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ON A RIGIDITY OF SOME MODULAR GALOIS DEFORMATIONS

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Abstract

Let F be a totally real field and $\boldsymbol{\rho} = (\rho_\lambda)_\lambda$ be a compatible system of two dimensional λ-adic representations of the Galois group of F. We assume that $\boldsymbol{\rho}$ has a residually modular λ-adic realization for some λ. In this paper, we consider local behaviors of modular deformations of λ-adic realizations of $\boldsymbol{\rho}$ at unramified primes. In order to control local deformations at specified unramified primes, we construct certain Hecke modules. Applying Kisin's Taylor-Wiles system, we obtain an $R = T$ type result supplemented with local conditions at specified unramified primes. As a consequence, we shall show a potential rigidity of some modular deformations of infinitely many λ-adic realizations of $\boldsymbol{\rho}$.

1. Introduction

Let F be a totally real field and S be a finite set of places of F which contains all infinite places. Let $G_{F,S}$ be the Galois group of the maximum extension of F which is unramified outside S. We take a rational prime $p > 2$ and we assume that all primes of F dividing p are contained in S. In this paper, we consider a two dimensional *modular* p-adic representation

$$\rho_{f,\lambda} : G_{F,S} \to \mathrm{GL}_2(K_{f,\lambda}).$$

Namely it is a continuous representation of $G_{F,S}$ associated to a Hilbert modular Hecke eigenform f over F; here $K_{f,\lambda}$ is the completion of the Hecke field K_f of f at a prime λ dividing p, and S is containing all primes dividing a level of f. In the elliptic modular (i.e. $F = \mathbf{Q}$) case, $\rho_{f,\lambda}$ is constructed by Eichler, Shimura and Deligne as is well known, and in the Hilbert modular (i.e. $F \neq \mathbf{Q}$) cases they are constructed by Shimura [18], Ohta [17], Carayol [2], Taylor [21], Blasius-Rogawski [1] and others. Moreover, the family $\boldsymbol{\rho}_f = (\rho_{f,\lambda})_\lambda$ of p-adic representations of f indexed by primes λ of K_f forms the regular and irreducible rank two strong compatible system; for the detail, see Introduction of [23].

As Fujiwara pointed out in Introduction of [9], modular p-adic representations play central roles in the theory of Galois representations, so it is important

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to reveal the conditions when a p-adic representation

$$\rho : G_{F,S} \to \mathrm{GL}_2(E)$$

is modular; here E is a finite extension of $\mathbf{Q}_p$. A piece of remarkable results is the theory of modularity lifting, which is developed by Wiles [25], Taylor-Wiles [24], Faltings [loc.cit., Appendix], Diamond [5], [6], Fujiwara [8], Kisin [13] and others. In his paper Kisin showed the modularity of a two dimensional p-adic representation $\rho : G_{F,S} \to \mathrm{GL}_2(E)$ which is potentially Barsotti-Tate at primes $\mathfrak{p}$ dividing p and whose mod p reduction $\bar{\rho}$ is associated to a Hilbert modular eigenform of parallel weight two. The important ingredient of Kisin's proof is the $R = T$ theorem ([13, Proposition (3.4.11)]), which states a structure theorem of the universal deformation ring of $\bar{\rho}$. We shall review it briefly.

Let $\mathcal{O}$ be the ring of integers of E. Taking a $G_{F,S}$-stable $\mathcal{O}$-lattice of the representation ρ we consider it as a continuous representation $\rho : G_{F,S} \to \mathrm{GL}_2(\mathcal{O})$. We denote by Σ the set of finite places $v \nmid p$ at which ρ is ramified; we also put $\Sigma_p := \Sigma \cup \{\mathfrak{p} : \mathfrak{p} \mid p\}$. Then the symbol R in "$R = T$" denotes the noetherian complete local $\mathcal{O}$-algebra

$$R = R_{F,S}^{\psi,\square} \otimes \tilde{R}_{\Sigma,p}^{\psi,\sigma,\square}$$

obtained by tensoring the noetherian complete local $\mathcal{O}$-algebra[1)]

$$\tilde{R}_{\Sigma,p}^{\psi,\sigma,\square} = \hat{\bigotimes}_{v\in\Sigma_p} \tilde{R}_v^{\psi,\sigma,\square}$$

to the universal framed deformation ring $R_{F,S}^{\psi,\square}$ of $\bar{\rho}$ with fixed determinant condition. Here v runs over all primes in Σ_p and $\tilde{R}_v^{\psi,\sigma,\square}$ denotes the suitable quotient of the framed local deformation ring of $\bar{\rho}|_{G_{F_v}}$; at primes dividing p we have to take further modification corresponding to the ordinarity data σ, see §2.3 and §2.5. We say a deformation of $\bar{\rho}$ is *of type* $\tilde{R}_{\Sigma,p}^{\psi,\sigma,\square}$ when its local restrictions at primes $v \in \Sigma_p$ are controlled by $\tilde{R}_{\Sigma,p}^{\psi,\sigma,\square}$. On the other hand, the symbol T means a certain p-adic local Hecke algebra, which controls framed deformations of $\bar{\rho}$ associated to modular forms; in this paper we will denote it by $\mathbf{T}^{\square}$. Then Kisin's $R = T$ theorem asserts that the surjective map

$$R \to \mathbf{T}^{\square}$$

obtained by $\bar{\rho}$ and σ has the p-power torsion kernel.

In this theorem, it seems that the local behavior at ramified primes are dominant for the deforming of $\bar{\rho}$. So we have a following simple question:

QUESTION 1.1. *In the situation when Kisin's $R = T$ is established, how much local deformations does it permit at unramified primes?*

In this paper, we give an answer of this question. The precise statement of the result is the following:

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1) The important ingredient of Kisin's theorem is that $\tilde{R}_{\Sigma,p}^{\psi,\sigma,\square}$ is a domain; see [13, (2.5), (3.4)].

THEOREM 1.2 (Theorem 3.15). *Let F be a totally real number field. Let K be a number field, Σ be a finite set of finite primes of F, $\mathcal{S}$ be a finite set of finite primes of K and*

$$\mathcal{R} := (K, \Sigma, \mathcal{S}, \{Q_v(X)\}_{v \notin \Sigma}, \boldsymbol{\rho} = (\rho_\lambda)_{\lambda \in |K|^\infty})$$

be a regular and irreducible rank two weakly pre-compatible system of λ-adic representations of G_F; for the definition, see §2.6. We assume that there is a finite character $\psi : G_F \to \mathcal{O}_K^\times$ such that, for any prime $\lambda \in |K|^\infty$ the determinant of ρ_λ is $\psi\varepsilon_{p_\lambda}$; here we denote by p_λ the residue characteristic of K_λ and by ε_{p_λ} the p_λ-adic cyclotomic character of G_F. We further assume that there is a prime $\lambda \notin \mathcal{S}$ lying over a rational prime $p = p_\lambda > 2$ such that the λ-adic realization

$$\rho = \rho_\lambda : G_F \to \mathrm{GL}_2(K_\lambda)$$

satisfies the conditions in [13, Theorem (3.5.5)]. *Namely,*

(1) *For any prime $\mathfrak{p}$ dividing p, the restriction $\rho|_{G_{F_\mathfrak{p}}}$ is potentially Barsotti-Tate*;
(2) *ρ is strongly residually modular* (*in the sense of Remark* 3.14);
(3) *$\bar{\rho}|_{G_{F(\zeta_p)}}$ is absolutely irreducible*;
(4) *If $p = 5$ and the projective image of* $\mathrm{Im}\,\bar{\rho}$ *is isomorphic to* $\mathrm{PGL}_2(\mathbf{F}_5)$, *then the kernel of the projectivization of $\bar{\rho}$ does not fix $F(\zeta_5)$.*

Then, for infinitely many primes $\mu \notin \mathcal{S}$ we have the following: After taking a suitable totally real solvable base change $F^\#$ of F (*and denoting objects of $F^\#$ by the same symbols*), *each $\mathcal{O}_{K_\mu}$-deformation ρ'_μ of*

$$\bar{\rho}_\mu : G_F \to \mathrm{GL}_2(\mathbf{F}_\mu)$$

of type $\tilde{R}^{\psi,\sigma_\mu,\square}_{\Sigma,p_\mu}$ has a field automorphism $\phi \in \mathrm{Aut}(\overline{K}_\mu)$ satisfying

$$(\rho'_\mu \otimes \overline{K}_\mu|_{G_{F_v}})^{\mathrm{ss}} \simeq \phi_*(\rho_\mu \otimes \overline{K}_\mu|_{G_{F_v}})^{\mathrm{ss}}$$

for almost all unramified primes v of ρ_μ. Here σ_μ is the ordinary data associated to ρ_μ and "ss" *denotes semi-simplifications.*

As a consequence, using the isomorphism criterion by the Chebotarev density theorem, we obtain the following result.

COROLLARY 1.3 (Corollary 3.17). *Let $\mathcal{R}$ be a regular and irreducible rank* 2 *weakly pre-compatible system of λ-adic representations of G_F satisfying conditions in Theorem* 1.2. *Then, for infinitely many primes μ of K, there is a totally real solvable base change $F^\#/F$ such that, after taking it all $\mathcal{O}_{K_\mu}$-deformations ρ_1, ρ_2 of $\bar{\rho}_\mu$ of type $\tilde{R}^{\psi,\sigma_\mu,\square}_{\Sigma,p_\mu}$ are isomorphic to each other modulo an automorphism of $\overline{K}_\mu$; namely, there is a field automorphism $\phi \in \mathrm{Aut}(\overline{K}_\mu)$ such that*

$$\rho_1 \otimes \overline{K}_\mu|_{G_{F^\#}} \simeq \phi_*(\rho_2 \otimes \overline{K}_\mu)|_{G_{F^\#}}.$$

The strategy of the main theorem (Theorem 1.2) is as follows: (0) Let $\mathcal{R}$ be a pre-compatible system as in the statement of Theorem 1.2. We put $E = K_\lambda$ and denote by $\mathcal{O}$ the ring of integers of E. We take a $G_{F,S}$-stable $\mathcal{O}$-lattice of ρ_λ and consider it as a continuous representation $\rho_\lambda : G_{F,S} \to \mathrm{GL}_2(\mathcal{O})$. Let f be a Hilbert modular Hecke eigenform of parallel weight two of F such that $\bar{\rho}_\lambda \simeq \bar{\rho}_{f,\lambda}$. Let $S_{2,\psi}^{M_2}(U, \mathbf{C})$ be the space of Hilbert modular forms over F of parallel weight 2, adelic level U and character ψ, which contains f. (1) Taking a suitable totally real base change due to Langlands, Skinner-Wiles and Kisin, we reduce the situation to the spacial case when Kisin's $R = T$ theorem is applicable. By the Jacquet-Langlands and Shimizu correspondence, we associates f to a (p-adic) quaternionic Hecke eigenform f^D of some totally definite quaternion algebra D over F. (2) Let $\mathcal{P}$ be a finite set of unramified primes of ρ_λ. We focus our attention to local behaviors of deformations of $\bar{\rho}_\lambda$ at specified unramified primes in $\mathcal{P}$. In order to observe these, we define the local conditions of framed deformations of $\bar{\rho}_\lambda$ which controls local deformations at all primes in $\mathcal{P}$. (3) We construct Hecke modules $M^{\mathcal{P}_*}$ and $M_{Q_n}^{\mathcal{P}_*}$ for all $n \geq 1$ to apply Kisin's Taylor-Wiles system supplementing with our local conditions. These Hecke modules will be constructed from representation spaces of modular Galois deformations of type $\tilde{R}_{\Sigma,p}^{\psi,\sigma,\square}$ satisfying our local conditions at primes in $\mathcal{P}$. Applying the $\mathcal{O}[\Delta_{Q_n}]$-freeness result of the localized spaces of quaternionic modular forms due to Taylor, we will obtain the important property that the augmented quotient of $M_{Q_n}^{\mathcal{P}_*}$ is isormophic to $M^{\mathcal{P}_*}$; see Proposition 3.10. (4) We then apply Kisin's Taylor-Wiles system to our Hecke modules. Then under the global condition

$$\dim_{\mathbf{C}} S_{2,\psi}^{M_2}(U, \mathbf{C}) < p \tag{1.0.1}$$

we obtain the result that, for any modular $\mathcal{O}$-deformation of $\bar{\rho}$ of type $\tilde{R}_{\Sigma,p}^{\psi,\sigma,\square}$ its local deformations at all primes in $\mathcal{P}$ are controlled by local deformations at ramified primes. We note that the Condition 1.0.1 enables us to control the growth of p-adic Hecke fields at Taylor-Wiles deformations. (5) We can take a further totally real base change such that there are infinitely many primes $\mu \notin \mathcal{S}$ satisfying the condition (1.0.1) for $p = p_\mu$, and hence our $R = T$ result is applicable for ρ_μ. We take $\mathcal{P}$ as a one-point set of an arbitrary prime $v \notin S$, and we obtain the result of Theorem 1.2. Moreover, as a consequence of Chebotarev's density theorem, a global rigidity result of deformations of $\bar{\rho}_\mu$ of type $\tilde{R}_{\Sigma,p_\mu}^{\psi,\sigma_\mu,\square}$ (Corollary 1.3) is obtained.

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2. Preliminaries

Notation and conventions. In this paper, for any global or local field K we denote by $\mathcal{O}_K$ the ring of integers of K.

Let F be a totally real field. Let $|F|$ (resp. $|F|^{\infty}$) be the set of all places (resp. all finite places) of F. We fix an algebraic closure $\bar{F}$ of F and let G_F be the absolutely Galois group. For any $v \in |F|^{\infty}$, we fix an algebraic closure $\bar{F}_v$ of the completion F_v of F at v and fix an embedding $\bar{F} \hookrightarrow \bar{F}_v$, so that we identify the absolutely Galois group $G_{F_v} := \mathrm{Gal}(\bar{F}_v/F_v)$ with a decomposition group of G_F at v. We denote by I_v the inertia subgroup of G_{F_v}. At a finite place v, we choose a uniformizer ϖ_v of F_v and we normalize the reciprocity map $\mathrm{Art}_v : F_v^{\times} \to G_{F_v}^{\mathrm{ab}}$ of the local class field theory as $\varpi_v \mapsto \mathrm{Frob}_v$, where Frob_v is the *geometric* Frobenius element at v. Let $\mathbf{A}_F$ (resp. $\mathbf{A}_F^{\infty}$) be the ring of adeles (resp. of finite adeles) of F. When the base field F is clear, we often omit the subscript F.

Let S be a finite set of places of F containing all infinite places and let $G_{F,S}$ be the Galois group of the maximal extension of F which is unramified outside S. For any prime $v \notin S$, we denote the unique lifting of Frob_v in $G_{F,S}$ by the same symbol.

For a rational prime p, let E be a finite extension field of $\mathbf{Q}_p$. When we fix E we often write $\mathcal{O}_E$ as $\mathcal{O}$ for simplicity. We denote by $\mathbf{F}$ the residue field of $\mathcal{O}_E$. We also fix an algebraic closure $\bar{E}$ of E and for any place v we choose an isomorphism of fields $\iota_v : \bar{E} \simeq \bar{F}_v$ given by the axiom of choice; we also fix an isomorphism $\iota_{\infty} : \bar{E} \simeq \mathbf{C}$. Let $\varepsilon : G_F \to \mathcal{O}^{\times}$ be the p-adic cyclotomic character.

Let $\mathfrak{AR}_{\mathcal{O}}$ be the category consisting of finite local artinian $\mathcal{O}$-algebras $(A, \mathfrak{m}_A)$ equipped with an isomorphism $A/\mathfrak{m}_A \xrightarrow{\simeq} \mathbf{F}$ and with $\mathfrak{m}_A$-adic topology. The morphisms of $\mathfrak{AR}_{\mathcal{O}}$ are local $\mathcal{O}$-algebra homomorphisms. Here we note that each $A \in \mathfrak{AR}_{\mathcal{O}}$ has finite cardinality, so that the topology of A is discrete and each morphism of $\mathfrak{AR}_{\mathcal{O}}$ is continuous. We also define the category $\widehat{\mathfrak{AR}}_{\mathcal{O}}$ consisting of projective limits of objects of $\mathfrak{AR}_{\mathcal{O}}$.

2.1. Kisin's Taylor-Wiles system. First of all, we recall the modified Taylor-Wiles system due to Kisin. We also refer to [26]. In the following of this section, we fix a rational odd prime $p > 2$ and fix a finite extension E of $\mathbf{Q}_p$. We fix a uniformizer λ of $\mathcal{O}$. Then the statement of Kisin's Taylor-Wiles system is as follows.

Proposition 2.1 ([13, Proposition (3.3.1)]). *Let B be a complete local and flat $\mathcal{O}$-algebra, satisfying that:*

- *B is a domain of relative dimension b over $\mathcal{O}$,*
- *$B[1/p]$ is formally smooth over E.*

Let R be a complete local B-algebra and H is a non-zero R-module. Suppose that there exist non-negative integers r and j such that for any integer $n \geq 1$, there exist the following commutative diagrams of $\mathcal{O}$-algebras

$$
\begin{array}{ccccc}
 & & B[[X_1,\dots,X_{r+j-b}]] & & \\
 & & \downarrow\!\!\!\downarrow & & \\
\mathcal{O}[[S_1,\dots,S_r,Y_1,\dots,Y_j]] & \longrightarrow & R_n & \longrightarrow & \mathrm{End}_{\mathcal{O}}(H_n) \\
 & & \downarrow\!\!\!\downarrow & & \downarrow \\
 & & R & \longrightarrow & \mathrm{End}_{\mathcal{O}}(H),
\end{array}
$$

where H_n is an R_n-module and the dashed arrow means a map between the images of R_n and R. We further assume the following:

(1) $R_n/(S_1,\dots,S_r)R_n \xrightarrow{\simeq} R$ *and* $H_n/(S_1,\dots,S_r)H_n \xrightarrow{\simeq} H$,

(2) $\mathrm{Ann}_{\mathcal{O}[[S_1,\dots,S_r,Y_1,\dots,Y_j]]}(H_n)$ *is contained in the ideal* $((S_i+1)^{p^n}-1)_{1\le i\le r}$,

(3) H_n *is finite free over* $\mathcal{O}[[S_1,\dots,S_r,Y_1,\dots,Y_j]]/\mathrm{Ann}_{\mathcal{O}[[S_1,\dots,S_r,Y_1,\dots,Y_j]]}(H_n)$.

Then, the map

$$R \to \mathrm{End}_{\mathcal{O}}(H)$$

has p-power torsion kernel.

2.2. Global deformation rings and Taylor-Wiles deformations. In the following we assume $p > 2$ and that the finite set S of places of F contains all primes $\mathfrak{p}$ dividing p. Let ε be the p-adic cyclotomic character and $\psi : G_{F,S} \to \mathcal{O}^\times$ be a continuous character of finite order. Let $\bar{\rho} : G_{F,S} \to \mathrm{GL}_2(\mathbf{F})$ be an absolutely irreducible continuous representation of determinatnt $\psi\varepsilon$ modulo λ. We denote by $V_{\mathbf{F}}$ the representation space of $\bar{\rho}$, and we fix an ordered basis $\beta_{\mathbf{F}}$ of $V_{\mathbf{F}}$.

Now we consider deformations of $\bar{\rho}$. We denote by $D^{\square}_{F,S}$ the groupoid over $\mathfrak{AR}_{\mathcal{O}}$ defined as follows: For any $A \in \mathfrak{AR}_{\mathcal{O}}$, objects of the category $D^{\square}_{F,S}(A)$ are triples (V_A, ϕ, β), where V_A is a free A-module of rank two provided with a continuous $G_{F,S}$-action $\rho_A : G_{F,S} \to \mathrm{Aut}_A(V_A)$, ϕ is a $G_{F,S}$-equivariant $\mathbf{F}$-linear isomorphism $\phi : V_A \otimes_A \mathbf{F} \xrightarrow{\simeq} V_{\mathbf{F}}$ and β is an ordered A-basis of V_A which is a lifting of the fixed basis $\beta_{\mathbf{F}}$ of $V_{\mathbf{F}}$. In particular we identify $\beta \otimes_A \mathbf{F}$ with $\beta_{\mathbf{F}}$ by the isomorphism ϕ. We call an object of $D^{\square}_{F,S}(A)$ a *framed deformation* of $\bar{\rho}$ to A. For a given morphism $f : A \to A'$ of $\mathfrak{AR}_{\mathcal{O}}$, a covering morphism $(V_A, \phi, \beta) \to (V_{A'}, \phi', \beta')$ of f is a $G_{F,S}$-equivariant A'-linear isomorphism $V_A \otimes_f A' \xrightarrow{\simeq} V_{A'}$ compatible with ϕ, ϕ' and which sends β to β'.

We also denote by $D_{F,S}$ the groupoid over $\mathfrak{AR}_{\mathcal{O}}$ obtained by forgetting the basis data. An object of the category $D_{F,S}(A)$ over $A \in \mathfrak{AR}_{\mathcal{O}}$ is called a *deformation* of $\bar{\rho}$ to A. As is well-known, $D^{\square}_{F,S}$ is always (i.e. without the irreducibility of $\bar{\rho}$) pro-represented by a complete noetherian local $\mathcal{O}$-algebra $R^{\square}_{F,S}$; see [20] for instance. Under the absolute irreducibility of $\bar{\rho}$, the groupoid $D_{F,S}$ is also pro-represented by a complete noetherian local $\mathcal{O}$-algebra $R_{F,S}$. Let $R^{\psi,\square}_{F,S}$ (resp. $R^{\psi}_{F,S}$) be the quotient of $R^{\square}_{F,S}$ (resp. $R_{F,S}$) corresponding the fixed determinant condition $\det \rho_A = \psi\varepsilon$.

When $H^0(G_{F,S}, \mathrm{ad}\,\bar{\rho}) = \mathbf{F}$, the algebra $R^{\psi,\square}_{F,S}$ is isomorphic to the formal power series ring over $R^{\psi}_{F,S}$; namely $R^{\square}_{F,S} \cong R_{F,S}[[X_1, \dots, X_j]]$, where j denotes $4\#\Sigma_p - 1$. This follows from the fact that the morphism $D^{\square}_{F,S} \to D_{F,S}$ is formally smooth (cf. [13, p. 1165]).

Next we consider local deformations. We take and fix a (finite) subset Σ of S which do not meet the set of primes $\mathfrak{p}$ above p. Put $\Sigma_p := \Sigma \cup \{\mathfrak{p} \mid p\}$. For any $v \in \Sigma_p$, we consider the groupoid $D^{\square}_{\bar{\rho}|_{G_{F_v}}}$ of framed deformations of $\bar{\rho}|_{G_{F_v}}$. This is pro-represented by a complete noetherian local $\mathcal{O}$-algebra $R^{\square}_v$. We denote by $R^{\psi,\square}_v$ the quotient of $R^{\square}_v$ corresponding the fixed determinant condition $\det \rho_A = \psi\varepsilon$.

We define

$$R^{\psi,\square}_{\Sigma} := \hat{\bigotimes}_{v \in \Sigma} R^{\psi,\square}_v, \quad R^{\psi,\square}_p := \hat{\bigotimes}_{\mathfrak{p} \mid p} R^{\psi,\square}_{\mathfrak{p}}$$

and

$$R^{\psi,\square}_{\Sigma,p} := R^{\psi,\square}_{\Sigma} \hat{\otimes} R^{\psi,\square}_p.$$

Here all complete tensor products are taken over $\mathcal{O}$.

In order to apply the Kisin's Taylor-Wlies system in §3.4, we will suppose the following assumptions for $\bar{\rho}$.

Condition 2.2 ([13, (3.2.3)]). *We assume the following conditions*:

(1) *$\bar{\rho}$ is totally odd, namely for any complex conjugate $c \in G_{F,S}$,* $\det \bar{\rho}(c) = -1$.
(2) *$\bar{\rho}$ is unramified at all primes v not dividing p.*
(3) *The restriction of $\bar{\rho}$ to $G_{F(\zeta_p)}$ is absolutely irreducible.*
(4) *If $p = 5$ and the projective image of* $\mathrm{Im}\,\bar{\rho}$ *is isomorphic to* $\mathrm{PGL}_2(\mathbf{F}_5)$, *then the kernel of the projectivization of $\bar{\rho}$ does not fix $F(\zeta_5)$.*
(5) *If $v \in S \backslash \Sigma_p$ is a finite place, then $N(v) \not\equiv 1 \bmod p$ and*

$$(2.2.1) \quad (1 + N(v))^2 \det \bar{\rho}(\mathrm{Frob}_v) - N(v) \cdot (\mathrm{Tr}\,\bar{\rho}(\mathrm{Frob}_v))^2 \not\equiv 0 \bmod \lambda.$$

Here $N(v)$ denotes the cardinality of $\mathcal{O}_{F_v}/\varpi_v$.

These conditions are used for the calculation of the Selmer groups. For the condition (4), see also the proof of [23, Lemma 2.5]; in particular, this condition holds when $[F(\zeta_5) : F] = 4$ (cf. [13, p. 1155]).

Next we recall Taylor-Wlies deformations for each $n \geq 1$. These deformations correspond to $\mathcal{O}[[S_1, \dots, S_r, Y_1, \dots, Y_j]]$-algebras R_n in Proposition 2.1. The following proposition shows the existence of the set Q_n of primes, which are needed to construct the ring R_n in Proposition 2.1.

Proposition 2.3 ([13, Proposition (3.2.5)]). *We assume that a continuous representation $\bar{\rho} : G_{F,S} \to \mathrm{GL}_2(\mathbf{F})$ is absolutely irreducible and satisfies the condition 2.2. Then, for any positive integer $n \geq 1$, there exists a finite set Q_n of primes of F which do not meet S and satisfying the following*:

- *for each* $v \in Q_n$, $N(v) \equiv 1 \bmod p^n$ *and* $\bar{\rho}(\mathrm{Frob}_v)$ *has distinct eigenvalues,*
- *the cardinality of* Q_n *is equal to* $r := \dim_{\mathbf{F}} H^1(G_{F,S}, \mathrm{ad}^0\,\bar{\rho}(1))$,
- *if we take* $S_{Q_n} := S \cup Q_n$ *then the global universal deformation ring* $R^{\psi,\square}_{F,S_{Q_n}}$ *is topologically generated by* $r - [F:\mathbf{Q}] + \#\Sigma_p - 1$ *elements over* $R^{\psi,\square}_{\Sigma,p}$.

Let $R^{\square}_{F,S_{Q_n}}$ (resp. $R_{F,S_{Q_n}}$) be the universal deformation ring which pro-represents the groupoid $D^{\square}_{F,S_{Q_n}}$ (resp. $D_{F,S_{Q_n}}$) and $R^{\psi,\square}_{F,S_{Q_n}}$ (resp. $R^{\psi}_{F,S_{Q_n}}$) be its quotient corresponding the fixed determinant condition $\det = \psi\varepsilon$. A deformation of $\bar{\rho}$ controlled by $R^{\psi}_{F,S_{Q_n}}$ is called a deformation of *Taylor-Wiles type*. For any prime $v \in Q_n$ we denote by Δ_v the maximal p-power quotient of $(\mathcal{O}_{F_v}/\varpi_v\mathcal{O}_{F_v})^{\times}$, and put $\Delta_{Q_n} := \prod_{v \in Q_n} \Delta_v$. We now define the $\mathcal{O}[\Delta_{Q_n}]$-algebra structure of $R^{\psi}_{F,S_{Q_n}}$ (so that of $R^{\psi,\square}_{F,S_{Q_n}}$) as in [4, §2.8]. In particular, the canonical map $R^{\psi,\square}_{F,S_{Q_n}} \to R^{\psi,\square}_{F,S}$ of the universal deformation rings gives an isomorphism

$$R^{\psi,\square}_{F,S_{Q_n}}/\mathfrak{a}_{\Delta_{Q_n}} R^{\psi,\square}_{F,S_{Q_n}} \cong R^{\psi,\square}_{F,S}, \tag{2.2.2}$$

where $\mathfrak{a}_{\Delta_{Q_n}}$ is the augmentation ideal of $\mathcal{O}[\Delta_{Q_n}]$.

2.3. Local deformation rings. In the application of Proposition 2.1 to the modularity lifting theorem, we take the domain B as the tensor product of local deformation rings with the suitable conditions. In this subsection, we describe these according to Kisin [12, §2] and [13, §2]. Firstly we consider the local deformation ring at a prime dividing p.

We assume $p > 2$ and fix a prime $\mathfrak{p}$ of F dividing p. We consider a continuous representation $\bar{\rho}_{\mathfrak{p}} : G_{F_{\mathfrak{p}}} \to \mathrm{GL}_2(\mathbf{F})$, whose representation space is denoted by $V_{\mathbf{F}}$. We assume that the determinant of $\bar{\rho}_{\mathfrak{p}}$ is the mod p cyclotomic character times the reduction of a character $\psi : G_{F_{\mathfrak{p}}} \to W(\mathbf{F})$.

We also assume $\bar{\rho}_{\mathfrak{p}}$ is flat, and consider its flat deformations. In this paper a representation $\rho_A : G_{F_{\mathfrak{p}}} \to \mathrm{Aut}_A(M) \simeq \mathrm{GL}_2(A)$ over a finite ring A is called *flat* if there is a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_{F_{\mathfrak{p}}}$ such that $M \otimes (\det \rho_A)^{-1}$ is isomorphic to $\mathcal{G}(\bar{F}_{\mathfrak{p}})$ as $\mathbf{Z}[G_{F_{\mathfrak{p}}}]$-modules. Let $R^{\mathrm{fl},\square}_{V_{\mathbf{F}}}$ be the quotient of the framed universal deformation ring $R^{\square}_{V_{\mathbf{F}}}$ of $\rho_{\mathfrak{p}}$ corresponding to the flatness condition.

Let $\mathbf{v}$ be the p-adic Hodge type corresponding to the condition that the determinant of the restriction of deformations ρ_A of $\bar{\rho}_{\mathfrak{p}}$ to the inertia $I_{\mathfrak{p}}$ is equal to the p-adic cyclotomic character. Kisin constructs the moduli of $\mathfrak{S}$-modules ([13, (2.1)]), so-called *Kisin modules*, and the projective morphism

$$\Theta^{\mathbf{v}}_{V_{\mathbf{F}},\xi} : \mathscr{GR}^{\mathbf{v}}_{V_{\mathbf{F}},\xi} \to \mathrm{Spec}\, R^{\mathrm{fl},\square}_{V_{\mathbf{F}}}; \tag{2.3.1}$$

see [13, §2]. Here ξ means the universal flat and framed deformation of $\bar{\rho}_{\mathfrak{p}}$. Let $\mathrm{Spec}\, R^{\mathbf{v},\square}_{V_{\mathbf{F}}}$ be the closure of the image of $\Theta^{\mathbf{v}}_{V_{\mathbf{F}},\xi}$. By studying $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbf{F}},\xi}$, Kisin showed important properties of $R^{\mathbf{v},\square}_{V_{\mathbf{F}}}$, so we shall review it.

If $\bar{\rho}_{\mathfrak{p}}$ is ordinary (resp. non-ordinary), then we denote by $R^{\mathrm{ord},\square}_{V_{\mathbf{F}}}$ (resp. $R^{\text{non-ord},\square}_{V_{\mathbf{F}}}$) the quotient of $R^{\mathbf{v},\square}_{V_{\mathbf{F}}}$ corresponding to the closure of the image of the connected component of $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbf{F}},\xi}$ consisting of ordinary modules (resp. non-ordinary modules). Moreover if $V_{\mathbf{F}}$ is ordinary and $V_{\mathbf{F}} \simeq \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix}$ with distinct unramified characters χ_1 and χ_2, then we define the quotient $R^{\mathrm{ord},\chi_i,\square}_{V_{\mathbf{F}}}$ to be the closure of the image of the connected component of $\mathscr{GR}^{\mathbf{v}}_{V_{\mathbf{F}},\xi}$ corresponding to χ_i (cf. [13, §2.4, 2.5]). Let Spec $\bar{R}^{\square}_{V_{\mathbf{F}}}$ be a connected spectrum of one of $R^{\mathbf{v},\square}_{V_{\mathbf{F}}}$, $R^{\mathrm{non\text{-}ord},\square}_{V_{\mathbf{F}}}$ or $R^{\mathrm{ord},\chi_i,\square}_{V_{\mathbf{F}}}$. Then, by [12, (2.5.15)], [10, Proposition 2.3] and [11, Main theorem], $\bar{R}^{\square}_{V_{\mathbf{F}}}$ is a domain of relative dimension $[F_{\mathfrak{p}} : \mathbf{Q}_p] + 4$ over $W(\mathbf{F})$ and flat over $W(\mathbf{F})$.

By [13, (2.4.17)], there exists the universal characteristic polynomial $P^{\mathbf{v}}_{V_{\mathbf{F}}}(X)$ of the linear map $\varphi^{[\kappa(\mathfrak{p}):\mathbf{F}_p]}$ on Kisin modules with coefficients in the ring of global sections $\Gamma(\mathscr{GR}^{\mathbf{v}}_{V_{\mathbf{F}},\xi}, O_{\mathscr{GR}^{\mathbf{v}}_{V_{\mathbf{F}},\xi}})$, which we denote by

$$P^{\mathbf{v}}_{V_{\mathbf{F}}}(X) = X^2 - t_{\mathfrak{p}}X + d_{\mathfrak{p}}. \tag{2.3.2}$$

Now let $\tilde{R}^{\square}_{V_{\mathbf{F}}}$ be the sub-$W(\mathbf{F})$-algebra of $\Gamma(\mathscr{GR}^{\mathbf{v}}_{V_{\mathbf{F}},\xi}, O_{\mathscr{GR}^{\mathbf{v}}_{V_{\mathbf{F}},\xi}})$ generated by the image of $\bar{R}^{\square}_{V_{\mathbf{F}}}$ under the map (2.3.1) and by the coefficients of $P^{\mathbf{v}}_{V_{\mathbf{F}}}(X)$. Then, $\tilde{R}^{\square}_{V_{\mathbf{F}}}$ is flat over $W(\mathbf{F})$, and the natural map $\bar{R}^{\square}_{V_{\mathbf{F}}} \to \tilde{R}^{\square}_{V_{\mathbf{F}}}$ is finite; moreover this map becomes an isomorphism after inverting p. Now we take a quotient $\bar{R}^{\psi,\square}_{V_{\mathbf{F}}}$ of $\bar{R}^{\square}_{V_{\mathbf{F}}}$ corresponding to the fixed determinant condition and let $\tilde{R}^{\psi,\square}_{V_{\mathbf{F}}}$ be the corresponding quotient of $\tilde{R}^{\square}_{V_{\mathbf{F}}}$. Since $\bar{R}^{\square}_{V_{\mathbf{F}}} \cong \bar{R}^{\psi,\square}_{V_{\mathbf{F}}}[[X]]$, we have:

Proposition 2.4 ([13, §2]). *The local algebra $\tilde{R}^{\psi,\square}_{V_{\mathbf{F}}}$ is a domain of relative dimension $[F_{\mathfrak{p}} : \mathbf{Q}_p] + 3$ over $W(\mathbf{F})$, which is flat over $W(\mathbf{F})$. Moreover, $\tilde{R}^{\psi,\square}_{V_{\mathbf{F}}}[1/p]$ is geometrically integral and formally smooth over $W(\mathbf{F})[1/p]$.*

Here we note that a $W(\mathbf{F})$-algebra R is called *geometrically integral* if for any finite extension E of $W(\mathbf{F})[1/p]$, the scalar extension $R[1/p] \otimes E$ is a domain ([13, p. 1165]). Thus, for any finite and totally ramified extension E of $W(\mathbf{F})[1/p]$, the similar argument holds for the scalar extension $\tilde{R}^{\psi,\square}_{V_{\mathbf{F}},\mathcal{O}_E} := \tilde{R}^{\psi,\square}_{V_{\mathbf{F}}} \otimes_{W(\mathbf{F})} \mathcal{O}_E$.

Next we take a prime v not dividing p. Let $\bar{\rho}_v : G_{F_v} \to \mathrm{GL}_2(\mathbf{F})$ be a continuous representation with representation space $V_{\mathbf{F}}$. As well as dividing p case, we obtain the suitable quotient of the universal framed deformation ring of $\bar{\rho}_v$.

Proposition 2.5 ([13, (2.6.7)]). *Let $\mathcal{O}$ be the ring of integers of a finite and totally ramified extension E of $W(\mathbf{F})[1/p]$ with a uniformizer λ. Let $\gamma : G_{F_v} \to \mathcal{O}^\times$ be an unramified character. We write $\psi = \gamma^2$ and assume* $\det \bar{\rho}_v \equiv \psi\varepsilon \bmod \lambda$. *Then there is a quotient $R^{\psi,\gamma,\square}_{V_{\mathbf{F}},\mathcal{O}}$ of $R^{\psi,\square}_{V_{\mathbf{F}}} \otimes_{W(\mathbf{F})} \mathcal{O}$ satisfying the following*:

(1) *A morphism $\xi : R^{\psi,\square}_{V_{\mathbf{F}},\mathcal{O}} \to \mathcal{O}$ of $\widehat{\mathfrak{AR}}_{\mathcal{O}}$ factors through $R^{\psi,\gamma,\square}_{V_{\mathbf{F}},\mathcal{O}}$ if and only if the associated E-representation V_ξ is an extension of $\gamma\varepsilon \otimes E$ by $\gamma \otimes E$.*
(2) *$R^{\psi,\gamma,\square}_{V_{\mathbf{F}},\mathcal{O}}$ is a domain of relative dimension 3 over $\mathcal{O}$.*
(3) *$R^{\psi,\gamma,\square}_{V_{\mathbf{F}},\mathcal{O}}[1/p]$ is geometrically integral and formally smooth over E.*

2.4. p-adic quaternionic modular forms. We summarize the notion of p-adic modular forms of a quaternion algebra over a totally real field. We only treat the parallel weight two case. For the general definition and the details, see [13, (3.1)] and [23, §1, 2].

We assume $p > 2$. Let D be a central quaternionic algebra over F which is ramified at all infinite places of F. We further assume that D is unramified at all primes $\mathfrak{p}$ over p. Let Σ be the set of all finite places of F on which D is ramified. Let o_D be a maximal order of D. For any finite place v of F, we put $(o_D)_v := o_D \otimes_{\mathcal{O}_F} \mathcal{O}_{F_v}$. Let $U = \prod_v U_v$ be a compact open subgroup of $(D \otimes_F \mathbf{A}_F^\infty)^\times$ which is contained in $\prod_v (o_D)_v^\times$ satisfying that: (i) for any $v \in \Sigma$, $U_v = (o_D)_v^\times$, (ii) for any $\mathfrak{p} \mid p$, $U_\mathfrak{p} = \mathrm{GL}_2(\mathcal{O}_{F_\mathfrak{p}})$.

Let E be a finite extension of $\mathbf{Q}_p$ and $\mathcal{O}$ the ring of integers of E. We fix a continuous character $\psi : (\mathbf{A}_F^\infty)^\times / F^\times \to \mathcal{O}^\times$. Let τ_{triv} be the trivial $\mathcal{O}$-linear representation of U of rank one, whose representation space is denoted by $W_{\tau_{\mathrm{triv}}}$. We assume that ψ satisfies the equation

$$\tau_{\mathrm{triv}}|_{U_v \cap \mathcal{O}_{F_v}^\times} = \psi^{-1}|_{U_v \cap \mathcal{O}_{F_v}^\times} \tag{2.4.1}$$

for any prime v. Since τ_{triv} is trivial on $U \cap \mathcal{O}_F^\times$, such a ψ exists (cf. [13, (3.1)]). We regard $W_{\tau_{\mathrm{triv}}}$ as a $U(\mathbf{A}_F^\infty)^\times$-module by τ_{triv} and ψ^{-1}.

We define a (p-*adic*) *quaternionic modular form* of parallel weight two, of level U and of character ψ to be a continuous function

$$f : D^\times \backslash (D \otimes_F \mathbf{A}_F^\infty)^\times \to W_{\tau_{\mathrm{triv}}} \cong \mathcal{O}$$

satisfying that:

(1) for any $u \in U$, $f(xu) = f(x)$ $(x \in (D \otimes_F \mathbf{A}_F^\infty)^\times)$,

(2) for any $z \in \mathbf{A}_F^\infty$, $f(xz) = \psi(z) f(x)$ $(x \in (D \otimes_F \mathbf{A}_F^\infty)^\times)$.

For the meaning of "parallel weight two", see [13, (3.1.9)]. Let

$$S_{2,\psi}^D(U, \mathcal{O})$$

be the space of modular forms of parallel weight two, of level U and of character ψ; when D is clear, we omit the symbol D.

Since $D^\times \backslash (D \otimes_F \mathbf{A}_F^\infty)^\times / U(\mathbf{A}_F^\infty)^\times$ is finite, if we write

$$(D \otimes_F \mathbf{A}_F^\infty)^\times = \coprod_{i \in I} D^\times t_i U(\mathbf{A}_F^\infty)^\times$$

for some representatives $t_i \in (D \otimes_F \mathbf{A}_F^\infty)^\times$ with index set I, then we have

$$S_{2,\psi}(U, \mathcal{O}) \xrightarrow{\simeq} \bigoplus_{i \in I} W_{\tau_{\mathrm{triv}}}^{(U(\mathbf{A}_F^\infty)^\times \cap t_i D^\times t_i^{-1})/F^\times}; \quad f \mapsto (f(t_i))_{i \in I}. \tag{2.4.2}$$

In particular, $S_{2,\psi}(U, \mathcal{O})$ is a free $\mathcal{O}$-module of finite rank. For any $\mathcal{O}$-algebra A, we will write $S_{2,\psi}(U, \mathcal{O}) \otimes_\mathcal{O} A$ as $S_{2,\psi}(U, A)$.

We denote by $S_{2,\psi}^{\mathrm{triv}}(U, A)$ the subspace of $S_{2,\psi}(U, A)$ consisting of functions which factor through the reduced norm of $(D \otimes_F \mathbf{A}_F^\infty)^\times$.

As [13, (3.1.2)] we assume the following:

CONDITION 2.6. *For any* $t \in (D \otimes_F \mathbf{A}_F^\infty)^\times$ *the cardinality of* $(U(\mathbf{A}_F^\infty)^\times \cap tD^\times t^{-1})/F^\times$ *is prime to* p.

As Kisin remarked, if U is sufficiently small then it is satisfied (cf. [13, p. 1147]). Using Condition 2.6, we can define a perfect pairing $\langle \cdot , \cdot \rangle_U$ on $S_{2,\psi}(U,\mathcal{O})$ by

$$\text{(2.4.3)} \quad \langle f, g \rangle_U = \sum_{i \in I} \psi(\det t_i)^{-1} \#((U(\mathbf{A}_F^\infty)^\times \cap t_i D^\times t_i^{-1})/F^\times)^{-1} f(t_i) g(t_i).$$

Now we define the Hecke algebras. We take a finite set S of places of F containing $\Sigma \cup \{\mathfrak{p} \mid p\} \cup \{v \mid U_v \neq (o_D)_v^\times\}$. We also let $S^p := S \backslash \{\mathfrak{p} \mid p\}$. For an $\mathcal{O}$-algebra A, let $\mathbf{T}_{S,A}^{\text{univ}}$ be the commutative ring $A[T_v, S_v]_{v \notin S}$. We choose a uniformizer ϖ_v of $\mathcal{O}_{F_v}$ for each $v \in |F|^\infty$ and we define an $\mathcal{O}$-algebra homomorphism

$$\text{(2.4.4)} \qquad \mathbf{T}_{S,\mathcal{O}}^{\text{univ}} \to \operatorname{End}_{\mathcal{O}}(S_{2,\psi}(U,\mathcal{O}))$$

by

$$\text{(2.4.5)} \quad T_v \mapsto \left[U \begin{pmatrix} 1 & 0 \\ 0 & \varpi_v \end{pmatrix} U \right] \quad \text{and} \quad S_v \mapsto \left[U \begin{pmatrix} \varpi_v & 0 \\ 0 & \varpi_v \end{pmatrix} U \right] \quad (v \notin S).$$

This is independent of the choice of ϖ_v. We denote the image of the map (2.4.4) by $\mathbf{T}'_{\psi,\mathcal{O}}(U)$ and the image of T_v and S_v via (2.4.4) by the same symbol. The action of $\mathbf{T}_{S,\mathcal{O}}^{\text{univ}}$ by the rule (2.4.5) is called the *standard* Hecke action. We can also define Hecke operator $\left[U \begin{pmatrix} 1 & 0 \\ 0 & \varpi_{\mathfrak{p}} \end{pmatrix} U \right]$ for each $\mathfrak{p} \mid p$ ([13, p. 1151]), which is also independent of the choice of $\varpi_{\mathfrak{p}}$. Let

$$\mathbf{T}_{S^p,\mathcal{O}}^{\text{univ}} \to \operatorname{End}_{\mathcal{O}}(S_{2,\psi}(U,\mathcal{O}))$$

be the $\mathcal{O}$-algebra homomorphism defined by (2.4.5) and

$$\text{(2.4.6)} \qquad T_{\mathfrak{p}} \mapsto \left[U \begin{pmatrix} 1 & 0 \\ 0 & \varpi_{\mathfrak{p}} \end{pmatrix} U \right]$$

for each $\mathfrak{p}$ dividing p. We denote by $\mathbf{T}_{\psi,\mathcal{O}}(U)$ the image of this. By definition, Hecke algebras $\mathbf{T}'_{\psi,\mathcal{O}}(U)$ and $\mathbf{T}_{\psi,\mathcal{O}}(U)$ are finite and flat over $\mathcal{O}$. For any $\mathcal{O}$-algebra A, we will write

$$\mathbf{T}_{\psi,A}^{\star}(U) := \mathbf{T}_{\psi,\mathcal{O}}^{\star}(U) \otimes_{\mathcal{O}} A,$$

where the symbol $\star$ means either $\emptyset$ (nothing) or $'$.

Now we shall recall some basic properties of eigenforms. For sufficiently large E, the space $S_{2,\psi}(U,E)$ has an E-basis consisting of eigenforms for $\mathbf{T}_{S,\mathcal{O}}^{\text{univ}}$

since the Hecke operators T_v, S_v for each $v \notin S$ are self-adjoint with respect to the pairing $\langle \cdot, \cdot \rangle_U$. For any eigenform f of $S_{2,\psi}(U, \mathcal{O})$ we can associate the $\mathcal{O}$-algebra map

$$\theta_f : \mathbf{T}'_{\psi,\mathcal{O}}(U) \to \overline{E}$$

determined by $T \mapsto \theta_f(T)$, where $\theta_f(T)$ is the eigenvalue of T associated to f. The image of this map is the ring $\mathcal{O}_{E_f}$ of integers of a finite extension E_f of E in $\overline{E}$. We denote by λ' the prime of E_f; when we consider f as a $\mathbf{C}$-valued function via $\imath_\infty : \overline{E} \simeq \mathbf{C}$, it corresponds to the prime of the Hecke field $K_f := \mathbf{Q}(\{\imath_\infty \theta_f(T_v), \imath_\infty \theta_f(S_v) \mid v \notin S\})$ of f, which is a number field, corresponding to an embedding $K_f \to \overline{E}$.

We also recall that a maximal ideal $\mathfrak{m}'$ of $\mathbf{T}'_{\psi,\mathcal{O}}(U)$ is called *Eisenstein* if $T_v - 2 \in \mathfrak{m}'$ for all but not finitely many primes v which split in some fixed abelian extension of F. For a maximal ideal $\mathfrak{m}$ of $\mathbf{T}_{\psi,\mathcal{O}}(U)$, it is called Eisenstein if $\mathfrak{m}' := \mathfrak{m} \cap \mathbf{T}'_{\psi,\mathcal{O}}(U)$ is.

At the end of this subsection, we shall recall the relation between spaces of p-adic quaternionic modular forms and cuspidal automorphic representations. For the detail, see [23, §1] and [13, (3.1.14)]. Put

$$S^D_{2,\psi,\overline{E}} := \varinjlim_U S^D_{2,\psi}(U, \overline{E}) \quad \text{and} \quad S^{D,\mathrm{triv}}_{2,\psi,\overline{E}} := \varinjlim_U S^{D,\mathrm{triv}}_{2,\psi}(U, \overline{E}),$$

where U runs over all compact open subgroups of $(D \otimes_F \mathbf{A}_F^\infty)^\times$. These spaces have the smooth actions of $(D \otimes_F \mathbf{A}_F^\infty)^\times$. Moreover, we have the following:

LEMMA 2.7 ([23, Lemma 1.3], [13, (3.1.14)]). *The space $S^D_{2,\psi,\overline{E}}$ is a semi-simple admissible representation of $(D \otimes_F \mathbf{A}_F^\infty)^\times$ and the U-invariant part of $S^D_{2,\psi,\overline{E}}$ is $S^D_{2,\psi}(U, \overline{E})$. Under the fixed isomorphism $\imath_\infty : \overline{E} \simeq \mathbf{C}$ we have*

$$S^D_{2,\psi,\overline{E}} \simeq S^{D,\mathrm{triv}}_{2,\psi,\overline{E}} \oplus \bigoplus_\Pi \imath_\infty(\Pi^\infty) \quad \text{and} \quad S^{D,\mathrm{triv}}_{2,\psi,\overline{E}} \simeq \bigoplus_\chi \overline{E}(\chi), \tag{2.4.7}$$

where Π runs over all cuspidal representations of $(D \otimes_F \mathbf{A}_F)^\times$ having weight 2 and central character ψ, and χ runs over all characters $(\mathbf{A}_F^\infty)^\times / F^\times_{\gg 0} \to \overline{E}^\times$ satisfying $\chi^2 = \psi$. Here $F^\times_{\gg 0}$ denotes the subgroup of $F^\times$ consisting of totally positive elements.

2.5. Galois representations associated to modular forms. We keep notation and assumptions in the previous subsection. In the following, we only consider the parallel weight two case. We take E sufficiently large.

By [17], [2] and [21], for any eigenform f of $S_{2,\psi}(U, \mathcal{O})$ we have a two dimensional continuous representation

$$\rho_f : G_{F,S} \to \mathrm{GL}_2(\mathcal{O}_{E_f})$$

which is characterized by

$$\mathrm{Tr}\,\rho_f(\mathrm{Frob}_v) = \theta_f(T_v), \quad \det \rho_f(\mathrm{Frob}_v) = \theta_f(S_v) N(v) \quad (v \notin S).$$

Moreover, by the theory of pseudo-representations, for any non-Eisenstein maximal ideal $\mathfrak{m}'$ of $\mathbf{T}'_{\psi,\mathcal{O}}(U)$ there exists a Galois representation

$$\rho_{\mathfrak{m}'} : G_{F,S} \to \mathrm{GL}_2(\mathbf{T}'_{\psi,\mathcal{O}}(U)_{\mathfrak{m}'}) \tag{2.5.1}$$

satisfying that:

- for any prime $v \notin S$, we have $\operatorname{Tr} \rho_{\mathfrak{m}'}(\mathrm{Frob}_v) = T_v$,
- the determinant of $\rho_{\mathfrak{m}'}$ is $\psi\varepsilon$,
- the mod $\mathfrak{m}'$ reduction $\bar{\rho}_{\mathfrak{m}'}$ of $\rho_{\mathfrak{m}'}$ is absolutely irreducible.

Furthermore, when we denote by $V_{\mathfrak{m}',E}$ the representation space of $\rho_{\mathfrak{m}'} \otimes E$ we have the following:

Lemma 2.8 ([13, Lemma (3.4.2)]). *Let $\mathfrak{p}$ be a primes of F above p.*

(1) *$V_{\mathfrak{m}',E}$ is a Barsotti-Tate representation at $\mathfrak{p}$. Moreover, the determinant of the restriction $V_{\mathfrak{m}',E}|_{I_\mathfrak{p}}$ is the p-adic cyclotomic character.*

(2) *The Hecke operator $T_\mathfrak{p} \in \mathbf{T}_{\psi,\mathcal{O}}(U)_{\mathfrak{m}'}$ is contained in $\mathbf{T}'_{\psi,\mathcal{O}}(U)_{\mathfrak{m}'}[1/p]$, and*

$$T_\mathfrak{p} = \operatorname{Tr}_{\mathbf{T}'_{\psi,\mathcal{O}}(U)_{\mathfrak{m}'}[1/p]\otimes_{\mathbf{Z}_p} W(\kappa(\mathfrak{p}))}(\varphi^{[\kappa(\mathfrak{p}):\mathbf{F}_p]} \mid D_{\mathrm{cris}}(V_{\mathfrak{m}',E})). \tag{2.5.2}$$

Here $\kappa(\mathfrak{p})$ is the residue field of $\mathfrak{p}$.

In the following, we put

$$\mathbf{T}' := \mathbf{T}'_{\psi,\mathcal{O}}(U)_{\mathfrak{m}'}.$$

If we denote by $R^{\psi}_{F,S}$ the universal deformation ring of $\bar{\rho}_{\mathfrak{m}'}$ with fixed determinant condition $\det = \psi\varepsilon$, then we have a morphism in $\widehat{\mathfrak{AR}}_{\mathcal{O}}$:

$$\varphi_{\mathfrak{m}'} : R^{\psi}_{F,S} \to \mathbf{T}'.$$

We call a deformation of $\bar{\rho}_{\mathfrak{m}'}$ which factors through $\varphi_{\mathfrak{m}'}$ a *modular deformation*.

We take a maximal ideal $\mathfrak{m}$ of $\mathbf{T}_{\psi,\mathcal{O}}(U)$ above $\mathfrak{m}'$, and put

$$\mathbf{T} := \mathbf{T}_{\psi,\mathcal{O}}(U)_{\mathfrak{m}}.$$

Next we define the Hecke algebra $\mathbf{T}_{Q_n}$ corresponding to modular deformations of Taylor-Wiles type. We prepare notation for levels of modular forms. For a prime $v \notin \Sigma$ and an integer $n \geq 1$, we define

$$U_0(\varpi_v^n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{F_v}) \,\middle|\, c \equiv 0 \bmod \varpi_v^n \right\},$$

$$U_1(\varpi_v^n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{F_v}) \,\middle|\, a \equiv 1,\ c \equiv 0 \bmod \varpi_v^n \right\}$$

and

$$U_{11}(\varpi_v^n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{F_v}) \,\middle|\, a \equiv d \equiv 1,\ c \equiv 0 \bmod \varpi_v^n \right\}.$$

For a quotient H_v of $(\mathcal{O}_{F_v}/\pi_v\mathcal{O}_{F_v})^\times$ we also define

$$U_{H_v}(\varpi_v^n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(\varpi_v^n) \,\middle|\, ad^{-1} \mapsto 1 \in H_v \right\}.$$

For any $n \geq 1$ we take a finite set Q_n of primes of F described in Proposition 2.3. For any prime $v \in Q_n$ we denote by Δ_v the maximal p-power quotient of $(\mathcal{O}_{F_v}/\varpi_v\mathcal{O}_{F_v})^\times$. Let $U_{Q_n} \subset U_{Q_n}^-$ be the subgroups of U defined as

$$U_{Q_n} := \prod_{v \in Q_n} U_{\Delta_v}(\varpi_v) \cdot \prod_{v \notin Q_n} U_v \quad \text{and} \quad U_{Q_n}^- := \prod_{v \in Q_n} U_0(\varpi_v) \cdot \prod_{v \notin Q_n} U_v.$$

When we write $\Delta_{Q_n} := \prod_{v \in Q_n} \Delta_v$ we have $U_{Q_n}^-/U_{Q_n} \cong \Delta_{Q_n}$. Put

$$S_{Q_n} := S \cup Q_n \quad \text{and} \quad S_{Q_n}^p := S_{Q_n} \backslash \{\mathfrak{p} : \mathfrak{p} \mid p\}.$$

We define the Hecke algebra $\mathbf{T}'_{\psi,\mathcal{O}}(U_{Q_n})$ (resp. $\mathbf{T}_{\psi,\mathcal{O}}(U_{Q_n})$) by the image of $\mathbf{T}^{\mathrm{univ}}_{S_{Q_n},\mathcal{O}}$ (resp. $\mathbf{T}^{\mathrm{univ}}_{S_{Q_n}^p,\mathcal{O}}$) in $\mathrm{End}(S_{2,\psi}(U_{Q_n},\mathcal{O}))$ via (2.4.5) (resp. (2.4.5) and (2.4.6)) for the level U_{Q_n}. Moreover, for the set Q_n we define the abstract Hecke algebra

$$\tilde{\mathbf{T}}^{\mathrm{univ}}_{S_{Q_n},\mathcal{O}} := \mathbf{T}^{\mathrm{univ}}_{S_{Q_n},\mathcal{O}}[u_v \mid v \in Q_n],$$

and define the Hecke algebra $\tilde{\mathbf{T}}'_{\psi,\mathcal{O}}(U_{Q_n})$ to be the image of the $\mathbf{T}^{\mathrm{univ}}_{S_{Q_n},\mathcal{O}}$-algebra map

$$\tilde{\mathbf{T}}^{\mathrm{univ}}_{S_{Q_n},\mathcal{O}} \to \mathrm{End}_{\mathcal{O}}(S_{2,\psi}(U_{Q_n},\mathcal{O}))$$

determined by

$$u_v \mapsto \mathrm{U}_{\varpi_v} := \left[U_{Q_n} \begin{pmatrix} 1 & 0 \\ 0 & \varpi_v \end{pmatrix} U_{Q_n} \right]$$

for all $v \in Q_n$. As well, we define $\tilde{\mathbf{T}}_{\psi,\mathcal{O}}(U_{Q_n})$ to be the sub-$\mathcal{O}$-algebra of the endomorphism ring of $S_{2,\psi}(U_{Q_n},\mathcal{O})$ generated by $\mathbf{T}_{\psi,\mathcal{O}}(U_{Q_n})$ and U_{ϖ_v} for all $v \in Q_n$. We also define the Diamond operator $\langle - \rangle : \Delta_{Q_n} \to \mathrm{End}(S_{2,\psi}(U_{Q_n},\mathcal{O}))$ as follows: for any prime $v \in Q$ and element $a \in \Delta_v$, we take a lifting $\tilde{a} \in \mathcal{O}_{F_v}$ of a and define $\langle a \rangle$ to be

$$\langle a \rangle := \left[U_{Q_n} \begin{pmatrix} 1 & 0 \\ 0 & \tilde{a} \end{pmatrix} U_{Q_n} \right].$$

Similarly we define $\tilde{\mathbf{T}}'_{\psi,\mathcal{O}}(U_{Q_n}^-)$ and $\tilde{\mathbf{T}}_{\psi,\mathcal{O}}(U_{Q_n}^-)$. For a maximal ideal $\mathfrak{m}$ of $\mathbf{T}_{\psi,\mathcal{O}}(U)$ we denote the ideal $\mathfrak{m} \cap \mathbf{T}^{\mathrm{univ}}_{S_{Q_n}^p,\mathcal{O}}$ of $\mathbf{T}^{\mathrm{univ}}_{S_{Q_n}^p,\mathcal{O}}$ by $\mathfrak{m}$ again. We take a maximal ideal $\mathfrak{m}_{Q_n}^-$ of the localization $\tilde{\mathbf{T}}_{\psi,\mathcal{O}}(U_{Q_n}^-)_{\mathfrak{m}}$ which corresponds to the choice of eigenvalues $\{\alpha_v\}_{v \in Q_n}$ of $\bar{\rho}_{\mathfrak{m}'}(\mathrm{Frob}_v)$ so that it coincides with the choice determining the $\mathcal{O}[\Delta_{Q_n}]$-structure of $R^{\psi}_{F,S_{Q_n}}$; for the detail, see [13, Lemma (3.4.6)]. We will write

$$\mathbf{T}_{Q_n}^- := \tilde{\mathbf{T}}_{\psi,\mathcal{O}}(U_{Q_n}^-)_{\mathfrak{m}_{Q_n}^-}. \tag{2.5.3}$$

Let $\mathfrak{m}_{Q_n}$ be the maximal ideal of $\tilde{\mathbf{T}}_{\psi,\mathcal{O}}(U_{Q_n})$ induced by $\mathfrak{m}_{Q_n}^{-}$. We also define

$$\mathbf{T}_{Q_n} := \tilde{\mathbf{T}}_{\psi,\mathcal{O}}(U_{Q_n})_{\mathfrak{m}_{Q_n}} \quad \text{and} \quad \mathbf{T}'_{Q_n} := \tilde{\mathbf{T}}'_{\psi,\mathcal{O}}(U_{Q_n})_{\mathfrak{m}'_{Q_n}}, \tag{2.5.4}$$

where $\mathfrak{m}'_{Q_n} := \mathfrak{m}_{Q_n} \cap \tilde{\mathbf{T}}'_{\psi,\mathcal{O}}(U_{Q_n})$. As Taylor constructed in [23, §3], we have the Galois representation

$$\rho_{\mathfrak{m}'_{Q_n}} : G_{F,S_{Q_n}} \to \mathrm{GL}_2(\mathbf{T}'_{Q_n}).$$

In particular, $\mathbf{T}'_{Q_n}$ is generated by semi-simple operators T_v and S_v for all $v \notin S_{Q_n}$, so that it is reduced; this follows from [23, Lemma 1.6] and its corollaries. If $\bar{\rho}_{\mathfrak{m}'_{Q_n}}$ is its residual representation, then $\bar{\rho}_{\mathfrak{m}'_{Q_n}} \simeq \bar{\rho}_{\mathfrak{m}'}$. In particular, $\rho_{\mathfrak{m}'_{Q_n}}$ is a modular deformation of $\bar{\rho}_{\mathfrak{m}'}$ of Taylor-Wiles type, i.e. a modular deformation of $\bar{\rho}_{\mathfrak{m}'}$ controlled by $R^{\psi}_{F,S_{Q_n}}$.

At the end of this section, we recall the local structure of $\rho_{\mathfrak{m}'}$ in the language of deformation theory. Let $\mathbf{F}$ be the residual field of $\mathbf{T}' = \mathbf{T}'_{\psi,\mathcal{O}}(U)_{\mathfrak{m}'}$ and

$$\bar{\rho}_{\mathfrak{m}'} : G_{F,S} \to \mathrm{Aut}_{\mathbf{F}}(V_{\mathbf{F}}) \simeq \mathrm{GL}_2(\mathbf{F})$$

be the residual representation of (2.5.1). Let $\sigma := (\sigma', \{\chi_{\mathfrak{p}}\}_{\mathfrak{p}\in\sigma'})$ be an ordinary data such that $\mathfrak{m}$ is σ-ordinary[2)]. In the following, for any $W(\mathbf{F})$-algebra R we denote by $R_{\mathcal{O}}$ the scalar extension $R \otimes_{W(\mathbf{F})} \mathcal{O}$.

For any prime v contained in Σ_p, we denote by $R^{\psi,\square}_{v,\mathcal{O}}$ the universal framed deformation ring of $V_{\mathbf{F}}|_{G_{F_v}}$ over $\mathcal{O}$ with fixed determinant $\psi\varepsilon|_{G_{F_v}}$.

For any $v \in \Sigma$, let γ_v be an unramified character such that $\gamma_v^2 = \psi|_{G_{F_v}}$ and let

$$\bar{R}^{\psi,\square}_v := R^{\psi,\gamma_v,\square}_{v,\mathcal{O}}$$

be the quotient of $R^{\psi,\square}_{v,\mathcal{O}}$ as in Proposition 2.5. Similarly, if $\mathfrak{p}$ is a prime dividing p then we define the quotient $\bar{R}^{\psi,\sigma,\square}_{\mathfrak{p}}$ of $R^{\psi,\square}_{\mathfrak{p},\mathcal{O}}$ to be $R^{\psi,\text{non-ord},\square}_{\mathfrak{p},\mathcal{O}}$ if $\mathfrak{p} \notin \sigma'$; and if $\mathfrak{p} \in \sigma'$ then we define $\bar{R}^{\psi,\sigma,\square}_{\mathfrak{p}}$ to be the quotient

- $R^{\psi,\text{ord},\chi_{\mathfrak{p}},\square}_{\mathfrak{p},\mathcal{O}}$ if $V_{\mathbf{F}}|_{G_{F_{\mathfrak{p}}}} \simeq \chi_1 \oplus \chi_{\mathfrak{p}}$ with unramified characters $\chi_1 \not\simeq \chi_{\mathfrak{p}}$,
- $R^{\psi,\text{ord},\square}_{\mathfrak{p},\mathcal{O}}$ otherwise.

Moreover, we denote by $\tilde{R}^{\psi,\sigma,\square}_{\mathfrak{p}}$ the $\bar{R}^{\psi,\sigma,\square}_{\mathfrak{p}}$-algebra described as in Proposition 2.4. Let $\bar{R}^{\psi,\square}_{\Sigma} := \hat{\otimes}_{v\in\Sigma} \bar{R}^{\psi,\square}_v$, $\bar{R}^{\psi,\sigma,\square}_p := \hat{\otimes}_{\mathfrak{p}|p} \bar{R}^{\psi,\sigma,\square}_{\mathfrak{p}}$, $\tilde{R}^{\psi,\sigma,\square}_p := \hat{\otimes}_{\mathfrak{p}|p} \tilde{R}^{\psi,\sigma,\square}_{\mathfrak{p}}$ and

$$\bar{R}^{\psi,\sigma,\square}_{\Sigma,p} := \bar{R}^{\psi,\square}_{\Sigma} \hat{\otimes} \bar{R}^{\psi,\sigma,\square}_p, \quad B := \tilde{R}^{\psi,\sigma,\square}_{\Sigma,p} := \bar{R}^{\psi,\square}_{\Sigma} \hat{\otimes} \tilde{R}^{\psi,\sigma,\square}_p.$$

Here we take completed tensor products over $\mathcal{O}$. Then $B = \tilde{R}^{\psi,\square}_{\Sigma,p}$ is a domain of relative dimension $b = [F:\mathbf{Q}] + 3\#\Sigma_p$ over $\mathcal{O}$ (see [13, Proof of (3.4.11), (3.4.12)]), and the local structures of $\rho_{\mathfrak{m}'_{Q_n}}$ and $\rho_{\mathfrak{m}'}$ are described as follows.

---

2) Accoridng to [13, (3.4.4)], we say that $\mathfrak{m}$ is σ-ordinary if it satisfies the following: (i) σ' is the subset of all primes $\mathfrak{p}$ of F dividing p satisfying $T_{\mathfrak{p}} \notin \mathfrak{m}$, and (ii) for each $\mathfrak{p} \in \sigma'$, $\chi_{\mathfrak{p}}$ is an unramified character of $G_{F_{\mathfrak{p}}}$ given as a one dimensional subspace of $V_{\mathbf{F}}$ as $G_{F_{\mathfrak{p}}}$-modules such that $T_{\mathfrak{p}} \equiv \chi_{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{p}}) \bmod \mathfrak{m}$. For any maximal ideal $\mathfrak{m}$ of $\mathbf{T}_{\psi,\mathcal{O}}(U)$, there is an ordinary data σ such that $\mathfrak{m}$ is σ-ordinary.

LEMMA 2.9 ([13, (3.4.9)]). *For any* $n \geq 1$, *put* $\mathbf{T}_{Q_n}^{\square} := \mathbf{T}_{Q_n} \otimes_{R_{F,S_{Q_n}}} R_{F,S_{Q_n}}^{\square}$. *We assume that* $\bar{\rho}_{\mathfrak{m}'_{Q_n}}$ *is unramified outside the set of primes* $\mathfrak{p}$ *dividing* p. *Then the morphism*

$$R_{\Sigma,p}^{\psi,\square} = \hat{\otimes}_{v\in\Sigma_p} R_v^{\psi,\square} \to \mathbf{T}_{Q_n}^{\square}$$

given by the restrictions $\rho_{\mathfrak{m}'_{Q_n}}|_{G_{F_v}}$ *at all* $v \in \Sigma_p$ *factor through* $\bar{R}_{\Sigma,p}^{\psi,\sigma,\square}$. *Moreover, the induced morphism*

$$\tilde{R}_{F,S_{Q_n}}^{\psi,\sigma,\square} := R_{F,S_{Q_n}}^{\psi,\square} \otimes_{R_{\Sigma,p}^{\psi,\square}} \tilde{R}_{\Sigma,p}^{\psi,\sigma,\square} \to \mathbf{T}_{Q_n}^{\square}$$

is surjective. The similar arguments in the case of the level U (*i.e. the case replacing* S_{Q_n}*with* S) *also hold.*

In this paper, we say that a framed deformation of $\bar{\rho}_{\mathfrak{m}'}$ has *type* $\tilde{R}_{\Sigma,p}^{\psi,\sigma,\square}$ if its local factors at $v \in \Sigma_p$ are controlled by $\tilde{R}_{\Sigma,p}^{\psi,\sigma,\square}$.

2.6. Compatible systems. We introduce the notion of compatible systems according to Taylor [23]. For any prime λ of a number field K, we denote by p_λ the rational prime lying under λ.

DEFINITION 2.10 (cf. [23, Introduction]). Let F be a number field and G_F the absolute Galois group of F. *A rank* 2 *weakly compatible system* of λ-adic representations of G_F is a data

$$\mathcal{R} = (K, \Sigma, \mathcal{S}, \{Q_v(X)\}_{v\notin\Sigma}, (\rho_\lambda)_{\lambda\in|K|^\infty}, \{n_1, n_2\})$$

consisting of:

(1) a number field K;
(2) a finite set Σ of primes of F;
(3) a finite set $\mathcal{S}$ of primes of K containing all primes λ dividing $\prod_{v\in\Sigma} N_{F/\mathbf{Q}}(v)$;
(4) a family of degree 2 monic polynomials $Q_v(X)$ in $K[X]$ indexed by $|F|^\infty \backslash \Sigma$;
(5) a family of continuous representations

$$\rho_\lambda : G_F \to \mathrm{GL}_2(K_\lambda)$$

with coefficients in the λ-adic completion K_λ of K, indexed by finite places $\lambda \in |K|^\infty$ satisfying that:
   (5-i) if $\lambda \notin \mathcal{S}$ and $v \notin \Sigma \cup \{\mathfrak{p} \mid p_\lambda\}$ then $\rho_\lambda|_{G_{F_v}}$ is unramified and the characteristic polynomial of $\rho_\lambda(\mathrm{Frob}_v)$ is $Q_v(X)$, where p_λ is the rational prime lying under λ;
   (5-ii) for all $\lambda \notin \mathcal{S}$ and all $v \in |F|^\infty$ satisfying $v \mid p_\lambda$, the local restriction $\rho_\lambda|_{G_{F_v}}$ is crystalline;
(6) n_1 and n_2 are integers such that, for any $\lambda \notin \mathcal{S}$ and any prime $\mathfrak{p} \mid p_\lambda$, the restriction $\rho_\lambda|_{G_{F_\mathfrak{p}}}$ is Hodge-Tate of weights $\{n_1, n_2\}$.

In this paper, a system without the data of Hodge-Tate weights $\{n_1, n_2\}$ and the conditions (5-ii), (6) is called *a rank* 2 *weakly pre-compatible system* of λ-adic representations of G_F. According to [loc.cit.], we say $\mathcal{R}$ is *regular* if $n_1 \neq n_2$ and ρ_λ is totally odd for one (and hence for all) primes λ; we say $\mathcal{R}$ is *reducible* if ρ_λ is absolutely reducible for one (and hence for all) primes λ, and otherwise $\mathcal{R}$ is called *irreducible*. $\mathcal{R}$ is *strongly compatible* if for any prime $v \in |F|^\infty$ there is a Weil-Deligne representation $\mathrm{WD}_v(\mathcal{R})$ of the Weil group W_{F_v} such that for each prime $\lambda \notin \mathcal{S} \cup \{\mu \mid p_v\}$ we have

$$\mathrm{WD}(\rho_\lambda|_{W_{F_v}})^{F\text{-ss}} \cong \mathrm{WD}_v(\mathcal{R}),$$

where the left hand side is the Frobenius semi-simplification of the Weil-Deligne representation associated to $\rho_\lambda|_{G_{F_v}}$.

3. Main result

3.1. Some deformation conditions. Let F be a totally real field and S be a finite set of primes of F containing all infinite places. Let $G_{F,S}$ be the Galois group of the maximal extension of F unramified outside S. We fix a rational odd prime p and assume that all primes $\mathfrak{p}$ dividing p are contained in S. Let $E/\mathbf{Q}_p$ be a finite extension of fields, let $\mathcal{O}$ be the ring of integers of E with uniformizer λ and let $\mathbf{F}$ be its residue field. Let

$$\bar{\rho} : G_{F,S} \to \mathrm{GL}_2(\mathbf{F})$$

be a continuous representation. We assume that its determinant is congruent to the p-adic cyclotomic character ε times a character $\psi : G_{F,S} \to \mathcal{O}^\times$ of finite order. In this subsection, we consider local deformations of $\bar{\rho}$ at an unramified prime.

We fix a prime v outside S, and denote by $\bar{\rho}_v$ the restriction of $\bar{\rho}$ to the local Galois group G_{F_v}. We denote by $V_\mathbf{F}$ the representation space of $\bar{\rho}_v$ and we fix an ordered $\mathbf{F}$-basis $\beta_\mathbf{F}$ of $V_\mathbf{F}$;

$$\bar{\rho}_v : G_{F_v} \to \mathrm{Aut}_\mathbf{F}(V_\mathbf{F}) \overset{\beta_\mathbf{F}}{\simeq} \mathrm{GL}_2(\mathbf{F}).$$

We take an unramified character $\gamma_v : G_{F_v} \to \mathcal{O}^\times$ of G_{F_v}. Let A be an object of $\mathfrak{AR}_\mathcal{O}$ and $\rho_{v,A} : G_{F_v} \to \mathrm{GL}_2(A)$ be an unramified deformation of $\bar{\rho}_v$. We consider the following conditions;

(P_{γ_v}) If $\varphi : \mathcal{O} \to A$ is the structure morphism, then $\rho_{v,A}$ is isomorphic to an extension of $(\det \rho_{v,A}) \otimes (\varphi_*\gamma_v)^{-1}$ by $\varphi_*\gamma_v$.

(P_{ss}) $\rho_{v,A}$ splits to the copies of an unramified character $\chi_{v,A}$; namely,

$$\rho_{v,A} \simeq \begin{pmatrix} \chi_{v,A} & 0 \\ 0 & \chi_{v,A} \end{pmatrix}.$$

We denote by $D^{\psi,\mathrm{un},\square}_{V_\mathbf{F},v}$ the groupoid over $\mathfrak{AR}_\mathcal{O}$ consisting of unramified framed deformations of $\bar{\rho}_v$ with fixed determinant $\psi\varepsilon|_{G_{F_v}}$. Let $\gamma_v : G_{F_v} \to \mathcal{O}^\times$ be an unramified character of G_{F_v}. We denote by $D^{\psi,\gamma_v,\square}_{V_\mathbf{F},v}$ (resp. $D^{\psi,\gamma_v\text{-ss},\square}_{V_\mathbf{F},v}$) the full subcategory of $D^{\psi,\mathrm{un},\square}_{V_{\mathbf{F}},v}$ consisting of unramified framed deformations satisfying the condition (P_{γ_v}) (resp. $(\mathrm{P}_{\mathrm{ss}})$). In the following, when we consider the groupoid $D^{\psi,\gamma_v\text{-ss},\square}_{V_{\mathbf{F}},v}$ or its objects we always assume the condition $\gamma_v^2 = \psi\varepsilon|_{G_{F_v}}$. By definition, we have the morphisms of groupoids $D^{\psi,\gamma_v,\square}_{V_{\mathbf{F}},v} \to D^{\psi,\mathrm{un},\square}_{V_{\mathbf{F}},v}$ and

$$D^{\psi,\gamma_v\text{-ss},\square}_{V_{\mathbf{F}},v} \to D^{\psi,\gamma_v,\square}_{V_{\mathbf{F}},v} \to D^{\psi,\mathrm{un},\square}_{V_{\mathbf{F}},v}$$

when $\gamma_v^2 = \psi\varepsilon|_{G_{F_v}}$. As usual, we extend these over $\widehat{\mathfrak{AR}}_{\mathcal{O}}$, and for any object A of $\widehat{\mathfrak{AR}}_{\mathcal{O}}$, denote by $|D^{\psi,\gamma_v,\square}_{V_{\mathbf{F}},v}|(A)$ (resp. $|D^{\psi,\gamma_v\text{-ss},\square}_{V_{\mathbf{F}},v}|(A)$) the set of isomorphism classes of objects of $D^{\psi,\gamma_v,\square}_{V_{\mathbf{F}},v}(A)$ (resp. $D^{\psi,\gamma_v\text{-ss},\square}_{V_{\mathbf{F}},v}(A)$).

Similarly we define the groupoids $D^{\psi,\gamma_v}_{V_{\mathbf{F}},v} \to D^{\psi,\mathrm{un}}_{V_{\mathbf{F}},v}$ and $D^{\psi,\gamma_v\text{-ss}}_{V_{\mathbf{F}},v} \to D^{\psi,\gamma_v}_{V_{\mathbf{F}},v} \to D^{\psi,\mathrm{un}}_{V_{\mathbf{F}},v}$ over $\widehat{\mathfrak{AR}}_{\mathcal{O}}$ by forgetting base data. For any object $(A,\mathfrak{m}_A)$ of $\widehat{\mathfrak{AR}}_{\mathcal{O}}$, we denote by $\mathrm{Hom}_{\bar{\rho}_v}(G_{F_v},\mathrm{GL}_2(A))$ the set of unramified 2-dimensional representations of G_{F_v} over A whose $\bmod \mathfrak{m}_A$ reduction is $\bar{\rho}_v$. Then $|D^{\psi,\mathrm{un}}_{V_{\mathbf{F}},v}|(A)$ is bijectively mapped to the set $\mathrm{Hom}_{\bar{\rho}_v}(G_{F_v},\mathrm{GL}_2(A))$ modulo strict equivalence[3)]. Here we denote by $G_{\mathbf{F}_v}$ the absolute Galois group of the residue field $\mathbf{F}_v$ of F_v.

LEMMA 3.1. *Let $\gamma_v : G_{F_v} \to \mathcal{O}^\times$ be an unramified character satisfying $\gamma_v^2 = \psi\varepsilon|_{G_{F_v}}$. Then the groupoid $D^{\psi,\gamma_v\text{-ss},\square}_{V_{\mathbf{F}},v}$ over $\widehat{\mathfrak{AR}}_{\mathcal{O}}$ is pro-representable. Namely, the functor $\widehat{\mathfrak{AR}}_{\mathcal{O}} \to$ Set defined by $A \mapsto |D^{\psi,\gamma_v\text{-ss},\square}_{V_{\mathbf{F}},v}|(A)$ is represented by an object $R^{\psi,\gamma_v\text{-ss},\square}_{V_{\mathbf{F}},v}$ of $\widehat{\mathfrak{AR}}_{\mathcal{O}}$.*

Proof of Lemma 3.1. First we note that

$$D^{\psi,\gamma_v\text{-ss},\square}_{V_{\mathbf{F}},v} \cong D^{\psi,\gamma_v\text{-ss}}_{V_{\mathbf{F}},v} \times_{D^{\psi,\mathrm{un}}_{V_{\mathbf{F}},v}} D^{\psi,\mathrm{un},\square}_{V_{\mathbf{F}},v}.$$

Thus, we may show that $D^{\psi,\gamma_v\text{-ss}}_{V_{\mathbf{F}},v}$ is pro-representable.

An object $\rho_{v,A}$ of $D^{\psi,\mathrm{un}}_{V_{\mathbf{F}},v}(A)$ satisfies the condition $(\mathrm{P}_{\mathrm{ss}})$ if and only if it can be written as

$$\rho_{v,A} : G_{\mathbf{F}_v} \to Z(\mathrm{GL}_2(A)) \cong A^\times,$$

where $Z(\mathrm{GL}_2(A))$ is the center of $\mathrm{GL}_2(A)$. So we have the functorial isomorphism

$$\begin{aligned}|D^{\psi,\gamma_v\text{-ss}}_{V_{\mathbf{F}},v}|(A) &\cong \mathrm{Hom}_{\bar{\rho}_v}(G_{\mathbf{F}_v}, Z(\mathrm{GL}_2(A)))/\text{strict equiv.}\\ &\cong \mathrm{Hom}_{\bar{\gamma}_v}(G_{\mathbf{F}_v}, A^\times),\end{aligned}$$

where $\bar{\gamma}_v : G_{\mathbf{F}_v} \to \mathbf{F}^\times$ is the mod λ reduction of γ_v. Since 1-dimensional deformations are pro-representable (cf. [16, 1.4]), $D^{\psi,\gamma_v\text{-ss}}_{V_{\mathbf{F}},v}$ is also. □

3) Let $\bar{\rho} : G \to \mathrm{GL}_n(\mathbf{F})$ be a continuous representation of a profinite group G. Then, two deformations ρ_A, $\rho'_A : G \to \mathrm{GL}_n(A)$ of $\bar{\rho}$ over $A \in \widehat{\mathfrak{AR}}_{\mathcal{O}}$ are strictly equivalent if there is a matrix $H \in \mathrm{Ker}(\mathrm{GL}_n(A) \to \mathrm{GL}_n(\mathbf{F}))$ such that $\rho'_A = H^{-1}\rho_A H$.

In order to prove that the groupoid $D^{\psi,\gamma_v,\square}_{V_{\mathbf{F}},v}$ is pro-representable, we introduce some notation according to [13, (2.6)]. First we define the category $\mathfrak{Aug}_{\mathcal{O}}$ consisting of pairs (A,I) where A is an $\mathcal{O}$-algebra and $I \subset A$ is a nilpotent ideal with $\mathfrak{m}_{\mathcal{O}}A \subset I$, and an arrow $(A,I) \to (B,J)$ of $\mathfrak{Aug}_{\mathcal{O}}$ are maps of rings $A \to B$ taking I into J; for the detail, see [13, (2.1)]. We remark that $\mathfrak{AR}_{\mathcal{O}}$ is a full-subcategory of $\mathfrak{Aug}_{\mathcal{O}}$. We can extend a groupoid $D_{V_{\mathbf{F}}}$ over $\mathfrak{AR}_{\mathcal{O}}$ to the groupoid over $\mathfrak{Aug}_{\mathcal{O}}$ in the standard way, see [loc.cit., (2.1.1)].

Next we define a groupoid $L^{\psi,\gamma_v,\square}_{V_{\mathbf{F}},v}$ over $\mathfrak{Aug}_{\mathcal{O}}$ as follows: For any object (A,I) of $\mathfrak{Aug}_{\mathcal{O}}$, $L^{\psi,\gamma_v,\square}_{V_{\mathbf{F}},v}(A,I)$ is the category consisting of pairs (V_A, L_A), where V_A is an object of $D^{\psi,\mathrm{un},\square}_{V_{\mathbf{F}},v}(A,I)$ and L_A is a projective sub-A-module of V_A of rank 1 on which G_{F_v} acts via γ_v, and which V_A/L_A is a projective A-module with G_{F_v}-action $\psi\varepsilon \otimes \gamma_v^{-1}$; we call such a submodule L_A a *γ_v-line* of V_A.

Lemma 3.2. *Let $\gamma_v : G_{F_v} \to \mathcal{O}^\times$ be an unramified character. The functor $|L^{\psi,\gamma_v,\square}_{V_{\mathbf{F}},v}| \to |D^{\psi,\mathrm{un},\square}_{V_{\mathbf{F}},v}|$ defined by $(V_A, L_A) \mapsto V_A$ is represented by the projective morphism $\Theta^{\mathrm{un}}_{V_{\mathbf{F}}} : \mathscr{L}^{\psi,\gamma_v,\square}_{V_{\mathbf{F}},v} \to \operatorname{Spec} R^{\psi,\mathrm{un},\square}_{V_{\mathbf{F}},v}$ of schemes over $\mathcal{O}$.*

This is an unramified analogue of [13, Lemma (2.6.2)]; in the proof replacing $D^{\psi,\square}_{V_{\mathbf{F}},v}$ with $D^{\psi,\mathrm{un},\square}_{V_{\mathbf{F}},v}$ we obtain the result.

We define the closed subscheme $\operatorname{Spec} R^{\psi,\gamma_v,\square}_{V_{\mathbf{F}},v}$ of $\operatorname{Spec} R^{\psi,\mathrm{un},\square}_{V_{\mathbf{F}},v}$ to be the scheme theoretic image of $\Theta^{\mathrm{un}}_{V_{\mathbf{F}}} : \mathscr{L}^{\psi,\gamma_v,\square}_{V_{\mathbf{F}},v} \to \operatorname{Spec} R^{\psi,\mathrm{un},\square}_{V_{\mathbf{F}},v}$ in Lemma 3.2. Then, by [13, Proposition (2.3.5)] we have the following:

Proposition 3.3. *The map $\xi : R^{\psi,\mathrm{un},\square}_{V_{\mathbf{F}},v} \to \mathcal{O}$ of $\widehat{\mathfrak{AR}}_{\mathcal{O}}$ factors through $R^{\psi,\gamma_v,\square}_{V_{\mathbf{F}},v}$ (resp. $R^{\psi,\gamma_v\text{-ss},\square}_{V_{\mathbf{F}},v}$ when $\gamma_v^2 = \psi\varepsilon|_{G_{F_v}}$) if and only if the associated E-representation (ρ_ξ, V_ξ) is unramified and an extension of $(\det \rho_\xi) \otimes (\gamma_v \otimes E_\lambda)^{-1}$ by $\gamma_v \otimes E$. (resp. $V_\xi \otimes E$ is the direct product of copies of such a $\gamma_v \otimes E$). In particular, $D^{\psi,\gamma_v,\square}_{V_{\mathbf{F}},v}$ is pro-represented by $R^{\psi,\gamma_v,\square}_{V_{\mathbf{F}},v}$.*

We note that this is a variation of [13, (2.6.6), (2.6.7)].

Proof of Proposition 3.3. First we assume $\gamma_v^2 = \psi\varepsilon|_{G_{F_v}}$. For the condition $(\mathrm{P}_{\mathrm{ss}})$, this is clear because the image $\rho_\xi(\mathrm{Frob}_v)$ at $v \in \mathcal{P}$ is a scalar matrix, which does not depend on lattices of V_ξ.

We now consider the condition (P_{γ_v}). We denote the composite $R^{\psi,\mathrm{un},\square}_{V_{\mathbf{F}},v} \xrightarrow{\xi} \mathcal{O} \hookrightarrow E$ by the same symbol ξ. For any noetherian complete local $\mathcal{O}$-algebra R and continuous local $\mathcal{O}$-algebra map $\eta : R \to E$, we denote by $R^\wedge_\eta$ the completion of $R \otimes_{\mathcal{O}} E$ along the kernel of $\eta \otimes 1$. Similarly, for any R-scheme X, we denote the completed fiber at η by $X^\wedge_\eta \to \operatorname{Spf} R^\wedge_\eta$.

Then, we know that $\xi : R^{\psi,\mathrm{un},\square}_{V_{\mathbf{F}},v} \to \mathcal{O} \subseteq E$ factors through the quotient $R^{\psi,\gamma_v,\square}_{V_{\mathbf{F}},v}$ if and only if the induced map $(R^{\psi,\mathrm{un},\square}_{V_{\mathbf{F}},v})^\wedge_\xi \to E$ factors through the corresponding quotient $(R^{\psi,\gamma_v,\square}_{V_{\mathbf{F}},v})^\wedge_\xi$.

On the other hand, by [13, Lemma (2.3.3)] and [loc.cit., Proposition (2.3.5)], the groupoid $D^{\psi,\mathrm{un},\square}_{V_\xi}$ over $\mathfrak{AR}_E$ of unramified framed deformations of V_ξ with fixed determinant $\psi\varepsilon|_{G_{F_v}}$ is pro-represented by $(R^{\psi,\mathrm{un},\square}_{V_{\mathbf{F}},v})^{\wedge}_{\xi}$. Then $\xi : (R^{\psi,\mathrm{un},\square}_{V_{\mathbf{F}},v})^{\wedge}_{\xi} \to E$ factors through $(R^{\psi,\gamma_v,\square}_{V_{\mathbf{F}},v})^{\wedge}_{\xi}$ if and only if its corresponding point lifts to $(\mathscr{L}^{\psi,\mathrm{un},\square}_{V_{\mathbf{F}},v})^{\wedge}_{\xi}$, which means that the associated E-representation V_{ξ} has a $\gamma_v \otimes E$-line. □

3.2. Preliminaries on the space of modular forms and Hecke algebras. We use notation in §2.4 and §2.5. In this subsection, we often use the fixed isomorphism $\iota_\infty : \overline{E} \simeq \mathbf{C}$ without the symbol ι_∞.

Let $\bar{\rho}_{\mathfrak{m}'} : G_{F,S} \to \mathrm{GL}_2(\mathbf{F})$ be the absolutely irreducible and modular residual representation. In this subsection, we assume that $\bar{\rho}_{\mathfrak{m}'}$ satisfies Condition 2.2. In order to construct the Hecke module H in Proposition 2.1 from the representation space of the modular deformations of $\bar{\rho}_{\mathfrak{m}'}$, we will take auxiliary primes as in [4]. In this subsection, we describe these.

Recall that Σ is the set of finite places of F at which D is ramified. We denote by S_{aux} the complement of $\Sigma_p \cup \{v \mid \infty\}$ in S. Let U_0 be the maximal compact open subgroup $\prod_{v \in |F|^\infty} (o_D)_v^\times$ of $(D \otimes_F \mathbf{A}_F^\infty)^\times$. Let f be a p-adic quaternionic Hecke eigenform whose residual representation is isomorphic to $\bar{\rho}_{\mathfrak{m}'}$; we assume that f has level U_0. We note that, under the assumption of Condition 2.2 (5) the representation ρ_f is unramified at each prime in S_{aux}; this follows from the result of [3], see the introduction of [7]. Thus, for each prime $\mathfrak{r}$ in S_{aux} the $\mathfrak{r}$-factor of the cuspidal representation π_f of $(D \otimes_F \mathbf{A}_F)^\times$ associated to f is an unramified principal series representation of $\mathrm{GL}_2(F_{\mathfrak{r}})$. We now define the compact open subgroup $U = \prod_v U_v$ contained in U_0 by putting

$$U_{\mathfrak{r}} := U_{11}(\varpi_{\mathfrak{r}}^2)$$

if $\mathfrak{r} \in S_{\mathrm{aux}}$ and $U_v := (U_0)_v$ otherwise. In the following, we consider f as an eigenform in $S_{2,\psi}(U, \mathcal{O})$.

As in the section 2, we denote by $\mathbf{T}'_{\psi,\mathcal{O}}(U)$ (resp. $\mathbf{T}_{\psi,\mathcal{O}}(U)$) the image of $\mathbf{T}^{\mathrm{univ}}_{S,\mathcal{O}}$ (resp. $\mathbf{T}^{\mathrm{univ}}_{S^p,\mathcal{O}}$) in $\mathrm{End}_{\mathcal{O}}(S_{2,\psi}(U,\mathcal{O}))$ by the standard Hecke action.

We shall define the Hecke algebra $\mathbf{T}'^{,\mathrm{aux}}_{\psi,\mathcal{O}}(U)$ (resp. $\mathbf{T}^{\mathrm{aux}}_{\psi,\mathcal{O}}(U)$) to be the sub-$\mathcal{O}$-algebra of $\mathrm{End}_{\mathcal{O}}(S_{2,\psi}(U,\mathcal{O}))$ generated by $\mathbf{T}'_{\psi,\mathcal{O}}(U)$ (resp. $\mathbf{T}_{\psi,\mathcal{O}}(U)$) and the Hecke operators

$$\mathrm{U}_{\varpi_{\mathfrak{r}}} := \left[U \begin{pmatrix} 1 & 0 \\ 0 & \varpi_{\mathfrak{r}} \end{pmatrix} U \right].$$

Let $\mathfrak{m}'$ be the maximal ideal of $\mathbf{T}'_{\psi,\mathcal{O}}(U)$ determined by f. We take the maximal ideal $\mathfrak{m}'^{,\mathrm{aux}}$ of $\mathbf{T}'^{,\mathrm{aux}}_{\psi,\mathcal{O}}(U)_{\mathfrak{m}'}$ generated by $\mathfrak{m}'$ and $\mathrm{U}_{\varpi_{\mathfrak{r}}}$ for all $\mathfrak{r} \in S_{\mathrm{aux}}$. We denote by $\mathbf{T}'^{,\mathrm{aux}}$ the localization of $\mathbf{T}'^{,\mathrm{aux}}_{\psi,\mathcal{O}}(U)$ by $\mathfrak{m}'^{,\mathrm{aux}}$, which is a noetherian complete local $\mathcal{O}$-algebra whose residue field is $\mathbf{F}$. We take a maximal ideal $\mathfrak{m}^{\mathrm{aux}}$ of the $\mathbf{T}'^{,\mathrm{aux}}$-algebra $\mathbf{T}^{\mathrm{aux}}_{\psi,\mathcal{O}}(U)_{\mathfrak{m}'^{,\mathrm{aux}}}$.

For any $n \geq 1$ we take the set Q_n of primes as in §2.5. As well as above, we define the Hecke algebra $\tilde{\mathbf{T}}'^{,\mathrm{aux}}_{\psi,\mathcal{O}}(U_{Q_n})$ (resp. $\tilde{\mathbf{T}}^{\mathrm{aux}}_{\psi,\mathcal{O}}(U_{Q_n})$) as sub-$\mathcal{O}$-algebras of $\mathrm{End}_{\mathcal{O}}(S_{2,\psi}(U_{Q_n},\mathcal{O}))$ generated by $\tilde{\mathbf{T}}'_{\psi,\mathcal{O}}(U_{Q_n})$ (resp. $\tilde{\mathbf{T}}_{\psi,\mathcal{O}}(U_{Q_n})$) and the operators

$\mathrm{U}_{\varpi_{\mathfrak{r}}}$ for all $\mathfrak{r} \in S_{\mathrm{aux}}$. Similarly we define the Hecke algebras $\tilde{\mathbf{T}}'^{,\mathrm{aux}}_{\psi,\mathcal{O}}(U^-_{Q_n})$ and $\tilde{\mathbf{T}}^{\mathrm{aux}}_{\psi,\mathcal{O}}(U^-_{Q_n})$. These Hecke algebras are all finite and flat over $\mathcal{O}$.

We take the maximal ideal $\mathfrak{m}^{-,\mathrm{aux}}_{Q_n}$ of $\tilde{\mathbf{T}}^{\mathrm{aux}}_{\psi,\mathcal{O}}(U^-_{Q_n})_{\mathfrak{m}^{\mathrm{aux}}}$ determined by the choice of eigenvalues of $\bar{\rho}_{\mathfrak{m}'}(\mathrm{Frob}_v)$ for all $v \in Q_n$; see the discussion in §2.5. We denote by $\mathfrak{m}^{\mathrm{aux}}_{Q_n}$ the maximal ideal of $\tilde{\mathbf{T}}^{\mathrm{aux}}_{\psi,\mathcal{O}}(U_{Q_n})$ induced by $\mathfrak{m}^{-,\mathrm{aux}}_{Q_n}$. We also denote by $\mathfrak{m}'^{,\mathrm{aux}}_{Q_n}$ the ideal $\mathfrak{m}^{\mathrm{aux}}_{Q_n} \cap \tilde{\mathbf{T}}'^{,\mathrm{aux}}_{\psi,\mathcal{O}}(U_{Q_n})$ of $\tilde{\mathbf{T}}'^{,\mathrm{aux}}_{\psi,\mathcal{O}}(U_{Q_n})$, and put

$$\mathbf{T}'^{,\mathrm{aux}}_{Q_n} := \tilde{\mathbf{T}}'^{,\mathrm{aux}}_{\psi,\mathcal{O}}(U_{Q_n})_{\mathfrak{m}'^{,\mathrm{aux}}_{Q_n}}, \quad \mathbf{T}^{\mathrm{aux}}_{Q_n} := \tilde{\mathbf{T}}^{\mathrm{aux}}_{\psi,\mathcal{O}}(U_{Q_n})_{\mathfrak{m}^{\mathrm{aux}}_{Q_n}}.$$

We also put $Q_0 := \emptyset$ for our convenience. Then, for any $n \geq 0$ we have the natural maps

$$\mathbf{T}'_{Q_n} \to \mathbf{T}'^{,\mathrm{aux}}_{Q_n} \quad \text{and} \quad \mathbf{T}_{Q_n} \to \mathbf{T}^{\mathrm{aux}}_{Q_n}.$$

Moreover, we have the following lemma.

Lemma 3.4. *Suppose that* $\bar{\rho}_{\mathfrak{m}'}$ *is unramified at each prime* $\mathfrak{r} \in S_{\mathrm{aux}}$ *and for any* $\mathfrak{r} \in S_{\mathrm{aux}}$ *the Condition* 2.2 (5) *is satisfied. Then*:

(1) *For any* $n \geq 0$, *the Hecke algebra* $\mathbf{T}'^{,\mathrm{aux}}_{Q_n}$ *is generated by Hecke operators* T_v, S_v *for all* $v \notin S_{Q_n}$ *and* U_{ϖ_v} *for all* $v \in Q_n$.

(2) *We have*

$$\mathbf{T}'^{,\mathrm{aux}}_{Q_n} \cong \mathbf{T}'_{Q_n} \quad \textit{and} \quad \mathbf{T}^{\mathrm{aux}}_{Q_n} \cong \mathbf{T}_{Q_n}.$$

The similar arguments hold for the level $U^-_{Q_n}$ *cases.*

Proof of Lemma 3.4. Put

$$\mathbf{T}'^{,\mathrm{aux}}_{Q_n,\bar{E}} := \mathbf{T}'^{,\mathrm{aux}}_{Q_n} \otimes_{\mathcal{O}} \bar{E}.$$

Since $\mathbf{T}'^{,\mathrm{aux}}_{Q_n}$ is flat over $\mathcal{O}$, the natural map $\mathbf{T}'^{,\mathrm{aux}}_{Q_n} \to \mathbf{T}'^{,\mathrm{aux}}_{Q_n,\bar{E}}$ is injective; moreover, when we put

$$H'^{,\mathrm{aux}}_{Q_n,\bar{E}} := S_{2,\psi}(U_{Q_n},\mathcal{O})_{\mathfrak{m}'^{,\mathrm{aux}}_{Q_n}} \otimes_{\mathcal{O}} \bar{E}$$

the natural map

$$\mathbf{T}'^{,\mathrm{aux}}_{Q_n,\bar{E}} \to \mathrm{End}_{\bar{E}}(H'^{,\mathrm{aux}}_{Q_n,\bar{E}})$$

is injective.

We shall show $\mathrm{U}_{\varpi_{\mathfrak{r}}} = 0$ over $H'^{,\mathrm{aux}}_{Q_n,\bar{E}}$. Let g be a $\mathbf{T}'_{Q_n}$-eigenform of $H'^{,\mathrm{aux}}_{Q_n,\bar{E}}$. Since ρ_g is unramified at $\mathfrak{r} \in S_{\mathrm{aux}}$, the restriction of the central character of $(\pi_g)_{\mathfrak{r}}$ to $\mathcal{O}^{\times}_{F_{\mathfrak{r}}}$ is trivial; it implies that

$$(\pi^{\infty}_g)_{\mathfrak{r}}^{U_{11}(\varpi^2_{\mathfrak{r}})} = (\pi^{\infty}_g)_{\mathfrak{r}}^{U_1(\varpi^2_{\mathfrak{r}})} = (\pi^{\infty}_g)_{\mathfrak{r}}^{U_0(\varpi^2_{\mathfrak{r}})}.$$

Moreover, when we write $(\pi_g)_{\mathfrak{r}} \cong \text{n-Ind}(\chi_{\mathfrak{r},1},\chi_{\mathfrak{r},2})$ as a normalized induced representation, the Hecke operator $\mathrm{U}_{\varpi_{\mathfrak{r}}}$ acting on the space $(\pi^{\infty}_g)_{\mathfrak{r}}^{U_0(\varpi^2_{\mathfrak{r}})}$ has the characteristic polynomial

$$X(X - N(\mathfrak{r})^{1/2}\chi_{\mathfrak{r},1}(\varpi_{\mathfrak{r}}))(X - N(\mathfrak{r})^{1/2}\chi_{\mathfrak{r},2}(\varpi_{\mathfrak{r}})); \tag{3.2.1}$$

see [23, Lemma 1.6] for instance. In particular, $(\pi_g^{\infty})_{\mathfrak{r}}^{U_0(\varpi_{\mathfrak{r}})}$ has the eigenspace of $\mathrm{U}_{\varpi_{\mathfrak{r}}}$ with eigenvalue 0, whose dimension is 1. It implies that $\mathrm{U}_{\varpi_{\mathfrak{r}}}$ annihilates g. Since $H'^{,\mathrm{aux}}_{Q_n,\bar{E}}$ has a $\mathbf{T}'_{Q_n}$-eigenbasis, by (2.4.7) we deduce that $\mathrm{U}_{\varpi_{\mathfrak{r}}}$ acts on $H'^{,\mathrm{aux}}_{Q_n,\bar{E}}$ as 0.

As a consequence, we obtain $\mathbf{T}'^{,\mathrm{aux}}_{Q_n} \cong \mathbf{T}'_{Q_n}$. The remainings are obtained from this immediately. □

Let $H_n = S_{2,\psi}(U_{Q_n}, \mathcal{O})_{\mathfrak{m}_{Q_n}}$ be the localized space of modular forms of level U_{Q_n}. Let g be a Hecke eigenform of $H_n \otimes \bar{E}$. Then, as mentioned above, at each prime $\mathfrak{r} \in S_{\mathrm{aux}}$ the $\mathfrak{r}$-factor of $(\pi_g^{\infty})^{U_{Q_n}}$ has dimension 1. On the other hand, for any prime $v \notin S_{\mathrm{aux}}$ the v-factor of $(\pi_g^{\infty})^{U_{Q_n}}$ is also of dimension 1, which is verified as follows: If $v \notin S_{Q_n}$ then π_g is unramified at v and $(U_{Q_n})_v$ is maximal. If $v \in Q_n$ then the v-factor of π_g is a principal series and it has two distinct 1-dimensional eigenspaces of U_{ϖ_v} (cf. [23, Lemma 1.6]). Thus, the local condition $\mathrm{U}_{\varpi_v} - \tilde{\alpha}_v \in \mathfrak{m}_{Q_n}$, where $\tilde{\alpha}_v \in \mathcal{O}$ is a lift of the chosen eigenvalue α_v of $\bar{\rho}_{\mathfrak{m}'}(\mathrm{Frob}_v)$, cuts out the 1-dimensional subspace of $\iota_{\infty}(\pi_g^{\infty})^{U_{Q_n}}$. As a consequence, the dimension of each Hecke eigenspace of the $\bar{E}$-vector space $H_n \otimes \bar{E}$ is less than or equal to 1.

Now we consider the augmented quotient. Let Δ_{Q_n} be the product of the maximal p-power quotient of $(\mathcal{O}_{F_v}/\varpi_v)^{\times}$ for all $v \in Q_n$. Recall that $U^-_{Q_n}/U_{Q_n} \cong \Delta_{Q_n}$ and $S_{2,\psi}(U_{Q_n}, \mathcal{O})$ becomes a Δ_{Q_n}-module by $a \mapsto \langle a \rangle$. Let

$$\sum_{a \in \Delta_{Q_n}} \langle a \rangle : S_{2,\psi}(U_{Q_n}, \mathcal{O}) \to S_{2,\psi}(U^-_{Q_n}, \mathcal{O}) \tag{3.2.2}$$

be the augmentation map; we note that this is $\tilde{\mathbf{T}}^{\mathrm{univ}}_{S_{Q_n},\mathcal{O}}$-equivariant. By [23, Lemma 2.3], the augmentation map (3.2.2) induces an isomorphism

$$S_{2,\psi}(U_{Q_n}, \mathcal{O})/\mathfrak{a}_{\Delta_{Q_n}} S_{2,\psi}(U_{Q_n}, \mathcal{O}) \xrightarrow{\simeq} S_{2,\psi}(U^-_{Q_n}, \mathcal{O}).$$

Moreover, $S_{2,\psi}(U_{Q_n}, \mathcal{O})$ is free over $\mathcal{O}[\Delta_{Q_n}]$.

For any subset Q of Q_n and an element $v \in Q$, let us denote by α_v the chosen eigenvalue of $\bar{\rho}_{\mathfrak{m}'}(\mathrm{Frob}_v)$ and by A_v the lifting of α_v in $\mathbf{T}'_{\psi,\mathcal{O}}(U^-_{Q-\{v\}})_{(\mathfrak{m}^-_{Q-\{v\}})'}$; cf. the discussion above (2.5.4). Then we have the $\mathcal{O}$-module map

$$\eta : S_{2,\psi}(U^-_{Q-\{v\}}, \mathcal{O})_{\mathfrak{m}^-_{Q-\{v\}}} \to S_{2,\psi}(U^-_{Q}, \mathcal{O})_{\mathfrak{m}^-_{Q}} \tag{3.2.3}$$

determined by

$$f \mapsto A_v f - \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} f.$$

Here $\begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} f$ is the right translation of f by the matrix $\begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix}$. This is an isomorphism, and it induces the isomorphism

$$\mathbf{T}^-_{Q} \xrightarrow{\simeq} \mathbf{T}^-_{Q-\{v\}} \tag{3.2.4}$$

sending U_{ϖ_v} to A_v for all $v \in Q_n$; see [23, Lemma 2.2]. Combining the localization of (3.2.2) with the isomorphisms (3.2.3) we obtain the following result:

PROPOSITION 3.5 ([23, Corollary 2.4]).
(1) *The augmentation map induces an isomorphism*

$$S_{2,\psi}(U_{Q_n},\mathcal{O})_{\mathfrak{m}_{Q_n}}/\mathfrak{a}_{\Delta_{Q_n}}S_{2,\psi}(U_{Q_n},\mathcal{O})_{\mathfrak{m}_{Q_n}}\xrightarrow{\simeq}S_{2,\psi}(U,\mathcal{O})_{\mathfrak{m}}.$$

This is compatible with the $\mathbf{T}^{\mathrm{univ}}_{S^p_{Q_n},\mathcal{O}}$*-algebra map* $\mathbf{T}_{Q_n}\to\mathbf{T}$ *sending* U_{ϖ_v} *to* A_v *for all* $v\in Q_n$ *and* $\langle a\rangle$ *to* 1 *for all* $a\in\Delta_{Q_n}$.
(2) $S_{2,\psi}(U_{Q_n},\mathcal{O})_{\mathfrak{m}_{Q_n}}$ *is free over* $\mathcal{O}[\Delta_{Q_n}]$.

In the following, we put

$$H_n^{\pm}:=S_{2,\psi}(U^{\pm}_{Q_n},\mathcal{O})_{\mathfrak{m}^{\pm}_{Q_n}}\quad\text{and}\quad H:=S_{2,\psi}(U,\mathcal{O})_{\mathfrak{m}},$$

where the symbol + means the U_{Q_n} case. Over the algebraic closure $\bar{E}$ of E we have a natural $\mathbf{T}^{\mathrm{univ}}_{S_{Q_n},\mathcal{O}}$-equivariant splitting

$$H_n^-\otimes\bar{E}\xrightarrow{\simeq}(H_n\otimes\bar{E})^{\Delta_{Q_n}}. \tag{3.2.5}$$

We take a $\mathbf{T}^{\mathrm{univ}}_{S,\mathcal{O}}$-eigenbasis $\mathcal{B}$ of $H\otimes\bar{E}$ and we denote by $\mathcal{B}_n^{\Delta}$ the image of $\mathcal{B}$ under the composition of (3.2.5) and (3.2.3).

By Proposition 3.5, $H_n\otimes\bar{E}$ is free over $\bar{E}[\Delta_{Q_n}]$. If we denote by d the $\mathcal{O}$-rank of H_n^- then we have a non-canonical isomorphism

$$H_n\otimes\bar{E}\simeq\bigoplus_{\chi\in\Delta^{\vee}_{Q_n}}\bar{E}(\chi)^{\oplus d},$$

where $\Delta^{\vee}_{Q_n}$ is the set of characters of Δ_{Q_n} with values in $\bar{E}^{\times}$, and $\bar{E}(\chi)$ is the representation space of the character χ. For each $\chi\in\Delta^{\vee}_{Q_n}$, the χ-eigenspace $(H_n\otimes\bar{E})(\chi)$ has a $\mathbf{T}'_{Q_n}$-eigenbasis. In particular, if χ_{triv} is the trivial character of Δ_{Q_n} then $(H_n\otimes\bar{E})(\chi_{\mathrm{triv}})=(H_n\otimes\bar{E})^{\Delta_{Q_n}}$ has an eigenbasis $\mathcal{B}_n^{\Delta}$. Moreover, we have a (non-canonical) decomposition

$$H_n\otimes\bar{E}=\bigoplus_{f\in\mathcal{B}_n^{\Delta}}\bigoplus_{\chi\in\Delta^{\vee}_{Q_n}}\bar{E}(\chi)\simeq\bigoplus_{f\in\mathcal{B}_n^{\Delta}}\bar{E}[\Delta_{Q_n}]. \tag{3.2.6}$$

Now we take a $\mathbf{T}'_{Q_n}$-eigenbasis $\mathcal{B}_n$ of $H_n\otimes\bar{E}$ containing $\mathcal{B}_n^{\Delta}$. Using the basis $\mathcal{B}_n$ we can take a decomposition (3.2.6) as $\mathbf{T}'_{Q_n}$-stable.

We shall consider the Galois conjugates of eigenforms and p-adic Hecke fields. We say eigenforms g_1, g_2 in $\mathcal{B}_n$ are Galois conjugate over E if σg_1 is a scalar multiple of g_2 for some $\sigma\in G_E$. Let $\tilde{\mathcal{B}}_{n/E}$ be the set of Galois conjugate classes of $\mathcal{B}_n$; then it is bijective to the set of maximal ideals of $\mathbf{T}'_{Q_n}[1/p]$, and for each $[g]\in\tilde{\mathcal{B}}_{n/E}$ we have $E_g\otimes_E\bar{E}=\prod_{h\in[g]}\bar{E}$.

Let Φ_n be the set of maximal ideals of $E[\Delta_{Q_n}]$. Then for each Galois conjugate class $[g]\in\tilde{\mathcal{B}}_{n/E}$ we can associate a maximal ideal $m=m(g)\in\Phi_n$ as the kernel of the composite

$$E[\Delta]\longrightarrow\mathbf{T}'_{Q_n}[1/p]\xrightarrow{\varphi_{g,E}}\bar{E}.$$

Moreover, the localization E_m of $E[\Delta_{Q_n}]$ by m, which is a finite extension of E, is contained in the Hecke field E_g of g. We put $a(g):=[E_g:E_m]$. Then (3.2.6) implies that, for each $m \in \Phi_n$ we have

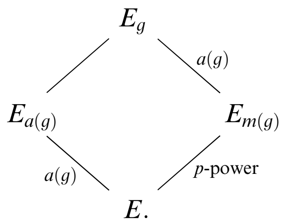
$$d[E_m : E] = \sum_{\substack{[g] \in \tilde{\mathcal{B}}_{n/E}, \\ m(g)=m}} [E_g : E] = \sum_{\substack{[g] \in \tilde{\mathcal{B}}_{n/E}, \\ m(g)=m}} a(g)[E_m : E].$$

In particular, $a(g) \le d$ for all $[g] \in \tilde{\mathcal{B}}_{n/E}$. Thus we have:

Lemma 3.6. *In the above situation, we further assume $d = \dim_E H \otimes E < p$. Then there is a finite extension E' of E such that, for any $n \ge 1$ and any eigenform $g \in \mathcal{B}_n$ the Hecke field E'_g of g over E' coincides with the localization $E'_{m(g)}$ of $E'[\Delta_{Q_n}]$ by the associated maximal ideal $m(g) = \mathrm{Ker}(\varphi_{g,E'}) \cap E'[\Delta_{Q_n}]$.*

In this lemma we only use the $E[\Delta_{Q_n}]$-freeness of $H_n \otimes E$, but not the $\mathcal{O}[\Delta_{Q_n}]$-freeness of H_n.

Proof of Lemma 3.6. For a moment, we assume that $\zeta_p \in E$; then for any $n \ge 2$ and any $m \in \Phi_n$ the extension E_m/E is wildly ramified. By assumption, for any $g \in \mathcal{B}_n$ we have a tamely ramified sub-extension $E_{a(g)}/E$ in E_g of degree $a(g)$ such that the composite of $E_{a(g)}$ and $E_{m(g)}$ is E_g:



We now take E' as the union of all finite and totally ramified extensions of E with degree $\le (p-1)d$. Then E' is an extension of E which contains ζ_p and E_g for all $g \in \mathcal{B}_1$. By the result of Krasner (cf. [14]) we know that the number of extensions of E of a fixed degree in $\bar{E}$ is finite. Thus E' is finite over E, and E' satisfies the condition in the statement. □

3.3. The Hecke modules. We continue to use notation and assumptions in §2.4, §2.5 and §3.2. In this subsection, for any $\mathcal{O}$-module M and any $\mathcal{O}$-algebra R we denote by M_R its scalar extension by R over $\mathcal{O}$. We assume that the residual representation $\bar{\rho} = \bar{\rho}_{\mathfrak{m}'}$ is associated to an eigenform f of $S_{2,\psi}(U, \mathcal{O})_{\mathfrak{m}'}$, whose automorphic representation π_f is special with exponential conductor 1 at each prime in Σ.

For any $n \ge 1$ we take a finite set Q_n of primes of F as above. For simplicity, $\mathbf{T}_{Q_n}^{\pm}$ denotes either $\mathbf{T}_{Q_n}$ or $\mathbf{T}_{Q_n}^{-}$. We also denote by $\mathbf{T}_{Q_n}^{\pm,\text{ß}}$ the image of the composite map

$$R_{F,S_{Q_n}} \to \tilde{R}_{F,S_{Q_n}}^{\psi,\sigma,\square} \to \mathbf{T}_{Q_n}^{\pm,\square}.$$

Then $\mathbf{T}_{Q_n}^{\pm,\text{ß}}$ is an object of $\widehat{\mathfrak{AR}}_{\mathcal{O}}$, whose maximal ideal is denoted by $\mathfrak{m}_{Q_n}^{\pm,\text{ß}}$. We have a modular deformation of $\bar{\rho}_{\mathfrak{m}'}$:

$$\rho_{Q_n}^{\pm,\text{ß}} : G_{F,S_{Q_n}} \to \mathrm{GL}_2(\mathbf{T}_{Q_n}^{\pm,\text{ß}}).$$

We denote by $V_{Q_n}^{\pm,\text{ß}}$ the representation space of $\rho_{Q_n}^{\pm,\text{ß}}$. Even in the level U case we can define the subalgebra $\mathbf{T}^{\text{ß}}$ of $\mathbf{T}$ similarly, and denote by $(\rho^{\text{ß}}, V^{\text{ß}})$ the deformation of $\bar{\rho}$ corresponding to $R_{F,S} \to \mathbf{T}^{\text{ß}}$.

Let $\mathcal{P}$ be a finite set of places which do not meet S. Let $\gamma = (\gamma_v)_{v \in \mathcal{P}}$ be a family of unramified local characters $\gamma_v : G_{F_v} \to \mathcal{O}^{\times}$. We shall define the Hecke modules corresponding to the local condition defined in §3.1. We consider the following conditions for deformations (ρ, ϕ) of $\bar{\rho}_{\mathfrak{m}'}$:

Condition 3.7. *For any $v \in \mathcal{P}$, there is a basis β_v such that the map $R_{V_{\mathbf{F}},v}^{\psi,\square} \to \mathcal{O}$ of $\widehat{\mathfrak{AR}}_{\mathcal{O}}$ corresponding to the local framed data $(\rho|_{G_{F_v}}, \phi, \beta_v)$ factors through $R_{V_{\mathbf{F}},v}^{\psi,\gamma_v,\square}$ (resp. $R_{V_{\mathbf{F}},v}^{\psi,\gamma_v\text{-ss},\square}$ with γ_v satisfying $\gamma_v^2 = \psi\varepsilon|_{G_{F_v}}$).*

By abuse of notation, we often denote the conditions in Condition 3.7 by (P_{γ}) or $(\mathrm{P}_{\mathrm{ss}})$ respectively; in the following (P_*) denotes the either condition.

Let $\mathcal{B}$ be a $\mathbf{T}_{S,\mathcal{O}}^{\mathrm{univ}}$-eigenbasis of $H \otimes \bar{E}$. We assume that it is contained in H. If we denote by

$$\Gamma = \mathrm{Aut}_{\mathbf{F}}(\mathcal{O})$$

the group of continuous automorphisms of $\mathcal{O}$ which associate the identity on $\mathbf{F}$, then it acts up to scalar multiples on $\mathcal{B}$ by the Galois conjugation. We denote by $\tilde{\mathcal{B}}_{/\Gamma}$ the set of Galois conjugate classes of $\mathcal{B}$ by Γ.

We denote by $\mathcal{B}(\mathcal{P}_*)$ the subset of $\mathcal{B}$ consisting of eigenforms which associate deformations of $\bar{\rho}$ satisfying the condition (P_*). We assume $\mathcal{B}(\mathcal{P}_*)$ is non-empty. The set $\mathcal{B}(\mathcal{P}_*)$ is stable under the Galois conjugation by Γ; we denote by $\tilde{\mathcal{B}}(\mathcal{P}_*)_{/\Gamma}$ the quotient set. For any conjugate class $[f]$ of $\mathcal{B}(\mathcal{P}_*)$, we denote by $\mathbf{T}_{[f]}^{\text{ß}}$ the image of the diagonal map

$$\mathbf{T}^{\text{ß}} \to \prod_{f' \in [f]} E$$

which associates T_v to the tuple of eigenvalues $(\theta_{f'}(T_v))_{f' \in [f]}$. We also denote by $I_{[f]}^{\text{ß}}$ the kernel of it. We put $V_{[f]}^{\text{ß}} := V^{\text{ß}} \otimes_{\mathbf{T}^{\text{ß}}} \mathbf{T}_{[f]}^{\text{ß}}$; this is a modular deformation of $\bar{\rho}$ corresponding to $R_{F,S} \to \mathbf{T}_{[f]}^{\text{ß}}$. By Lemma 2.8 we have the $\mathbf{T}^{\text{ß}}$-algebra map

$$\mathbf{T} \to \mathbf{T}_{[f]}^{\text{ß}}[1/p] \hookrightarrow \mathrm{End}_E(V_{[f]}^{\text{ß}}[1/p]) \tag{3.3.1}$$

determined by

$$T_{\mathfrak{p}} \mapsto \mathrm{Tr}_{\mathbf{T}_{[f]}^{\text{ß}}[1/p] \otimes W(\kappa(\mathfrak{p}))}(\varphi^{[\kappa(\mathfrak{p}):\mathbf{F}_p]} \mid D_{\mathrm{cris}}(V_{[f]}^{\text{ß}}[1/p]))$$

for all $\mathfrak{p} \mid p$. We define the Hecke algebra $\mathbf{T}_{[f]}$ to be the image of (3.3.1) and define the Hecke module $M_{[f]}$ to be

$$M_{[f]} := V_{[f]}^{\text{ß}} \otimes_{\mathbf{T}_{[f]}^{\text{ß}}} \mathbf{T}_{[f]}.$$

By construction, $M_{[f]}$ is free of rank 2 over $\mathbf{T}_{[f]}$ and in particular torsion free. We then put

$$M^{\mathcal{P}_*} := \bigoplus_{[f] \in \tilde{\mathcal{B}}(\mathcal{P}_*)_{/\Gamma}} M_{[f]}.$$

For any $n \geq 1$ we define $\mathbf{T}_{Q_n}^{-,\mathcal{P}_*}$ and $M_{Q_n}^{-,\mathcal{P}_*}$ as follows: Firstly we take a finite set Q_n of primes of F given in Proposition 2.3 so that it does not meet $S \cup \mathcal{P}$. For any $[f] \in \tilde{\mathcal{B}}(\mathcal{P}_*)_{/\Gamma}$, we denote by $I_{Q_n,[f]}^{-,\text{ß}}$ the inverse image of $I_{[f]}^{\text{ß}}$ under the isomorphism $\mathbf{T}_{Q_n}^{-} \xrightarrow{\simeq} \mathbf{T}$ induced by (3.2.4), and define

$$\mathbf{T}_{Q_n,[f]}^{-,\text{ß}} := \mathbf{T}_{Q_n}^{-,\text{ß}} / I_{Q_n,[f]}^{-,\text{ß}}, \quad V_{Q_n,[f]}^{-,\text{ß}} := V_{Q_n}^{-,\text{ß}} \otimes_{\mathbf{T}_{Q_n}^{-,\text{ß}}} \mathbf{T}_{Q_n,[f]}^{-,\text{ß}}.$$

As well as the level U case, we define the Hecke algebra $\mathbf{T}_{Q_n,[f]}^{-}$ to be the image of

$$\mathbf{T}_{Q_n}^{-} \to \operatorname{End}_E(V_{Q_n,[f]}^{-,\text{ß}}[1/p])$$

and the Hecke module $M_{Q_n,[f]}^{-}$ as

$$M_{Q_n,[f]}^{-} := V_{Q_n,[f]}^{-,\text{ß}} \otimes_{\mathbf{T}_{Q_n,[f]}^{-,\text{ß}}} \mathbf{T}_{Q_n,[f]}^{-}.$$

The isomorphisms (3.2.3) and (3.2.4) induce the natural isomorphisms

$$\mathbf{T}_{Q_n,[f]}^{-} \xrightarrow{\simeq} \mathbf{T}_{[f]} \quad \text{and} \quad M_{Q_n,[f]}^{-} \xrightarrow{\simeq} M_{[f]},$$

which are compatible with each other. We note that the later is $G_{F,S}$-equivariant. Now we put

$$M_{Q_n}^{-,\mathcal{P}_*} := \bigoplus_{[f] \in \tilde{\mathcal{B}}(\mathcal{P}_*)_{/\Gamma}} M_{Q_n,[f]}^{-};$$

then we have a $G_{F,S}$-equivariant isomorphism of Hecke modules

$$M_{Q_n}^{-,\mathcal{P}_*} \xrightarrow{\simeq} M^{\mathcal{P}_*}. \tag{3.3.2}$$

In the following we further assume the condition that, for any $n \geq 1$ and any eigenform g of $H_n \otimes \bar{E}$ we have

$$E_g = E_{m(g)};$$

here $m(g)$ is the kernel of $E[\Delta_{Q_n}] \longrightarrow \mathbf{T}'_{Q_n}[1/p] \xrightarrow{\varphi_{g,E}} \bar{E}$, cf. Lemma 3.6. Then Galois conjugates of eigenforms of $H_n \otimes \bar{E}$ over E are equivalent to that of characters of Δ_{Q_n}. Now we fix a $\mathbf{T}'_{Q_n}$-stable decomposition (3.2.6) and denote by $\mathcal{B}_n^{\Delta}(\mathcal{P}_*)$ the image of $\mathcal{B}(\mathcal{P}_*)$ under the composition of (3.2.5) and (3.2.3). Then, for any $f \in \mathcal{B}_n^{\Delta}(\mathcal{P}_*)$ we have a non-canonical isomorphism

$$\bar{E}[\Delta_{Q_n}] \simeq \bigoplus_{\chi \in \Delta_{Q_n}^{\vee}} \bar{E} g(f,\chi), \tag{3.3.3}$$

where $g(f,\chi)$ is a $\mathbf{T}'_{Q_n}$-eigenform in H_n on which Δ_{Q_n} acts as χ; this isomorphism is obtained from the irreducible decomposition of the regular representation $\bar{E}[\Delta_{Q_n}]$ of Δ_{Q_n}.

We denote by $\tilde{\mathcal{B}}_n^{\Delta}(\mathcal{P}_*)_{/\Gamma}$ the set of Galois conjugate classes of $\mathcal{B}_n^{\Delta}(\mathcal{P}_*)$ by Γ; we also choose its representative system $(f^j)_j$. For any eigenform in $\mathcal{B}_n^{\Delta}(\mathcal{P}_*)$ we take above isomorphism (3.3.3) as follows: Firstly, for each representative eigenform f^j of $\tilde{\mathcal{B}}_n^{\Delta}(\mathcal{P}_*)_{/\Gamma}$ we fix above isomorphism (3.3.3) so as $g(f^j,1) = f^j$, $g(f^j,\chi) \neq g(f^k,\chi)$ if $f^j \neq f^k$, and for each $f = \sigma f^j$ we take (3.3.3) as $g(f,\chi) = \sigma g(f^j,\sigma^{-1}\chi)$.

For any $f \in \mathcal{B}_n^{\Delta}(\mathcal{P}_*)$ we put

$$\mathcal{B}_n(f) := \{g(f,\chi) \mid \chi \in \Delta_{Q_n}^{\vee}\}$$

and denote by $\tilde{\mathcal{B}}_n(f)_{/E}$ the set of Galois conjugate classes of $\mathcal{B}_n(f)$ over E. Then, under our assumption we have

$$\prod_{[g] \in \tilde{\mathcal{B}}_n(f)_{/E}} E_g = \prod_{m \in \Phi_n} E_m \cong E[\Delta_{Q_n}].$$

Moreover we have the natural $\mathcal{O}[\Delta_{Q_n}]$-algebra map

$$\mathbf{T}_{Q_n}^{\beta} \to \prod_{[g] \in \tilde{\mathcal{B}}_n(f)_{/E}} E_g \cong E[\Delta_{Q_n}]. \tag{3.3.4}$$

By the following discussion, the image of this map is isomorphic to $\mathcal{O}[\Delta_{Q_n}]$.

Lemma 3.8. *Let Δ be a finite abelian p-group and R a finite local sub-$\mathcal{O}[\Delta]$-algebra of $E[\Delta]$ whose maximal ideal contains λ. If R has a specializing local map $\varphi : R \to \mathcal{O}$ of local $\mathcal{O}$-algebras which sends Δ to $\{1\}$ and the maximal ideal of R is generated by $\mathfrak{a}_{\Delta}$ and λ, then R is isomorphic to $\mathcal{O}[\Delta]$.*

As usual, *a local map $A \to B$ of local $\mathcal{O}$-algebras* means a ring homomorphism between local $\mathcal{O}$-algebras A and B, which maps the maximal ideal of A into that of B.

Proof of Lemma 3.8. We note that $\mathcal{O}[\Delta]$ is a local ring. By assumption, the ring R is torsion free and it has an injective local map of local $\mathcal{O}$-algebras

$$i : \mathcal{O}[\Delta] \hookrightarrow R,$$

which becomes an isomorphism after inverting p. We shall prove it is an isomorphism. By Nakayama's lemma, it is sufficient to show that i becomes an isomorphism after taking modulo $\mathfrak{m}_{\mathcal{O}[\Delta]}$.

By assumption, we have

$$R/\mathfrak{m}_{\mathcal{O}[\Delta]}R = R/\mathfrak{m}_R \cong \mathcal{O}/\lambda = \mathbf{F},$$

which proves the lemma. □

Let $\mathcal{T}$ be the image of (3.3.4). $\mathcal{T}$ is finite and faithful as a $\mathcal{O}[\Delta_{Q_n}]$-module, which is generated by tuples

$$T_v|_{\mathcal{T}} := (\theta_g(T_v))_{[g]\in\tilde{\mathcal{B}}_n(f)_{/E}} \qquad (v \notin S_{Q_n}),$$

$$\mathrm{U}_{\varpi_v}|_{\mathcal{T}} := (\theta_g(\mathrm{U}_{\varpi_v}))_{[g]\in\tilde{\mathcal{B}}_n(f)_{/E}} \quad (v \in Q_n).$$

On the other hand, since each eigenform of

$$H_{Q_n} \otimes_{\mathcal{O}[\Delta_{Q_n}]} \mathbf{F} \cong H \otimes_{\mathcal{O}} \mathbf{F}$$

has the same eigenvalue in $\mathbf{F}$ for all operators in $\mathbf{T}_{Q_n}^{\text{ß}}$, we have

$$T_v|_{\mathcal{T}} \equiv \theta_f(T_v)\cdot(1,\dots,1) \bmod \mathfrak{m}_{\mathcal{O}[\Delta_{Q_n}]} \quad (v \notin S_{Q_n}),$$

$$\mathrm{U}_{\varpi_v}|_{\mathcal{T}} \equiv \alpha_v\cdot(1,\dots,1) \bmod \mathfrak{m}_{\mathcal{O}[\Delta_{Q_n}]} \qquad (v \in Q_n).$$

Thus $\mathcal{T}/\mathfrak{m}_{\mathcal{O}[\Delta_{Q_n}]}\mathcal{T} = \mathbf{F}\cdot(1,\dots,1)$, and so $\mathcal{O}[\Delta_{Q_n}] \simeq \mathcal{T}$ by Lemma 3.8.

We denote by

$$\rho_{f,\Delta_{Q_n}} : G_{F,S_{Q_n}} \to \mathrm{GL}_2(\mathcal{O}[\Delta_{Q_n}])$$

the deformation associated to (3.3.4) and by $V_{f,\Delta_{Q_n}}$ its representation space. Let

$$\varphi_{f,\Delta_{Q_n}} : \mathbf{T}_{Q_n} \to \mathrm{End}_E(V_{f,\Delta_{Q_n}}[1/p])$$

be the $\mathbf{T}_{Q_n}^{\text{ß}}$-algebra map determined by

$$T_{\mathfrak{p}} \mapsto \mathrm{Tr}_{E[\Delta_{Q_n}]\otimes W(\kappa(\mathfrak{p}))}(\varphi^{[\kappa(\mathfrak{p}):\mathbf{F}_p]} \mid D_{\mathrm{cris}}(V_{f,\Delta_{Q_n}}[1/p]))$$

for all $\mathfrak{p} \mid p$. The image of the map $\varphi_{f,\Delta_{Q_n}}$ is contained in the center of

$$\mathrm{End}_{E[\Delta_{Q_n}]}(V_{f,\Delta_{Q_n}}[1/p]) \simeq M_2(E[\Delta_{Q_n}]).$$

Moreover, we have the $\mathbf{T}_{Q_n}^{\text{ß}}$-algebra map

$$\theta_{f,\Delta_{Q_n}} : \mathbf{T}_{Q_n} \xrightarrow{\varphi_{f,\Delta_{Q_n}}} E[\Delta_{Q_n}] \xrightarrow{\text{aug.}} E$$

which maps $T_{\mathfrak{p}}$ to

$$\mathrm{Tr}_{E\otimes W(\kappa(\mathfrak{p}))}(\varphi^{[\kappa(\mathfrak{p}):\mathbf{F}_p]} \mid D_{\mathrm{cris}}(V_f[1/p]))$$

for all $\mathfrak{p} \mid p$. The image of $\theta_{f,\Delta_{Q_n}}$ is a local sub-$\mathcal{O}$-algebra of E whose residue field is $\mathbf{F}$, so that it coincides with $\mathcal{O}$. By Lemma 3.8 again, we have:

Lemma 3.9. *The image of* $\varphi_{f,\Delta_{Q_n}} : \mathbf{T}_{Q_n} \to E[\Delta_{Q_n}]$ *is* $\mathcal{O}[\Delta_{Q_n}]$.

For each Γ-conjugate class $[f]$ of $\mathcal{B}_n^{\Delta}(\mathcal{P}_*)$, we denote by $\mathbf{T}_{Q_n,[f]}^{\text{ß}}$ the image of the diagonal map

$$\prod_{f'\in[f]} \varphi_{f',\Delta_{Q_n}} : \mathbf{T}_{Q_n}^{\text{ß}} \to \prod_{f'\in[f]} \mathcal{O}[\Delta_{Q_n}]$$

and denote by $V_{Q_n,[f]}^{\text{ß}}$ the modular deformation corresponding to $R_{F,S_{Q_n}} \to \mathbf{T}_{Q_n,[f]}^{\text{ß}}$. We then define the Hecke algebra $\mathbf{T}_{Q_n,[f]}$ to be the image of the $\mathbf{T}_{Q_n}^{\text{ß}}$-algebra map

$$\mathbf{T}_{Q_n} \to \mathbf{T}_{Q_n,[f]}^{\text{ß}}[1/p] \hookrightarrow \mathrm{End}_E(V_{Q_n,[f]}^{\text{ß}}[1/p])$$

determined by

$$T_{\mathfrak{p}} \mapsto \mathrm{Tr}_{\mathbf{T}^{\text{ß}}_{Q_n,[f]}[1/p]\otimes W(\kappa(\mathfrak{p}))}(\varphi^{[\kappa(\mathfrak{p}):\mathbf{F}_p]} \mid D_{\mathrm{cris}}(V^{\text{ß}}_{Q_n,[f]}[1/p])),$$

and define the Hecke module $M_{Q_n,[f]}$ to be

$$M_{Q_n,[f]} := V^{\text{ß}}_{Q_n,[f]} \otimes_{\mathbf{T}^{\text{ß}}_{Q_n,[f]}} \mathbf{T}_{Q_n,[f]};$$

this is a free $\mathbf{T}_{Q_n,[f]}$-module of rank 2. Finally we put

$$M^{\mathcal{P}_*}_{Q_n} := \bigoplus_{[f]\in \tilde{\mathcal{B}}^{\Delta}_n(\mathcal{P}_*)_{/\Gamma}} M_{Q_n,[f]}.$$

Proposition 3.10. *Let $\mathcal{P}$ be a finite set of places which do not meet S and $\gamma = (\gamma_v)_{v\in\mathcal{P}}$ be a family of unramified local characters $\gamma_v : G_{F_v} \to \mathcal{O}^\times$. For each prime $v \in \mathcal{P}$ we take either deformation condition* (P_γ) *or* (P_{ss}) *of Condition* 3.7; *when we consider the condition* (P_{ss}) *we always assume that for all $v \in \mathcal{P}$ the character γ_v satisfies $\gamma_v^2 = \psi\varepsilon|_{G_{F_v}}$. We also assume that the localized space $H = S_{2,\psi}(U,\mathcal{O})_{\mathfrak{m}}$ satisfies*

$$\text{(3.3.5)} \qquad \mathrm{rank}_{\mathcal{O}}\, H < p,$$

and that there is an eigenform f of $H \otimes \overline{E}$ which satisfies (P_*) *for all $v \in \mathcal{P}$. Then, for sufficiently large E we have the following*:

(1) *The augmentation map induces an isomorphism*

$$M^{\mathcal{P}_*}_{Q_n}/\mathfrak{a}_{\Delta_{Q_n}} M^{\mathcal{P}_*}_{Q_n} \xrightarrow{\simeq} M^{\mathcal{P}_*}.$$

(2) $M^{\mathcal{P}_*}_{Q_n}$ *is free over* $\mathcal{O}[\Delta_{Q_n}]$.

Let $f \in H$ be an eigenform satisfying the conditions (P_*) for all $v \in \mathcal{P}$ and $\tilde{f}$ the corresponding Hilbert modular eigenform. Let $\tilde{U}$ be the level of $\tilde{f}$. As each Hecke eigenspace of H has dimension 1 we have

$$\mathrm{rank}_{\mathcal{O}}\, H \le \dim_{\mathbf{C}} S^{M_2}_{2,\psi}(\tilde{U},\mathbf{C}).$$

Thus the condition (3.3.5) follows from the global condition $\dim_{\mathbf{C}} S^{M_2}_{2,\psi}(\tilde{U},\mathbf{C}) < p$, which is introduced in (1.0.1).

Proof of Proposition 3.10. Put $\Delta := \Delta_{Q_n}$ and $d' = \#\mathcal{B}(\mathcal{P}_*)$ for simplicity. It is sufficient to show the case when $\mathcal{B}(\mathcal{P}_*)$ consists of single Γ-orbit; we shall put $\mathcal{B}(\mathcal{P}_*) = \{f_0,\ldots,f_{d'-1}\} = [f_0]$.

By assumption $M^{\mathcal{P}_*}$ is a nonzero module. Enlarging E if necessary we may assume that:

- Galois conjugates of each eigenform $f \in \mathcal{B}(\mathcal{P}_*)$ over E are only itself;
- we can take an extension of E in Lemma 3.6 as itself.

As mentioned at (3.3.2) we have the $G_{F,S}$-equivariant isomorphism

$$M^{-,\mathcal{P}_*}_{Q_n} \xrightarrow{\simeq} M^{\mathcal{P}_*},$$

which is compatible with the isomorphism of Hecke algebras $\mathbf{T}_{Q_n}^{-,\mathcal{P}_*} \xrightarrow{\simeq} \mathbf{T}^{\mathcal{P}_*}$ sending U_{ϖ_v} to A_v for all $v \in Q_n$.

We shall show that the augmented quotient of $M_{Q_n}^{\mathcal{P}_*}$ is $M_{Q_n}^{-,\mathcal{P}_*}$. Identifying $\mathcal{B}_n^{\Delta}(\mathcal{P}_*)$ with $\mathcal{B}(\mathcal{P}_*)$, we have

$$M_{Q_n}^{\mathcal{P}_*}[1/p] = V_{Q_n,E}^{\text{ß},\mathcal{P}_*} \simeq \bigoplus_{f_j \in \mathcal{B}_n^{\Delta}(\mathcal{P}_*)} E[\Delta]^{\oplus 2}. \tag{3.3.6}$$

We consider the (normalized) augmentation map

$$\frac{1}{\#\Delta}\sum_{a\in\Delta}\langle a\rangle : V_{Q_n,E}^{\text{ß},\mathcal{P}_*} \to V_{Q_n,E}^{-,\text{ß},\mathcal{P}_*}. \tag{3.3.7}$$

Then it induces an isomorphism $V_{Q_n,E}^{\text{ß},\mathcal{P}_*}/\mathfrak{a}_\Delta V_{Q_n,E}^{\text{ß},\mathcal{P}_*} \xrightarrow{\simeq} V_{Q_n,E}^{-,\text{ß},\mathcal{P}_*}$. Moreover, we have the following commutative diagram:

$$\begin{array}{ccc} \mathbf{T}_{Q_n} & \longrightarrow & \mathrm{End}_{E[\Delta]}(V_{Q_n,E}^{\text{ß},\mathcal{P}_*}) \\ {\scriptstyle \text{aug.}}\downarrow & & \downarrow{\scriptstyle \text{induced by (3.3.7)}} \\ \mathbf{T}_{Q_n}^{-} & \longrightarrow & \mathrm{End}_{E}(V_{Q_n,E}^{-,\text{ß},\mathcal{P}_*}). \end{array} \tag{3.3.8}$$

Here "aug." means the augmentation map. In particular we obtain the surjective map of Hecke algebras

$$\mathbf{T}_{Q_n}^{\mathcal{P}_*} \to \mathbf{T}_{Q_n}^{-,\mathcal{P}_*} \tag{3.3.9}$$

which sends $\langle a\rangle$ to 1 for all $a \in \Delta$. We also note that the right vertical arrow in (3.3.8) is equivalent to the augmentation map of matrix algebras

$$M_{2d'}(E[\Delta]) \to M_{2d'}(E).$$

The kernel of (3.3.9) is $\mathfrak{a}_\Delta M_{2d'}(E[\Delta]) \cap \mathbf{T}_{Q_n}^{\mathcal{P}_*}$, which contains $\mathfrak{a}_\Delta \mathbf{T}_{Q_n}^{\mathcal{P}_*}$. We shall prove that it coincides with $\mathfrak{a}_\Delta \mathbf{T}_{Q_n}^{\mathcal{P}_*}$. Let x be an element of $\mathfrak{a}_\Delta M_{2d'}(E[\Delta]) \cap \mathbf{T}_{Q_n}^{\mathcal{P}_*}$. Then, under the decomposition (3.3.6), the element x can be written as

$$\left(\begin{array}{cc|c|cc} \varphi_{f_0,\Delta}(t) & & & & \\ & \varphi_{f_0,\Delta}(t) & & & \\ \hline & & \ddots & & \\ & & \ddots & & \\ \hline & & & \varphi_{f_{d'-1},\Delta}(t) & \\ & & & & \varphi_{f_{d'-1},\Delta}(t) \end{array}\right)$$

for some $t \in \mathbf{T}_{Q_n}$. By Lemma 3.9 the component $\varphi_{f_j,\Delta}(t)$ is contained in

$$\mathfrak{a}_\Delta E[\Delta] \cap \mathcal{O}[\Delta] = \mathfrak{a}_\Delta$$

for each j. We shall write $\varphi_{f_0,\Delta}(t) = \sum_{a\in\Delta}(a-1)y_a$ for some $y_a \in \mathcal{O}[\Delta]$. Since each f_j in $\mathcal{B}_n^{\Delta}(\mathcal{P}_*)$ is written as $f_j = \sigma_j f_0$ for some $\sigma_j \in \Gamma$ we can decompose $\varphi_{f_j,\Delta}$

as $\varphi_{f_j,\Delta} = \sigma_j \circ \varphi_{f_0,\Delta}$. Thus $\varphi_{f_j,\Delta}(t)$ can be expressed as $\varphi_{f_j,\Delta}(t) = \sum_{a\in\Delta}(a-1)\sigma y_a$. It implies that x is the image of the element

$$\sum_{a\in\Delta}(a-1)i(z_a)$$

of $\mathbf{T}_{Q_n}$, where each z_a is a (unique) element of $\mathcal{O}[\Delta]$ whose image under the composite

$$\mathcal{O}[\Delta] \xrightarrow{\ i\ } \mathbf{T}_{Q_n} \xrightarrow{\varphi_{f_0,\Delta}} \mathcal{O}[\Delta]$$

is y_a. Thus x is contained in $\mathfrak{a}_\Delta \mathbf{T}_{Q_n}^{\mathcal{P}_*}$. Therefore we have

$$\mathfrak{a}_\Delta M_{2d'}(E[\Delta]) \cap \mathbf{T}_{Q_n}^{\mathcal{P}_*} = \mathfrak{a}_\Delta \mathbf{T}_{Q_n}^{\mathcal{P}_*},$$

and we obtain $\mathbf{T}_{Q_n}^{\mathcal{P}_*}/\mathfrak{a}_\Delta \mathbf{T}_{Q_n}^{\mathcal{P}_*} \xrightarrow{\simeq} \mathbf{T}_{Q_n}^{-,\mathcal{P}_*}$. Since $M_{Q_n}^{\pm,\mathcal{P}_*}$ is free over $\mathbf{T}_{Q_n}^{\pm,\mathcal{P}_*}$, we deduce that

$$M_{Q_n}^{\mathcal{P}_*}/\mathfrak{a}_\Delta M_{Q_n}^{\mathcal{P}_*} \xrightarrow{\simeq} M_{Q_n}^{-,\mathcal{P}_*}.$$

Now we prove the freeness. We note that $M_{Q_n}^{\pm,\mathcal{P}_*}[1/p] = V_{Q_n}^{\pm,\text{ß},\mathcal{P}_*}[1/p]$ since $\mathbf{T}_{Q_n}^{\pm,\mathcal{P}_*}[1/p] = \mathbf{T}_{Q_n}^{\pm,\text{ß},\mathcal{P}_*}[1/p]$. Thus we have an equation

$$\dim_E(M_{Q_n}^{\mathcal{P}_*}[1/p]) = 2(\#\Delta \cdot \#\mathcal{B}(\mathcal{P}_*)) = \#\Delta \cdot \dim_E(M_{Q_n}^{-,\mathcal{P}_*}[1/p]).$$

By Lemma 3.11 below we obtain the result. □

Lemma 3.11. *Let $\mathcal{O}$ be a complete discrete valuation ring of mixed characteristic $(0,p)$. Let Δ be a finite abelian p-group. Let M be a finitely generated $\mathcal{O}[\Delta]$-module. We assume the following*:

(1) *M is free over $\mathcal{O}$.*
(2) *$M/\mathfrak{a}_\Delta M$ is a free $\mathcal{O}$-module of finite rank. Here $\mathfrak{a}_\Delta$ is the augmentation ideal of $\mathcal{O}[\Delta]$.*
(3) *We have the equation for $\mathcal{O}$-ranks*:

$$\operatorname{rank}_{\mathcal{O}} M = \#\Delta \cdot \operatorname{rank}_{\mathcal{O}}(M/\mathfrak{a}_\Delta M).$$

Then M is free over $\mathcal{O}[\Delta]$.

Proof of Lemma 3.11. First note that $\mathcal{O}[\Delta]$ is a local ring. Let $(\bar{x}_i)_{1\le i\le m}$ be a $\mathcal{O}$-basis of $M/\mathfrak{a}_\Delta M$. By Nakayama's lemma, we have a lifting $(x_i)_i$ of $(\bar{x}_i)_i$, which generates M over $\mathcal{O}[\Delta]$. We note that $(x_i)_i$ is a linearly independent system over $\mathcal{O}$. By construction, $(tx_i)_{1\le i\le m, t\in\Delta}$ is an $\mathcal{O}$-generating system of M and we have the inequality

$$\operatorname{rank}_{\mathcal{O}} M \le \#(tx_i)_{1\le i\le m, t\in\Delta} \le \#\Delta \cdot m.$$

By assumption, this is an equation. Thus $(tx_i)_{i,t\in\Delta}$ is a $\mathcal{O}$-basis of M and we know that $(x_i)_i$ is a $\mathcal{O}[\Delta]$-basis of M. □

3.4. $R = T$ **type result.** We use the notation in §2.5. We also keep assumptions in the previous sections.

Let $\mathcal{P}$ be a finite set of places which do not meet S. Let $\gamma = (\gamma_v)_{v\in\mathcal{P}}$ be a family of local unramified characters $\gamma_v : G_{F_v} \to \mathcal{O}^\times$. Let (P_*) be either condition (P_γ) or $(\mathrm{P}_{\mathrm{ss}})$. We assume that there is an eigenform $f \in S_{2,\psi}(U,\mathcal{O})_{\mathfrak{m}'}$ satisfying Condition 3.7 with respect to $\mathcal{P}$ and γ; when we consider the condition $(\mathrm{P}_{\mathrm{ss}})$, we will assume that for any primes $v \in \mathcal{P}$ the local character γ_v satisfies $\gamma_v^2 = \psi\varepsilon|_{G_{F_v}}$. Let $\rho_f : G_{F,S} \to \mathrm{GL}_2(\mathcal{O})$ be the deformation of $\bar{\rho}_{\mathfrak{m}'}$ corresponding to f. Let $\sigma = (\sigma', \{\chi_{\mathfrak{p}}\}_{\mathfrak{p}\in\sigma'})$ be the ordinary data given by ρ_f. Let $\mathfrak{m}$ be a maximal ideal of $\mathbf{T}_{\psi,\mathcal{O}}(U)$ determined by f. For any $n \geq 1$ we take a finite set Q_n of primes of F given in Proposition 2.3 so that it does not meet $S \cup \mathcal{P}$. We write $Q_0 := \emptyset$ for our convenience, and for any $n \geq 0$ we consider the Hecke module $M_{Q_n}^{\mathcal{P}_*}$ defined in §3.3. We put

$$M_{Q_n}^{\mathcal{P}_*,\square} := M_{Q_n}^{\mathcal{P}_*} \otimes_{R_{F,S_{Q_n}}} R_{F,S_{Q_n}}^{\square}.$$

As usual we will write $M^{\mathcal{P}_*,\square} = M_{Q_0}^{\mathcal{P}_*,\square}$. The following proposition, which is the key of our result, is a variation of [13, Proposition (3.4.11)].

Proposition 3.12. *Let $\bar{\rho} : G_{F,S} \to \mathrm{GL}_2(\mathbf{F})$ be an absolutely irreducible continuous representation satisfying Coniditon 2.2. We assume that $\bar{\rho} \simeq \bar{\rho}_{\mathfrak{m}'}$ for some modular representation (2.5.1). Let $\mathcal{P}$ be a finite set of primes of F which do not meet S and let $\gamma = (\gamma_v)_{v\in\mathcal{P}}$ be a family of local unramified characters $\gamma_v : G_{F_v} \to \mathcal{O}^\times$. Let (P_*) be either condition (P_γ) or $(\mathrm{P}_{\mathrm{ss}})$ of Condition 3.7 with respect to $(\mathcal{P},\gamma)$; when we consider the condition $(\mathrm{P}_{\mathrm{ss}})$ we further assume that for any $v \in \mathcal{P}$ the character γ_v satisfies $\gamma_v^2 = \psi\varepsilon|_{G_{F_v}}$. We also assume that there is an eigenform $f \in S_{2,\psi}(U,\mathcal{O})_{\mathfrak{m}'}$ satisfying (P_*) and that the inequality (3.3.5) holds for the maximal ideal $\mathfrak{m}$ of $\mathbf{T}_{\psi,\mathcal{O}}(U)_{\mathfrak{m}'}$ associated to the ordinary data σ determined by ρ_f. Then, the kernel of the map*

$$\tilde{R}_{F,S}^{\psi,\sigma,\square} \to \mathrm{End}_{\mathcal{O}}(M^{\mathcal{P}_*,\square})$$

is p-power torsion. In particular, if some modular deformation satisfies (P_*) on a finite set $\mathcal{P}$ of unramifimed primes, then every modular deformation of type $\tilde{R}_{\Sigma,p}^{\psi,\sigma,\square}$ also satisfies (P_*) on $\mathcal{P}$.

Proof of Proposition 3.12. Let $B = \tilde{R}_{\Sigma,p}^{\psi,\sigma,\square}$, $R = \tilde{R}_{F,S}^{\psi,\sigma,\square}$, $R_n = \tilde{R}_{F,S_{Q_n}}^{\psi,\sigma,\square}$, $H = M^{\mathcal{P}_*,\square}$ and $H_n = M_{Q_n}^{\mathcal{P}_*,\square}$. By our hypothesis, H is a non-zero module. Then by (2.2.2) and Proposition 3.10, we can apply Proposition 2.1 and we obtain the result. □

3.5. A rigidity of deformations of a mod p **Galois representation associated to a Hilbert modular form.** Let F be a totally real number field, f be a Hilbert modular Hecke-eigen cuspform over F. Let λ be a prime of the Hecke field K_f of f above a rational prime p, $K_{f,\lambda}$ be its λ-adic completion and $\mathcal{O}_{f,\lambda}$ be the ring of integers of $K_{f,\lambda}$. Then, we have a two dimensional λ-adic representation

$$\rho_{f,\lambda} : G_F \to \mathrm{GL}_2(K_{f,\lambda}).$$

Let Σ be the finite set of primes of F at which π_f is ramified; we assume that each prime in Σ does not divide p. If we denote by S a finite set of places of F containing Σ, all primes dividing p and all infinite places, then $\rho_{f,\lambda}$ is unramified outside S. Namely it factors through the Galois group $G_{F,S}$ of the maximum extension of F which is unramified outside S. Let $V_{f,\lambda}$ be the representation space of $\rho_{f,\lambda}$. Taking its G_F-stable lattice $T_{f,\lambda}$ of $V_{f,\lambda}$, we have a continuous representation (denoting by the same symbol)

$$\rho_{f,\lambda} : G_{F,S} \to \mathrm{GL}_2(\mathcal{O}_{f,\lambda}).$$

When we denote by $\mathbf{F}_\lambda$ the residue field of $\mathcal{O}_{f,\lambda}$, composing $\rho_{f,\lambda}$ with the natural reduction map, we have the continuous representation $G_F \to \mathrm{GL}_2(\mathbf{F}_\lambda)$, whose semi-simplification is independent of choice of lattices $T_{f,\lambda}$, and which is denoted by

$$\bar{\rho}_{f,\lambda} : G_{F,S} \to \mathrm{GL}_2(\mathbf{F}_\lambda). \tag{3.5.1}$$

By Ribet's work described in [22], $\rho_{f,\lambda}$ is irreducible for all λ and $\bar{\rho}_{f,\lambda}$ is absolutely irreducible for almost all λ. Thus we take λ so that $\bar{\rho}_{f,\lambda}$ is absolutely irreducible.

In the following, we suppose f has parallel weight two. By the results of Carayol, Faltings and Taylor (cf. [22, Theorems 1.2, 1.4]), the family $\boldsymbol{\rho}_f = (\rho_{f,\lambda})_{\lambda \in |K_f|^\infty}$ of λ-adic representations fit into a regular and irreducible rank 2 weakly compatible system

$$\mathcal{R}_f := (K_f, \Sigma, \mathcal{S}, \{Q_{f,v}(X)\}_{v \notin \Sigma}, \boldsymbol{\rho}_f, \{0,1\}),$$

where $\mathcal{S}$ is the (finite) set of primes λ of K_f which divides the level of f, or λ at which the semi-simplification of $\bar{\rho}_{f,\lambda}$ is absolutely reducible. In particular, by [22, Proposition 1.6] for any primes $\lambda \notin \mathcal{S}$ the λ-adic realization $\rho_{f,\lambda}$ is Barsotti-Tate. Moreover, the local-global compatibilities of Langlands correspondences for $\rho_{f,\lambda}$'s are none other than that $\mathcal{R}_f$ is strongly compatible.

LEMMA 3.13. *Let F be a totally real field of even degree and f be a Hilbert modular Hecke-eigen cuspform of parallel weight two. Let $\mathcal{R}_f$ be the regular and irreducible strong compatible system associated to f. Let $\mathcal{L}$ be the set of all primes λ of K_f lying over a rational prime p_λ satisfying that*:

(1) *the* mod λ *representation $\bar{\rho}_{f,\lambda}$ is absolutely irreducible*,

(2) $F \cap \mathbf{Q}(\zeta_{p_\lambda}) = \mathbf{Q}$,

(3) $p_\lambda \equiv 1 \bmod 4$.

Then, for all but not finitely many primes $\lambda \in \mathcal{L}$, $\bar{\rho}_{f,\lambda}$ remains absolutely irreducible after restricting to the subgroup $G_{F(\zeta_{p_\lambda})}$.

Proof of Lemma 3.13. We note that such a $\mathcal{L}$ is an infinite set. As Kisin remarked in [13, p. 1155], by the condition (2) the absolute irreducibility of $\bar{\rho}_{f,\lambda}|_{G_{F(\zeta_{p_\lambda})}}$ is equivalent to that of $\bar{\rho}_{f,\lambda}|_{G_L}$, where $L = F(\sqrt{(-1)^{(p_\lambda-1)/2}p_\lambda})$. Moreover, by the condition (3), L is totally real, and so by [22, Proposition 3.1] the modular Galois representation

$$\rho_{f,\lambda}|_{G_L} \cong \rho_{\mathrm{BC}_{L/F}(\pi_f),\lambda'}$$

is irreducible for all $\lambda \in \mathcal{L}$; here $\mathrm{BC}_{L/F}(\pi_f)$ means the base change of π_f via L/F. Therefore, by [loc.cit., Proposition 3.3], for all but not finitely many primes $\lambda \in \mathcal{L}$, the restriction $\bar{\rho}_{f,\lambda}|_{G_L}$ must be absolutely irreducible. □

In order to state our main theorem, we prepare the following notion of residual modularity.

Remark 3.14 ([13, (3.5.4)], [26, §4]). Let $\mathcal{O}$ be the ring of integers of a finite extension $E/\mathbf{Q}_p$ and $\rho : G_{F,S} \to \mathrm{GL}_2(\mathcal{O})$ be a continuous representation. We call ρ is *strongly residually modular* of parallel weight two if there exists a Hilbert modular Hecke-eigen cuspform f over F of parallel weight two such that[4)]:

- There is an isomorphism $\bar{\rho} \simeq \bar{\rho}_{f,\lambda'}$, where λ' is a suitable prime of K_f,
- The automorphic representation π_f of $\mathrm{GL}_2(\mathbf{A}_F)$ generated by f is not special at any place dividing p,
- For any $\mathfrak{p} \mid p$, the restriction $\rho_{f,\lambda'}|_{G_{F_\mathfrak{p}}}$ is potentially ordinary if and only if $\rho|_{G_{F_\mathfrak{p}}}$ is.

We note that for almost all v, π_f is not special at v; this follows from the fact that automorphic representations π are admissible, and so that π is unramified at almost all v.

Theorem 3.15. *Let F be a totally real number field. Let K be a number field, Σ be a finite set of finite primes of F, $\mathcal{S}$ be a finite set of finite primes of K and*

$$\mathcal{R} := (K, \Sigma, \mathcal{S}, \{Q_v(X)\}_{v \notin \Sigma}, \boldsymbol{\rho} = (\rho_\lambda)_{\lambda \in |K|^\infty})$$

be a regular and irreducible rank 2 weakly pre-compatible system of λ-adic representations of G_F. We assume that there is a finite character $\psi : G_F \to \mathcal{O}_K^\times$ such that, for any primes $\lambda \in |K|^\infty$ the determinant of ρ_λ is $\psi\varepsilon_{p_\lambda}$; here we denote by p_λ the residue characteristic of K_λ. We further assume that there is a prime $\lambda \notin \mathcal{S}$ lying over a rational prime $p = p_\lambda > 2$ such that the λ-adic realization

$$\rho = \rho_\lambda : G_F \to \mathrm{GL}_2(K_\lambda)$$

satisfies the conditions in [13, Theorem (3.5.5)]. *Namely,*

(1) *For any prime $\mathfrak{p}$ dividing p, the restriction $\rho|_{G_{F_\mathfrak{p}}}$ is potentially Barsotti-Tate*;
(2) *ρ is strongly residually modular* (*in the sense of Remark* 3.14);
(3) *$\bar{\rho}|_{G_{F(\zeta_p)}}$ is absolutely irreducible*;
(4) *If $p = 5$ and the projective image of* $\mathrm{Im}\,\bar{\rho}$ *is isomorphic to* $\mathrm{PGL}_2(\mathbf{F}_5)$, *then the kernel of the projectivization of $\bar{\rho}$ does not fix $F(\zeta_5)$.*

[4)] We use the definition of [26, §4] which is slightly modified from [13, (3.5.4)].

Then, for infinitely many primes $\mu \notin S$ we have the following: After taking a suitable totally real solvable base change $F^{\#}$ of F (and denoting objects of $F^{\#}$ by the same symbols), each $\mathcal{O}_{K_\mu}$-deformation ρ'_μ of

$$\bar{\rho}_\mu : G_F \to \mathrm{GL}_2(\mathbf{F}_\mu)$$

of type $\tilde{R}^{\psi,\sigma_\mu,\square}_{\Sigma,p_\mu}$ has a field automorphism $\phi \in \mathrm{Aut}(\overline{K}_\mu)$ satisfying

$$(\rho'_\mu \otimes \overline{K}_\mu|_{G_{F_v}})^{\mathrm{ss}} \simeq \phi_*(\rho_\mu \otimes \overline{K}_\mu|_{G_{F_v}})^{\mathrm{ss}}$$

for almost all unramified primes v of ρ_μ. Here σ_μ is the ordinary data associated to ρ_μ.

In [13, Theorem (3.5.5)], Kisin proved the modularity of ρ satisfying conditions (1)–(4) as above. As a consequence, after a suitable base change the restriction of $\boldsymbol{\rho}$ can be fit into an irreducible strongly compatible system of Hodge-Tate weights $\{0,1\}$. Since the proof of Theorem 3.15 is based on his proof, we shall review it briefly (with a slight modification for our proof).

Let $\mathcal{O}$ be the ring of integers of $E = K_\lambda$. Let f be a Hilbert modular eigenforms of F of parallel weight two satisfying $\bar{\rho} \simeq \bar{\rho}_{f,\lambda'}$. Let Σ be the set of primes at which ρ is ramified and not dividing p. By base change arguments ([15], [19, Main Theorem] and [13, Lemmas (3.5.2), (3.5.3)]), replacing F with its suitable totally real and solvable extension and f with its base change, we may prove the modularity under the following conditions for ρ:

(1) $\rho|_{G_{F_\mathfrak{p}}}$ is Barsotti-Tate at all primes $\mathfrak{p}$ dividing p,
(2) If $\mathfrak{p}$ divides p and $\rho|_{G_{F_\mathfrak{p}}}$ is ordinary, then $\bar{\rho}|_{G_{F_\mathfrak{p}}}$ is either indecomsable or has the trivial image,
(3) $\bar{\rho}$ is unramified outside the primes dividing p,
(4) If $v \in \Sigma$ then the restriction $\rho|_{I_v}$ is unipotent,
(5) $[F:\mathbf{Q}]$ is even and $\bar{\rho}|_{G_{F(\zeta_p)}}$ remains absolutey irreducible;

and conditions for the pair f and Σ:

(i) π_f is unramified at all $v \notin \Sigma$; in particular at all primes $\mathfrak{p}$ dividing p,
(ii) π_f is special with conductor 1 at v if $v \in \Sigma$.

Then $\bar{\rho}$ satisfies (1)–(4) of Condition 2.2. Moreover, by [4, Lemma 4.11] we can take an auxiliary prime $\mathfrak{r}$ so that Condition 2.2 (5) is satisfied for the set

$$S := \Sigma \cup \{\mathfrak{p} \mid p\} \cup \{v \mid \infty\} \cup \{\mathfrak{r}\}.$$

By further base change if necessary, we may assume that the cardinality of Σ is even. Twisting by a character we assume that ρ and $\rho_{f,\lambda'}$ have the same determinant. Now let D be the quaternion algebra over F which is ramified at $\Sigma \cup \{v \mid \infty\}$. We fix a maximal order o_D of D and define a compact open subgroup $U_0 = \prod_v (U_0)_v$ of $(D \otimes_F \mathbf{A}_F^\infty)^\times$ by $(U_0)_v := (o_D)_v^\times$. By the Jacquet-Langlands and Shimizu correspondence, after enlarging $\mathcal{O}$ if necessary, there is a Hecke eigenform f^D of $S^D_{2,\psi}(U_0,\mathcal{O})$ corresponding to f. Here ψ is determined by the central character of f. We define a subgroup $U = \prod_v U_v$ by setting[5)]

$$U_{\mathfrak{r}} := U_{11}(\varpi_{\mathfrak{r}}^2)$$

and $U_v := (U_0)_v$ for all $v \neq \mathfrak{r}$. Taking $\mathfrak{r}$ as $N(\mathfrak{r})$ is sufficiently large, we can assume U satisfies Condition 2.6. We now regard f^D as an eigenform of $S^D_{2,\psi}(U, \mathcal{O})$. Then we take the maximal ideal $\mathfrak{m}$ of $\mathbf{T}_{\psi,\mathcal{O}}(U)$ associated to f^D. We take the set S of places of F as above. Then, by [13, (3.4.11)] we have a surjective map

$$\tilde{R}_{F,S}^{\psi,\sigma,\square} \to \mathbf{T}^{\square}$$

with p-power torsion kernel, where $\mathbf{T}^{\square} := \mathbf{T} \otimes_{R_{F,S}} R_{F,S}^{\square}$. In particular ρ is modular, namely there is an eigenform $g \in S^D_{2,\psi}(U, \mathcal{O})$ such that ρ is associated to g.

Proof of Theorem 3.15. Let

$$\tilde{U}_0 = \prod_{v \in \Sigma} U_0(\varpi_v) \times \prod_{v \notin \Sigma} \mathrm{GL}_2(\mathcal{O}_{F_v})$$

be the compact open subgroup of $\mathrm{GL}_2(\mathbf{A}_F^{\infty})$. Put

$$d_0 := \prod_{v \in \Sigma} \#(\mathrm{GL}_2(\mathcal{O}_{F_v})/U_0(\varpi_v));$$

here $\mathrm{GL}_2(\mathcal{O}_{F_v})/U_0(\varpi_v)$ is bijective to the set of cyclic subgroups of $(\mathcal{O}_{F_v}/\varpi_v)^2$ isomorphic to $\mathcal{O}_{F_v}/\varpi_v$. We note that d_0 is greater than or equal to the cardinality of the finite set $\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbf{A}_F^{\infty})/\tilde{U}_0$. We will use d_0 to bound the dimension of the localized space of modular forms.

Let $\mu \notin \mathcal{S}$ be a prime of K lying over a rational odd prime p_μ such that: (i) $p_\mu > \max(5, d_0)$ and (ii) the restriction of $\bar{\rho}_\mu$ to $G_{F(\zeta_{p_\mu})}$ remains absolutely irreducible; by Lemma 3.13 such μ are infinitely many exist.

Let K_μ be the μ-adic completion of K and $\mathcal{O}_\mu$ be the ring of integers of K_μ. We consider μ-adic quaternionic modular forms $S^D_{2,\psi}(U_0, \mathcal{O}_\mu)$.

Now we know that $\rho_\mu = \rho_{g,\mu}$ satisfies the following conditions:

(1) the local restriction $\rho_\mu|_{G_{F_\mathfrak{p}}}$ is Barsotti-Tate at each prime $\mathfrak{p}$ dividing p_μ,
(2) If $v \notin \Sigma_{p_\mu}$ then ρ_μ is unramified at v, and if $v \in \Sigma$ then the restriction $\rho_\mu|_{I_v}$ is unipotent,
(3) $[F : \mathbf{Q}]$ is even and $\bar{\rho}_\mu|_{G_{F(\zeta_{p_\mu})}}$ remains absolutely irreducible.

In particular, for any prime v in Σ the restriction of $\bar{\rho}_\mu$ to the inertia I_v is

$$\bar{\rho}_\mu|_{I_v} \sim \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},$$

5) In [13] Kisin take $U_{\mathfrak{r}}$ as $U_{11}(\varpi_{\mathfrak{r}})$. We refer to the method of [4] for our proof.

where a is the additive character of I_v which factors through $t_v : I_v \to \mathbf{Z}_{p_\mu}(1)$. We take a totally real cyclic extension $F^\#/F$ of degree p_μ such that:

- the restriction of ρ_μ to $G_{F^\#(\zeta_{p_\mu})}$ remains absolutely irreducible,
- each prime $v \in \Sigma$ is ramified at $F^\#/F$, so that $\bar{\rho}_\mu|_{G_{F^\#}}$ is unramified outside the primes dividing p_μ.

Since p_μ is an odd prime, the base change

$$\Pi_g := \mathrm{BC}_{F^\#/F}(\pi_g)$$

of π_g is a cuspidal representation of $\mathrm{GL}_2(\mathbf{A}_{F^\#})$ (cf. [15]). Moreover, $\rho_\mu|_{G_{F^\#}}$ satisfies (1)–(3) as above and the condition that:

(4) $\bar{\rho}_\mu|_{G_{F^\#}}$ is unramified outside primes dividing p_μ.

In the following, we replace objects of F with that of $F^\#$ lying over those, and denote these as $(-)^\#$. Then the set $\Sigma^\#$ of primes of $F^\#$ lying over Σ has even cardinality.

We take a totally definite quaternion algebra $D^\#$ over $F^\#$ which is ramified exactly at $\Sigma^\#$. We fix a maximal order $o_{D^\#}$ of $D^\#$ and define a subgroup of $(D^\# \otimes \mathbf{A}_{F^\#}^\infty)^\times$ as $U_0^\# := \prod_v (o_{D^\#})_v^\times$. As remarked at (2.4.7), under the fixed isomorphism $\iota_\mu : \bar{K}_\mu \simeq \mathbf{C}$ we have the decomposition

$$S_{2,\psi,\bar{K}_\mu}^{D^\#} / S_{2,\psi,\bar{K}_\mu}^{D^\#,\mathrm{triv}} \simeq \bigoplus_{\Pi} \iota_\mu(\Pi^\infty),$$

where Π runs over all cuspidal representations of $(D^\# \otimes_{F^\#} \mathbf{A}_{F^\#})^\times$ having weight 2 and central character ψ. By assumption, the finite part of Π_g occurs in the decomposition, and we have $(\Pi_g^\infty)^{U_0^\#} \neq 0$. We take an eigenform $g^\#$ in $S_{2,\psi}^{D^\#}(U_0^\#, \mathcal{O}_\mu)$ which generates Π_g.

We take a prime $\mathfrak{r}_\mu \notin \Sigma_{p_\mu}^\# = \Sigma^\# \cup \{\mathfrak{p} \mid p_\mu\}$ as before and define a subgroup

$$U^\# := U_{11}(\varpi_{\mathfrak{r}_\mu}^2) \times \prod_{v \nmid \mathfrak{r}_\mu} (o_{D^\#})_v^\times;$$

moreover we may take it so that $U^\#$ satisfies Condition 2.6 and the set

$$S^\# := \Sigma_{p_\mu}^\# \cup \{v \mid \infty\} \cup \{\mathfrak{r}_\mu\}$$

satisfies Condition 2.2 (5). We regard $g^\#$ as an eigenform in $S_{2,\psi}^{D^\#}(U^\#, \mathcal{O}_\mu)$. Let $\mathfrak{m}^\#$ be the maximal ideal of $\mathbf{T}_{\psi,\mathcal{O}_\mu}(U^\#)$ determined by $g^\#$ and

$$H^\# := S_{2,\psi}^{D^\#}(U^\#, \mathcal{O}_\mu)_{\mathfrak{m}^\#}$$

the corresponding localized space.

Lemma 3.16. *Under the above notation, the dimension of $H^\# \otimes \bar{K}_\mu$ satisfies*

$$\dim_{\bar{K}_\mu} H^\# \otimes \bar{K}_\mu \leq d_0.$$

Proof of Lemma 3.16. Let $\tilde{D}^\#$ be a quaternion algebra over $F^\#$ which is ramified exactly at all infinite places. By the Jacquet-Langlands-Shimizu correspondence, for any eigenform h in $H^{\#} \otimes \bar{K}_\mu$ there is a cuspidal representation $\tilde{\Pi}$ of $(\tilde{D}^{\#} \otimes_{F^{\#}} \mathbf{A}_{F^{\#}})^\times$ such that

$$\mathrm{JL}_{D^{\#}}(\Pi_h) \cong \mathrm{JL}_{\tilde{D}^{\#}}(\tilde{\Pi}).$$

If we take the compact open subgroup $\tilde{U}_0^{\#}$ of $\mathrm{GL}_2(\mathbf{A}_{F^{\#}}^\infty)$ to be

$$\tilde{U}_0^{\#} := \prod_{w \in \Sigma^{\#}} U_0(\varpi_w) \times \prod_{w \notin \Sigma^{\#}} \mathrm{GL}_2(\mathcal{O}_{F_w^{\#}}),$$

then $(\tilde{\Pi}^\infty)^{\tilde{U}_0^{\#}} \neq 0$. Thus, for any eigenform $h \in H^{\#} \otimes \bar{K}_\mu$ we have an eigenform of

$$S_{2,\psi}^{\tilde{D}^{\#}}(\tilde{U}_0^{\#}, \bar{K}_\mu).$$

Since each Hecke eigenspace of $H^{\#} \otimes \bar{K}_\mu$ has dimension one (cf. the discussion of Proof of Lemma 3.4), we have

$$\dim_{\bar{K}_\mu} H^{\#} \otimes \bar{K}_\mu \leq \dim_{\bar{K}_\mu} S_{2,\psi}^{\tilde{D}^{\#}}(\tilde{U}_0^{\#}, \bar{K}_\mu).$$

Moreover, by assumption the dimension of $S_{2,\psi}^{\tilde{D}^{\#}}(\tilde{U}_0^{\#}, \bar{K}_\mu)$ is less than or equal to

$$\begin{aligned}\#(\mathrm{GL}_2(F^{\#})\backslash \mathrm{GL}_2(\mathbf{A}_{F^{\#}}^\infty)/\tilde{U}_0^{\#}) &\leq \prod_{v^{\#} \in \Sigma^{\#}} \#(\mathrm{GL}_2(\mathcal{O}_{F^{\#}_{v^{\#}}})/U_0(\varpi_{v^{\#}})) \\ &= \prod_{v \in \Sigma} \#(\mathrm{GL}_2(\mathcal{O}_{F_v})/U_0(\varpi_v)) = d_0,\end{aligned}$$

which completes the proof. □

We back to the proof of the theorem. By this lemma, the localized space $H^{\#} = S_{2,\psi}^{D^{\#}}(U^{\#}, \mathcal{O}_\mu)_{\mathfrak{m}^{\#}}$ satisfies the inequality (3.3.5);

$$\mathrm{rank}_{\mathcal{O}_\mu} H^{\#} < p_\mu.$$

Let v_0 be a prime of $F^{\#}$ not contained in $S^{\#}$. Let $\mathcal{P}$ be the one-point set consisting of v_0. We enlarge K_μ so that the matrix $\rho_\mu(\mathrm{Frob}_{v_0})$ is triangularizable, and take an unramified local character $\gamma_{v_0} : G_{F^{\#}_{v_0}} \to \mathcal{O}_\mu^\times$ such that

$$\rho_\mu|_{G_{F^{\#}_{v_0}}} \simeq \begin{pmatrix} v_*\gamma_{v_0} & * \\ 0 & \psi\varepsilon_{p_\mu} \cdot v_*\gamma_{v_0}^{-1} \end{pmatrix},$$

where v is the structure map $\mathcal{O}_\mu \longrightarrow R_{F^{\#},S^{\#}} \xrightarrow{\varphi_{\rho_\mu}} \mathcal{O}_\mu$. We now apply Proposition 3.12 for $g^{\#} \in H^{\#}$ and the condition (P_*) with respect to $(\{v_0\}, \gamma_{v_0})$. Then we obtain the result that, after enlarging K_μ, every $\mathcal{O}_\mu$-deformation ρ'_μ of $\bar{\rho}_\mu$ of type $\tilde{R}_{\Sigma^{\#},p_\mu}^{\psi,\sigma_\mu,\square}$ satisfies this condition. Namely, we have

$$\rho'_\mu|_{G_{F^{\#}_{v_0}}} \simeq \begin{pmatrix} v'_*\gamma_{v_0} & * \\ 0 & \psi\varepsilon_{p_\mu} \cdot v'_*\gamma_{v_0}^{-1} \end{pmatrix},$$

where ν' is the structure morphism of the $\mathcal{O}_\mu$-rational point $\varphi_{\rho'_\mu}$ of $R^{\psi,\sigma_\mu,\square}_{F^\#,S^\#}$. Since $\nu,\nu' \in \mathrm{Aut}_{\mathbf{F}}(\mathcal{O}_\mu)$, taking $\phi = \nu' \circ \nu^{-1}$ we obtain

$$(\rho'_\mu|_{G_{F^\#_{v_0}}})^{\mathrm{ss}} \simeq (\phi_*\rho_\mu|_{G_{F^\#_{v_0}}})^{\mathrm{ss}},$$

where "ss" means the semi-simplification of the representation. Since ν and ν' are independent from the choice of v_0, we complete the proof. □

Moreover, since the way of taking $v = v_0 \notin S^\#$ is arbitrary, by the Chebotarev's density theorem we know that $\rho'_\mu \otimes \overline{K}_\mu$ is equivalent to $\rho_\mu \otimes_\phi \overline{K}_\mu$. Consequently, we have:

Corollary 3.17. *Let $\mathcal{R}$ be a regular and irreducible rank 2 weakly pre-compatible system of λ-adic representations of G_F satisfying conditions in Theorem 3.15. Then, for infinitely many primes μ of K, there is a suitable totally real solvable base change $F^\#/F$ such that, after taking it all $\mathcal{O}_{K_\mu}$-deformations ρ_1, ρ_2 of $\bar{\rho}_\mu$ of type $\tilde{R}^{\psi,\sigma_\mu,\square}_{\Sigma,p_\mu}$ are isomorphic to each other modulo an automorphism of $\overline{K}_\mu$; namely, there is a field automorphism $\phi \in \mathrm{Aut}(\overline{K}_\mu)$ such that*

$$\rho_1 \otimes \overline{K}_\mu|_{G_{F^\#}} \simeq \phi_*(\rho_2 \otimes \overline{K}_\mu)|_{G_{F^\#}}.$$

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