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GEOMETRIC INVARIANTS OF 5/2-CUSPIDAL EDGES

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Dedicated to Professor Takashi Nishimura on the occasion of his 60th birthday.

Abstract

We introduce two invariants called the *secondary cuspidal curvature* and the *bias* on 5/2-cuspidal edges, and investigate their basic properties. While the secondary cuspidal curvature is an analog of the cuspidal curvature of (ordinary) cuspidal edges, there are no invariants corresponding to the bias. We prove that the product (called the *secondary product curvature*) of the secondary cuspidal curvature and the limiting normal curvature is an intrinsic invariant. Using this intrinsicity, we show that any real analytic 5/2-cuspidal edges with non-vanishing limiting normal curvature admit non-trivial isometric deformations, which provides the extrinsicity of various invariants.

1. Introduction

In this paper, we study local differential geometric properties of curves and surfaces with singular points. Since we look at local properties, we essentially deal with map-germs: $(\boldsymbol{R},0) \to (\boldsymbol{R}^2,0)$ and $(\boldsymbol{R}^2,0) \to (\boldsymbol{R}^3,0)$. We consider invariants under an action that is a diffeomorphism on the source space and an orientation preserving isometry of the target Euclidean space: $\boldsymbol{R}^2$ and $\boldsymbol{R}^3$. In the case of curves, we take a representative and identify a map-germ $(\boldsymbol{R},0) \to (\boldsymbol{R}^2,0)$ with a curve $I \to \boldsymbol{R}^2$, where I is an open interval including the origin of $\boldsymbol{R}$. Similarly, in the case of surfaces, we take a representative and identify a map-germ $(\boldsymbol{R}^2,0) \to (\boldsymbol{R}^3,0)$ with a surface $U \to \boldsymbol{R}^3$, where U is an open neighborhood of the origin in $\boldsymbol{R}^2$. We mainly deal with 5/2-cusps and 5/2-cuspidal edges in this paper.

The *ordinary cusp* or *3/2-cusp* is a map-germ $(\boldsymbol{R},0) \to (\boldsymbol{R}^2,0)$ which is diffeomorphic ($\mathscr{A}$-equivalent) to the map-germ $t \mapsto (t^2,t^3)$ at the origin. It is known that the 3/2-cusp is the most frequently appearing singularity on plane curves. A *cuspidal edge* is a map-germ $(\boldsymbol{R}^2,0) \to (\boldsymbol{R}^3,0)$ which is $\mathscr{A}$-equivalent

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to the map-germ $(u,v) \mapsto (u, v^2, v^3)$ at the origin (Figure 1, right), where two map-germs $f, g : (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}^n, 0)$ are *$\mathscr{A}$-equivalent* if there exist diffeomorphisms $\phi_s : (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}^m, 0)$ and $\phi_t : (\boldsymbol{R}^n, 0) \to (\boldsymbol{R}^n, 0)$ such that $\phi_t \circ f \circ \phi_s = g$. By definition, the image of a cuspidal edge is diffeomorphic to a direct product of a 3/2-cusp with an interval ($\{(x, y, z) \mid y^3 - z^2 = 0\}$), and its differential geometric properties are well studied. In [27] (see also [22]), the *cuspidal curvature* for 3/2-cusps is defined. Roughly speaking, the cuspidal curvature measures whether a 3/2-cusp is narrower or wider. For cuspidal edges, the *singular curvature* and the *limiting normal curvature* are introduced in [20], and their geometric meanings are studied.

A 5/2-*cusp* (respectively, 5/2-*cuspidal edge*) is a map-germ $(\boldsymbol{R}, 0) \to (\boldsymbol{R}^2, 0)$ (respectively, $(\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$) which is $\mathscr{A}$-equivalent to the map-germ $t \mapsto (t^2, t^5)$ (respectively, $(u,v) \mapsto (u, v^2, v^5)$) at the origin (Figure 1, left). A 5/2-cusp is also called a *rhamphoid cusp*. Although 5/2-cuspidal edges do not generically appear, it has been pointed out that they naturally appear in various differential geometric situations [6, 10, 19]. For 5/2-cusps, the cuspidal curvature vanishes. Hence, to measure the width of 5/2-cusps, we need to consider higher order invariants. In this paper, we define two curvatures on 5/2-cusps in addition to the invariants we mentioned above, which are the *secondary cuspidal curvature* and the *bias* of cusps. The secondary cuspidal curvature is an analog of cuspidal curvature of 3/2-cusps, but as we will see in Section 2.1, there is no corresponding notion of bias for 3/2-cusps. Using these invariants, the secondary cuspidal curvature and the bias, we also define two curvatures for 5/2-cuspidal edges.

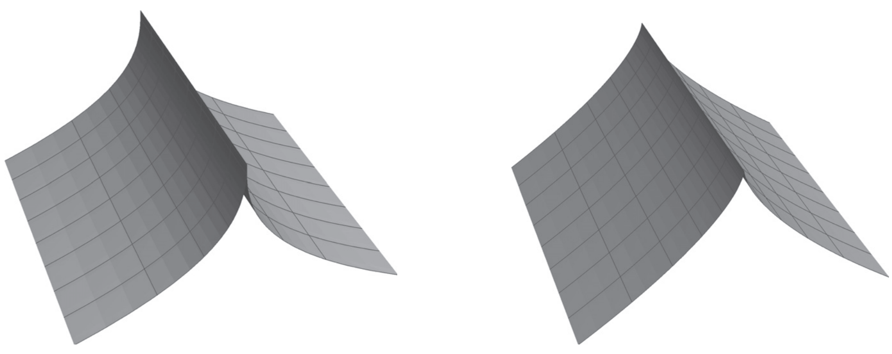


FIGURE 1. The standard 5/2-cuspidal edge (left) and cuspidal edge (right).

On the other hand, one fundamental problem is to determine the intrinsicity and extrinsicity of invariants. It is proved that some basic invariants such as the singular curvature and the product curvature are intrinsic in [20, 15], and they have various applications. For example, the intrinsicity of the product curvature is used to prove existence of isometric deformations of real analytic cuspidal edges with non-vanishing limiting normal curvature in [16] and [8]. See [4] for other applications. In [3, 7], several geometric invariants of cross caps are proved to be intrinsic or extrinsic. In this paper, we determine whether the above invariants of 5/2-cuspidal edges are intrinsic or extrinsic, proving the existence of isometric deformations of real analytic 5/2-cuspidal edges with non-vanishing limiting normal curvature as in [16] and [8].

This paper is organized as follows. In Section 2, we define the secondary cuspidal curvature and the bias for 5/2-cusps, and study their geometric properties. In Section 3, we deal with 5/2-cuspidal edges and define two invariants for them. As an example, in Section 3.6, we calculate the invariants on the conjugate surfaces of spacelike Delaunay surfaces. In Section 4, we prove that the product (called the *secondary product curvature*) of the secondary cuspidal curvature and the limiting normal curvature is an intrinsic invariant. Using this intrinsicity, we show the existence of isometric deformations of real analytic 5/2-cuspidal edges with non-vanishing limiting normal curvature, which yields the extrinsicity of various invariants, see Table 2. Finally, in Section 5, we provide an intrinsic formulation of 5/2-cuspidal edges as a singular point of a positive semi-definite metric, called the Kossowski metric. Using an argument similar to that in Section 4, we prove the existence of isometric realizations of Kossowski metrics with intrinsic 5/2-cuspidal edges.

2. Invariants of 5/2-cusps

In this section, we discuss the geometric properties of 5/2-cusps.

2.1. Invariants of 5/2-cusps. Let $\gamma : (\boldsymbol{R}, 0) \to (\boldsymbol{R}^2, 0)$ be a map-germ, and $\gamma'(0) = 0$. We say that γ is of *A-type* if $\gamma''(0) \neq 0$. Let $\gamma : (\boldsymbol{R}, 0) \to (\boldsymbol{R}^2, 0)$ be an A-type map-germ. The *cuspidal curvature* for γ at 0 is defined by

$$\omega(\gamma, 0) = \frac{\det(\gamma''(0), \gamma'''(0))}{|\gamma''(0)|^{5/2}},$$

which measures a kind of wideness of γ at 0 ([22]). We may abbreviate $\omega(\gamma, 0)$ as $\omega(\gamma)$, or ω, in some cases. It is well known that an A-type map-germ γ is a 3/2-cusp if and only if $\det(\gamma''(0), \gamma'''(0)) \neq 0$, and hence $\omega \neq 0$.

Let $\gamma : (\boldsymbol{R}, 0) \to (\boldsymbol{R}^2, 0)$ be an A-type map-germ with $\det(\gamma''(0), \gamma'''(0)) = 0$. Then there exists $l \in \boldsymbol{R}$ such that

$$\gamma'''(0) = l\gamma''(0).$$

Then the *secondary cuspidal curvature* for γ at 0 is defined by

$$\omega_r(\gamma, 0) = \frac{\det(\gamma''(0), 3\gamma^{(5)}(0) - 10l\gamma^{(4)}(0))}{|\gamma''(0)|^{7/2}}.$$

We abbreviate $\omega_r = \omega_r(\gamma) = \omega_r(\gamma, 0)$ as well. By a direct calculation, one can see that ω_r does not depend on the parameter of γ. The following criterion for 5/2-cusp is known [19]:

FACT 2.1. *Let $\gamma : (\boldsymbol{R}, 0) \to (\boldsymbol{R}^2, 0)$ be a map-germ with $\gamma'(0) = 0$. Then γ is a 5/2-cusp if and only if*
(1) $\det(\gamma''(0), \gamma'''(0)) = 0$,
(2) $3\det(\gamma''(0), \gamma^{(5)}(0))\gamma''(0) - 10\det(\gamma''(0), \gamma^{(4)}(0))\gamma'''(0) \neq (0, 0)$.

By the condition (2), $\gamma''(0) \neq 0$. When γ is of A-type at 0, the conditions (1) and (2) are written as follows. By (1), there exists $l \in \boldsymbol{R}$ such that $\gamma'''(0) = l\gamma''(0)$, and then (2) is written as

$$\det(\gamma''(0), 3\gamma^{(5)}(0) - 10l\gamma^{(4)}(0)) \neq 0.$$

Thus an A-type germ γ is a 5/2-cusp if and only if $\omega = 0$ and $\omega_r \neq 0$.

Next we define the bias of cusps. Let $\gamma : (\boldsymbol{R}, 0) \to (\boldsymbol{R}^2, 0)$ be an A-type map-germ which is not a 3/2-cusp (i.e., $\omega = 0$). Then

$$b(\gamma, 0) = \frac{\det(\gamma''(0), \gamma^{(4)}(0))}{|\gamma''(0)|^3}$$

does not depend on the parameter, and it is called the *bias* of cusps. We abbreviate $b = b(\gamma) = b(\gamma, 0)$ as well. Let γ be an A-type germ. A line

$$\left\{u \lim_{t\to 0} \frac{\gamma'(t)}{|\gamma'(t)|};\ u \in \boldsymbol{R}\right\} = \{u\gamma''(0);\ u \in \boldsymbol{R}\}$$

passing through $\gamma(0) = 0$ is called the *tangent line* of γ at 0. We set two images of γ as

$$\gamma_+ = \gamma((0, \varepsilon)), \quad \gamma_- = \gamma((-\varepsilon, 0)),$$

for $\varepsilon > 0$. We have the following proposition.

PROPOSITION 2.2. *Let γ be an A-type germ with $\omega = 0$. If $b \neq 0$, then for a sufficiently small $\varepsilon > 0$, the images γ_+ and γ_- lie on the same side of the tangent line of γ. Moreover, if γ is a 5/2-cusp and $b = 0$, then for a sufficiently small $\varepsilon > 0$, the images γ_+ and γ_- lie on both sides of the tangent line of γ.*

Proof. By rotating γ and by a parameter change, we may assume that

$$\gamma = \left(\frac{t^2}{2}, \frac{t^4}{4!}\gamma_4 + \frac{t^5}{5!}\gamma_5(t)\right), \tag{2.1}$$

where $\gamma_4 \in \boldsymbol{R}$ and $\gamma_5(t)$ is a smooth function. Then $b = \gamma_4$ and $\omega_r = 3\gamma_5(0)$. Since the tangent line is the horizontal axis, the claim of the proposition is obvious by these observations. $\square$

One can easily see that for $3/2$-cusps, the images γ_+ and γ_- always lie on both sides of the tangent line of γ. Thus there is no similar notion of bias for $3/2$-cusps. If an A-type map-germ γ with $\omega = 0$ satisfies $b = 0$, then γ is said to be *balanced* (see Figure 2).

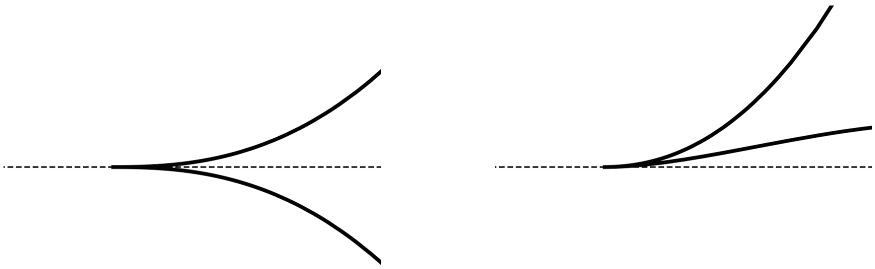


FIGURE 2. The left figure shows a balanced $5/2$-cusp (i.e., $b = 0$), and the right one is non-balanced (i.e., $b \neq 0$). The dotted lines are the tangent lines at each singular point. As we have shown in Proposition 2.2, the image of a balanced $5/2$-cusp extends over the two domains separated by the tangent line.

2.2. Behavior of the curvature function. Let s_g be the arclength function $s_g(t) = \int_0^t |\gamma'(t)|\, dt$ of an A-type germ $\gamma : (\boldsymbol{R}, 0) \to (\boldsymbol{R}^2, 0)$. It is shown that $(s(t) :=)\ \mathrm{sgn}(t)\sqrt{|s_g(t)|}$ is C^∞-differentiable and $s'(0) \neq 0$ ([24, Theorem 1.1]). Thus one can take $s(t)$ as a parameter, which is called the *half-arclength parameter* [24]. We have the following proposition.

PROPOSITION 2.3. *Let $\gamma : (\boldsymbol{R}, 0) \to (\boldsymbol{R}^2, 0)$ be a $5/2$-cusp, and t a parameter. Let κ be the curvature defined everywhere except $t = 0$. Then $\tilde{\kappa} = \mathrm{sgn}(t)\kappa$ is a C^∞ function, and*

$$\tilde{\kappa}(0) = \frac{b}{3}, \quad \frac{d}{ds}\tilde{\kappa}(0) = \frac{\sqrt{2}}{24}\omega_r$$

holds, where s is the half-arclength parameter.

Proof. We may assume that γ is given by the form (2.1) without loss of generality. Then

$$\frac{\det(\gamma', \gamma'')}{|\gamma'|^3} = \frac{\dfrac{\gamma_4}{3}t^3 + \dfrac{\gamma_5(0)}{8}t^4 + O(5)}{\left|t^6 + \dfrac{\gamma_4^2}{12}t^{10} + O(11)\right|^{1/2}} = \mathrm{sgn}(t)\left(\frac{\gamma_4}{3} + \frac{\gamma_5(0)}{8}t + O(2)\right)$$

holds. Here, $O(n)$ stands for the terms whose degrees are greater than or equal to n. On the other hand, by $|\gamma'| = |t\sqrt{1 + t^4\gamma_4^2/36 + O(5)}| = |t + \gamma_4^2 t^4/72 + O(5)|$

and $s_g = t^2(1/2 + O(4))$, it holds that $s = t\sqrt{1/2 + O(4)}$ and $dt/ds = \sqrt{2}$ at 0. The proposition is then obvious from the above calculations. $\square$

See [2] for another treatment of curvatures of curves with singularities.

2.3. Projection of space curves. Let $\Gamma : (\boldsymbol{R}, 0) \to (\boldsymbol{R}^3, 0)$ be a regular space curve, and let t be an arclength parameter of Γ, and $\boldsymbol{e}$, $\boldsymbol{n}$, $\boldsymbol{b}$ the Frenet frame. We set the orthogonal projection of Γ to the normal plane $(\boldsymbol{e}(0))^{\perp}$ at 0 by

$$\gamma(t) = \Gamma(t) - \langle \Gamma(t), \boldsymbol{e}(0) \rangle \boldsymbol{e}(0).$$

Note that $\gamma'(0) = 0$. Then γ at 0 is A-type if and only if $\kappa(0) \neq 0$, where κ is the curvature of Γ. We assume that γ is A-type (i.e., $\kappa(0) \neq 0$). Since

$$\omega(\gamma, 0) = \frac{\tau(0)}{\sqrt{\kappa(0)}},$$

γ at 0 is a 3/2-cusp if and only if $\tau(0) \neq 0$, where τ is the torsion of Γ. If γ is A-type but not a 3/2-cusp (i.e., $\kappa(0) \neq 0$, $\tau(0) = 0$), then

$$b(\gamma, 0) = \frac{\tau'}{\kappa}(0), \quad \omega_r(\gamma, 0) = \frac{-\kappa'\tau' + 3\kappa\tau''}{\kappa^{5/2}}(0).$$

Thus, under the assumption $\kappa(0) \neq 0$, γ is a 3/2-cusp if and only if $\tau(0) \neq 0$, and γ is not a 3/2-cusp and non-balanced if and only if $\tau(0) = 0$, $\tau'(0) \neq 0$, and γ is a balanced 5/2-cusp if and only if $\tau(0) = \tau'(0) = 0$, $\tau''(0) \neq 0$.

3. Invariants of $5/2$-cuspidal edges

In this section, we discuss the geometric properties of 5/2-cuspidal edges.

3.1. Frontals. Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a map-germ. We call f a *frontal* if there exists a map $\nu : (\boldsymbol{R}^2, 0) \to S^2$ satisfying $\langle df(X), \nu \rangle = 0$ for any $X \in T_p\boldsymbol{R}^2$ and $p \in (\boldsymbol{R}^2, 0)$, where S^2 stands for the unit sphere in $\boldsymbol{R}^3$. We call ν a unit normal vector field of f. A frontal is called a *front* if (f, ν) is an immersion. Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a frontal, and ν a unit normal vector field of f. We set

$$\lambda = \det(f_u, f_v, \nu) \tag{3.1}$$

by taking a coordinate system (u, v), with $f_u = \partial f/\partial u$, $f_v = \partial f/\partial v$. We call λ a *signed area density function*. By the definition, $S(f) = \lambda^{-1}(0)$, where $S(f)$ is the set of singularities of f. A singular point p of f is said to be *non-degenerate* if $d\lambda(p) \neq 0$. If p is a non-degenerate singular point, then $S(f)$ near p is a regular curve. Let p be a singular point satisfying rank $df_p = 1$, then there exists a non-vanishing vector field η on a neighborhood U of p such that $\langle \eta_q \rangle_{\boldsymbol{R}} = \ker df_q$ for $q \in S(f) \cap U$. We call η a *null vector field*. We note that the notions of non-degeneracy and null vector field are introduced in [12]. We remark that a non-degenerate singular point satisfies rank $df_p = 1$. A non-degenerate singular point p of f is called *first kind* (respectively, *second kind*) if η_p is transverse to $S(f)$ at p (respectively, η_p is tangent to $S(f)$ at p). It is well-known that a singular point of the first kind on a front is a cuspidal edge ([12, Proposition 1.3], see also [21, Corollary 2.5]).

3.2. Basic invariants for singular points of the first kind. In [20], the singular curvature and the limiting normal curvature are defined for cuspidal edges, namely singular points of fronts of the first kind. In [14, 15], the cuspidal curvature and the cusp-directional torsion are defined. These definitions are also valid for singular points of frontals of the first kind.

Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a frontal and ν a unit normal vector field. Let 0 be a singular point of the first kind. Taking a parametrization $\gamma : (\boldsymbol{R}, 0) \to (\boldsymbol{R}^2, 0)$ of $S(f)$, the *singular curvature* κ_s and the *limiting normal curvature* κ_ν are defined by

$$\kappa_s(t) = \operatorname{sgn}(d\lambda(\eta)) \frac{\det(\hat{\gamma}', \hat{\gamma}'', \nu \circ \gamma)}{|\hat{\gamma}'|^3}(t), \quad \kappa_\nu(t) = \frac{\langle \hat{\gamma}'', \nu \circ \gamma \rangle}{|\hat{\gamma}'|^2}(t),$$

respectively ([20]), where (γ', η) is taken to be positively oriented. Let ξ be a vector field on $(\boldsymbol{R}^2, 0)$ such that ξ_q is tangent to $S(f)$ at each point $q \in S(f)$, and let η be a null vector field. Then the *cuspidal curvature* κ_c and the *cusp-directional torsion* or the *cuspidal torsion* κ_t are defined by

$$\kappa_c(t) = \left. \frac{|\xi f|^{3/2} \det(\xi f, \eta^2 f, \eta^3 f)}{|\xi f \times \eta^2 f|^{5/2}} \right|_{(u,v)=\gamma(t)}, \tag{3.2}$$

$$\kappa_t(t) = \left. \left(\frac{\det(\xi f, \eta^2 f, \xi\eta^2 f)}{|\xi f \times \eta^2 f|^2} - \frac{\det(\xi f, \eta^2 f, \xi^2 f)\langle \xi f, \eta^2 f \rangle}{|\xi f|^2 |\xi f \times \eta^2 f|^2} \right) \right|_{(u,v)=\gamma(t)}, \tag{3.3}$$

where $\zeta^i f$ stands for the i'th order directional derivative of f by a vector field ζ. The invariant κ_c measures a kind of "wideness" of the singularity. Furthermore, it is shown that $\kappa_\Pi = \kappa_c \kappa_\nu$ is an intrinsic invariant. See Section 4 for the definition of the intrinsicity and the extrinsicity of invariants. See [15] for details. One can easily see that $\kappa_c(0) \neq 0$ if and only if f is a front at 0. It is known that for two cuspidal edges f and g, if their invariants κ_s, κ_s', κ_ν, κ_ν', κ_c, κ_t coincide at 0, then there exists a coordinate system such that 3-jets $j^3 f$ and $j^3 g$ coincide at 0 ([14, Theorem 6.1]), where $' = d/dt$ and t is an arclength parameter. In [3, 4], intrinsicities and extrinsicities of these invariants are investigated. See [18, 1, 11] for another approach to investigating cuspidal edges, and [25] for other applications of the above invariants (see also [26]).

3.3. Criterion and invariants for 5/2-cuspidal edges. First, we review the criterion for 5/2-cuspidal edges given in [6, Theorem 4.1]. In order to do that, we recall the following fact:

Fact 3.1 ([6, Lemma 4.2]). *Let* $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ *be a frontal-germ such that* 0 *is a singular point of the first kind. Let* ξ, η *be vector fields on* $(\boldsymbol{R}^2, 0)$ *such that the restriction* $\xi|_{S(f)}$ *is tangent to* $S(f)$, *and* η *is a null vector field. Then there exists a null vector field* $\tilde{\eta}$ *such that*

$$\langle \xi f, \tilde{\eta}^2 f \rangle(0) = \langle \xi f, \tilde{\eta}^3 f \rangle(0) = 0 \tag{3.4}$$

holds. Moreover, if $\det(\xi f, \eta^2 f, \eta^3 f)(0) = 0$, *there exists the constant* $l \in \boldsymbol{R}$ *such that*

$$\tilde{\eta}^3 f(0) = l \tilde{\eta}^2 f(0) \tag{3.5}$$

holds.

This fact is also obtained as a corollary of Lemma 3.4. Then the criterion for 5/2-cuspidal edges is given as follows:

Proposition 3.2 (Criterion for 5/2-cuspidal edges, [6, Theorem 4.1]). *The frontal-germ* $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ *is a* 5/2-*cuspidal edge if and only if*

(1) $\eta\lambda(0) \neq 0$,
(2) $\det(\xi f, \eta^2 f, \eta^3 f) = 0$ *on* $S(f)$,
(3) $\det(\xi f, \tilde{\eta}^2 f, 3\tilde{\eta}^5 f - 10 l \tilde{\eta}^4 f)(0) \neq 0$.

Here, ξ *is a vector field on* $(\boldsymbol{R}^2, 0)$ *such that the restriction* $\xi|_{S(f)}$ *is tangent to* $S(f)$, *and* η *is a null vector field. Furthermore,* $\tilde{\eta}$ *is a null vector field and* $l \in \boldsymbol{R}$ *is the constant given in Fact* 3.1.

The condition (1) implies that 0 is a singular point of the first kind. Moreover, by [15, Proposition 3.11], the condition (2) implies that f is not a front:

Fact 3.3 ([15, Proposition 3.11]). *Let* $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ *be a frontal-germ such that* 0 *is a singular point of the first kind. Take vector fields* ξ, η *on* $(\boldsymbol{R}^2, 0)$ *such that the restriction* $\xi|_{S(f)}$ *is tangent to* $S(f)$, *and* η *is a null vector field. Let* $q \in S(f)$ *be a singular point of the first kind. Then,*

$$\det(\xi f, \eta^2 f, \eta^3 f)(q) \neq 0$$

holds if and only if f *is a front at* $q \in S(f)$.

Proof. A frontal-germ f at a non-degenerate singular point $q \in S(f)$ is a front if and only if $\eta\nu(q) \neq 0$, where η is a null vector field and ν is a unit normal vector field. Firstly we show that the condition (2) is equivalent to $\det(\xi f, \nu, \eta\nu) = 0$. Since $\langle \nu, \xi f \rangle = 0$ and $\langle \nu, \eta^2 f \rangle = -\langle \eta\nu, \eta f \rangle = 0$, we see that ν is parallel to $\xi f \times \eta^2 f$ on $S(f)$. Thus

$$\begin{aligned} |\xi f \times \eta\eta f|^2 \det(\xi f, \nu, \eta\nu) &= \det(\xi f, \xi f \times \eta^2 f, \eta(\xi f \times \eta^2 f)) \\ &= \det(\xi f, \xi f \times \eta^2 f, \eta\xi f \times \eta^2 f + \xi f \times \eta^3 f). \end{aligned} \tag{3.6}$$

Since $[\eta, \xi]$ is a vector field and $df(T_p)$ is generated by ξ_p $(p \in S(f))$, the derivative $[\eta, \xi]f = \eta\xi f - \xi\eta f$ is parallel to ξf, and $\eta f = 0$ on $S(f)$. Since ξ is tangent to $S(f)$, $\xi\eta f = 0$ on $S(f)$. Hence $\eta\xi f$ is parallel to ξf. Thus the left-hand side of (3.6) is equal to

$$\det(\xi f, \xi f \times \eta^2 f, \xi f \times \eta^3 f)(t). \tag{3.7}$$

Since $\det(a, a \times b, a \times c) = |a|^2 \det(a, b, c)$ for vectors $a, b, c \in \boldsymbol{R}^3$, (3.7) is a non-zero multiple of $\det(\xi f, \eta^2 f, \eta^3 f)(t)$. Thus (2) is equivalent to $\det(\xi f, v, \eta v) = 0$. One can write $\eta v = \alpha\xi f + \beta v$. Then $\beta = \langle \eta v, v\rangle = 0$. On the other hand, $|\xi f|^2 \alpha = \langle \eta v, \xi f\rangle$. Since $\langle v, \xi f\rangle(u, v) = \langle v, \eta f\rangle(u, v) = 0$ for any (u, v), it holds that $\langle \eta v, \xi f\rangle(u, v) + \langle v, \eta\xi f\rangle(u, v) = 0$, $\langle \xi v, \eta f\rangle(u, v) + \langle v, \xi\eta f\rangle(u, v) = 0$ for any (u, v). Since $[\eta, \xi]$ is a vector field, $[\eta, \xi]f$ is parallel to ξf at 0. Thus $\langle v, [\eta, \xi]f\rangle(0) = 0$. Hence we have $|\xi f|^2 \alpha = \langle \eta v, \xi f\rangle = -\langle v, \eta\xi f\rangle = -\langle v, \xi\eta f\rangle = \langle \xi v, \eta f\rangle = 0$. This completes the proof. □

To define the invariants of 5/2-cuspidal edges, we prepare the following lemma:

Lemma 3.4. *Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a frontal-germ such that 0 is a singular point of the first kind. Assume that each singular point $q \in S(f)$ is of the first kind. Let $\gamma(t)$ be a parametrization of $S(f)$ such that $\gamma(0) = 0$, and let ξ be a vector field on $(\boldsymbol{R}^2, 0)$ such that the restriction $\xi|_{S(f)}$ is tangent to $S(f)$. Then, there exists a null vector field $\tilde{\eta}$ such that*

$$\langle \xi f, \tilde{\eta}^2 f\rangle(\gamma(t)) = \langle \xi f, \tilde{\eta}^3 f\rangle(\gamma(t)) = 0 \tag{3.8}$$

holds along $\gamma(t)$. Moreover, if f is not a front at each $q \in S(f)$, then there exists a function $l(t)$ such that

$$\tilde{\eta}^3 f(\gamma(t)) = l(t)\tilde{\eta}^2 f(\gamma(t)) \tag{3.9}$$

holds along $\gamma(t)$.

Proof. We take a coordinate system (u, v) satisfying $S(f) = \{v = 0\}$, $\eta = \partial_v$. Set

$$\tilde{\eta} = \alpha(u, v)\partial_u + \partial_v \quad (\alpha(u, v) = v(\alpha_1(u) + \alpha_2(u)v)), \tag{3.10}$$

where we set

$$\alpha_1(u) = -\left.\frac{\langle f_{vv}, f_u\rangle}{\langle f_u, f_u\rangle}\right|_{v=0}, \quad \alpha_2(u) = -\left.\frac{3\alpha_1(u)\langle f_{uv}, f_u\rangle + \langle f_{vvv}, f_u\rangle}{2\langle f_u, f_u\rangle}\right|_{v=0}.$$

We can check that $\langle f_u, \tilde{\eta}^2 f\rangle(u, 0) = \langle f_u, \tilde{\eta}^3 f\rangle(u, 0) = 0$. With respect to the second assertion, we have $\det(\xi f, \tilde{\eta}^2 f, \tilde{\eta}^3 f) = 0$ along $S(f)$, by Fact 3.3. Hence, there exist functions $l(t)$, $\bar{l}(t)$ such that

$$\tilde{\eta}^3 f(\gamma(t)) = l(t)\tilde{\eta}^2 f(\gamma(t)) + \bar{l}(t)\xi f(\gamma(t)).$$

Since $\langle \xi f, \tilde{\eta}^3 f\rangle(\gamma(t)) = 0$, we have $\bar{l}(t) = 0$. □

COROLLARY 3.5. *Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a frontal-germ such that 0 is a singular point of the first kind. Assume that each singular point $q \in S(f)$ is of the first kind. Then, there exists a coordinate system (u, v) such that $S(f) = \{v = 0\}$, and*

$$f_v = 0, \quad \langle f_u, f_{vv} \rangle = \langle f_u, f_{vvv} \rangle = 0$$

along the u-axis.

Proof. We take a coordinate system (u, v) satisfying $S(f) = \{v = 0\}$, $\eta = \partial_v$. Set $\tilde{\eta}$ as in (3.10). Then there exists a coordinate system (x, y) such that $x = u$ and ∂_y is parallel to $\tilde{\eta}$. Since $\langle f_x, \tilde{\eta}^2 f \rangle(x, 0) = \langle f_x, \tilde{\eta}^3 f \rangle(x, 0) = 0$, we have $\langle f_x, f_{yy} \rangle(x, 0) = \langle f_x, f_{yyy} \rangle(x, 0) = 0$. Hence (x, y) is the desired coordinate system. □

Let us assume that $S(f)$ is oriented, and let ξ be a vector field such that the restriction $\xi|_{S(f)}$ is tangent to $S(f)$ agreeing with the orientation of $S(f)$, and let η be a null vector field so that (ξ, η) is positively oriented. We take a null vector field $\tilde{\eta}$ which satisfies the condition (3.4), and $(\xi, \tilde{\eta})$ is positively oriented. Assuming f is not a front at 0, then by Fact 3.1, there exists a number $l \in \boldsymbol{R}$ such that $\tilde{\eta}^3 f(0, 0) = l \tilde{\eta}^2 f(0, 0)$. We define two real numbers at 0, respectively, by

$$r_b = \frac{|\xi f(0,0)|^2 \det(\xi f(0,0), \tilde{\eta}^2 f(0,0), \tilde{\eta}^4 f(0,0))}{|\xi f(0,0) \times \tilde{\eta}^2 f(0,0)|^3},$$

$$r_c = \frac{|\xi f(0,0)|^{5/2} \det(\xi f(0,0), \tilde{\eta}^2 f(0,0), 3\tilde{\eta}^5 f(0,0) - 10 l \tilde{\eta}^4 f(0,0))}{|\xi f(0,0) \times \tilde{\eta}^2 f(0,0)|^{7/2}}.$$

LEMMA 3.6. *The two real numbers r_b, and r_c do not depend on the choices of ξ and $\tilde{\eta}$.*

Proof. We take the coordinate system (u, v) given in Corollary 3.5. We set

$$\xi = \alpha_1(u, v)\partial_u + \alpha_2(u, v)\partial_v, \quad \bar{\eta} = \alpha_3(u, v)\partial_u + \alpha_4(u, v)\partial_v,$$

where $\alpha_i(u, v)$ $(i = 1, 2, 3, 4)$ is a smooth function such that $\alpha_1, \alpha_4 > 0$, $\alpha_3(u, 0) = 0$. By the assumption (3.4), $(\alpha_3)_v = (\alpha_3)_{vv} = 0$ holds on the u-axis. By a straightforward calculation,

$$\begin{aligned}
\xi f &= \alpha_1 f_u, \\
\bar{\eta}^2 f &= \alpha_4^2 f_{vv}, \\
\bar{\eta}^3 f &= \alpha_4^2 (3(\alpha_4)_v f_{vv} + \alpha_4 f_{vvv}), \\
\bar{\eta}^4 f &= \alpha_4^4 f_{vvvv} + f_u * + f_{vv} *
\end{aligned}$$

hold on the u-axis. Thus

$$\frac{|f_\xi|^2 \det(\xi f, \bar{\eta}^2 f, \bar{\eta}^4 f)}{|\xi f \times \bar{\eta}^2 f|^3} = \frac{\det(\alpha_1 f_u, \alpha_4^2 f_{vv}, \alpha_4^4 f_{vvvv})}{|\alpha_1 f_u|\,|\alpha_4^2 f_{vv}|^3} = \frac{\det(f_u, f_{vv}, f_{vvvv})}{|f_u|\,|f_{vv}|^3}$$

holds at 0, which shows the independence of r_b. By the above calculation, if $f_{vvv} = lf_{vv}$, then $\bar{\eta}^3 f = (3(\alpha_4)_v + \alpha_4 l)\bar{\eta}^2 f$. Moreover, we see

$$\bar{\eta}^5 f = \alpha_4^4(10(\alpha_4)_v f_{vvvv} + \alpha_4 f_{vvvvv}).$$

Hence, setting $\bar{l} := 3(\alpha_4)_v + \alpha_4 l$,

$$\begin{aligned}
&\frac{|\xi f|^{5/2} \det(\xi f, \bar{\eta}^2 f, 3\bar{\eta}^5 f - 10\bar{l}\bar{\eta}^4 f)}{|\xi f \times \bar{\eta}^2 f|^{7/2}} \\
&\quad = \frac{\det(\alpha_1 f_u, \alpha_4^2 f_{vv}, 3\alpha_4^4(10(\alpha_4)_v f_{vvvv} + \alpha_4 f_{vvvvv}) - 10(3(\alpha_4)_v + \alpha_4 l)\alpha_4^4 f_{vvvv})}{|\alpha_1 f_u|\,|\alpha_4^2 f_{vv}|^{7/2}} \\
&\quad = \frac{\det(f_u, f_{vv}, 3f_{vvvvv} - 10lf_{vvvv})}{|f_u|\,|f_{vv}|^{7/2}}
\end{aligned}$$

holds at 0, which shows the independence of r_c. $\square$

Definition 3.7. Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a frontal-germ such that 0 is a singular point of the first kind. Assume that each singular point is of the first kind and is not a front. For example, 5/2-cuspidal edges satisfy this assumption. Then, by Lemma 3.4, we have $\tilde{\eta}^3 f(\gamma(t)) = l(t)\tilde{\eta}^2 f(\gamma(t))$, where $\gamma(t)$ is a parametrization of $S(f)$, and $\tilde{\eta}$ is a null vector field satisfying $\langle \xi f, \tilde{\eta}^2 f\rangle(\gamma(t)) = \langle \xi f, \tilde{\eta}^3 f\rangle(\gamma(t)) = 0$. Then we define $r_b(t)$, $r_c(t)$ as

$$r_b(t) = \left.\frac{|\xi f|^2 \det(\xi f, \tilde{\eta}^2 f, \tilde{\eta}^4 f)}{|\xi f \times \tilde{\eta}^2 f|^3}\right|_{(u,v)=\gamma(t)}, \tag{3.11}$$

$$r_c(t) = \left.\frac{|\xi f|^{5/2} \det(\xi f, \tilde{\eta}^2 f, 3\tilde{\eta}^5 f - 10l\tilde{\eta}^4 f)}{|\xi f \times \tilde{\eta}^2 f|^{7/2}}\right|_{(u,v)=\gamma(t)}, \tag{3.12}$$

respectively. The invariant $r_b(t)$ is called the *bias*, and $r_c(t)$ is called the *secondary cuspidal curvature*. We also define

$$r_\Pi(p) := \kappa_\nu(p) r_c(p)$$

for a singular point $p = 0$, which is called the *secondary product curvature*.

In [17], the bias r_b and the secondary cuspidal curvature r_c are used to investigate the cuspidal cross caps.

3.4. Geometric meanings. Here we study geometric meanings of the invariants r_b and r_c.

Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a frontal and 0 a singular point of the first kind. Moreover, let $\gamma(t)$ $(\gamma(0) = 0)$ be a parametrization of $S(f)$, and we set $\hat{\gamma}(t) := f(\gamma(t))$. Since 0 is a singular point of the first kind, $\hat{\gamma}'(0) \neq 0$, where $' = d/dt$. Denote by Π_f the normal plane $(\hat{\gamma}'(0))^{\perp}$ of $\hat{\gamma}'(0)$ passing through 0. We call Π_f the *normal plane* of f passing through 0.

Proposition 3.8. *Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a frontal and 0 a singular point of the first kind. Assume that f is not a front. Let r_b (respectively, r_c) be the bias (respectively, the secondary cuspidal curvature) of the frontal f at 0. Denote by Π_f the normal plane of f passing through 0. Then,*

- *the slice of f by the normal plane Π_f is an image of an A-type map-germ $\hat{c} : (\boldsymbol{R}, 0) \to (\Pi_f, 0)$. Moreover,*
- *if we denote by $b(\hat{c}, 0)$ (respectively, $\omega_r(\hat{c}, 0)$) the bias of cusps (respectively, the secondary cuspidal curvature) of $\hat{c}$ at 0 as a plane curve in Π_f, then we have*

$$r_b = b(\hat{c}, 0), \quad r_c = \omega_r(\hat{c}, 0).$$

Proof. Let $\gamma(t)$ $(\gamma(0) = 0)$ be a parametrization of $S(f)$, and we set $\hat{\gamma}(t) := f(\gamma(t))$. The slice of f by the normal plane $\Pi_f = (\hat{\gamma}'(0))^{\perp}$ is given by

$$C = \{(u, v);\ \langle f(u, v), \hat{\gamma}'(0) \rangle = 0\},$$

where $' = d/dt$. We take a coordinate system satisfying $S(f) = \{v = 0\}$, $\eta = \partial_v$ and $\langle f_u, f_{vv} \rangle = \langle f_u, f_{vvv} \rangle = 0$ on $S(f)$ (Corollary 3.5). Then we see that $\langle f(u, v), \hat{\gamma}'(0) \rangle_u \neq 0$ at 0. Thus we can take a parametrization of C as $c(v) = (c_1(v), v)$. We set $\hat{c} = f \circ c$. We remark that since $\langle \hat{c}(v), \hat{\gamma}'(0) \rangle = 0$, it holds that $c_1'(0) = 0$. Furthermore, since $f_v(u, 0) = 0$, it holds that $f_{uv}(u, 0) = f_{uuv}(u, 0) = f_{uuuv}(u, 0) = 0$. Then we have

$$\hat{c}''(0) = f_{vv}(0, 0) + c_1''(0) f_u(0, 0), \quad \hat{c}'''(0) = f_{vvv}(0, 0) + c_1'''(0) f_u(0, 0).$$

Since $\langle \hat{c}(v), \hat{\gamma}'(0) \rangle = 0$, it holds that $c_1''(0) = -\langle f_u(0, 0), f_{vv}(0, 0) \rangle = 0$ and $c_1'''(0) = -\langle f_u(0, 0), f_{vvv}(0, 0) \rangle = 0$. Furthermore, since

$$\hat{c}^{(4)}(0) = f_{vvvv}(0, 0) + c_1^{(4)}(0) f_u(0, 0), \quad \hat{c}^{(5)}(0) = f_{vvvvv}(0, 0) + c_1^{(5)}(0) f_u(0, 0),$$

we see that

$$b(\hat{c}, 0) = \frac{\det(f_u, f_{vv}, f_{vvvv})}{|f_{vv}|^3}(0, 0) = r_b,$$

$$\omega_r(\hat{c}, 0) = \frac{\det(f_u, f_{vv}, 3f_{vvvvv} - 10 l f_{vvvv})}{|f_{vv}|^{7/2}}(0, 0) = r_c. \qquad \square$$

3.5. Normal form for $5/2$**-cuspidal edges.** In [14], a normal form for cuspidal edges is given. See also [1]. We have the following.

Proposition 3.9. *Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a 5/2-cuspidal edge. Then there exist a coordinate system (u, v) on $(\boldsymbol{R}^2, 0)$ and an isometry $\Phi : (\boldsymbol{R}^3, 0) \to (\boldsymbol{R}^3, 0)$ such that*

$$\Phi \circ f(u,v) = \left(u, \sum_{i=2}^{5} \frac{a_i}{i!} u^i + \frac{v^2}{2}, \sum_{i=2}^{5} \frac{b_{i0}}{i!} u^i + \sum_{i=1}^{3} \frac{b_{i2}}{i!} u^i v^2 \right. \\ \left. + \frac{b_{14}}{4!} uv^4 + \sum_{i=4}^{5} \frac{b_{0i}}{i!} v^i \right) + h(u,v), \tag{3.13}$$

where $a_i \in \boldsymbol{R}$ $(i = 2, \dots, 5)$, $b_{ij} \in \boldsymbol{R}$ $(i + j \le 5)$ are constants satisfying $b_{05} \neq 0$, and $h(u, v)$ consists of the terms whose degrees are greater than or equal to 6, of the form

$$(0, u^6 h_1(u), u^6 h_2(u) + u^4 v^2 h_3(u) + u^2 v^4 h_4(u) + uv^5 h_5(u) + v^6 h_6(u, v)).$$

Although this proposition can be shown by the same method as in the proof of [14, Theorem 3.1], we give a proof in Appendix A for the sake of completeness. Under this normal form, the invariants defined above can be computed as

- $(\kappa_v(0), \kappa_v'(0), \kappa_v''(0), \kappa_v'''(0)) = (\boxed{b_{20}}, \boxed{b_{30}} - 2a_2 b_{12}, \boxed{b_{40}} - 4a_3 b_{12} - 2a_2^2 b_{20} - 2a_2 b_{22} - 3b_{20}^3 - 4b_{12}^2 b_{20}, \boxed{b_{50}} + 14a_2^3 b_{12} - 7a_2^2 b_{30} - 6a_4 b_{12} - 6a_3 b_{22} - 12b_{12} b_{20} b_{22} - 12b_{12}^2 b_{30} - 19b_{20}^2 b_{30} + a_2(-6a_3 b_{20} - 2b_{32} + 24b_{12}^3 + 32b_{20}^2 b_{12}))$,
- $(\kappa_s(0), \kappa_s'(0), \kappa_s''(0), \kappa_s'''(0)) = (\boxed{a_2}, \boxed{a_3} + 2b_{12} b_{20}, \boxed{a_4} - 4a_2(b_{12}^2 + b_{20}^2) + 2b_{20} b_{22} + 4b_{12} b_{30} - 3a_2^3, \boxed{a_5} - a_2^2(8b_{12} b_{20} + 19a_3) - 2a_3(6b_{12}^2 + 5b_{20}^2) - 3a_2(4b_{12} b_{22} + 5b_{20} b_{30}) - 24b_{20} b_{12}^3 + 2b_{12}(3b_{40} - 13b_{20}^3) + 6b_{22} b_{30} + 2b_{20} b_{32})$,
- $(\kappa_t(0), \kappa_t'(0), \kappa_t''(0)) = (\boxed{2b_{12}}, \boxed{2b_{22}} - a_2 b_{20}, \boxed{2b_{32}} + 4a_2^2 b_{12} - a_3 b_{20} - 2a_2 b_{30} - 16b_{12}^3 - 8b_{20}^2 b_{12})$,
- $(r_b(0), r_b'(0)) = (\boxed{b_{04}}, \boxed{b_{14}} - 12a_2 b_{12})$,
- $r_c(0) = \boxed{3b_{05}}$,

and $\kappa_c \equiv 0$, where the prime means differentiation with respect to the arclength parameter of $\hat{\gamma}$. Looking at the boxed entries, we have the following proposition.

Proposition 3.10. *Let f, g be germs of 5/2-cuspidal edges. If their invariants κ_v, κ_v', κ_v'', κ_v''', κ_s, κ_s', κ_s'', κ_s''', κ_t, κ_t', κ_t'', r_b, r_b', r_c coincide at 0, then there exist a coordinate system (u, v) and an isometry A of $\boldsymbol{R}^3$ such that*

$$j_0^5 f(u,v) = j_0^5 (A \circ g)(u,v),$$

where $j_0^5 f(u, v)$ stands for the 5-jet of f with respect to (u, v) at 0.

Moreover, a parametrization of $f(S(f))$ as a space curve is given by $f(u, 0)$. Since b_{04}, b_{14}, b_{05} do not appear in $f(u, 0)$, they also do not appear in the curvature κ and the torsion τ of $f(u,0)$. Thus we believe that the invariants r_b, r_c for 5/2-cuspidal edges were not paid attention to before.

3.6. Invariants of 5/2-cuspidal edges on conjugate surfaces. We denote by $\boldsymbol{R}^3_1$ the Lorentz-Minkowski 3-space with signature $(-,+,+)$. A *spacelike Delaunay surface with axis* ℓ is a surface in $\boldsymbol{R}^3_1$ such that the first fundamental form (that is, the induced metric) is positive definite, it is of constant mean curvature (*CMC*, for short), and it is invariant under the action of the group of motions in $\boldsymbol{R}^3_1$ which fixes each point of the line ℓ. Such spacelike Delaunay surfaces are classified and they have *conelike singularities* (see [5], for details).

As in the case of CMC surfaces in $\boldsymbol{R}^3$, for a given (simply-connected) spacelike CMC surface in $\boldsymbol{R}^3_1$, there exists a spacelike CMC surface called the *conjugate*. Any conjugate surface of a spacelike Delaunay surface is a spacelike *helicoidal* CMC surface[1], and it is shown in [6] that such spacelike helicoidal CMC surfaces have 5/2-cuspidal edges. We remark that spacelike zero-mean-curvature surfaces (i.e., *maximal* surface) never admit 5/2-cuspidal edges (cf. [6], see also [28]).

In this section, we compute the invariants r_b and r_c of 5/2-cuspidal edges on such spacelike helicoidal CMC surfaces, regarding them as surfaces in $\boldsymbol{R}^3$. More precisely, setting

$$\delta(u) = (u^2 + k + 1)^2 - 4k, \tag{3.14}$$

a non-totally-umbilical spacelike Delaunay surface with timelike axis is given by

$$f_{\mathrm{Del}}(u,v) = \frac{1}{2H}\left(\int_0^u \frac{\tau^2 + k - 1}{\sqrt{\delta(\tau)}}\,d\tau,\ u\cos(2Hv),\ u\sin(2Hv)\right)$$

for some constant $k \in \boldsymbol{R}$ ($k \neq 1$), where H is the mean curvature (see [6] for more details). Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3_1, 0)$ be a spacelike helicoidal CMC surface which is given as a conjugate surface of the Delaunay surface f_{Del}. Setting $\Delta(u) := \delta(u) - u^4$, such an f can be written as follows (cf. [6]):

(1) If $-1 < k < 1$ or $1 < k$, then f is congruent to

$$f_T(u,v) = \left(\psi + \frac{1-k}{2H(1+k)}\phi,\ \rho\cos\phi,\ \rho\sin\phi\right), \tag{3.15}$$

where

$$\rho(u) := \frac{\sqrt{\Delta(u)}}{2H(k+1)}, \quad \psi(u) := \int_0^u \frac{\sqrt{2(1+k)}\tau^4}{H\sqrt{\delta(\tau)}\Delta(\tau)}\,d\tau,$$

$$\phi(u,v) := \int_0^u \frac{\sqrt{2(1+k)}(1-k)\tau^2}{\sqrt{\delta(\tau)}\Delta(\tau)}\,d\tau - \sqrt{\frac{1+k}{2}}v.$$

---

[1] A *helicoidal surface* is a surface which is invariant under a non-trivial one-parameter subgroup of the isometry group of $\boldsymbol{R}^3_1$.

(2) If $k < -1$, then f is congruent to

$$f_S(u,v) = \left(\rho \sinh \phi,\, \rho \cosh \phi,\, \psi + \frac{k-1}{2H(1+k)}\phi\right), \tag{3.16}$$

where

$$\rho(u) := -\frac{\sqrt{\Delta(u)}}{2H(1+k)}, \quad \psi(r) := \int_0^u \frac{\sqrt{2(-k-1)}\tau^4}{H\sqrt{\delta(\tau)}\Delta(\tau)}\,d\tau,$$

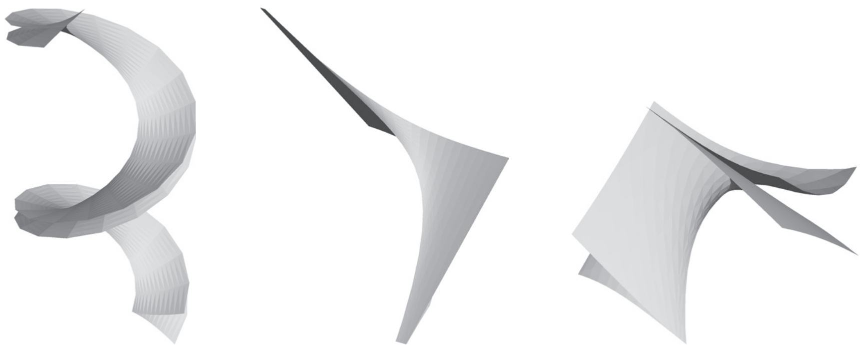
$$\phi(u,v) := -\int_0^u \frac{\sqrt{2(-k-1)}(1-k)\tau^2}{\sqrt{\delta(\tau)}\Delta(\tau)}\,d\tau - \sqrt{\frac{-k-1}{2}}v.$$

(3) If $k = -1$, then f is congruent to

$$\begin{aligned} f_L(u,v) = {} & (\psi - \rho - \rho\phi^2, -2\rho\phi, \psi + \rho - \rho\phi^2) \\ & + H\left(\frac{\phi^3}{3} + \phi, \phi^2, \frac{\phi^3}{3} - \phi\right), \end{aligned} \tag{3.17}$$

where $\rho(u) := u/2$,

$$\psi(u) := \int_0^u \frac{\tau^2(\sqrt{\tau^4+4}+\tau^2)}{4H^2\sqrt{\tau^4+4}}\,d\tau, \quad \phi(u,v) := \int_0^u \frac{\sqrt{\tau^4+4}+\tau^2}{2H\sqrt{\tau^4+4}}\,d\tau + v.$$



$k = 2$ ($f_T(u,v)$ given in (3.15)) $k = -2$ ($f_S(u,v)$ given in (3.16)) $k = -1$ ($f_L(u,v)$ given in (3.17))

FIGURE 3. Spacelike helicoidal CMC surfaces ($H = 1/2$) having 5/2-cuspidal edges in Lorentz-Minkowski 3-space $\boldsymbol{R}^3_1$. These surfaces are conjugates of spacelike Delaunay surfaces with timelike axis. See [6] for more details.

Here, we consider the case of $f = f_T(u,v)$ given in (3.15). Similar computations can be applied in the cases of f_S and f_L given in (3.16) and (3.17), respectively. For simplification, we may assume that $H > 0$.

Since $(f_T)_u(0,v) = 0$, the singular set $S(f_T)$ is given by $S(f_T) = \{u = 0\}$ and $\eta = \partial_u$ gives a null vector field. Since the map $\nu : (\boldsymbol{R}^2, 0) \to S^2$ defined by

$$\nu = \frac{1}{\sqrt{2\Delta}\sqrt{\delta - (k+1)u^2}}(\sqrt{\delta\Delta}, -\sqrt{2(k+1)}u^3 \cos\phi - \sqrt{\delta}(k-1)\sin\phi,$$
$$-\sqrt{2(k+1)}u^3 \sin\phi + \sqrt{\delta}(k-1)\cos\phi)$$

is a unit normal vector field along f_T (cf. Section 3.1), f_T is a frontal. Then we can check that $\eta\lambda(0,v) = -1/(2H^2\sqrt{k+1})$ $(\neq 0)$ holds, where λ is the signed area density function (cf. (3.1)). Thus, we have that f_T satisfies (1) in Proposition 3.2. Set $\xi(u,v) := \partial_v$ and

$$\tilde{\eta}(u,v) := \partial_u - \frac{2\,\mathrm{sign}(k-1)}{(k-1)^2}u^2\partial_v$$

(cf. (3.10)). Then we can check that $\langle \xi f_T(0,v), \tilde{\eta}^2 f_T(0,v)\rangle = 0$ and $\tilde{\eta}^3 f_T(0,v) = 0$. Hence, f_T satisfies (2) in Proposition 3.2. Moreover, the constant l is 0. Then, by a straightforward calculation, we have

$$\det(\xi f_T, \tilde{\eta}^2 f_T, \tilde{\eta}^5 f)(0,v) = -\frac{24}{H^2|k-1|^3}\ (\neq 0).$$

Therefore, f_T satisfies (3) in Proposition 3.2, and hence f_T has 5/2-cuspidal edges along $\gamma(v) = (0,v)$. The invariants are calculated as

$$r_c(0,v) = \frac{72H^{3/2}\sqrt{k+1}}{\sqrt{|k-1|}}, \quad r_b(0,v) = 0.$$

Similarly, in the case of $k < -1$, the invariants of f_S given in (3.16) are calculated as

$$r_c(0,v) = \frac{72H^{3/2}\sqrt{-k-1}}{\sqrt{1-k}\cosh\left(\dfrac{\sqrt{-k-1}v}{\sqrt{2}}\right)},$$

$$r_b(0,v) = 6\sqrt{2}H\frac{(1+k)\sinh\left(\dfrac{\sqrt{-k-1}v}{\sqrt{2}}\right)}{(1-k)\cosh^2\left(\dfrac{\sqrt{-k-1}v}{\sqrt{2}}\right)},$$

and in the case of $k = -1$, the invariants of f_L given in (3.17) are calculated as

$$r_c(0,v) = -\frac{72\sqrt{H}}{1+v^2}, \quad r_b(0,v) = \frac{6\sqrt{2}v}{H(1+v^2)^2}.$$

4. Intrinsicity and extrinsicity of invariants

Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a map-germ. The *induced metric* or the *first fundamental form* of f is the metric on $(\boldsymbol{R}^2, 0)$ defined by $f^*\langle\, , \rangle$. A function $I : (\boldsymbol{R}^2, 0) \to \boldsymbol{R}$, or $I : S(f) \to \boldsymbol{R}$, is an *invariant* if I does not depend on the choice of coordinate system on the source. An invariant $I : (\boldsymbol{R}^2, 0) \to \boldsymbol{R}$, or $I : S(f) \to \boldsymbol{R}$, is *intrinsic* if it can be represented by a C^∞ function of E, F, G and their derivatives, where

$$E = \langle f_u, f_u \rangle, \quad F = \langle f_u, f_v \rangle, \quad G = \langle f_v, f_v \rangle,$$

and (u, v) is a coordinate defined in terms of the first fundamental form $f^*\langle\, , \rangle$. An invariant $I : (\boldsymbol{R}^2, 0) \to \boldsymbol{R}$, or $I : S(f) \to \boldsymbol{R}$, is *extrinsic* if there exists a map $\tilde{f}$ such that the first fundamental form of $\tilde{f}$ is the same as f, but I does not coincide. In [15, 4], it is determined whether some invariants of cuspidal edges are intrinsic or extrinsic (cf. [3] for invariants of cross caps). In this section, we show the bias r_b is extrinsic.

4.1. Intrinsic criterion for $5/2$-cuspidal edges.

Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a frontal-germ and 0 a non-degenerate singular point. Here, we shall show that the $\mathscr{A}$-equivalence class of $5/2$-cuspidal edges can be determined intrinsically among frontal-germs with non-zero limiting normal curvature $\kappa_\nu \neq 0$ (Theorem 4.4, Corollary 4.5).

Definition 4.1. Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a frontal-germ such that 0 is a singular point of the first kind. A coordinate system (u, v) around 0 is called *adjusted at* 0 if $f_v(0, 0) = 0$. A coordinate system (u, v) which is adjusted at 0 is called *normally-adjusted at* 0 if (u, v) is compatible with the orientation of $(\boldsymbol{R}^2, 0)$, $E(0, 0) = 1$ and $\lambda_v(0, 0) = 1$.

The existence of such a normally-adjusted coordinate system can be verified by the existence of *normalized strongly adapted coordinate systems*[2] [4, Definition 2.24, Proposition 2.25] (cf. [20, Lemma 3.2] and [15, Definition 3.7]).

It was proved in [15, Corollary 3.14] that the Gaussian curvature K and the mean curvature H can be extended smoothly across $5/2$-cuspidal edges. Then, we set

$$H_\eta := H_v(0, 0), \quad K_\eta := K_v(0, 0), \tag{4.1}$$

where (u, v) is a coordinate system normally-adjusted at 0. We call K_η (respectively, H_η) the *null-derivative Gaussian curvature* (respectively, the *null-derivative mean curvature*) of $5/2$-cuspidal edge at 0. We shall prove that the definitions

[2] A coordinate system (u, v) centered at $(0, 0)$ is called *normalized strongly adapted* if the singular set is given by the u-axis, ∂_v gives the null vector field along the u-axis, $f_v(u, 0) = 0$, $|f_u(u, 0)| = |f_{vv}(u, 0)| = 1$, $\langle f_u(u, 0), f_{vv}(u, 0) \rangle = 0$ and $\langle f_u(u, v), f_v(u, v) \rangle = 0$ hold.

of null-derivative Gaussian and mean curvature do not depend on the choice of normally-adjusted coordinate systems, as follows.

LEMMA 4.2. *If two coordinate systems* (u,v) *and* (U,V) *are normally-adjusted at* 0, *then*

$$U_u = 1, \quad U_v = 0, \quad V_v = 1 \tag{4.2}$$

holds at $(0,0)$. *Moreover, the definitions of null-derivative Gaussian and mean curvatures* H_η, K_η *are independent of the choice of the coordinate system normally-adjusted at* 0.

Proof. Since $f_v = f_V = 0$ at $(0,0)$,

$$f_v = U_v f_U + V_v f_V = U_v f_U$$

yields $U_v(0,0) = 0$. Since $(u,v) \mapsto (U,V)$ is orientation-preserving, $J := U_u V_v - U_v V_u$ is positive-valued. In particular, $J(0,0) = U_u(0,0)V_v(0,0) > 0$ holds. Setting $\lambda := \det(f_u, f_v, \nu)$ and $\Lambda := \det(f_U, f_V, \nu)$, we have $\lambda = J\Lambda$. Then

$$\lambda_v = J_v\Lambda + J\Lambda_v = J_v\Lambda + J(\Lambda_U U_v + \Lambda_V V_v)$$

holds, and evaluating this at $(0,0)$ we have

$$1 = U_u(0,0)V_v^2(0,0), \tag{4.3}$$

which yields $U_u(0,0) > 0$. Since $J = U_u V_v > 0$ at $(0,0)$, $V_v(0,0) > 0$ holds. Moreover, by $1 = E = \langle f_u, f_u \rangle = U_u^2 \langle f_U, f_U \rangle = U_u^2$ at $(0,0)$, we have $U_u(0,0) = 1$. Substituting this into (4.3), $V_v(0,0) = 1$ holds. Hence we have (4.2). Moreover, then

$$\frac{\partial}{\partial v} = U_v \frac{\partial}{\partial U} + V_v \frac{\partial}{\partial V} = \frac{\partial}{\partial V}$$

holds at $(0,0)$. In particular, the definition of H_η, K_η as in (4.1) is independent of choice of the coordinate system normally-adjusted at 0. □

Since the Gaussian curvature K and the definition of normally-adjusted coordinate systems are intrinsic, the null-derivative Gaussian curvature K_η is an intrinsic invariant for 5/2-cuspidal edges. Now, we shall check the relationships amongst K, H, K_η, H_η and other invariants.

LEMMA 4.3. *Let* $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ *be a germ of a* 5/2*-cuspidal edge. Then the Gaussian curvature* K *and the mean curvature* H *of* f *satisfy*

$$K = \frac{1}{3}\kappa_\nu r_b - \kappa_t^2, \tag{4.4}$$

$$H = \frac{1}{2}\kappa_\nu + \frac{1}{6}r_b \tag{4.5}$$

along the singular set, respectively. Moreover, the null-derivative Gaussian curvature K_η and the null-derivative mean curvature H_η of f satisfy

$$K_\eta = \frac{1}{24} r_\Pi, \tag{4.6}$$

$$H_\eta = \frac{1}{48} r_c \tag{4.7}$$

along the singular set, respectively.

Proof. By Proposition 3.9, without loss of generality, we may assume that f is given by the form in (3.13). A direct calculation yields

$$\kappa_t(0) = 2b_{12}, \quad r_b(0) = b_{04}, \quad \kappa_\nu(0) = b_{20}, \quad r_c(0) = 3b_{05}, \quad r_\Pi(0) = 3b_{20}b_{05}$$

and

$$K = \frac{1}{3} b_{20} b_{04} - 4b_{12}^2, \quad H = \frac{1}{2} b_{20} + \frac{1}{6} b_{04}$$

hold at $(0,0)$. Hence, (4.4) and (4.5) hold. On the other hand, since the coordinate system (u,v) of $f(u,v)$ given by the form in (3.13) is normally-adjusted at $(0,0)$, we have $H_v = H_\eta$ and $K_v = K_\eta$ at $(0,0)$. By a direct calculation, we have that

$$H_v(0,0) = \frac{1}{16} b_{05}, \quad K_v(0,0) = \frac{1}{8} b_{20} b_{05},$$

and hence, (4.6) and (4.7) hold. □

THEOREM 4.4. *For 5/2-cuspidal edges, the secondary product curvature r_Π is an intrinsic invariant.*

Proof. By (4.6) in Lemma 4.3 and the fact that K_η is intrinsic, r_Π is intrinsic as well. □

Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a frontal-germ such that 0 is a non-degenerate singular point. If $\kappa_\nu(0) \neq 0$, then f is called *non-ν-flat.* The following corollary implies that the $\mathscr{A}$-equivalence class of 5/2-cuspidal edges can be determined intrinsically amongst non-ν-flat frontal-germs.

COROLLARY 4.5. *Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a frontal-germ such that 0 is a singular point of the first kind. Assume that f is non-ν-flat. Then, f at 0 is a 5/2-cuspidal edge if and only if $\kappa_\Pi = 0$ along $S(f)$ and $r_\Pi(0) \neq 0$.*

Proof. By the definitions of κ_c given in (3.2), r_c given in (3.12) and the criterion (Proposition 3.2), f at 0 is a 5/2-cuspidal edge if and only if $\kappa_c = 0$ along $S(f)$ and $r_c(0) \neq 0$. Therefore, imposing the non-ν-flatness $\kappa_\nu \neq 0$, we have that 0 is a non-ν-flat 5/2-cuspidal edge if and only if $\kappa_\Pi = 0$ along $S(f)$ and $r_\Pi(0) \neq 0$. □

Following Corollary 4.5, we give a definition of *intrinsic* 5/2-*cuspidal edges* for singular points of a certain metric, called the Kossowski metric in Section 5 (cf. Definition 5.3).

4.2. Isometric deformations of 5/2-cuspidal edges. The following fact is a direct conclusion of [8, Theorem B]:

FACT 4.6 ([8]). *Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be an analytic frontal-germ such that 0 is a singular point of the first kind, and $\gamma : (\boldsymbol{R}, 0) \to (\boldsymbol{R}^2, 0)$ a singular curve. Assume that f has non-vanishing limiting normal curvature. Then, for given analytic functions germs $\omega(t)$, $\tau(t)$ at $t = 0$, there exists an analytic frontal-germ $g = g_{\omega,\tau}$ such that*

(1) *the first fundamental form of $g_{\omega,\tau}$ coincides with that of f,*
(2) *the limiting normal curvature function of $g_{\omega,\tau}$ along γ coincides with $e^{\omega(t)}$ for a suitable choice of a unit normal vector field, and*
(3) *$\tau(t)$ gives the torsion function of $\hat{\gamma}_g(t)$, where $\hat{\gamma}_g(t) := g \circ \gamma(t)$.*

The possibilities for congruence classes of such a g are at most two unless τ vanishes identically. On the other hand, if τ vanishes identically (i.e., $\hat{\gamma}_g$ is a planar curve), then the congruence class of g is uniquely determined.

Using Fact 4.6, we shall prove the following, which is an analog of a result of [16, Theorem A] and [8, Corollary D].

THEOREM 4.7 (Isometric deformation of 5/2-cuspidal edges). *Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a germ of an analytic 5/2-cuspidal edge with non-vanishing limiting normal curvature, and let $\kappa_s(t)$ be the singular curvature function along the singular curve $\gamma(t)$. Take a germ of an analytic regular space curve $\sigma(t)$ such that its curvature function $\kappa(t)$ satisfies*

$$\kappa > |\kappa_s|$$

at 0. Then, there exists a germ of an analytic 5/2-cuspidal edge $g_\sigma : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ with non-vanishing limiting normal curvature such that

(1) *the first fundamental form of g_σ coincides with that of f,*
(2) *the singular image $g_\sigma \circ \gamma$ coincides with σ.*

The possibilities for congruence classes of such a g_σ are at most two unless τ vanishes identically. On the other hand, if τ vanishes identically (i.e., σ is a planar curve), then the congruence class of g_σ is uniquely determined.

Proof. Set $\omega(t)$ as

$$\omega(t) := \frac{1}{2} \log(\kappa(t)^2 - \kappa_s(t)^2).$$

Let $\tau(t)$ be the torsion function of $\sigma(t)$. By Fact 4.6, there exists an analytic frontal-germ $g_\sigma := g_{\omega,\tau} : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ such that the items (1)–(3) in Fact 4.6

hold. Thus, it suffices to show that g_σ has a 5/2-cuspidal edge at 0. Since the first fundamental form of f coincides with that of g_σ, the product curvature κ_Π and the secondary product curvature r_Π of f coincide with those of g_σ, respectively. Therefore, by Corollary 4.5, we have that $g_\sigma : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ has a 5/2-cuspidal edge at 0. □

In [8], the following is also proved.

FACT 4.8 ([8, Corollary E]). *Let f_0, f_1 be two analytic frontal germs with non-degenerate singularities whose limiting normal curvatures do not vanish. Suppose that they are isometric to each other. Then there exists a continuous* 1*-parameter family of frontal germs g_t $(0 \le t \le 1)$ satisfying the following properties*:

(1) *$g_0 = f_0$ and $g_1 = f_1$,*
(2) *g_t is isometric to g_0,*
(3) *the limiting normal curvature of each g_t does not vanish.*

Moreover, if both f_0 and f_1 are germs of cuspidal edges, swallowtails or cuspidal cross caps, then so are g_t for $0 \le t \le 1$.

By this fact and Corollary 4.5, we also have the following result analogous to [8, Corollary E].

COROLLARY 4.9. *Let f_0, f_1 be two analytic germs of* 5/2*-cuspidal edges whose limiting normal curvatures do not vanish. Suppose that they are isometric to each other. Then there exists a continuous* 1*-parameter family of germs g_t of* 5/2*-cuspidal edges $(0 \le t \le 1)$ satisfying the following properties*:

(1) *$g_0 = f_0$ and $g_1 = f_1$,*
(2) *g_t is isometric to g_0,*
(3) *the limiting normal curvature of each g_t does not vanish.*

Proof. By Fact 4.8, there exists a continuous 1-parameter family of frontal germs g_t $(0 \le t \le 1)$ such that the items (1)–(3) in Fact 4.8 hold. Since the limiting normal curvature of each g_t does not vanish and g_t is isometric to g_0 for each $t \in [0, 1]$, Corollary 4.5 yields that g_t has 5/2-cuspidal edges. Hence, the family $\{g_t\}_{t \in [0,1]}$ is the desired one. □

4.3. Extrinsicity of invariants. Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a germ of a non-ν-flat 5/2-cuspidal edge, and let $\gamma : (\boldsymbol{R}, 0) \to (\boldsymbol{R}^2, 0)$ be a germ of a singular curve of f. Let $\hat{\gamma}$ be the regular curve in $\boldsymbol{R}^3$ given by $\hat{\gamma} := f \circ \gamma$, with arclength parameter t. Set $\boldsymbol{e}(t) := \hat{\gamma}'(t)$ and $\boldsymbol{b}(t) := -\boldsymbol{e}(t) \times \hat{\nu}(t)$, where $\hat{\nu}(t) := \nu(\gamma(t))$. Then $\{\boldsymbol{e}, \boldsymbol{b}, \hat{\nu}\}$ is an orthonormal frame along γ. Remark that, in general, $\boldsymbol{b}$ may not coincide with the binormal vector field of $\hat{\gamma}$ as a space curve. Moreover we have

$$\hat{\gamma}'' = \kappa_s \boldsymbol{b} + \kappa_\nu \hat{\nu}, \quad \boldsymbol{b}' = -\kappa_s \boldsymbol{e} + \kappa_t \hat{\nu}, \quad \hat{\nu}' = -\kappa_\nu \boldsymbol{e} - \kappa_t \boldsymbol{b}. \tag{4.8}$$

Let κ, τ be the curvature and torsion functions of $\hat{\gamma}$, respectively. Substituting (4.8) into $\kappa^2\tau = \det(\hat{\gamma}', \hat{\gamma}'', \hat{\gamma}''')$, we have the following.

LEMMA 4.10. *It holds that*

$$\kappa = \sqrt{\kappa_s^2 + \kappa_\nu^2}, \quad \tau = \frac{\kappa_s'\kappa_\nu - \kappa_s\kappa_\nu'}{\kappa^2} - \kappa_t.$$

In particular, if $\kappa_s(t) = 0$ along $\gamma(t)$, then $\kappa_t(t) = -\tau(t)$ holds.

As a corollary of Theorem 4.7, we prove the extrinsicity of the limiting normal curvature κ_ν (Corollary 4.11), the cuspidal torsion κ_t (Corollary 4.12), and the bias r_b (Corollary 4.13). We remark that the proof of Corollary 4.11 is analogous to that of [16, Corollary D].

COROLLARY 4.11. *For 5/2-cuspidal edges, the limiting normal curvature κ_ν is an extrinsic invariant.*

Proof. Let us take a real-analytic germ of a non-ν-flat 5/2-cuspidal edge $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$. Denote by $\kappa(t)$ and $\tau(t)$ the curvature and torsion of $\hat{\gamma}(t) := f(\gamma(t))$, respectively. By Fact 4.6, for a given analytic function $\omega(t)$, there exists a non-ν-flat real-analytic frontal-germ $g_{\omega,\tau}$ such that $g_{\omega,\tau}$ is isometric to f, the limiting normal curvature function of $g_{\omega,\tau}$ is $e^{\omega(t)}$, and $\tau(t)$ gives the torsion function of $g_{\omega,\tau}(\gamma(t))$. Moreover, by Corollary 4.5, $g_{\omega,\tau}$ is a 5/2-cuspidal edge. Since we can choose $\omega(t)$ arbitrarily, the limiting normal curvature is extrinsic. □

COROLLARY 4.12. *For 5/2-cuspidal edges, the cuspidal torsion κ_t is an extrinsic invariant.*

Proof. Let us take a real-analytic germ of a non-ν-flat 5/2-cuspidal edge $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ satisfying $\kappa_s \equiv 0$ along the singular curve $\gamma(t)$. Denote by $\kappa(t)$ and $\tau(t)$ the curvature and torsion of $\hat{\gamma}(t) := f(\gamma(t))$, respectively. By Lemma 4.10, $\tau(t) = -\kappa_t(t)$. Take an arbitrary analytic function $\tilde{\tau}(t)$. Then, by the fundamental theorem of space curves, there exists an analytic regular space curve $\sigma(t)$ in $\boldsymbol{R}^3$ whose curvature and torsion functions are given by $\kappa(t)$ and $\tilde{\tau}(t)$, respectively. Applying Theorem 4.7 to $\sigma(t)$, there exists a real-analytic germ of a non-ν-flat 5/2-cuspidal edge $g_\sigma : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ such that g_σ is isometric to f and σ gives the image of the singular set of g_σ. Since $\kappa_s \equiv 0$ along the singular curve $\gamma(t)$, Lemma 4.10 yields that the cuspidal torsion of g_σ is $-\tilde{\tau}(t)$. Since we can choose $\tilde{\tau}(t)$ arbitrarily, the cuspidal torsion is extrinsic. □

We remark that an analytic non-ν-flat 5/2-cuspidal edge f satisfying $\kappa_s \equiv 0$ along $S(f)$ exists. In fact, by rotating the plane curve $(x(t), z(t)) := (1 + t^5, t^2)$ with respect to the z-axis, we have such an example.

COROLLARY 4.13. *For $5/2$-cuspidal edges, the bias r_b is an extrinsic invariant.*

Proof. Let us take a non-ν-flat real-analytic $5/2$-cuspidal edge satisfying $\kappa_s \equiv 0$ along the singular curve $\gamma(t)$. Moreover, assuming $\tau \equiv 0$, then by Lemma 4.10, it holds that $\kappa_t \equiv 0$. By Lemma 4.3,

$$K(\gamma(t)) = \frac{1}{3} r_b(t) \kappa_\nu(t).$$

Let $k \geq 0$ be a non-negative real number. Since $\kappa_\nu \neq 0$, then by Theorem 4.7, there exists a family $\{g^k\}_{k\geq 0}$ of real-analytic germs of $5/2$-cuspidal edges such that, for each $k \geq 0$, g^k is non-ν-flat, g^k has the same first fundamental form of f, and the curvature function $\kappa^k(t)$ of $g^k(\gamma(t))$ is given by $\kappa^k(t) = \kappa(t) + k$ and the torsion is 0. Since f and g^k have the same first fundamental form for each $k \geq 0$, the singular curvature $\kappa_s^k(t)$ of g^k vanishes identically along $\gamma(t)$. Thus the limiting normal curvature of g^k is $\kappa_\nu^k(t) = \kappa(t) + k$ (> 0). Hence the bias r_b^k for g^k is given by

$$r_b^k(t) = 3\frac{K(\gamma(t))}{\kappa_\nu^k} = 3\frac{K(\gamma(t))}{\kappa(t) + k}.$$

In particular, the bias is extrinsic. □

Remark 4.14. The secondary cuspidal curvature r_c is also extrinsic, since r_Π is intrinsic (Theorem 4.4), κ_ν is extrinsic (Corollary 4.11), and r_c is written as $r_c = r_\Pi/\kappa_\nu$ when $\kappa_\nu \neq 0$. Moreover, the product $\kappa_\nu r_b$ is also extrinsic, since $\kappa_\nu r_b = 3(K + \kappa_t^2)$ holds by (4.4) and κ_t is extrinsic (Corollary 4.12). Furthermore, by a proof similar to that of Corollary 4.12, we can prove that the cuspidal torsion κ_t for cuspidal edges is also extrinsic.

4.4. Summary of intrinsicity and extrinsicity. We can summarize the intrinsicity and extrinsicity as follows. As seen in Section 2.1, the corresponding invariant of the bias of cusps r_b does not exist for cuspidal edges.

invariants	κ_s	κ_ν	κ_t	κ_c	$\kappa_\Pi = \kappa_c\kappa_\nu$
int/ext	intrinsic	extrinsic	extrinsic	extrinsic	intrinsic

Table 1. Intrinsicity and extrinsicity for cuspidal edges.

invariants	κ_s	κ_ν	κ_t	r_b	r_c	$\kappa_\nu r_b$	$\kappa_\nu r_b - 3\kappa_t^2$	$r_\Pi = \kappa_\nu r_c$
int/ext	int	ext	ext	ext	ext	ext	int	int

Table 2. Intrinsicity (int) and extrinsicity (ext) for $5/2$-cuspidal edges. Here, we remark that the intrinsicity of the invariant in the seventh slot can be verified by the identity $\kappa_\nu r_b - 3\kappa_t^2 = 3K$ (cf. (4.4)). With respect to the eighth slot, see Theorem 4.4.

5. Isometric realizations of intrinsic 5/2-cuspidal edges

In this section, we deal with 5/2-cuspidal edge singularities without ambient spaces. We give a definition of intrinsic 5/2-cuspidal edges for singular points of Kossowski metrics, and prove the existence of their isometric realizations (Theorem 5.7) as in [16] and [8].

First, we briefly introduce the basic properties of Kossowski metrics. Further systematic treatments of Kossowski metrics are given in [4, 23, 8]. Let ds^2 be a germ of a positive semi-definite metric on $(\boldsymbol{R}^2, 0)$. Assume that 0 is a singular point of ds^2, that is, ds^2 is not positive-definite at 0. Denote by $S(ds^2)$ the set of singular points. A non-zero tangent vector $\boldsymbol{v}$ at 0 is called a *null vector* at 0 if $ds^2(\boldsymbol{v}, \boldsymbol{x}) = 0$ holds for every tangent vector $\boldsymbol{x}$ at 0. A local coordinate neighborhood $(U; u, v)$ is called *adjusted* at 0 if $\partial_v = \partial/\partial v$ gives a null vector at $(0,0)$.

If $(U; u, v)$ is a local coordinate neighborhood adjusted at 0, then $F = G = 0$ holds at $(0,0)$, where

$$ds^2 = E\,du^2 + 2F\,dudv + G\,dv^2. \tag{5.1}$$

A singular point 0 is called *admissible* if there exists an local coordinate neighborhood $(U; u, v)$ adjusted at 0 such that $E_v = 2F_u$, $G_u = G_v = 0$ hold at $(0,0)$.

DEFINITION 5.1 (Kossowski metric). If each singular point is admissible, and there exists a smooth function λ defined on a neighborhood $(U; u, v)$ of 0 such that

$$EG - F^2 = \lambda^2$$

on U, and $d\lambda \neq 0$ holds at $(0,0)$, then ds^2 is called a (germ of a) *Kossowski metric*, where E, F, G are smooth functions on U satisfying (5.1). Moreover, if we can choose E, F, G, and λ to be analytic functions, then the Kossowski metric is called *analytic*.

As shown in [4], the first fundamental form of a frontal-germ $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ whose singular points are all non-degenerate is a Kossowski metric.

Let ds^2 be a germ of a Kossowski metric having a singular point at 0. By the condition $d\lambda \neq 0$ at $(0,0)$, the implicit function theorem yields that there exists a regular curve $\gamma(t)$ $(|t| < \varepsilon)$ in the uv-plane (called the *singular curve*) parametrizing $S(ds^2)$. Then there exists a smooth non-zero vector field η such that η_q gives a null vector for each $q \in S(ds^2)$ near $(0,0)$. We call η a *null vector field*.

DEFINITION 5.2. If η is transversal to $S(ds^2)$ at 0, the singular point 0 is called *type* I (or an A_2 *point*).

For a Kossowski metric ds^2 induced from a frontal-germ f, type I singular points of ds^2 correspond to singular points of the first kind of f.

According to [4, Proposition 2.25], for a type I singular point, there exists a coordinate system $(U; u, v)$ centered at 0, such that
- the singular set $S(ds^2)$ is given by the u-axis,
- ∂_v gives the null vector field,
- $F = 0$ on U, and
- $E(u, 0) = 1$, $E_v(u, 0) = G_v(u, 0) = 0$, $G_{vv}(u, 0) = 2$

hold, where E, F, G are smooth functions as in (5.1). Such a coordinate system is called a *normalized strongly adapted coordinate system*. Since $G_{vv}(u, 0) = 2$ is equivalent to $\lambda_v(u, 0) = \pm 1$, by changing $v \mapsto -v$ if necessary, we may assume that $\lambda_v(u, 0) = 1$. (Hence, in the case of Kossowski metrics induced from frontals in $\boldsymbol{R}^3$, the normalized strongly adapted coordinate systems are normally-adjusted, cf. Definition 4.1.)

We shall review the definition of the product curvature for type I singular points defined in [4]. Let $(U; u, v)$ be a normalized strongly adapted coordinate system centered at a type I singular point 0. Denote by K the Gaussian curvature of ds^2 on $U \setminus \{v = 0\}$. By [4, Proposition 2.27], $vK(u, v)$ is a smooth function on U. Then

$$\tilde{\kappa}_\Pi := \lim_{v \to 0} vK(u, v)$$

does not depend on the choice of the normalized strongly adapted coordinate system satisfying $\lambda_v(0, 0) = 1$, and is called the *product curvature*.

Now, assume that $\tilde{\kappa}_\Pi$ vanishes along the u-axis. Then, K is a bounded smooth function on U, and

$$\tilde{K}_\eta := \lim_{v \to 0} K_v(u, v)$$

does not depend on the choice of the normalized strongly adapted coordinate system satisfying $\lambda_v(0, 0) = 1$. We call $\tilde{K}_\eta$ the *secondary product curvature* or the *null-derivative Gaussian curvature*.

Definition 5.3. Let ds^2 be a germ of a Kossowski metric at a type I singular point 0. If the product curvature $\tilde{\kappa}_\Pi$ vanishes along $S(ds^2)$, and the secondary product curvature $\tilde{K}_\eta$ does not vanish at 0, then the singular point 0 is called an *intrinsic* $5/2$-*cuspidal edge*.

The following lemma is a direct conclusion of Lemma 4.3 and Corollary 4.5.

Lemma 5.4. *Let* $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ *be a non-*ν*-flat frontal-germ having a singular point* 0 *of the first kind. Denote by* ds^2 *the first fundamental form of* f. *Then*, f *at* 0 *is a* $5/2$*-cuspidal edge if and only if* 0 *is an intrinsic* $5/2$*-cuspidal edge* (*as a singular point of the Kossowski metric* ds^2).

We remark that the assumption of the non-ν-flatness cannot be removed, since there exists a cuspidal edge with vanishing limiting normal curvature such that the corresponding singular points of ds^2 are intrinsic $5/2$-cuspidal edges.

Example 5.5. Let $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ be a map-germ defined by $f(u, v) = (u, u^2 + v^2, v^3 + v^4)$. The first fundamental form ds^2 is written as

$$ds^2 = (4u^2 + 1)\,du^2 + 8uv\,du dv + v^2(4 + v^2(4v + 3)^2)\,dv^2.$$

We can check that f is a front with a unit normal $\nu(u, v) = \hat{\lambda}^{-1}(2uv(4v + 3), -v(4v + 3), 2)$, where we set $\hat{\lambda} := \sqrt{4 + v^2(4u^2 + 1)(4v + 3)^2}$. Since the signed area density function λ is written as $\lambda = v\hat{\lambda}$, the u-axis gives the singular set $\{(u, 0);\, u \in \boldsymbol{R}\}$. As every singular point $(u, 0)$ is of the first kind, f is a cuspidal edge. The limiting normal curvature $\kappa_\nu(u)$ is identically zero along the u-axis, so the product curvature is too. The Gaussian curvature K is given by $K = -4(4v + 3)(8v + 3)/\hat{\lambda}^4$, which satisfies $K_v(u, 0) = -9$. Hence, the corresponding singular points of ds^2 are intrinsic $5/2$-cuspidal edges, although f is a cuspidal edge.

Kossowski [13] proved a realization theorem of Kossowski metrics which admit only singular points satisfying $K\,dA \neq 0$. In [8], a realization theorem of Kossowski metrics at an arbitrary singular point is proved. In the following Fact 5.6, we introduce the realization theorem, which is a restricted version of [8, Theorem B] so that $\gamma(t)$ is chosen to be the singular curve

FACT 5.6 (cf. [8, Theorem B]). *Let ds^2 be a germ of an analytic Kossowski metric on $(\boldsymbol{R}^2, 0)$, and let $\gamma(t)$ $(|t| < \varepsilon)$ be a singular curve passing through a singular point $0 = \gamma(0)$. Assume that 0 is a type* I *singular point of ds^2. Then, for given analytic function-germs $\omega(t)$, $\tau(t)$ at $t = 0$, there exists an analytic frontal-germ $f = f_{\omega,\tau} : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ satisfying the following properties*:

(1) *ds^2 is the first fundamental form of f,*

(2) *the limiting normal curvature function germ along the singular curve γ coincides with $e^{\omega(t)}$ for a suitable choice of a unit normal vector field ν,*

(3) *$\tau(t)$ gives the torsion function germ of $\hat{\gamma}(t) := f \circ \gamma(t)$.*

The possibilities for the congruence classes of such an f are at most two. Moreover, if τ vanishes identically (*i.e.*, *$\hat{\gamma}$ is a planar curve*), *then the congruence class of f is uniquely determined.*

Using Fact 5.6 and an argument similar to that of Theorem 4.7, we have the following realization theorem of Kossowski metrics with intrinsic $5/2$-cuspidal edges with prescribed singular images, which is an analogous to a result of [16, Theorem 12] for cuspidal edges and [8, Corollary D] for cuspidal cross caps.

THEOREM 5.7. *Let ds^2 be a germ of an analytic Kossowski metric on $(\boldsymbol{R}^2, 0)$. Assume that 0 is an intrinsic $5/2$-cuspidal edge. Take a germ of an analytic regular space curve $\sigma(t)$ such that its curvature function $\kappa(t)$ satisfies*

$$\kappa > |\kappa_s|$$

at 0, *where* κ_s *is the singular curvature of* ds^2 *along the singular curve* γ. *Then there exists a germ of an analytic* $5/2$-*cuspidal edge* $f_\sigma : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ *with non-vanishing limiting normal curvature such that*

(1) *the first fundamental form of* f_σ *coincides with* ds^2,

(2) *the singular image* $f_\sigma \circ \gamma$ *coincides with* σ.

The possibilities for congruence classes of such an f_σ *are at most two unless* τ *vanishes identically. On the other hand, if* τ *vanishes identically* (*i.e.*, σ *is a planar curve*), *then the congruence class of* f_σ *is uniquely determined.*

Proof. Set $\omega(t)$ to be $\omega(t) := \log \sqrt{\kappa(t)^2 - \kappa_s(t)^2}$. Let $\tau(t)$ be the torsion function of $\sigma(t)$. By Fact 5.6, there exists an analytic frontal-germ $f_\sigma := f_{\omega,\tau} : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ such that the items (1)–(3) in Fact 5.6 hold. Thus, it suffices to show that f_σ has a $5/2$-cuspidal edge at 0. Since the first fundamental form of f_σ coincides with ds^2, the product curvature κ_Π and the secondary product curvature r_Π of f_σ coincide with those of ds^2, respectively. Therefore, by Corollary 4.5, we have that $f_\sigma : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ has a $5/2$-cuspidal edge at 0. $\square$

Remark 5.8. We may suppose that $\sigma(t)$ is defined for $|t| < \varepsilon$. By Theorem 5.7, there exists a frontal $f_\pm : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ having a $5/2$-cuspidal edge at $p = \sigma(0)$ such that f_- is isometric to f_+ and $\sigma(t) = f_\pm \circ \gamma(t)$. On the other hand, reversing the orientation of $\sigma(t)$, there exists a frontal $g_\pm : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ having a $5/2$-cuspidal edge at $p = \sigma(0)$ such that g_- is isometric to g_+ and $\sigma(-t) = g \circ \gamma(-t)$. Thus if σ is not planar, there are totally four distinct $5/2$-cuspidal edges f_+, f_-, g_+ and g_- with the common first fundamental form whose image of the singular curve coincides with $\sigma((-\varepsilon, \varepsilon))$ in general, see [9] for details.

Appendix A. Proofs of propositions

A.1. Proof of Proposition 3.9. We show the following proposition, which is a normal form of a singular point of the first kind.

Proposition A.1. *Let* $f : (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^3, 0)$ *be a frontal and* 0 *a singular point of the first kind. Then there exist a coordinate system* (u, v) *and an isometry* A *of* $\boldsymbol{R}^3$ *such that*

$$A \circ f(u, v) = (u, a_2(u) + v^2/2, a_3(u) + v^2 b_3(u, v))$$

for some functions a_2, a_3, b_3. *If* 0 *is a* $5/2$-*cuspidal edge*, b_3 *has the form* $b_3 = c_3(u) + v^2 c_4(u) + v^3 c_5(u, v)$ *for some functions* c_3, c_4, c_5.

Proof. Let ν be a unit normal vector field along f. Since rank $df_0 = 1$, by an isometry A on $\boldsymbol{R}^3$, we may assume $df_0(X) = (*, 0, 0)$ for any $X \in T_0\boldsymbol{R}^2$ and

$\nu(0,0) = (0,0,1)$, where $*$ stands for a real number. Since 0 is a singular point of the first kind, $S(f)$ is a regular curve in $(\boldsymbol{R}^2, 0)$, and η is transversal to $S(f)$. Thus there exists a coordinate system $(\bar{u}, \bar{v})$ satisfying $S(f) = \{\bar{v} = 0\}$ and $\eta = \partial_{\bar{v}}$. Since $f_{\bar{u}}(0,0) = (a,0,0)$ $(a \neq 0)$, setting $u = f_1(\bar{u}, \bar{v})$, $v = \bar{v}$, the coordinate system (u, v) satisfies

$$\text{(A.1)} \qquad \begin{aligned} &f(u,v) = (u, f_2(u,v), f_3(u,v)),\\ &(f_2)_u(0,0) = (f_3)_u(0,0) = 0, \quad \nu(0,0) = (0,0,1), \end{aligned}$$

where $f_1(u,v)$ is the first component of f. Since $f_v(u,0) = 0$, there exist functions a_2, a_3, b_2, b_3 such that $f_i(u,v) = a_i(u) + v^2 b_i(u,v)/2$ $(i = 2,3)$. Since 0 is non-degenerate, $\lambda_v(0,0) \neq 0$. Thus $\det(f_u, f_{vv}, \nu)(0,0) = b_2(0,0) \neq 0$. Setting $\tilde{u} = u$, $\tilde{v} = v\sqrt{|b_2(u,v)|}$, (A.1) is

$$f(u,\tilde{v}) = (u, a_2(u) \pm \tilde{v}^2/2, a_3(u) + \tilde{v}^2 \tilde{b}_3(u,\tilde{v})).$$

This shows the first assertion.

If 0 is a 5/2-cuspidal edge, then by Lemma 3.2, $\det(f_u, f_{vv}, f_{vvv})(u,0) = 0$ holds. Thus we have the second assertion. □

Proposition 3.9 is now obvious by Proposition A.1.

A.2. Proof of Proposition 3.2.

Proof of Proposition 3.2. To show Proposition 3.2, firstly we show the independence of the condition on the choice of the vector fields. Obviously, the condition (1) does not depend on the choice of the vector fields. Since the condition (2) is equivalent to f not being a front (Fact 3.3), the condition (2) does not depend on the choice of the vector fields. Moreover, by the proof of Lemma 3.6, we see the independence of the condition (3) on the choice of the vector fields.

By Proposition A.1, we may assume that f is written in the form $f(u,v) = (u, v^2, v^5 c_5(u,v))$. There exist functions c_6, c_7 such that $c_5(u,v) = c_6(u,v^2) + v c_7(u,v^2)$. Considering $\Phi_1 \circ f(u,v)$, where $\Phi_1(X,Y,Z) = (X, Y, Z - Y^3 c_7(X,Y))$, we may assume that f has the form $f(u,v) = (u, v^2, v^5 c_6(u,v^2))$. Then a pair of vector fields $\xi = \partial_u$, $\eta = \partial_v$ satisfies the condition of Proposition 3.2 with (3.4), and we see that $l = 0$. By condition (3) of Proposition 3.2, we see $c_6(0,0) \neq 0$. We set $\Phi_2(X,Y,Z) = (X, Y, Z/c_6(X,Y))$. Then $\Phi_2 \circ f = (u, v^2, v^5)$, which shows the assertion. □

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