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A WEAK COHERENCE THEOREM AND REMARKS TO THE OKA THEORY

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Abstract

The proofs of K. Oka's Coherence Theorems are based on Weierstrass' Preparation (division) Theorem. Here we formulate and prove a Weak Coherence Theorem without using Weierstrass' Preparation Theorem, but only with *power series expansions*: The proof is almost of linear algebra. Nevertheless, this simple *Weak Coherence Theorem* suffices to give other proofs of the Approximation, Cousin I/II, and Levi's (Hartogs' Inverse) Problems even in simpler ways than those known, as far as the domains are non-singular; they constitute the main basic part of the theory of several complex variables.

The new approach enables us to complete the proofs of those problems in quite an elementary way without Weierstrass' Preparation Theorem or the cohomology theory of Cartan–Serre, nor L^2-$\bar{\partial}$ method of Hörmander.

We will also recall some new historical facts that Levi's (Hartogs' Inverse) Problem of general dimension $n \geq 2$ was, in fact, solved by K. Oka in 1943 (unpublished) and by S. Hitotsumatsu in 1949 (published in Japanese), whereas it has been usually recognized as proved by K. Oka 1953, by H. J. Bremermann and by F. Norguet 1954, independently.

1. Introduction and a weak coherence

K. Oka [29], [30] proved three fundamental coherence theorems for

(i) the sheaf $\mathcal{O} := \mathcal{O}_{\mathbf{C}^n}$ of germs of holomorphic functions on $\mathbf{C}^n$,

(ii) the ideal sheaf $\mathcal{I}\langle A\rangle$ of an analytic subset A of an open subset of $\mathbf{C}^n$,

(iii) the normalization of the structure sheaf of a complex space,

where for the second, H. Cartan [4] gave his own proof based on Oka [29] (cf. [17] Chap. 9). We simply call $\mathcal{I}\langle A\rangle$ a (resp. non-singular) *geometric ideal sheaf* of a (resp. non-singular) analytic subset A (cf., e.g., [17] Chap. 6). Oka's Coherence has played a fundamental role in modern Mathematics, so that it

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led to the notion of ringed spaces due to H. Cartan, and developed by J.-P. Serre, R. Remmert, H. Grauert and A. Grothendieck (cf. [6] p. 162). The proofs of the above coherence theorems rely on Weierstrass' Preparation (division) Theorem.

The aim of this paper is to formulate a *Weak Coherence Theorem* (Theorem 1.2 below), which we prove *not with Weierstrass' Preparation Theorem*, but *only with power series expansions*, and then to apply it to prove the Approximation Problem, Cousin I/II Problems, $\bar{\partial}$-equation (for functions), holomorphic extensions (interpolations), and Levi's (Hartogs' Inverse) Problem for unramified Riemann domains (over $\mathbf{C}^n$)[1] (see Theorem 4.11 and §4.3); they constitute the main basic part of the theory of several complex variables. The proofs are even simpler than those in the standard references (cf., e.g., Gunning–Rossi [10], Grauert–Remmert [9], Hörmander [12], Noguchi [17]).

Note that the present approach enables us to complete the proofs of those problems in quite an elementary way without Weierstrass' Preparation Theorem or the cohomology theory[2] of Cartan–Serre, nor L^2-$\bar{\partial}$ method of Hörmander. The present paper came out from the study of the degree structure of a generator system of a coherent analytic sheaf by [16], and was inspired by Oka's unpublished paper [33].

Let Ω denote a domain of $\mathbf{C}^n$ with the structure sheaf $\mathcal{O} = \mathcal{O}_\Omega$. For a holomorphic function $f \in \mathcal{O}(\Omega)$ in Ω we write $\underline{f} \in \Gamma(\Omega, \mathcal{O})$ for the induced sheaf-section of $\mathcal{O}$ and $\underline{f}_z$ for the germ of f at $z \in \Omega$. Let $\mathcal{F}$ be an analytic sheaf on Ω (i.e., a sheaf of $\mathcal{O}$-modules over Ω), and let $\xi_j \in \Gamma(\Omega, \mathcal{F})$, $1 \le j \le q$, be finitely many sections on Ω. Then the relation sheaf $\mathcal{R}(\xi_1, \dots, \xi_q)$ of $\{\xi_j\}_{j=1}^q$ is a subsheaf of $\mathcal{O}^q$ consisting of those germ-vectors $(\underline{f_1}_z, \dots, \underline{f_q}_z) \in \mathcal{O}_z^q$ such that

$$\underline{f_1}_z \xi_1(z) + \cdots + \underline{f_q}_z \xi_q(z) = 0, \quad z \in \Omega. \tag{1.1}$$

Now we formulate:

THEOREM 1.2 (Weak Coherence). *Let $S \subset \Omega$ be a complex submanifold.*[3]
(i) *The non-singular geometric ideal sheaf $\mathcal{I}\langle S\rangle$ is locally finite.*
(ii) *Let $\{\underline{\sigma_j} \in \Gamma(\Omega, \mathcal{I}\langle S\rangle) : 1 \le j \le N\}$ be a finite generator system of $\mathcal{I}\langle S\rangle$ on Ω with $\sigma_j \in \mathcal{O}(\Omega)$: i.e.,*

$$\mathcal{I}\langle S\rangle = \sum_{j=1}^{N} \mathcal{O} \cdot \underline{\sigma_j}.$$

Then, the relation sheaf $\mathcal{R}(\underline{\sigma_1}, \dots, \underline{\sigma_N})$ is locally finite.

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[1] In the present, "over $\mathbf{C}^n$" will be abbreviated, unless necessary.

[2] We use only the 1-cocyle class space $H^1(*, \mathcal{O})$ as a complex vector space.

[3] A complex submanifold is not necessarily connected in this paper.

We give a proof of this theorem in §2. In §3 we will apply it to prove Oka's Jôku-Ikô[4] (see Lemma 3.11), and then we will give a unified proof for Cousin I/II Problems, and $\bar{\partial}$-equation for functions in §4 (Theorem 4.11) by combining the Weak Coherence Theorem 1.2 with a method of *cuboid induction on dimension*; these yield $H^1(\Omega, \mathcal{O}) = 0$ for a holomorphically convex domain Ω (Lemma 4.20), which suffices to derive Oka's Heftungslemma or Grauert's finiteness theorem for $\mathcal{O}$ on a strongly pseudoconvex domain (Theorem 4.21). In §4.3 we finally give the solution of Levi's (Hartogs' Inverse) Problem for unramified Riemann domains.

2. Proof of Theorem 1.2

(i) We take an arbitrary point $a \in \Omega$.

Case of $a \notin S$: Since S is closed, there is a neighborhood $U \subset \Omega$ of a with $U \cap S = \emptyset$. Then,

$$\mathscr{I}\langle S\rangle_x = \mathcal{O}_x = 1 \cdot \mathcal{O}_x, \quad {}^{\forall}x \in U,$$

and therefore, $\{1\}$ is a finite generator system of $\mathscr{I}\langle S\rangle_x$ on U.

Case of $a \in S$: There is a holomorphic local coordinate neighborhood U of a with $z = (z_1, \ldots, z_n)$ such that

$$a = (0, \ldots, 0) \in U = \mathrm{P}\Delta(0; (r_j)), \tag{2.1}$$

$$S \cap U = \{z = (z_j) \in U : z_1 = \cdots = z_q = 0\} \quad (1 \leq {}^{\exists}q \leq n),$$

where $\mathrm{P}\Delta(0; (r_j)) = \{(z_j) \in \mathbf{C}^n : |z_j| < r_j,\ 1 \leq j \leq n\}$ is a polydisk with center at 0. Let $\underset{b}{f} \in \mathscr{I}\langle S\rangle_b$ $(b \in U \cap S)$ be any element. With the coordinate system (z_j) we write $b = (b_j) = (0, \ldots, 0, b_{q+1}, \ldots, b_n)$. The function f is represented by a unique power series expansion, $f(z) = \sum_{\nu \in \mathbf{Z}_+^n} c_\nu (z-b)^\nu$, which decomposes to

$$\begin{aligned} f(z) &= \sum_{\nu=(\nu_1,\nu')\in\mathbf{Z}_+^n,\,\nu_1>0} c_\nu (z-b)^\nu + \sum_{\nu=(\nu_1,\nu')\in\mathbf{Z}_+^n,\,\nu_1=0} c_\nu (z-b)^\nu \\ &= \left(\sum_{\nu=(\nu_1,\nu')\in\mathbf{Z}_+^n,\,\nu_1>0} c_\nu z_1^{\nu_1-1} (z'-b')^{\nu'} \right) z_1 + \sum_{\nu'\in\mathbf{Z}_+^{n-1}} c_{0\nu'} (z'-b')^{\nu'}. \end{aligned}$$

Here we put $\nu' = (\nu_2, \ldots, \nu_n)$, $z' = (z_2, \ldots, z_n)$, and $b' = (b_2, \ldots, b_n)$. Setting

[4] This is a method or a principle of K. Oka all through his series of papers [21]–[31] such that to solve a problem on a difficult domain one embeds the domain into a higher dimensional polydisk, extends the problem on the polydisk, and then solves it by making use of the simple shape of the polydisk (cf. [17]).

$$h_1(z_1, z') = \left(\sum_{\nu=(\nu_1,\nu')\in \mathbf{Z}_+^n,\, \nu_1>0} c_\nu z_1^{\nu_1-1} (z'-b')^{\nu'} \right),$$

$$g_1(z') = \sum_{\nu'\in \mathbf{Z}_+^{n-1}} c_{0\nu'} (z'-b')^{\nu'},$$

we have

$$f(z_1, z') = h_1(z_1, z') \cdot z_1 + g_1(z'). \tag{2.2}$$

For $g_1(z')$ we apply a similar decomposition with respect to variable z_2, so that

$$g_1(z') = h_2 \cdot z_2 + g_2(z''), \quad z'' = (z_3, \ldots, z_n).$$

Repeating this process, we get

$$f(z) = \sum_{j=1}^{q} h_j(z) \cdot z_j + g_q(z_{q+1}, \ldots, z_n).$$

If $z_1 = \cdots = z_q = 0$, then $f(z) = 0$, and so $g_q(z_{q+1}, \ldots, z_n) = 0$. Therefore,

$$f(z) = \sum_{j=1}^{q} h_j(z) \cdot z_j.$$

Thus,

$$\mathscr{I}\langle S\rangle|_U = \sum_{j=1}^{q} \mathcal{O}_U \cdot \underline{z_j}. \tag{2.3}$$

(ii) We begin with the following lemma:

LEMMA 2.4. *With the natural complex coordinate system* $z = (z_1, \ldots, z_n) \in \mathbf{C}^n$ *we consider a relation sheaf* $\mathscr{R}_p$ $(1 \le p \le n)$ *defined by*

$$\underline{f_1}_z \underline{z_1}_z + \cdots + \underline{f_p}_z \underline{z_p}_z = 0, \quad \underline{f_j}_z \in \mathcal{O}_z. \tag{2.5}$$

Then $\mathscr{R}_p$ *is finitely generated on* $\mathbf{C}^n$ *by*

$$T_{ij} = (0, \ldots, 0, \overset{i\text{-th}}{-\underline{z_j}}, 0, \ldots, 0, \overset{j\text{-th}}{\underline{z_i}}, 0, \ldots, 0), \quad 1 \le i < j \le p. \tag{2.6}$$

We call T_{ij} $(1 \le i < j \le p)$ of (2.6) the *trivial solutions* of (2.5) or of $\mathscr{R}_p$. In the case of $p = 1$, we set the trivial solution to be 0 as a convention.

Proof of Lemma 2.4. We use induction on $p \ge 1$. The case of $p = 1$ is clear.

Assuming that the case of $p-1$ $(p \geq 2)$ holds, we consider the case of p. Set

$$\Sigma = \{(z_1, \ldots, z_n) : z_1 = \cdots = z_p = 0\},$$

and let $a \in \mathbf{C}^n$ be an arbitrary point. If $a = (a_j) \notin \Sigma$, there is an $a_j \neq 0$ $(1 \leq j \leq p)$, to say, $a_1 \neq 0$. In a neighborhood V of a, $z_1 \neq 0$. Then, (2.5) is solvable with respect to $\underline{f_1}_z$:

$$\underline{f_1}_z = -\underline{f_2}_z \cdot \frac{\underline{z_2}_z}{\underline{z_1}_z} - \cdots - \underline{f_p}_z \cdot \frac{\underline{z_p}_z}{\underline{z_1}_z}, \quad {}^\forall\underline{f_j}_z \in \mathcal{O}_z \ \ (2 \leq j \leq p),\ z \in V.$$

It follows that with $z \in V$,

$$\begin{aligned}(\underline{f_j}_z) &= \left(-\sum_{j=2}^{p} \underline{f_j}_z \cdot \frac{\underline{z_j}_z}{\underline{z_1}_z}, \underline{f_2}_z, \ldots, \underline{f_p}_z \right) \\ &= \sum_{j=2}^{p} \frac{\underline{f_j}_z}{\underline{z_1}_z} \cdot (-\underline{z_j}_z, 0, \ldots, 0, \overset{j\text{-th}}{\underline{z_1}_z}, 0, \ldots, 0) \\ &= \sum_{j=2}^{p} -\frac{\underline{f_j}_z}{\underline{z_1}_z} \cdot T_{1j}(z) \in \sum_{j=2}^{p} \mathcal{O}_z \cdot T_{1j}(z).\end{aligned} \tag{2.7}$$

Therefore, $\mathscr{R}_p$ is generated by the trivial solutions $\{T_{1j}\}_{2 \leq j \leq p}$ on V.

If $a \in \Sigma$, we decompose an element $(\underline{f_j}_a) \in \mathscr{R}_{pa}$ in a polydisk neighborhood U of a as in (2.2):

$$f_j(z_1, z') = h_j(z_1, z')z_1 + g_j(z'), \quad z' = (z_2, \ldots, z_n), \quad 1 \leq j \leq p.$$

For $z \in U$ one gets

$$\begin{aligned}(\underline{f_j}_z) - \sum_{j=2}^{p} \underline{h_j}_z T_{1j}(z) &= \left(\underline{g_1}_z + \sum_{j=1}^{p} \underline{h_j}_z \underline{z_j}_z, \underline{g_2}_z, \ldots, \underline{g_p}_z \right) \\ &= (\tilde{\underline{g}}_{1z}, \underline{g_2}_z, \ldots, \underline{g_p}_z).\end{aligned} \tag{2.8}$$

Here, $\tilde{\underline{g}}_{1z} = \underline{g_1}_z + \sum_{j=1}^{p} \underline{h_j}_z \underline{z_j}_z$. Since $(\tilde{\underline{g}}_{1z}, \underline{g_2}_z, \ldots, \underline{g_p}_z) \in \mathscr{R}_{p_z}$,

$$\tilde{\underline{g}}_{1z} \underline{z_1}_z + \underline{g_2}_z \underline{z_2}_z + \cdots + \underline{g_p}_z \underline{z_p}_z = 0.$$

The second term and so forth of the right-hand side of the equation above do not contain variable z_1, and so $\tilde{\underline{g}}_{1z} = 0$ is deduced. Thus,

$$\underline{g_2}_z \underline{z_2}_z + \cdots + \underline{g_p}_z \underline{z_p}_z = 0.$$

This is the case of $p-1$ after changing the indices of variables. Therefore, the induction hypothesis implies that $(0, \underline{g_2}_z, \ldots, \underline{g_p}_z)$ is represented as a linear sum of $T_{ij}(z)$, $2 \leq i < j \leq p$, with coefficients in $\mathcal{O}_z$. Combining this with (2.8), we see

that $(\underline{f_j}_z)$ is represented as a linear sum of $T_{ij}(z)$, $1 \le i < j \le p$, with coefficients in $\mathcal{O}_z$. $\triangle$

Continued proof of (ii). Set $\mathscr{R} = \mathscr{R}(\underline{\sigma_1}, \dots, \underline{\sigma_N})$. We consider the relation

$$\underline{f_1}_z \underline{\sigma_1}_z + \cdots + \underline{f_N}_z \underline{\sigma_N}_z = 0, \quad \underline{f_j}_z \in \mathcal{O}_z. \tag{2.9}$$

We set the trivial solutions of this equation as follows:

$$\tau_{ij} = (\dots, \overset{i\text{-th}}{-\underline{\sigma_j}}, \dots, \overset{j\text{-th}}{\underline{\sigma_i}}, \dots), \quad 1 \le i < j \le N.$$

We take an arbitrary point $a \in \Omega$. If $a \notin S$, then some $\sigma_j(a) \neq 0$, to say, $\sigma_1(a) \neq 0$. As in (2.7), one sees that $\mathscr{R}$ is generated by $\{\tau_{1j}\}_{j=2}^{N}$ on a neighborhood of a.

If $a \in S$, we take a holomorphic local coordinate system $z = (z_1, \dots, z_n)$ in a polydisk neighborhood $P\Delta$ as in (2.1):

$$a = (0, \dots, 0),$$

$$S \cap P\Delta = \{(z_1, \dots, z_n) \in P\Delta : z_1 = \cdots = z_q = 0\} \quad (1 \le {}^{\exists} q \le n).$$

It follows from (2.3) and the assumption that

$$\mathscr{I}\langle S \rangle|_{P\Delta} = \sum_{j=1}^{q} \mathcal{O}_{P\Delta} \cdot \underline{z_j} = \sum_{j=1}^{N} \mathcal{O}_{P\Delta} \cdot \underline{\sigma_j}|_{P\Delta}.$$

Thus, we may assume without loss of generality that

$$\sigma_j = z_j, \quad 1 \le j \le q \ \text{ (on } P\Delta),$$

$$\sigma_i = \sum_{j=1}^{q} a_{ij} z_j, \quad a_{ij} \in \mathcal{O}(P\Delta),\ q+1 \le i \le N \ \text{ (on } P\Delta).$$

Set

$$\phi_i = (-\underline{a_{i1}}, \dots, -\underline{a_{iq}}, 0, \dots, 0, \overset{i\text{-th}}{1}, 0, \dots, 0) \in \Gamma(P\Delta, \mathscr{R}), \quad q+1 \le i \le N. \tag{2.10}$$

We deduce from (2.9) with $z \in P\Delta$ that

$$\left(\underline{f_1}_z + \sum_{i=q+1}^{N} \underline{f_i}_z \underline{a_{i1}}_z\right) \underline{z_1}_z + \cdots + \left(\underline{f_q}_z + \sum_{i=q+1}^{N} \underline{f_i}_z \underline{a_{iq}}_z\right) \underline{z_q}_z = 0. \tag{2.11}$$

By Lemma 2.4,

$$\left(\underline{f_1}_z + \sum_{i=q+1}^{N} \underline{f_i}_z \underline{a_{i1}}_z, \dots, \underline{f_q}_z + \sum_{i=q+1}^{N} \underline{f_i}_z \underline{a_{iq}}_z, 0, \dots, 0\right)$$

is a linear sum of $\tau_{jk}(z)$, $1 \le j < k \le q$, with coefficients in $\mathcal{O}_z$. Therefore there are $\underline{b_{jk}}_z \in \mathcal{O}_z$, $1 \le j < k \le q$, such that

$$\sum_{1\le j<k\le q} \underline{b_{jk}}_z \tau_{jk}(z) = \left(\underline{f_1}_z + \sum_{i=q+1}^{N} \underline{f_i}_z \underline{a_{i1}}_z, \dots, \underline{f_q}_z + \sum_{i=q+1}^{N} \underline{f_i}_z \underline{a_{iq}}_z, 0, \dots, 0\right) \tag{2.12}$$

$$= (\underline{f_1}_z, \dots, \underline{f_q}_z, 0, \dots, 0) + \sum_{i=q+1}^{N} \underline{f_i}_z (\underline{a_{i1}}_z, \dots, \underline{a_{iq}}_z, 0, \dots, 0).$$

By making use of (2.10) we get

$$(\underline{f_1}_z, \dots, \underline{f_q}_z, \dots, \underline{f_N}_z) = \sum_{1\le j<k\le q} \underline{b_{jk}}_z \tau_{jk}(z) + \sum_{i=q+1}^{N} \underline{f_i}_z \phi_i(z). \tag{2.13}$$

Thus, $\mathscr{R}$ is generated on $\mathrm{P}\Delta$ by

$$\tau_{jk},\ \phi_i, \quad 1 \le j < k \le q,\ q+1 \le i \le N. \tag{2.14}$$

This finishes the proof. □

Remark 2.15. (i) In the Weak Coherence Theorem 1.2 it is a point to assume that $\{\sigma_j\}_{j=1}^N$ is a generator system of $\mathscr{I}\langle S\rangle$; otherwise, the proof above does not work even if S is non-singular.

(ii) It is an advantage of the above method to the general First Coherence Theorem of Oka that we have an explicit system of generators (2.14).

3. Oka's Jôku-Ikô

The term "Jôku-Ikô" was used by K. Oka since he wrote the first paper of the series in 1936 and retained this principle all through his works ([21]–[31]) (see footnote 4)): The aim of the present section is to prove Oka's Jôku-Ikô, Lemma 3.11 below only by making use of Theorem 1.2 combined with Cousin's integral (3.7). The technics may be essentially similar to those in some references, e.g., Nishino [14] and Noguchi [17], but they are not in a suitable form for our purpose.

3.1. Syzygy for non-singular geometric ideal sheaves

We begin with:

DEFINITION 3.1. A cuboid E is a bounded open or closed subset of $\mathbf{C}^n$ with the boundary parallel to the real and imaginary axes of $z = (z_1, \dots, z_n) \in \mathbf{C}^n$. In the case of $n = 1$, E is called a rectangle. When E is a closed cuboid, we allow the widths of some edges to degenerate to 0, and call the number of edges of E of positive widths the dimension of E, denoted by $\dim E$.

Let $\Omega \subset \mathbf{C}^n = \mathbf{C}^{n-1} \times \mathbf{C}$ be a domain and let $E', E'' \Subset \Omega$[5] be two closed cuboids as follows (cf. Fig. 1): There are a closed cuboid $F \Subset \mathbf{C}^{n-1}$ and two

[5] The symbol "$\Subset$" stands for that the inclusion is relatively compact.

adjacent closed rectangles $E'_n, E''_n \Subset \mathbf{C}$ sharing a side ℓ, and

$$E' = F \times E'_n, \quad E'' = F \times E''_n, \quad \ell = E'_n \cap E''_n. \tag{3.2}$$

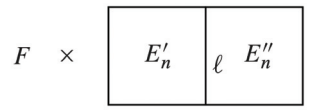

FIGURE 1. Adjacent closed cuboids.

We now recall:

LEMMA 3.3 (Cartan's Merging Lemma). *Let $E', E'' \Subset \Omega$ be adjacent closed cuboids as in* (3.2), *and let $\mathscr{F}$ be an analytic sheaf on Ω. Let $\{\sigma'_j \in \Gamma(E', \mathscr{F}) : 1 \le j \le p'\}$ (resp. $\{\sigma''_k \in \Gamma(E'', \mathscr{F}), 1 \le k \le p''\}$) be a finite generator system of $\mathscr{F}$ on E' (resp. E'').*[6)]

Moreover, assume that there are holomorphic functions $a_{jk}, b_{kj} \in \mathcal{O}(E' \cap E'')$, $1 \le j \le p'$, $1 \le k \le p''$, such that

$$\sigma'_j = \sum_{k=1}^{p''} \underline{a_{jk}} \cdot \sigma''_k, \quad \sigma''_k = \sum_{j=1}^{p'} \underline{b_{kj}} \cdot \sigma'_j \quad (\text{on } E' \cap E'').$$

Then, there exists a merged finite generator system $\{\sigma_l \in \Gamma(E' \cup E'', \mathscr{F}) : 1 \le l \le p' + p''\}$ on $E' \cup E''$.

This is due to H. Cartan 1940; a rather simplified proof of it can be found in [17], "Added at galley-proof".

LEMMA 3.4 (Oka's Syzygy). *Let $E \Subset \mathbf{C}^n$ be a closed cuboid.*

(i) *Every locally finite analytic sheaf $\mathscr{F}$ defined on E (i.e., in a neighborhood of E) has a finite generator system on E.*

(ii) *Let $\mathscr{F}$ be an analytic sheaf on E with a finite generator system $\{\sigma_j\}_{1 \le j \le N}$ on E such that the relation sheaf $\mathscr{R}(\sigma_1, \ldots, \sigma_N)$ is locally finite.*

Then for every section $\sigma \in \Gamma(E, \mathscr{F})$ there are holomorphic functions $a_j \in \mathcal{O}(E)$, $1 \le j \le N$, such that

$$\sigma = \sum_{j=1}^{N} \underline{a_j} \cdot \sigma_j \quad (\text{on } E). \tag{3.5}$$

[6)] This means that they are defined so in some neighborhoods of E' and E'', respectively; this expression is the same through the paper.

Proof. The proof is carried out in the same way as in [14], or [17] Lemma 4.3.7 except for the use of the vanishing $H^1(U, \mathcal{O}) = 0$ for an affine convex cylinder domain $U \subset \mathbf{C}^n$, which we replace by Cousin's integral (3.7) as follows. Suppose that E is a closed cuboid such that

$$E = F \times \{z_n : |\Re z_n| \leq T,\ |\Im z_n| \leq \theta\}, \quad T > 0,\ \theta \geq 0. \tag{3.6}$$

Set $E_0 = F \times \{z_n : \Re z_n = 0,\ |\Im z_n| \leq \theta\}$, and let $\varphi(z', z_n) \in \mathcal{O}(E_0)$. Then there is a small $\delta > 0$ such that $\varphi(z', z_n)$ is defined on

$$F \times \{z_n : |\Re z_n| \leq \delta,\ |\Im z_n| \leq \theta + \delta\}.$$

Set

$$\begin{aligned}
\ell &= \{z_n : \Re z_n = 0,\ -\theta - \delta \leq \Im z_n \leq \theta + \delta\},\\
E_1 &= F \times \{z_n : -T \leq \Re z_n \leq \delta,\ |\Im z_n| \leq \theta\},\\
E_2 &= F \times \{z_n : -\delta \leq \Re z_n \leq T,\ |\Im z_n| \leq \theta\},
\end{aligned}$$

where ℓ is positively oriented as $\Im z_n$ increases. We define Cousin's integral of $\varphi(z', z_n)$ along ℓ by

$$\Phi(z', z_n) = \frac{1}{2\pi i}\int_\ell \frac{\varphi(z', \zeta_n)}{\zeta_n - z_n} d\zeta_n. \tag{3.7}$$

Then $\Phi(z', z_n)$ is holomorphic on $(E_1 \cup E_2)\backslash(F \times \ell)$. After analytic continuations we obtain $\Phi_j(z', z_n) \in \mathcal{O}(E_j)$ $(j = 1, 2)$ satisfying

$$\Phi_1(z', z_n) - \Phi_2(z', z_n) = \varphi(z', z_n), \quad (z', z_n) \in E_1 \cap E_2. \tag{3.8}$$

We call this the *Cousin decomposition* of $\varphi(z', z_n)$.

The rest is the same as in the proof of [17] Lemma 4.3.7. □

By the Weak Coherence Theorem 1.2 and Lemma 3.4 we have:

THEOREM 3.9 (Syzygy for $\mathcal{I}\langle S\rangle$). *Let S be a complex submanifold of a neighborhood of a closed cuboid E $(\Subset \mathbf{C}^n)$.*

(i) *$\mathcal{I}\langle S\rangle$ has a finite generator system on E.*

(ii) *Let $\{\sigma_j\}_{1\leq j\leq N}$ be a finite generator system of $\mathcal{I}\langle S\rangle$ on E with $\sigma_j \in \mathcal{O}(E)$. Then for every $\underline{\sigma} \in \Gamma(E, \mathcal{I}\langle S\rangle)$ $(\sigma \in \mathcal{O}(E))$ there are holomorphic functions $a_j \in \mathcal{O}(E)$, $1 \leq j \leq N$, such that*

$$\sigma = \sum_{j=1}^{N} a_j \cdot \sigma_j \quad (\text{on } E). \tag{3.10}$$

3.2. Oka's Jôku-Ikô

Let P be an open cuboid in $\mathbf{C}^n$, and let $S \subset \mathrm{P}$ be a complex submanifold. The following is fundamental in the Oka theory.

LEMMA 3.11 (Oka's Jôku-Ikô). *Let $E \Subset \mathrm{P}$ be a closed cuboid. Then for every holomorphic function g on $E \cap S$ ($\Subset S$)*[7] *there exists an element $G \in \mathcal{O}(E)$ satisfying*

$$G|_{E\cap S} = g|_{E\cap S}.$$

Here, the equality holds in a neighborhood of $E \cap S$ in S.[8]

Remark. We call G above a *solution* on E.

Proof. Notice that in the case of $E \cap S = \emptyset$, G can be any holomorphic function on E, and the statement is true. We use induction on $\dim E$.

(a) Case of $\dim E = 0$: Since E consists of one point, the assertion is clear.

(b) Case of $\dim E = v$ $(v \geq 1)$ with the induction hypothesis that the case of $\dim E = v - 1$ is true: By Theorem 3.9 (i) there is a finite generator system $\{\underline{\sigma_j}\}_{j=1}^{N}$ of $\mathscr{I}\langle S\rangle$ on a neighborhood W ($\subset \mathrm{P}$) of E with $\sigma_j \in \mathcal{O}(W)$.

We may assume that E is taken as in (3.6). We set

$$E_t = \{z = (z', z_n) \in E : \Re z_n = t\}, \quad -T \leq t \leq T. \tag{3.12}$$

Since $\dim E_t = v - 1$, the induction hypothesis implies that there is a solution $G_t \in \mathcal{O}(E_t)$ satisfying $G_t|_{S\cap E_t} = g|_{S\cap E_t}$. By the Heine–Borel Theorem there is a finite partition

$$-T = t_0 < t_1 < \cdots < t_L = T, \tag{3.13}$$

$$E_\alpha := \{z = (z', z_n) \in E : t_{\alpha-1} \leq \Re z_n \leq t_\alpha\}, \quad 1 \leq \alpha \leq L,$$

such that there are solutions $G_\alpha \in \mathcal{O}(E_\alpha)$ satisfying

$$G_\alpha|_{S\cap E_\alpha} = g|_{S\cap E_\alpha}.$$

Therefore, $\underline{G_{\alpha+1} - G_\alpha} \in \Gamma(E_\alpha \cap E_{\alpha+1}, \mathscr{I}\langle S\rangle)$. It follows from Theorem 3.9 (ii) that there are $a_{\alpha j} \in \mathcal{O}(E_\alpha \cap E_{\alpha+1})$ $(1 \leq j \leq N)$ satisfying

$$G_{\alpha+1} - G_\alpha = \sum_{j=1}^{N} a_{\alpha j}\sigma_j \quad (\text{on } E_\alpha \cap E_{\alpha+1}). \tag{3.14}$$

By the Cousin decomposition (3.7) of $a_{\alpha j}$ we write

$$a_{\alpha j} = b_{\alpha j} - b_{\alpha+1 j} \quad (\text{on } E_\alpha \cap E_{\alpha+1}), \quad b_{\alpha j} \in \mathcal{O}(E_\alpha),\ b_{\alpha+1 j} \in \mathcal{O}(E_{\alpha+1}). \tag{3.15}$$

Then,

$$G_\alpha + \sum_{j=1}^{N} b_{\alpha j}\sigma_j = G_{\alpha+1} + \sum_{j=1}^{N} b_{\alpha+1 j}\sigma_j \quad (\text{on } E_\alpha \cap E_{\alpha+1}). \tag{3.16}$$

---

[7] With this writing we mean that g is a holomorphic function in a neighborhood V of $E \cap S$ in S. The notation will be used in sequel.

[8] The formulation of this lemma and the proof below should be new.

Thus this yields a solution $H_{\alpha+1}$ on $E_\alpha \cup E_{\alpha+1}$; for this procedure we say that we merge the solutions G_α and $G_{\alpha+1}$ to obtain a solution $H_{\alpha+1}$ on $E_\alpha \cup E_{\alpha+1}$.

Starting from $\alpha = 1$, we merge G_1 and G_2 to obtain a solution H_2 on $E_1 \cup E_2$. We then merge H_2 and G_3 to obtain a solution H_3 on $E_1 \cup E_2 \cup E_3$. Repeating this procedure up to $\alpha = L-1$, we obtain a solution H_L on $E = \bigcup_{\alpha=1}^{L} E_\alpha$, and set $G = H_L$: This finishes the proof of Lemma 3.11. □

Remark 3.17. We call the above induction argument *cuboid induction on dimension*, which will be used furthermore in the sequel.

It is well-known that Oka's Jôku-Ikô Lemma 3.11 immediately implies (cf., e.g., [17] Lemma 4.4.17):

THEOREM 3.18 (Runge–Weil–Oka Approximation). *Let $\Delta \Subset \Omega$ be an analytic polyhedron of a domain Ω ($\subset \mathbf{C}^n$). Then every holomorphic function on the closure $\overline{\Delta}$ is uniformly approximated on $\overline{\Delta}$ by elements of $\mathcal{O}(\Omega)$.*

4. Cousin I/II, $\bar{\partial}$, extension and Levi's (Hartogs' Inverse) Problems

The aim of this section is to show how the result obtained in the previous section is applied to solve the titled problems.

4.1. Cousin I/II, and $\bar{\partial}$-equation

We will give one unified proof to all of the three problems. We recall them: Let $\Omega \subset \mathbf{C}^n$ be a domain, let $\Omega = \bigcup_{\alpha\in\Lambda} U_\alpha$ be an open covering, and let $\mathscr{M}(U_\alpha)$ denote the set of all meromorphic functions in U_α.

I (**Cousin I**) For given $f_\alpha \in \mathscr{M}(U_\alpha)$ ($\alpha \in \Lambda$) satisfying $f_\alpha - f_\beta \in \mathcal{O}(U_\alpha \cap U_\beta)$ (Cousin-I data), find $F \in \mathscr{M}(\Omega)$ (called a *solution*) with $F|_{U_\alpha} - f_\alpha \in \mathcal{O}(U_\alpha)$ for all $\alpha \in \Lambda$.

II (**Cousin II, Oka Principle**) Here we assume that U_α are simply-connected. Let $f_\alpha \in \mathscr{M}^*(U_\alpha)$ ($\alpha \in \Lambda$) be locally non-zero meromorphic functions satisfying

(a) $f_\alpha/f_\beta \in \mathcal{O}^*(U_\alpha \cap U_\beta)$ (nowhere vanishing holomorphic functions) (Cousin-II data),

(b) (Topological condition) there are nowhere vanishing continuous functions $\psi_\alpha \in \mathscr{C}^*(U_\alpha)$ with $\psi_\alpha/\psi_\beta = f_\beta/f_\alpha$ on $U_\alpha \cap U_\beta$.

Find $F \in \mathscr{M}^*(\Omega)$ with $F|_{U_\alpha}/f_\alpha \in \mathcal{O}^*(U_\alpha)$ for all $\alpha \in \Lambda$.

We may take a continuous branch $h_\alpha := \log \psi_\alpha$ in each U_α. It follows that

$$h_\alpha|_{U_\alpha\cap U_\beta} - h_\beta|_{U_\alpha\cap U_\beta} \in \mathcal{O}(U_\alpha \cap U_\beta). \tag{4.1}$$

The problem to find the solution F above is reduced to the following problem:

($\star$) For a given family $\{h_\alpha\}$ of continuous functions satisfying (4.1), find a continuous function H (called a solution) in Ω such that for every U_α

$$H|_{U_\alpha} - h_\alpha \in \mathcal{O}(U_\alpha).$$

III ($\bar{\partial}$**-Equation**) For a given C^∞-$(0,1)$-form u on Ω with $\bar{\partial}u = 0$, find a C^∞-function g (called a *solution*) on Ω with $\bar{\partial}g = u$.

Locally, by Dolbeault's lemma, there is a solution f of this problem in a neighborhood of a point of Ω. Thus, there are an open covering $\{U_\alpha\}_{\alpha\in\Lambda}$ of Ω and C^∞-functions g_α on U_α such that $\bar{\partial}g_\alpha = u|_{U_\alpha}$. Then, the present problem is equivalent to find a C^∞-function G (*solution*) on Ω with $G|_{U_\alpha} - g_\alpha \in \mathcal{O}(U_\alpha)$ for all $\alpha \in \Lambda$.

CONVENTION. For a unified treatment for the above problems, we introduce an "*argument* χ" representing one of I–III above: Problem-χ means one of Problems I–III above, where Problem-II means Problem ($\star$), and a χ-solution means a *solution* of the corresponding Problem-χ.

Remark 4.2. If Ψ is so obtained in Cousin-II Problem above, then $F_1 = f_\alpha e^{\log\psi_\alpha - \Psi} \in \mathscr{M}^*(\Omega)$ satisfies the required property for F. Then we have a homotopy,

$$F_t = f_\alpha e^{\log\psi_\alpha - t\Psi}, \quad 0 \le t \le 1,$$

from the topological assumed solution $F_0 (= f_\alpha\psi_\alpha)$ to an aimed analytic (meromorphic) solution F_1.

Remark 4.3. The common property of Problem-χ that we will use is the following: If f and f' are two solutions of Problem-χ on an open set U in general, then $f - f' \in \mathcal{O}(U)$.

We begin with:

LEMMA 4.4. *Let* P *be an open cuboid in* $\mathbf{C}^n$ *and let* S *be a complex submanifold of* P. *We consider Problem-*χ *defined on* S. *Let* $E \Subset$ P *be a closed cuboid. Then there is a* χ*-solution on* $E \cap S (\Subset S)$.[9)]

Proof. We use cuboid induction on dimension.

(a) Case of $\dim E = 0$: It is clear by definition.

(b) Case of $\dim E = \nu$ ($\nu \ge 1$) with the induction hypothesis that the case of $\dim E = \nu - 1$ holds: Without loss of generality we may assume that E is given as in (3.6), and let E_t be as in (3.12). Since $\dim E_t = \nu - 1$, the induction hypothesis implies the existence of a χ-solution Φ_t on $E_t \cap S$ ($\Subset S$). Then, by

[9)] Cf. footnote 7) at p. 9.

the Heine-Borel Theorem there are a partition of $[-T, T]$, E_α $(1 \le \alpha \le L)$ as in (3.13), and χ-solutions Φ_α on $E_\alpha \cap S (\Subset S)$.

If $E_\alpha \cap E_{\alpha+1} \cap S \neq \emptyset$, we say that E_α and $E_{\alpha+1}$ is pairwise connected on S. It is sufficient to prove the existence of a χ-solution for each maximal sequence of E_α pairwise connected on S,

$$E_{\alpha_0} \cup E_{\alpha_0+1} \cup \cdots \cup E_{\alpha_1}. \tag{4.5}$$

For simplicity we suppose that $\alpha_0 = 1$. It follows from Remark 4.3 that for $1 \le \alpha \le \alpha_1$

$$\Phi_{\alpha+1} - \Phi_\alpha \in \Gamma(E_\alpha \cap E_{\alpha+1} \cap S, \mathcal{O}_S). \tag{4.6}$$

By Oka's Jôku-Ikô Lemma 3.11, there is a holomorphic function $H_\alpha \in \mathcal{O}(E_\alpha \cap E_{\alpha+1})$ such that

$$H_\alpha|_{E_\alpha \cap E_{\alpha+1} \cap S} = \Phi_{\alpha+1} - \Phi_\alpha. \tag{4.7}$$

By the Cousin decomposition of H_α as in (3.7) we have $\tilde{H}_\alpha \in \mathcal{O}(E_\alpha)$ and $\tilde{H}_{\alpha+1} \in \mathcal{O}(E_{\alpha+1})$ such that

$$H_\alpha = \tilde{H}_\alpha - \tilde{H}_{\alpha+1} \quad (\text{on } E_\alpha \cap E_{\alpha+1}). \tag{4.8}$$

We infer from (4.8) and (4.7) that

$$\Phi_\alpha + \tilde{H}_\alpha|_{E_\alpha \cap S} = \Phi_{\alpha+1} + \tilde{H}_{\alpha+1}|_{E_{\alpha+1} \cap S} \quad \text{on } E_\alpha \cap E_{\alpha+1} \cap S \ (\Subset S). \tag{4.9}$$

Note that $\Phi_\alpha + \tilde{H}_\alpha|_{E_\alpha \cap S}$ (resp. $\Phi_{\alpha+1} + \tilde{H}_{\alpha+1}|_{E_{\alpha+1} \cap S}$) is a χ-solution on $E_\alpha \cap S (\Subset S)$ (resp. $E_{\alpha+1} \cap S (\Subset S)$). Thus, from (4.9) we obtain a merged χ-solution $\Psi_{\alpha+1}$ on $(E_\alpha \cup E_{\alpha+1}) \cap S (\Subset S)$ from Φ_α and $\Phi_{\alpha+1}$.

Now, from Φ_1 and Φ_2 we obtain a merged χ-solution Ψ_2 on $(E_1 \cup E_2) \cap S (\Subset S)$. We then obtain a merged χ-solution Ψ_3 on $(E_1 \cup E_2 \cup E_3) \cap S (\Subset S)$ from Ψ_2 and Φ_3, and so on; we obtain a χ-solution on $(\bigcup_{\alpha=1}^{\alpha_1} E_\alpha) \cap S (\Subset S)$. □

Definition 4.10. A complex manifold M is said to be *holomorphically convex* if for every compact subset $K \Subset M$ the holomorphically convex hull of K defined by

$$\hat{K}_M = \{a \in M : |f(a)| \le \max\{|f(x)| : x \in K\},\ f \in \mathcal{O}(M)\}$$

is again compact.

Theorem 4.11. *Let Ω be a holomorphically convex domain. Then Problem-χ on Ω has a χ-solution on Ω.*

Proof. We take an increasing sequence of analytic polyhedra of Ω,

$$\Delta_1 \Subset \Delta_2 \Subset \Delta_3 \Subset \cdots, \quad \bigcup_{\nu=1}^{\infty} \Delta_\nu = \Omega. \tag{4.12}$$

For each ν we let $\phi_\nu : \overline{\Delta}_\nu \to \overline{\mathrm{P}\Delta}_\nu$ be the Oka map (a holomorphic proper embedding) of $\overline{\Delta}_\nu$ into a closed polydisk $\overline{\mathrm{P}\Delta}_\nu$, which extends from a neighborhood U_ν of $\overline{\Delta}_\nu$ into a polydisk, biholomorphic to an open cuboid P_ν ($\supseteq \overline{\mathrm{P}\Delta}_\nu$). Then, the image $\phi_\nu(U_\nu)$ is a complex submanifold of P_ν. We identify U_ν with the image $\phi_\nu(U_\nu)$.

By Lemma 4.4 there is a χ-solution G_ν on every $\overline{\Delta}_\nu$. Put $F_1 = G_1$ on $\overline{\Delta}_1$. Suppose that χ-solutions F_ν on $\overline{\Delta}_\nu$, $1 \le \nu \le \mu$, are determined so that

$$\|F_{\nu+1} - F_\nu\|_{\overline{\Delta}_\nu} < \frac{1}{2^\nu}, \quad 1 \le \nu \le \mu. \tag{4.13}$$

Let $G_{\mu+1}$ be a χ-solution on $\overline{\Delta}_{\mu+1}$. Since $G_{\mu+1}|_{\overline{\Delta}_\mu} - F_\mu \in \mathcal{O}(\overline{\Delta}_\mu)$, by Theorem 3.18 there is an element $h_{\mu+1} \in \mathcal{O}(\overline{\Delta}_{\mu+1})$ with

$$\|G_{\mu+1}|_{\overline{\Delta}_\mu} - F_\mu - h_{\mu+1}\|_{\overline{\Delta}_\mu} < \frac{1}{2^{\mu+1}}.$$

Setting $F_{\mu+1} = G_{\mu+1} - h_{\mu+1}$, we see that (4.13) holds up to $\mu+1$. Inductively, we have χ-solutions F_ν on $\overline{\Delta}_\nu$ satisfying (4.13), and the series

$$F = F_\mu + \sum_{\nu=\mu}^{\infty} (F_{\nu+1} - F_\nu)$$

converges locally uniformly and the limit gives rise to a χ-solution on Ω. □

Remark 4.14. As easily seen, the above proof of Theorem 4.11 works on Stein manifolds, which is defined by:

DEFINITION 4.15. A complex manifold M is said to be *Stein* if the following conditions are satisfied.

(i) M satisfies the second countability axiom:
(ii) (*Holomorphic separation*) For every two distinct two point $a, b \in M$ there is a holomorphic function $f \in \mathcal{O}(M)$ on M with $f(a) \neq f(b)$:
(iii) For every point $a \in M$ there are holomorphic functions $f_j \in \mathcal{O}(M)$, $1 \le j \le n = \dim_a M$, such that $df_1(a) \wedge \cdots \wedge df_n(a) \neq 0$:
(iv) M is holomorphically convex (see Definition 4.10).

4.2. Extension problem

By means of the Weak Coherence Theorem 1.2 we consider the extension problem (interpolation problem) from a complex submanifold in a holomorphically convex domain.

THEOREM 4.16. *Let $\Omega \subset \mathbf{C}^n$ be a holomorphically convex domain and let $S \subset \Omega$ be a complex submanifold. Then the restriction map*

$$F \in \mathcal{O}(\Omega) \to F|_S \in \mathcal{O}(S)$$

is a surjection.

Proof. We take analytic polyhedra $\Delta_\nu \Subset \Omega$ and Oka maps $\phi_\nu : \overline{\Delta}_\nu (\Subset U_\nu) \to \overline{\mathrm{P}\Delta}_\nu (\Subset \mathrm{P}_\nu)$ $(\nu = 1, 2, \ldots)$ as in the proof of Theorem 4.11. By Theorem 3.9 (i) there is a finite generator system $\{\underline{\sigma}_{\nu j}\}_{j=1}^{N_\nu}$ of $\mathscr{I}\langle S \cap \mathrm{P}_\nu \rangle$ on each $\overline{\mathrm{P}\Delta}_\nu$ $(\Subset \mathrm{P}_\nu)$, where U_ν is identified with $\phi_\nu(U_\nu)$.

Let $f \in \mathcal{O}(S)$ be any element. By Oka's Jôku-Ikô Lemma 3.11 there are $G_\nu \in \mathcal{O}(\overline{\mathrm{P}\Delta}_\nu)$ with $G_\nu|_{\overline{\Delta}_\nu \cap S} = f|_{\overline{\Delta}_\nu \cap S}$ $(\nu = 1, 2, \ldots)$.

We set $F_1 = G_1|_{\overline{\Delta}_1}$. Suppose that $F_\nu \in \mathcal{O}(\overline{\Delta}_\nu)$, $1 \le \nu \le \mu$, are determined so that

$$F_\nu = f|_{\overline{\Delta}_\nu \cap S}, \quad \|F_{\nu+1} - F_\nu\|_{\overline{\mathrm{P}\Delta}_\nu} < \frac{1}{2^\nu}, \quad 1 \le \nu \le \mu - 1. \tag{4.17}$$

For $\nu = \mu + 1$ we first note that $(G_{\mu+1}|_{\overline{\Delta}_\mu} - F_\mu)|_{\overline{\Delta}_\mu \cap S} = 0$. By Lemma 3.11 there is an element $H_\mu \in \mathcal{O}(\overline{\mathrm{P}\Delta}_\mu)$ with $H_\mu|_{\overline{\Delta}_\mu} = G_{\mu+1}|_{\overline{\Delta}_\mu} - F_\mu$. Since $\underline{H}_\mu \in \Gamma(\overline{\mathrm{P}\Delta}_\nu, \mathscr{I}\langle S \rangle)$, by Theorem 3.9 (ii) there are $h_{\mu j} \in \mathcal{O}(\overline{\mathrm{P}\Delta}_\mu)$, $1 \le j \le N_{\mu+1}$, such that

$$H_\mu = \sum_{j=1}^{N_{\mu+1}} h_{\mu j} \cdot \sigma_{\mu+1 j} \quad \text{on } \overline{\mathrm{P}\Delta}_\mu.$$

Restricting this to $\overline{\Delta}_\nu$, we have

$$G_{\mu+1}|_{\overline{\Delta}_\mu} = F_\mu + \sum_{j=1}^{N_{\mu+1}} h_{\mu j} \cdot \sigma_{\mu+1 j}|_{\overline{\Delta}_\mu}.$$

Approximating $h_{\mu j}$ sufficiently close by $\tilde{h}_{\mu j} \in \mathcal{O}(\Omega)$ on $\overline{\Delta}_\mu$ (Theorem 3.18), and setting

$$F_{\mu+1} = G_{\mu+1} - \sum_{j=1}^{N_{\mu+1}} \tilde{h}_{\mu j} \cdot \sigma_{\mu+1 j} \in \mathcal{O}(\overline{\Delta}_{\mu+1}),$$

we have

$$F_{\mu+1}|_{\overline{\Delta}_{\mu+1} \cap S} = f|_{\overline{\Delta}_{\mu+1} \cap S}, \quad \|F_{\mu+1} - F_\mu\|_{\overline{\Delta}_\mu} < \frac{1}{2^\mu}.$$

Then the series

$$F = F_\mu + \sum_{\nu=\mu}^{\infty} (F_{\nu+1} - F_\nu)$$

converges locally uniformly to the limit $F \in \mathcal{O}(\Omega)$ with $F|_S = f$. □

Remark 4.18. The above proof of Theorem 4.16 works on Stein manifolds.

4.3. Levi's (Hartogs' Inverse) Problem

4.3.1. *Levi's (Hartogs' Inverse) Problem.* We first recall some basic terminologies (see, e.g., [9], [12], [17] for more details). Let M be a connected complex manifold.

An upper semi-continuous function ψ on M is said to be *plurisubharmonic* if every restriction of ψ to a 1-dimensional complex submanifold of any holomorphic local chart of M is subharmonic. If ψ is of C^2-class, it is plurisubharmonic if and only if the hermitian matrix

$$\left(\frac{\partial^2\psi}{\partial z_j\bar{\partial}z_k}\right) \geq 0 \quad \text{(semi-positive definite)},$$

where $z=(z_j)$ is a holomorphic local coordinate system of M; moreover, if

$$\left(\frac{\partial^2\psi}{\partial z_j\bar{\partial}z_k}\right) \gg 0 \quad \text{(positive definite)},$$

then ψ is said to be *strongly plurisubharmonic*.

If M carries a continuous plurisubharmonic exhaustion ψ[10], then M is said to be *pseudoconvex*. Note that there are several equivalent definitions of being "pseudoconvex" (cf. [10], [9], [12], [14], [17], etc.).

A relatively compact domain $\Omega \Subset M$ is said to be *strongly pseudoconvex* if for every boundary point $b \in \partial\Omega$ there are a neighborhood U of b in M and a strongly plurisubharmonic function ϕ on U satisfying

$$\Omega \cap U = \{x \in U : \phi(x) < 0\}.$$

It is known that a strongly pseudoconvex domain is pseudoconvex.

If M is connected and there is a locally finite holomorphic map $\pi : M \to \mathbf{C}^n$ with $\dim M = n$, then we call M a Riemann domain, in general; furthermore, if π is (resp. not) locally biholomorphic, we call M a (resp. ramified) *unramified Riemann domain*; in this case, M carries a distance function induced from the euclidean metric on $\mathbf{C}^n$ through π, so that M satisfies the second countability axiom.

It had been known that a Stein manifold is pseudoconvex: Levi's (Hartogs' Inverse) Problem had asked originally the converse for univalent domains of $\mathbf{C}^n$. K. Oka extended the problem for Riemann domains. (There was a necessity to do so (cf., e.g., [17] §5.1).)

4.3.2. *Oka's method.* Notice that Oka's Jôku-Ikô Lemma 3.11 is sufficient to deduce *Oka's Heftungslemma* which, together with a method of an integral equation and the construction of a plurisubharmonic exhaustion on a pseudoconvex unramified Riemann domain, implies Levi's (Hartogs' Inverse) Problem (cf. Oka [26], [27], [32], [31], Andreotti-Narasimhan [1], Nishino [14]):

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[10] A functions ψ is called an exhaustion if $\{x \in M : \psi(x) < c\} \Subset M$ for all $c \in \mathbf{R}$.

THEOREM 4.19 (Oka, 1941/42/43/53; cf. §5). *Let M be an unramified Riemann domain. If M is pseudoconvex, then M is Stein.*

4.3.3. *Grauert's method.* In 1958 H. Grauert [8] gave another proof of Theorem 4.19 by proving the finite dimensionality of the first cohomology of coherent sheaves which was inspired by the Cartan–Serre Theorem for coherent sheaves on compact analytic spaces.[11] We shall observe that the Weak Coherence Theorem 1.2 suffices for Grauert's method to prove Theorem 4.19.

We use the first Čech cohomology $H^1(*, \mathcal{O})$. The following immediately follows from Theorem 4.11:

LEMMA 4.20. (i) *If Ω be a holomorphically convex domain of $\mathbf{C}^n$, then $H^1(\Omega, \mathcal{O}) = 0$.*

(ii) *Let M be a complex manifold with the second countability axiom, and let $\mathcal{U} = \{U_\nu\}$ be a Stein covering of M (i.e., every U_ν is open and Stein). Then we have*

$$H^1(\mathcal{U}, \mathcal{O}) \cong H^1(M, \mathcal{O}).$$

Then we can apply Grauert's bumping method [8] to prove:

THEOREM 4.21 (Grauert). *Let $\Omega \Subset M$ be a relatively compact domain of a complex manifold M with strongly pseudoconvex boundary. Then we have*

$$\dim_{\mathbf{C}} H^1(\Omega, \mathcal{O}) < \infty.$$

Proof of Theorem 4.19. (i) The key of the proof is to show that a strongly pseudoconvex domain $\Omega \Subset M$ is Stein. Let $b \in \partial\Omega$ be a boundary point. By the definition of strong pseudoconvexity there are a neighborhood U of b in M and a quadratic polynomial $P_b(z_1, \ldots, z_n)$ such that $U \cap \overline{\Omega} \cap \{P_b = 0\} = \{b\}$. Then there is a little bit larger strongly pseudoconvex domain $\Omega_\varepsilon \supset\supset \Omega$ such that $\Omega_\varepsilon \cap \{P_b = 0\}$ is closed in Ω_ε. We consider Cousin-I data $(U_0, f_{\nu 0})$ and $(U_1, f_{\nu 1})$ of Ω_ε such that for $\nu = 1, 2, \ldots,$

$$f_{\nu 0} = \frac{1}{P_b(z)^\nu} \quad \text{on } U_0 = \Omega_\varepsilon \cap U,$$

$$f_{\nu 1} = 0 \qquad \text{on } U_1 = \Omega_\varepsilon \backslash \{P_b = 0\}.$$

Theorem 4.21 applied to these Cousin-I data on Ω_ε yields a meromorphic function F_b on Ω_ε such that F_b is holomorphic in U_1, and in U_0, F_b is written as

$$F_b(z) = \frac{c_\nu}{P_b(z)^\nu} + \cdots + \frac{c_1}{P_b(z)} + \text{holomorphic term}, \tag{4.22}$$

[11] Cf. the footnote of [8] p. 466. The proof relies on L. Schwartz's finiteness theorem, whose rather simple, short and complete proof is found in [5] and [17] pp. 313–315.

where $c_\mu \in \mathbf{C}$ $(1 \leq \mu \leq \nu)$ and $c_\nu \neq 0$. Thus $F_b|_\Omega \in \mathcal{O}(\Omega)$ and Ω being holomorphically convex is deduced.

(ii) To show the holomorphic separation of Ω, we take two distinct points $Q_1, Q_2 \in \Omega$. We may assume that $\pi(Q_1) = \pi(Q_2) = a \in \mathbf{C}^n$. Let $\phi(t)$, $t \geq 0$, be any affine linear curve with $\phi(0) = a$. Then there is a unique lifting $\phi_j(t) \in \Omega$, $j = 1, 2$, such that $\phi_j(0) = Q_j$ and $\pi \circ \phi_j(t) = \phi(t)$. Since Ω is relatively compact, $\phi_j(t)$ hits the boundary $\partial\Omega$. We may assume that $\phi_1(t)$ hits $\partial\Omega$ first with $t = T \in \mathbf{R}$, so that $\phi_j([0, T]) \subset \overline{\Omega}$ $(j = 1, 2)$ and $\phi_1(T) \in \partial\Omega$. Note that $\phi_1(T) \neq \phi_2(T)$. With setting $b = \phi_1(T)$ we have by (4.22) a meromorphic function F_b in Ω_e which is holomorphic in Ω.

We consider the Taylor expansions of F_b at Q_1 and Q_2 in coordinates $(z_1, \ldots, z_n)$. Since F_b has a pole at $\phi_1(T)$ and no pole at $\phi_2(T)$, those two expansions must be different. Therefore, there is some partial differential operator $\partial^\alpha = \partial^{|\alpha|}/\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}$ with a multi-index α such that

$$\partial^\alpha F_b(Q_1) \neq \partial^\alpha F_b(Q_2).$$

Since $\partial^\alpha F_b$ is holomorphic in Ω, this finishes the proof. □

Remark 4.23. (i) The idea of the proof above is inspired by Oka's unpublished paper in 1943 ([33]). Note that it is subtle how to deal with the holomorphic separation of an unramified Riemann domain. In Hörmander [12] the holomorphic separation is included in the definition of Riemann domains. In [17] Chap. 7 we used Grauert's Theorem 4.21 for a non-singular geometric ideal sheaves, which can be also deduced from our Weak Coherence Theorem, but then it involves a sheaf-cohomological argument. In Gunning–Rossi [10] and in Nishino [14] they gave their own proofs.

(ii) The proof presented above provides a complete proof of Levi's (Hartogs' Inverse) Problem without the sheaf cohomology theory nor L^2-$\bar{\partial}$ method.

5. Historical remarks

Here, cf. [17] Chap. 9 "On Coherence" and [19] for more details, and cf., e.g., Lieb [13] for the general background.

Oka's Theorem 4.19 was first proved for univalent domains $\Omega \subset \mathbf{C}^2$ by Oka [26] (announcement) in 1941, and the full paper [27] was published in 1942 with a comment of the validity for $n \geq 3$.

In 1943 Oka proved Theorem 4.19 for unramified Riemann domains of general dimension ≥ 2 in a series of research reports of p. 109 in total, sent to Teiji Takagi: The reports were written in Japanese and unpublished (see [32], [33]). He remarked this fact three times, first in his survey note [28] (1949), VIII [30] (1951) and IX [31] (1953). The paper [28] has not been referred very much, but it should have a significant interest, for it was written during the submission of VII [29] before the publication; he had sent it to H. Cartan 1948 one year ago

(through the hands of S. Kakutani and A. Weil), and in return he had received Cartan's conjectural (or experimental) paper [3] referred in [28]. He surveyed the state of the development of analytic function theory of several variables at the time.

In the 1943 reports to Takagi he did not use Weierstrass' Preparation Theorem, but he was writing a primitive form of the notion of coherence and non-reduced structures of analytic subsets; the study later led to the notion "*idéaux de domaines indéterminés*", coherence in 1948 ([29]). The key of Oka's proof of Theorem 4.19 was his "*Heftungslemma*". In [26] and [27] he proved Heftungslemma by Weil's integral, but in 1943 ([32] no. 1, [33]) he replaced Weil's integral by simple Cauchy's integral, proving "Oka's Jôku-Ikô" for unramified Riemann domains.

In 1949 S. Hitotsumatsu [11] written in Japanese gave a proof of Oka's Heftungslemma by Weil's integral to solve Levi's (Hartogs' Inverse) Problem in general dimension $n \geq 2$; here he gave no argument of plurisubharmonic exhaustions on pseudoconvex unramified Riemann domains, and so the result might hold only for univalent domains.

In 1953 Oka [31] proved Theorem 4.19 above by making use of his First and Second Coherence Theorems obtained in [29]: the Third Coherence Theorem was not used there.

In 1954 Bremermann [2] and Norguet [20] independently proved Theorem 4.19 for univalent domains $\Omega \subset \mathbf{C}^n$ with general $n \geq 2$, generalizing Oka's Heftungslemma by means of Weil's integral, similarly to Hitotsumatsu [11].

Concluding Remark (*Oka's Problem*). It is interesting to learn that Oka invented and proved three fundamental coherence theorems by means of Weierstrass' Preparation Theorem in order to treat the pseudoconvexity problem on singular ramified Riemann domains. Levi's (Hartogs' Inverse) Problem for ramified Riemann domains has a counter-example (Fornæss [7]), but in the same time there is a positive case for which Levi's (Hartogs' Inverse) Problem is affirmative ([18]). Because of the above historical facts we may call the following question

Oka's Problem: *What is necessary and/or sufficient for the validity of Levi's* (*Hartogs' Inverse*) *Problem on a ramified Riemann domain* X (*over* $\mathbf{C}^n$)?:

This is open even when X is non-singular.

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