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A TOPOLOGICAL CHARACTERIZATION OF THE STRONG DISK PROPERTY ON OPEN RIEMANN SURFACES

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Abstract

In this paper, we give a topological characterization of a subdomain G of an open Riemann surface R which has the strong disk property. Namely, we show that the domain G satisfies the strong disk property in R if and only if the canonical homomorphism $\pi_1(G) \to \pi_1(R)$ is injective.

1. Introduction

Let R be a Stein space and let G be an open set of R. If G is Stein and Runge in R, then G satisfies the *strong disk property* in R, that is, if $\varphi : \overline{\Delta} \to R$ is a continuous map holomorphic on Δ such that $\varphi(\partial\Delta) \subset G$, then we have that $\varphi(\overline{\Delta}) \subset G$, where Δ is the open unit disk in $\mathbf{C}$ (see Abe [1, Proposition 1]). The converse of this fact is not true in general. In fact, for every natural number $n \geq 2$, there exists a connected open set G of $\mathbf{C}^n$ such that G satisfies the strong disk property but is not Runge in $\mathbf{C}^n$ (see Abe [1, Theorem 7]).

On the other hand, by the theorem of Carathéodory, an open set G satisfies the strong disk property in $\mathbf{C}$ if and only if G is Runge in $\mathbf{C}$. Moreover, an open set G of a planar open Riemann surface R satisfies the strong disk property in R if and only if every connected component of G is Runge in R (see Abe-Nakamura [3, Theorem 3.3]). In the present paper, we consider the same property on open Riemann surfaces which are not necessarily planar.

Let R be an open Riemann surface and let G be a *domain* in R, that is, a connected open set of R. We prove that G satisfies the strong disk property in R if and only if the canonical homomorphism $\pi_1(G) \to \pi_1(R)$ is injective (see Theorem 3.1). As a corollary (Corollary 4.1), we may show that every Runge domain in an open Riemann surface R satisfies the strong disk property in R.

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It gives an alternative proof of Abe-Nakamura [3, Proposition 2.6]. We also prove that an open Riemann surface R is planar if and only if for every domain G in R the condition that G satisfies the strong disk property in R implies the condition that G is Runge in R (see Corollary 4.2). It answers the problem in Abe-Nakamura [3, Problem 3.5].

2. Preliminaries

An open set G of an open Riemann surface R is said to be *Runge* in R if for every $f \in \mathcal{O}(G)$, for every compact set K of G, and for every $\varepsilon > 0$, there exists $h \in \mathcal{O}(R)$ such that $|f - h| < \varepsilon$ on K. We have the following characterization of a Runge open set of an open Riemann surface, which is originally due to Behnke-Stein [5] (see Kusunoki [6, Theorem 6.10] and Mihalache [7, Theorem 5.1]).

Proposition 2.1. *Let R be an open Riemann surface and G an open set of R. Then, the following three conditions are equivalent.*
(1) *G is Runge in R.*
(2) *The canonical homomorphism $H_1(G, \mathbf{Z}) \to H_1(R, \mathbf{Z})$ is injective.*
(3) *No connected component of $R \backslash G$ is compact.*

Let $\Delta := \{\zeta \in \mathbf{C} \mid |\zeta| < 1\}$ be the unit disk. An open set G of a Riemann surface R is said to have the *strong disk property* in R if G satisfies the following condition: if $\varphi : \overline{\Delta} \to R$ is a continuous map holomorphic on Δ such that $\varphi(\partial\Delta) \subset G$, then $\varphi(\overline{\Delta}) \subset G$ (see Abe et al. [2], Abe [1], and Abe-Nakamura [3]).

We rephrase Proposition 2.6 in [3] in terms of open Riemann surfaces as follows.

Proposition 2.2. *Let R be an open Riemann surface and G an open set of R. If every connected component of G is Runge in R, then G satisfies the strong disk property in R.*

A Riemann surface R is said to be *planar* if R is biholomorphic to a domain of the Riemann sphere. We have the following characterization of a Runge domain in a planar open Riemann surface.

Proposition 2.3 (see [3, Theorem 3.3]). *Let R be a planar open Riemann surface and G an open set of R. Then, the following two conditions are equivalent.*
(1) *G satisfies the strong disk property in R.*
(2) *Every connected component of G is Runge in R.*

3. Main theorem

Let R be an open Riemann surface and G a domain of R. For a base point p_0 in G, $\iota_* : \pi_1(G, p_0) \to \pi_1(R, p_0)$ denotes the homomorphism given by the inclusion $\imath : G \hookrightarrow R$. We show that the strong disk property of G in R is characterized by $\imath_*$.

Theorem 3.1. *A subdomain G of an open Riemann surface R has the strong disk property in R if and only if $\imath_*$ is injective.*

Proof. We may assume that the open Riemann surface R is hyperbolic, that is, Δ is the universal covering surface of R. Our proof still works for $R = \mathbf{C}$ or $\mathbf{C}^*$. Let $\pi : \Delta \to R$ be the universal covering map and $\tilde{G} \subset \Delta$ a connected component of $\pi^{-1}(G)$. We take a Fuchsian group Γ so that $\Delta/\Gamma = R$.

First, we shall show "only if" part. Suppose that $\imath_*$ is not injective. Then there exists an element $[\gamma]$ in $\pi_1(G, p_0)$ such that $\imath_*([\gamma]) = [\{p_0\}] \in \pi_1(R, p_0)$ while $[\gamma] \neq [\{p_0\}]$ in $\pi_1(G, p_0)$. Let $\tilde{\gamma}$ be a connected component of $\pi^{-1}(\gamma)$ contained in $\tilde{G}$. Since γ is homotopic to a trivial curve in R, $\tilde{\gamma}$ is a closed curve in $\tilde{G}$ passing through a point $\tilde{p}_0$ in $\pi^{-1}(p_0)$. Furthermore, $\tilde{\gamma}$ is not homotopic to a trivial curve in $\tilde{G}$. Indeed, if it is homotopic to a trivial curve in $\tilde{G}$, the homotopy is projected to a homotopy in G between γ and a trivial curve. It is a contradiction. Thus, we see that $\tilde{G}$ is not simply connected in Δ. Hence, there exists a compact connected component $\tilde{E}$ of $\Delta \backslash \tilde{G}$.

Let $\tilde{G}'$ be a connected component of $\pi^{-1}(G)$. We show that $\tilde{E} \cap \tilde{G}' = \emptyset$. It is obvious if $\tilde{G}' = \tilde{G}$. Suppose that $\tilde{G}' \neq \tilde{G}$. Then, there exists a non-trivial element $g \in \Gamma$ such that $\tilde{G}' = g(\tilde{G})$. If $\tilde{E} \cap \tilde{G}' \neq \emptyset$, then $\tilde{G}' \subset \tilde{E}$ since $\tilde{G}$ and $\tilde{G}'$ are mutually disjoint. Hence we have $g(\tilde{E}) \subset \tilde{E}$. However, it is absurd because Γ acts properly discontinuously on Δ while $g^n(\tilde{E}) \subset \tilde{E}$ holds for the compact set $\tilde{E}$ for every n. Thus, we conclude that $\tilde{E} \cap \tilde{G}' = \emptyset$ and $\pi(\tilde{E}) \cap G = \emptyset$.

Let $\tilde{\alpha}$ be a simple closed curve in $\tilde{G}$ which surrounds $\tilde{E}$. We take a Riemann map $F : \Delta \to D(\tilde{\alpha})$ from Δ onto a Jordan domain $D(\tilde{\alpha})$ bounded by $\tilde{\alpha}$. Then, $f := \pi \circ F$ is a holomorphic map from Δ to R with homeomorphic extension on $\overline{\Delta}$ and $f(\partial\Delta) = \pi(\tilde{\alpha}) \subset G$. However, $f(\Delta) \not\subset G$ since $f(\Delta) \supset \pi(\tilde{E})$. Thus, we see that G does not have the strong disk property in R.

Next we shall show "if" part. Suppose that $\imath_*$ is injective. If $\pi_1(G, p_0)$ is the trivial group, then G is a simply connected domain and any connected component of $\pi^{-1}(G)$ is simply connected in Δ. If $\pi_1(G, p_0)$ is not the trivial group, then any connected component $\tilde{G}$ of $\pi^{-1}(G)$ is also simply connected in Δ. Indeed, if $\tilde{G}$ is not simply connected, then $\Delta \backslash \tilde{G}$ has a compact connected component $\tilde{E}$. Let $\tilde{\alpha}$ be a simple closed curve in $\tilde{G}$ surrounding $\tilde{E}$. Then, $\alpha := \pi(\tilde{\alpha}) \subset G$ is homotopic to a trivial curve in R because $\tilde{\alpha}$ is a closed curve. It is also homotopic to a trivial curve in G since $\imath_*$ is injective. However, it is absurd because $\tilde{\alpha}$ is not homotopic to a trivial curve in $\tilde{G}$. Thus, we see that $\tilde{G}$ is simply connected in Δ. Therefore, in any case $\tilde{G}$ is simply connected.

Let $f : \overline{\Delta} \to R$ be a continuous map which is holomorphic in Δ. Suppose that $\gamma := f(\partial\Delta) \subset G$. We may take a lift $F : \overline{\Delta} \to \Delta$ of f such that $\tilde{\gamma} := F(\partial\Delta) \subset \tilde{G}$. Since $\tilde{G}$ is simply connected, the closed curve $\tilde{\gamma}$ is homotopic to a trivial curve in $\tilde{G}$. Let ζ be a point in $\Delta \backslash \pi^{-1}(G)$. Then, the winding number of $\tilde{\gamma} =$

$F(\partial\Delta)$ for ζ is zero for any lift F of f. Indeed, since $\tilde{\gamma}$ is homotopic to a trivial curve, there exists a continuous map $H : [0,1] \times [0,1] \to \tilde{G}$ such that

(i) $H(0,t) = \tilde{\gamma}(t)$ and $H(1,t) \equiv \tilde{p}_0 \in \tilde{G}$ for every $t \in [0,1]$.

(ii) $H(s,0) = H(s,1)$ for every $s \in [0,1]$.

Let $\tilde{\gamma}_s$ be the closed curve given by $H(s,\cdot) : [0,1] \to \tilde{G}$. Since $\tilde{\gamma}_s \not\ni \zeta$ for every $s \in [0,1]$, the winding number of $\tilde{\gamma}_s$ for ζ depends continuously on $s \in [0,1]$ and it is a constant. Noting that $\tilde{\gamma}_1 = \{\tilde{p}_0\}$, we verify that the winding number of $\tilde{\gamma}_0 = \tilde{\gamma}$ for ζ is zero. From the argument principle, we see that there is no point $z \in \Delta$ such that $F(z) = \zeta$. Hence, we conclude that $F(\Delta) \subset \tilde{G}$ and $f(\Delta) \subset G$. Namely, G has the strong disk property in R. □

4. Corollaries

In the first part of the proof of Theorem 3.1, we found a compact connected component $\tilde{E}$ of $\Delta \backslash \pi^{-1}(G)$ if ι_* is not injective. Then, $E := \pi(\tilde{E})$ is a connected component of $R \backslash G$ and compact. This argument is an alternative proof of Proposition 2.2 when G is a domain.

Corollary 4.1. *Let G be a domain in an open Riemann surface R. If G is a Runge domain, then G has the strong disk property in R.*

Theorem 3.1 also solves a problem posed in [3] as follows.

Corollary 4.2. *Let R be an open Riemann surface. Then, the Runge property and the strong disk property are the same for any domain in R if and only if R is planar.*

Proof. Suppose that R is planar and G is a proper domain of R. We may assume that R is a domain in $\mathbf{C}$. If G is a Runge domain, then G has the strong disk property in R from Corollary 4.1. If G is not a Runge domain, then there exists a connected component E of $R \backslash G$ which is compact in R. Here, we claim that there exists a simple closed curve α in G such that the Jordan domain D_α bounded by α which contains E is in R.

Indeed, as an open Riemann surface, G admits a sequence of regular subregions $\{G_n\}_{n=1}^{\infty}$ such that $\overline{G_n} \subset G_{n+1}$ and $G = \bigcup_{n=1}^{\infty} G_n$ ([4, II. 12D Theorem]), where a subregion D of G is regular if it is a relatively compact region bounded by a finite number of simple closed analytic curves and every connected component of $G \backslash D$ is not compact. Moreover, we may assume that each boundary component of G_n is a dividing curve (cf. [6, p. 184]).

Let E_n be the connected component of $\mathbf{C} \backslash G_n$ containing E. Then, $E' := \bigcap_{n=1}^{\infty} E_n$ is connected in $\mathbf{C} \backslash G$ which contains E. Since E is compact in R, we see that $E' \subset R$ and $E' = E$. Hence, $E_N \subset R$ if N is sufficiently large. From our assumption, each E_n is simply connected domain bounded by a simple closed analytic curve. Thus, we see that ∂E_N becomes our desired curve α of the claim.

Let $f : \Delta \to D_\alpha$ be a Riemann map onto D_α. The map f admits a homeomorphic extension $f : \overline{\Delta} \to \overline{D_\alpha}$. Hence $f(\partial\Delta) = \alpha \subset G$. On the other hand,

$f(\Delta) = D_\alpha \subset R$ but $f(\Delta) \not\subset G$ because $f(\Delta) \supset E$. Thus, we verify that G does not have the strong disk property in R. Therefore, we conclude that the Runge property and the strong disk property are the same in R.

Next, we suppose that R is not planar. Then, there exists a non-trivial simple closed curve α in R such that α satisfies the following conditions.

(1) $R \backslash \alpha$ consists of two components R_1 and R_2.

(2) R_1 is relatively compact in R (and α is homologous to zero).

Let G be an annular neighborhood of α. It follows from (2) that G is not Runge domain. On the other hand, G has the strong disk property in R because α is non-trivial and $\iota_* : \pi_1(G, p_0) \cong \mathbf{Z} \to \pi_1(R, p_0)$ is injective. Hence, we have a domain which is not Runge but has the strong disk property for a non-planar surface R. □

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