar{z}}\frac{\partial J_{t,h}}{\partial z}.$$

Assume that w belongs to $J_{t,h}^{-1}(0) \cap \left(\frac{\partial J_{t,h}}{\partial z}\right)^{-1}(0)$. By the definition of $G_{t,h}$, $G_{t,h}(w)$ is equal to 0. Since $\frac{\partial J_{t,h}}{\partial z}(w) = \frac{1}{2}\left(\frac{\partial J_{t,h}}{\partial x}(w) - i\frac{\partial J_{t,h}}{\partial y}(w)\right) = 0$, $\frac{\partial(\Re G_{t,h}, J_{t,h})}{\partial(x, y)}(w)$ and $\frac{\partial(\Im G_{t,h}, J_{t,h})}{\partial(x, y)}(w)$ are equal to 0. So w belongs to G'. Since $f_{t,h}$ is an excellent map, G' is empty by Proposition 1. This is a contradiction. Thus the intersection of $J_{t,h}^{-1}(0)$ and $\left(\frac{\partial J_{t,h}}{\partial z}\right)^{-1}(0)$ is empty. Let U be a sufficiently small neighborhood of the origin. Suppose that $z \in U$ satisfies $z \notin \left(\frac{\partial J_{t,h}}{\partial z}\right)^{-1}(0)$ and $G_{t,h}(z) = -2i\left(\frac{\partial f}{\partial z}(z) + t\frac{\partial h}{\partial z}(z)\right)\overline{\frac{\partial J_{t,h}}{\partial z}}(z) + 2ti\frac{\partial h}{\partial \bar{z}}(z)\frac{\partial J_{t,h}}{\partial z}(z) = 0$. Then z satisfies

$$\frac{\partial f}{\partial z}(z) + t\frac{\partial h}{\partial z}(z) = \frac{\frac{\partial J_{t,h}}{\partial z}(z)}{\overline{\frac{\partial J_{t,h}}{\partial z}}(z)}t\frac{\partial h}{\partial \bar{z}}(z),$$

$$\left|\frac{\partial f_{t,h}}{\partial z}(z)\right| = \left|\frac{\partial f_{t,h}}{\partial \bar{z}}(z)\right|.$$

Thus z is a singularity of $f_{t,h}$, i.e., $z \in J_{t,h}^{-1}(0)$. Then the number of cusps of $f_{t,h}|_U$ is equal to the number of $\left\{z \in U \mid G_{t,h}(z) = 0, \frac{\partial J_{t,h}}{\partial z}(z) \neq 0\right\}$ by Proposition 1. We define

$$\delta = \begin{cases} 1 & \left|\frac{\partial h}{\partial z}(0)\right| > \left|\frac{\partial h}{\partial \bar{z}}(0)\right| \\ -1 & \left|\frac{\partial h}{\partial z}(0)\right| < \left|\frac{\partial h}{\partial \bar{z}}(0)\right|. \end{cases}$$

THEOREM 3. *Let $f_{t,h}$ be a perturbation of a complex polynomial f which satisfies the condition* (i) *and the condition* (ii). *Then the number of cusps of $f_{t,h}|_U$ is greater than or equal to $k - \delta$, where k is the multiplicity of f at the origin.*

Proof. Note that $m_s(G_{0,h}, 0) = m_s\left(\left(\frac{\partial f}{\partial z}\right)^2\overline{\frac{\partial^2 f}{\partial z \partial z}}, 0\right) = k$. By [7, Proposition 16], we have

$$\begin{aligned} k = m_s(G_{0,h}, 0) &= \sum_{\alpha \in G_{t,h}^{-1}(0)} m_s(G_{t,h}, \alpha) \\ &= \sum_{\beta \in G_{t,h}^{-1}(0), (\partial J_{t,h}/(\partial z))(\beta) \neq 0} m_s(G_{t,h}, \beta) + \sum_{\gamma \in (\partial J_{t,h}/(\partial z))^{-1}(0)} m_s(G_{t,h}, \gamma). \end{aligned}$$

Set

$$\deg \tilde{G} = \sum_{\beta \in G_{t,h}^{-1}(0), (\partial J_{t,h}/(\partial z))(\beta) \neq 0} m_s(G_{t,h}, \beta) \quad \text{and}$$

$$\deg \tilde{J} = \sum_{\gamma \in (\partial J_{t,h}/(\partial z))^{-1}(0)} m_s(G_{t,h}, \gamma).$$

By the condition (ii), the number of cusps of $f_{t,h}|_U$ is greater than or equal to $\deg \tilde{G}$. By the definition of $J_{t,h}$, we have

$$\sum_{\gamma \in (\partial J_{t,h}/(\partial z))^{-1}(0)} m_s\left(\frac{\partial J_{t,h}}{\partial z}, \gamma\right) = m_s\left(\frac{\partial J_{0,h}}{\partial z}, 0\right) = m_s\left(\frac{\partial^2 f}{\partial z \partial z} \overline{\frac{\partial f}{\partial z}}, 0\right) = -1.$$

Since $\frac{\partial f}{\partial z}(0) = 0$ and $\left|\frac{\partial h}{\partial z}(0)\right| \neq \left|\frac{\partial h}{\partial \bar{z}}(0)\right|$, $\deg \tilde{J}$ is equal to δ. Thus $\deg \tilde{G}$ is equal to $k - \delta$. □

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