R. HIRAKAWA
KODAI MATH. J.
42 (2019), 476–495

ON THE FAMILY OF RIEMANN SURFACES WITH TETRAHEDRAL GROUP ACTION

Ryota Hirakawa

Abstract

This is the second of our series of papers to solve Mutsuo Oka's problems concerning our polyhedral construction of degenerations of Riemann surfaces. Oka posed *globalization problem* of our degenerations and *determination problem* of the defining equation of a Riemann surface appearing in our construction—which is equipped with the standard tetrahedral group action (i.e. topologically equivalent to the tetrahedral group action on the cable surface of the tetrahedron). A joint work with S. Takamura solved the first problem. In this paper, we solve the second one—in an unexpected way: an algebraic curve with the standard tetrahedral group action turns out to be *not* unique: a sporadic one (hyperelliptic) and a 1-parameter family of non-hyperelliptic curves. We study their properties. At first glance they are 'independent', but actually intricately connected—we show that at one special value in this family, a degeneration whose monodromy is a hyperelliptic involution occurs, and the sporadic hyperelliptic curve emerges after the stable reduction (*hyperelliptic jump*). This jumping phenomenon seems deeply related to the moduli geometry and is possibly universal for other families of curves with finite group actions. Based on this observation, we pose *stably-connectedness problem.*

Contents

1. Problems and results

Concerning our polyhedral construction of degenerations of Riemann surfaces, Mutsuo Oka raised two problems at the symposium "Contact structure, singularity, differential equation and related topics" at Kochi (2014):

2010 *Mathematics Subject Classification*. Primary 14H37; Secondary 14D06, 14H10.

Key words and phrases. Riemann surface, Group action, Moduli space, Stable reduction, Degenerating family.

Received October 22, 2018; revised February 11, 2019.

I. *Globalize the above degenerations in a natural way.*

II. *What is the defining equation of such a Riemann surface?*

We solved Problem I in the joint work [12] with S. Takamura. In this paper, we solve Problem II. It however turns out that such a Riemann surface is not unique but forms a 1-parameter non-hyperelliptic family together with a sporadic hyperelliptic one. We explicitly describe this family, and in terms of stable reduction reveal the relationship between this family and the sporadic one. We moreover describe the image of this family under the moduli map.

Let Σ be an orientable real surface obtained by thickening the edges of a polyhedron (Figure 1.1). We say that Σ is the *cable surface* of the polyhedron—the genus of the cable surface of the n-hedron is $n-1$.

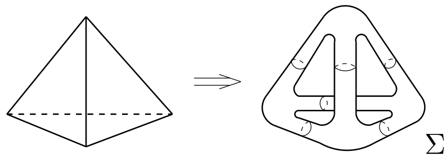

FIGURE 1.1

The automorphism group G of the polyhedron naturally acts on Σ orientation-preservingly. Kerckhoff's theorem [13] ensures the existence of a complex structure on Σ such that G acts holomorphically. In this paper, we consider the cable surface of the tetrahedron (*tetra surface*); its genus is 3. We may regard this Riemann surface as an algebraic curve. Noting that any (non-hyperelliptic) curve of genus 3 is realized as a plane algebraic curve in $\mathbf{P}^2$. M. Oka asked:

PROBLEM. *Determine the defining equation of such a curve. Moreover is this curve hyperelliptic or not?* (The same problem may be considered for any regular polyhedron, but it is subtle—for which the cable surface, being of genus ≥ 4, is not necessarily a plane curve, so may not be defined by a *single* equation.)

The complete classification of *full* automorphism groups of genus 3 curves is known ([4] for non-hyperelliptic ones, [10] for hyperelliptic ones); this however does not give the solution of the above problem—in fact the tetrahedral group may not be the full automorphism group of a curve in question. Moreover we must take into account the topological types of group actions: the action must be topologically equivalent to the standard tetrahedral group action on the cable surface Σ.

REFORMULATION. The tetrahedral group $\mathfrak{T}$ permutes the four vertices of the tetrahedron, which induces an isomorphism $\mathfrak{T} \cong \mathfrak{A}_4$ (alternating group of degree

4). A curve with a tetrahedral group action may be thus called an $\mathfrak{A}_4$-curve. If moreover the tetrahedral group action is topologically equivalent to the standard one, that is, the natural tetrahedral group action on Σ, then the $\mathfrak{A}_4$-curve is said to be *of tetra type*. M. Oka's problem is then reformulated as:

PROBLEM. Determine all genus 3 $\mathfrak{A}_4$-curves of tetra type.

We will show that:

SOLUTION (Theorem 4.9 (1)). *The genus* 3 $\mathfrak{A}_4$-*curves of tetra type are as follows*:

(H) *The hyperelliptic curve B defined by $y^2 = x^8 + 14x^4 + 1$ in* $\mathbf{C}^2$ (more precisely, compactify this curve in $\mathbf{P}^1 \times \mathbf{P}^1$ and then resolve its singularities, which yields B; refer to [9] p. 254 for this procedure).

(NH) *The non-hyperelliptic curves C_t $(t \in \mathbf{C}\backslash\{\pm 2, -1\})$ in $\mathbf{P}^2$ given by*

$$x^4 + y^4 + z^4 + t(x^2y^2 + y^2z^2 + z^2x^2) = 0.$$

(Note: All degree 4 curves are non-hyperelliptic ([11] p. 315, Exercise 3.2 (c)).)

We will actually show much more. Observe first that the $\mathfrak{A}_4$-actions on B and C_t are *a priori* 'independent' and moreover these curves are unrelated (as seen from their defining equations). This is however *not* the case; there exists an analytic deformation from B to C_s $(s = (t-2)^2)$ that is compatible with $\mathfrak{A}_4$-action (we say an "$\mathfrak{A}_4$-deformation"). The construction of this deformation is carried out by stable reduction (so B and C_t are said to be *stably connected*). We will also show that the singularities of the complex surface $S = \{C_t\}_{t \in \mathbf{C}}$ are eight A_1-singularities and they arise as the quotient under a hyperelliptic involution.

In the theory of algebraic curves, the classification of automorphism groups of curves (of fixed genus) is usually carried out separately for hyperelliptic curves or non-hyperelliptic curves; then there often appears a pair of a hyperelliptic G-curve X and a family of non-hyperelliptic G-curves Y_t (where G is a finite group) such that these G-actions are topologically equivalent (examples of such pairs indeed appear in the list of S. Hirose in his talk at the symposium "Algebraic topology around transformation groups" at RIMS, 2017). Based on our results, we pose the following:

STABLY-CONNECTEDNESS PROBLEM. *Are X and Y_t connected via a G-deformation? Are they related via stable reduction?*

We plan to discuss this in our subsequent paper.

Main results

We state our main results explicitly:

MAIN THEOREM. (1) *The genus* 3 $\mathfrak{A}_4$*-curves of tetra type are exhausted by*:

(i) *a hyperelliptic curve* $B : y^2 = x^8 + 14x^4 + 1$ *and*

(ii) *non-hyperelliptic curves* $C_t : x^4 + y^4 + z^4 + t(x^2y^2 + y^2z^2 + z^2x^2) = 0$ *in* $\mathbf{P}^2$, *where* $t \in \mathbf{C}\backslash\{\pm 2, -1\}$ (Theorem 4.9 (1)). *Here* $C_{\pm 2}$, C_{-1} *are excluded, because they are singular* (Lemma 5.4; see also Figure 1.2):

- C_2 *is* $\mathbf{P}^1$ *of multiplicity* 2.
- C_{-2} *consists of four* $\mathbf{P}^1$*'s and any two of them intersect at one point.*
- C_{-1} *consists of two* $\mathbf{P}^1$*'s intersecting at four points.*

(2) *Let S be a complex surface defined by*

$$S := \{(x, y, z, t) \in \mathbf{P}^2 \times \mathbf{C} : x^4 + y^4 + z^4 + t(x^2y^2 + y^2z^2 + z^2x^2) = 0\}$$

and $p : S \to \mathbf{C}$ *be the projection* $(x, y, z, t) \mapsto t$; *so* $C_t = p^{-1}(t)$. *Then the singularities of* S *are eight* A_1*-singularities and they lie on* C_2 (Theorem 4.9 (2)).

(3) *Take a sufficiently small disk* Δ *centered at* $t = 2$ *in* $\mathbf{C}$ *and set* $W := p^{-1}(\Delta)$. *Let* $\mathfrak{r} : M \to W$ *be the minimal resolution of the singularities. Then* $\pi := p \circ \mathfrak{r} : M \to \Delta$ *is a degeneration of smooth curves whose monodromy is a hyperelliptic involution* (Proposition 4.11).

(4) *Let* $p'' : N \to \Delta$ *be the* $\mathbf{Z}_2$*-stable reduction of* $p : W \to \Delta$ *via the base change* $\Delta \to \Delta$, $t - 2 \mapsto (t - 2)^2$. *Then the central fiber of* p'' *is* B (Theorem 4.9 (3)) *and the natural* $\mathbf{Z}_2$*-action on* B *is a hyperelliptic involution with* $B/\mathbf{Z}_2 = C_2$ (see Corollary 4.5).

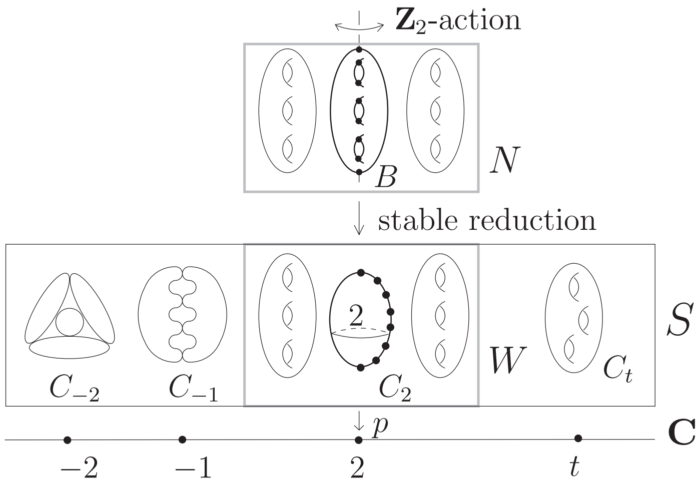

FIGURE 1.2. The eight bold points on C_2 are A_1-singularities.

Remark 1.1. The family of curves C_t is also studied by other researchers: Kuribayashi–Sekita [15], which is subsequently used in our discussion, and

Alwaleed and Sakai [2], which classified the 2-Weierstrass points on C_t and determined the numbers of flexes and sextactic points.

Description of the moduli map. Let $\mathscr{M}_3$ be the moduli space of Riemann surfaces of genus 3 and $\bar{\mathscr{M}}_3$ be its Deligne–Mumford compactification. Consider the moduli map $f : \mathbf{C}\backslash\{2\} \to \bar{\mathscr{M}}_3$ of the family $\{C_t\}_{t\in\mathbf{C}\backslash\{2\}}$. As $t \to 2$, $f(t) = [B]$, so f is bounded, thus naturally extends to a holomorphic map $f : \mathbf{C} \to \bar{\mathscr{M}}_3$. Set $\operatorname{Im} f := f(\mathbf{C})$. Then:

(1) f is injective except for two values $t = \dfrac{-3 \pm 3\sqrt{-7}}{4}$, for which C_t are the Klein curve ([15] Theorem 2 p. 121). Moreover $\operatorname{Im} f$ intersects *transversally* at the point corresponding to the Klein curve (this is shown by using linear quotient family; see [20] for details).
(2) $\operatorname{Im} f$ intersects the hyperelliptic locus in $\mathscr{M}_3$ at one point $f(2) = [B]$ (from Main theorem (4)).
(3) $\operatorname{Im} f$ intersects the boundary of $\mathscr{M}_3$ at $f(-2)$ and $f(-1)$, which correspond to the stable curves C_{-2} and C_{-1}.

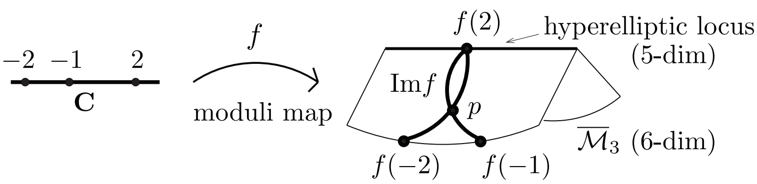

Figure 1.3. The point p corresponds to the Klein curve.

Exotic $\mathfrak{S}_4$-action. Each C_t actually admits a larger group action than $\mathfrak{A}_4$. Indeed the symmetric group $\mathfrak{S}_4$ acts on it (see [4] Table p. 10). Since C_t is homeomorphic to the cable surface Σ of the tetrahedron, this $\mathfrak{S}_4$-action is transformed to Σ. On the other hand, besides the automorphism group $\mathfrak{T}$ of the tetrahedron, the full automorphism group $\hat{\mathfrak{T}}$ (which contains orientation-reversing automorphisms) also acts on Σ, and this group is isomorphic to $\mathfrak{S}_4$. It is thus plausible that the previous $\mathfrak{S}_4$-action coincides with this $\mathfrak{S}_4$-action. However this is *not* the case, because the former contains *no* orientation-reversing automorphisms (as it is holomorphic). Thus Σ has two distinct $\mathfrak{S}_4$-actions: the standard one by $\hat{\mathfrak{T}}$ and the *exotic* one from the $\mathfrak{S}_4$-action on C_t.

Acknowledgments. I would like to express my gratitude to Professor Mutsuo Oka for calling my attention to the problems in the present work. I would like to express my gratitude to Professor Ryushi Goto for warm encouragement throughout this work. I also would like to thank Professors Tadashi Ashikaga, Akira Ohbuchi, Shigeru Takamura and Takeshi Harui for many valuable comments and fruitful discussion. I would like to thank Professor Takayuki Okuda

for pointing out that as a byproduct of our result, a geometric realization of the decomposition of the hyperelliptic involution of genus 3 into 13 Dehn twists is obtained, which may be applied to construct an interesting Lefschetz fibration (this will be discussed elsewhere).

2. Description of tetrahedral group action

Let Σ be the cable surface of the tetrahedron, on which the tetrahedral group $\mathfrak{T}$ naturally acts. Thanks to Kerckhoff's theorem [13], we may give a complex structure to it such that the $\mathfrak{T}$-action is holomorphic. We determine the branch data of the quotient map $\psi : \Sigma \to \Sigma/\mathfrak{T}$. We first review terminology with the intension of fixing notation.

Note first that ψ is a $|\mathfrak{T}|$-fold covering.

- For $y \in \Sigma/\mathfrak{T}$, if $\#\psi^{-1}(y) < |\mathfrak{T}|$, then y is a *branch point* (with branch index $|\mathfrak{T}|/\#\psi^{-1}(y)$).
- If $y \in \Sigma/\mathfrak{T}$ is a branch point, then $x \in \psi^{-1}(y)$ is a *ramification point* (with ramification index $|\mathfrak{T}|/\#\psi^{-1}(y)$).

A ramification point is alternatively characterized as a point with non-trivial stabilizer. For a point $x \in \Sigma$, its stabilizer $\mathfrak{T}_x$ is a subgroup of $\mathfrak{T}$ given by

$$\mathfrak{T}_x := \{g \in \mathfrak{T} : gx = x\}.$$

Now for $y \in \Sigma/\mathfrak{T}$, take $x \in \psi^{-1}(y)$. Then $\mathfrak{T}$ acts transitively on the points of $\psi^{-1}(y)$ while $\mathfrak{T}_x$ fixing x. Thus $\psi^{-1}(y) \cong \mathfrak{T}/\mathfrak{T}_x$ (as sets), and $\#\psi^{-1}(y) = |\mathfrak{T}|/|\mathfrak{T}_x|$. Hence $|\mathfrak{T}|/\#\psi^{-1}(y) = |\mathfrak{T}|/|\mathfrak{T}|/|\mathfrak{T}_x| = |\mathfrak{T}_x|$.

We thus obtain:

LEMMA 2.1. *The ramification index of x is $|\mathfrak{T}_x|$. Thus*:

$$\#\psi^{-1}(y) < |\mathfrak{T}| \text{ (i.e. } y \textit{ is a branch point}) \Leftrightarrow 1 < |\mathfrak{T}_x| \Leftrightarrow \mathfrak{T}_x \neq \{1\}.$$

Remark 2.2. $|\mathfrak{T}_x|$ is independent of the choice of $x \in \psi^{-1}(y)$. In fact for another $x' \in \psi^{-1}(y)$, $\mathfrak{T}_x$ and $\mathfrak{T}_{x'}$ are conjugate: There exists $g \in \mathfrak{T}$ such that $x' = gx$, for which $\mathfrak{T}_{x'} = g\mathfrak{T}_x g^{-1}$.

Take a ramification point $x \in \Sigma$. Then to each conjugate $g\mathfrak{T}_x g^{-1}$ $(g \in \mathfrak{T})$ of $\mathfrak{T}_x$, a ramification point $y = gx$ is associated; note that $\mathfrak{T}_y = g\mathfrak{T}_x g^{-1}$ and the ramification index $|\mathfrak{T}_y|$ of y is equal to $|\mathfrak{T}_x|$. Now denote by H the conjugacy class $\{g\mathfrak{T}_x g^{-1} : g \in \mathfrak{T}\}$ of $\mathfrak{T}_x$. The ramification points associated with the subgroups in this conjugacy class are called *H-ramification points.*

DEFINITION 2.3. Let $y_1, y_2, \dots, y_l$ be the branch points of ψ, and for each $y_i \in \Sigma/\mathfrak{T}$, let $e_i := |\mathfrak{T}_x|$ $(x \in \psi^{-1}(y_i))$ be the *branch index* of y_i. Then the tuple $(\text{genus}(\Sigma/\mathfrak{T}); e_1, e_2, \dots, e_l)$ is called the *branch data* (*signature*) of ψ (or of the $\mathfrak{T}$-action on Σ).

The ramification points of $\Sigma \to \Sigma/\mathfrak{T}$ are the points of Σ with nontrivial stabilizers (Lemma 2.1). To determine such points, identify $\mathfrak{T}$ with the alternating group $\mathfrak{A}_4$ under the canonical isomorphism induced from the permutation of the vertices of the tetrahedron. Here the (proper) nontrivial subgroups of $\mathfrak{A}_4$ are $\mathbf{Z}_2$, $\mathbf{Z}_3$ and $\mathbf{Z}_2 \times \mathbf{Z}_2$ up to conjugation. Among them, $\mathbf{Z}_2$ and $\mathbf{Z}_3$ are stabilizers of some points of Σ. In fact $\mathbf{Z}_2$ acts as a $1/2$-rotation fixing four points as illustrated in Figure 2.1 (there are three conjugate $\mathbf{Z}_2$'s in $\mathfrak{A}_4$) and $\mathbf{Z}_3$ acts as a $1/3$-rotation fixing two points as illustrated in Figure 2.2 (there are four conjugate $\mathbf{Z}_3$'s in $\mathfrak{A}_4$), while $\mathbf{Z}_2 \times \mathbf{Z}_2$ fixes *no* point (as a whole group) and is generated by a pair of $1/2$-rotations (there are three conjugate $\mathbf{Z}_2 \times \mathbf{Z}_2$'s in $\mathfrak{A}_4$ respectively generated by (1) and (2), (2) and (3), or (3) and (1) in Figure 2.1). The total number of $\mathbf{Z}_2$-ramification points are $4 \times 3 = 12$ and the total number of $\mathbf{Z}_3$-ramification points are $2 \times 4 = 8$. The ramification index of each $\mathbf{Z}_2$-ramification point is $|\mathbf{Z}_2| = 2$ and the ramification index of each $\mathbf{Z}_3$-ramification point is $|\mathbf{Z}_3| = 3$.

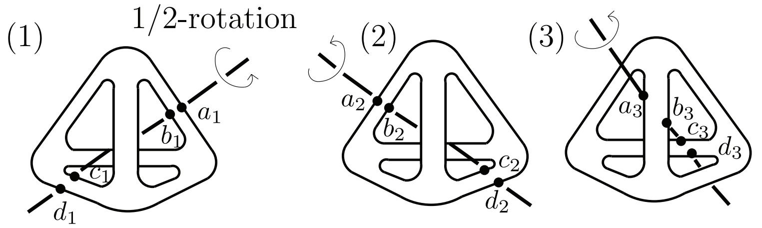

FIGURE 2.1

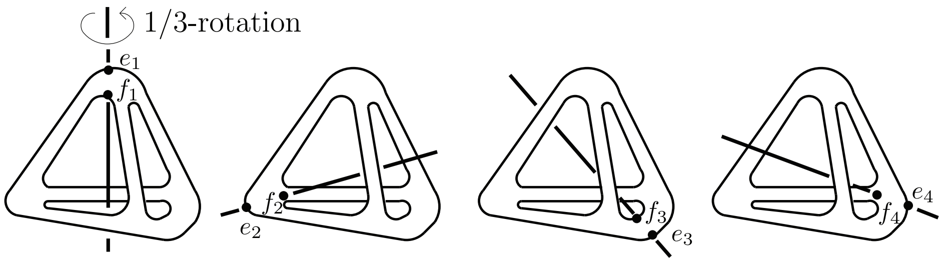

FIGURE 2.2

The images of the ramification points under the quotient map $\psi : \Sigma \to \Sigma/\mathfrak{T}$ are the *branch points*. Note:

- a_i, b_i, c_i, d_i $(i = 2, 3)$ are identified with a_1, b_1, c_1, d_1 respectively.
- a_1 (resp. b_1) is identified with d_1 (resp. c_1) via a $1/2$-rotation as illustrated in Figure 2.3.

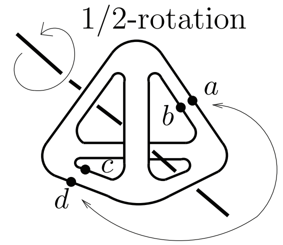

FIGURE 2.3

Hence the images of the $\mathbf{Z}_2$-ramification points are two points $\bar{a}_1$ and $\bar{b}_1$ (with branch index 2).

Next under $\psi : \Sigma \to \Sigma/\mathfrak{T}$, e_i, f_i $(i = 2, 3, 4)$ are identified with e_1, f_1 respectively. Hence the images of the $\mathbf{Z}_3$-ramification points are two points $\bar{e}_1$ and $\bar{f}_1$ (with branch index 3).

We summarize the above as follows:

LEMMA 2.4. *The quotient map* $\Sigma \to \Sigma/\mathfrak{T}$ *has four branch points with branch indices* $(2, 2, 3, 3)$.

We next show that $\Sigma/\mathfrak{T} \cong \mathbf{P}^1$ by applying the Riemann–Hurwitz formula:

$$\chi(\Sigma) = |\mathfrak{T}|\chi(\Sigma/\mathfrak{T}) - \sum_{p \in \mathscr{R}} (e_p - 1), \tag{2.1}$$

where $\mathscr{R}$ is the set of the ramification points and e_p is its ramification index of $p \in \mathscr{R}$. In the present case, $\chi(\Sigma) = -4$, $|\mathfrak{T}| = 12$ and $\sum_p (e_p - 1) = 12(2 - 1) + 8(3 - 1) = 28$. Thus from (2.1), $\chi(\Sigma/\mathfrak{T}) = -2$, implying that $\Sigma/\mathfrak{T} \cong \mathbf{P}^1$. This with Lemma 2.4 yields the following:

PROPOSITION 2.5. *Let* Σ *be the cable surface of the tetrahedron, on which the tetrahedral group* $\mathfrak{T}$ *acts. Then* $\Sigma/\mathfrak{T} \cong \mathbf{P}^1$ *and the quotient map* $\Sigma \to \Sigma/\mathfrak{T}$ *has four branch points with branch indices* $(2, 2, 3, 3)$. (Thus the branch data of the $\mathfrak{T}$-action on Σ is $(0; 2, 2, 3, 3)$.)

We regard the branch points on $\Sigma/\mathfrak{T}$ as "marked points"; observe that the complex structure on $\Sigma/\mathfrak{T}$ ($\cong \mathbf{P}^1$) with four marked points admits a 1-parameter family of deformations (caused by moving one point among the four points—three points on $\mathbf{P}^1$ are normalized as 0, 1, ∞ under some element of $PSL_2(\mathbf{C})$). Varying one branch point (in $\mathbf{P}^1 \setminus \{\text{other branch points}\}$) yields a family of *topologically equivalent* coverings. The complex structures on the covering spaces are given by the pull back of the complex structures on $\Sigma/\mathfrak{T}$ with four marked points via the quotient map $\Sigma \to \Sigma/\mathfrak{T}$. We thus obtain a 1-parameter family of complex structures on Σ with the same covering transformation group, that is, $\mathfrak{T}$ (their branch data remain $(0; 2, 2, 3, 3)$). We formalize this as follows:

COROLLARY 2.6 (Non-rigidity). *Let Σ be the cable surface of the tetrahedron, on which the tetrahedral group $\mathfrak{T}$ acts. Give a complex structure to Σ such that the $\mathfrak{T}$-action is holomorphic, and regard Σ as a Riemann surface with $\mathfrak{T}$-action. Then Σ admits a "$\mathfrak{T}$-action preserving" 1-parameter deformation—there exists a 1-parameter family of Riemann surfaces with $\mathfrak{T}$-actions starting from Σ such that their branch data remain $(0; 2, 2, 3, 3)$.*

3. Defining equations of $\mathfrak{A}_4$-curves of genus 3

In what follows, unless otherwise mentioned, all curves are of genus 3. We identify the tetrahedral group $\mathfrak{T}$ with the alternating group $\mathfrak{A}_4$ under the canonical isomorphism (recall: $\mathfrak{T}$ permutes the four vertices of the tetrahedron, which induces $\mathfrak{T} \cong \mathfrak{A}_4$). A curve C with $\mathfrak{A}_4$-action (i.e. $\mathfrak{A}_4 \subset \mathrm{Aut}(C)$) is called an $\mathfrak{A}_4$-*curve*. The aim of this section is to show the following:

THEOREM 3.1. *The $\mathfrak{A}_4$-curves of genus 3 are as follows*: (i) *There is a unique hyperelliptic one*: $B : y^2 = x^8 + 14x^4 + 1$. (ii) *Non-hyperelliptic ones form a 1-parameter family* $C_t : x^4 + y^4 + z^4 + t(x^2y^2 + y^2z^2 + z^2x^2) = 0$ $(t \in \mathbf{C}\setminus\{\pm 2, -1\})$.

Note first that: *If C is an $\mathfrak{A}_4$-curve, then $\mathfrak{A}_4 \subset \mathrm{Aut}(C)$, so $|\mathrm{Aut}(C)|$ is divisible by $|\mathfrak{A}_4| = 12$.* We thus consider curves C such that $|\mathrm{Aut}(C)|$ is divisible by 12. We separate into hyperelliptic case and non-hyperelliptic case:

(H) The list of hyperelliptic curves C such that $|\mathrm{Aut}(C)|$ is divisible by 12 is as follows ([10] p. 118 Table 1):

Aut	$\lvert\mathrm{Aut}\rvert$	C
D_{12}	12	H1 : $y^2 = x(x^6 + tx^2 + 1)$
U_6	24	H2 : $y^2 = x(x^6 - 1)$
$\mathbf{Z}_2 \times \mathfrak{S}_4$	48	H3 : $y^2 = x^8 + 14x^4 + 1$

Table 3.1. $\mathfrak{S}_n$: symmetric group of degree n. $U_6 := \langle a, b : a^2 = b^{12} = abab^7 = 1\rangle$ (or $\langle a, b : a^2, b^{12}, abab^7\rangle$ in [5] p. 272 Table 2, 3.e). $D_{2n} := \langle a, b : a^n = b^2 = abab = 1\rangle$: dihedral group of order $2n$. NOTE: The presentation $U_6 = \langle a, b : a^2, b^6, abab^4\rangle$ in [10] p. 118 seems a typo, because for which $|U_6| \neq 24$ (but $|U_6| = 6$).

Here:

- H1 and H2 are *not* $\mathfrak{A}_4$-curves, as $\mathfrak{A}_4 \not\subset \mathrm{Aut}(\mathrm{H1})$ $(= D_{12})$ and $\mathfrak{A}_4 \not\subset \mathrm{Aut}(\mathrm{H2})$ $(= U_6)$ by Lemma 3.3 below.
- H3 is an $\mathfrak{A}_4$-curve, indeed $\mathfrak{A}_4 \subset \mathbf{Z}_2 \times \mathfrak{S}_4$. (Note: The Galois group of $x^8 + 14x^4 + 1$ is $\mathfrak{S}_4$, see [14] p. 58.)

This confirms (i) of Theorem 3.1.

(NH) The list of non-hyperelliptic curves C such that $|\mathrm{Aut}(C)|$ is divisible by 12 is as follows ([4] Theorem 16 p. 10)—note that any non-hyperelliptic curve of genus 3 is realized as a quadric in $\mathbf{P}^2$:

Aut	$\lvert\mathrm{Aut}\rvert$	C
$\mathfrak{S}_4$	24	NH1 : $x^4 + y^4 + z^4 + t(x^2y^2 + y^2z^2 + z^2x^2) = 0$, where $t \in \mathbf{C} \setminus \left\{0, \frac{-3 \pm 3\sqrt{-7}}{2}\right\}$. Precisely: $t = \pm 2, -1$ are also excluded, as NH1 for them are singular (Lemma 5.6).
$\mathbf{Z}_4 \circledcirc \mathfrak{A}_4 \cong SL_2(\mathbf{F}_3) \rtimes \mathbf{Z}_2$	48	NH2 : $x^4 + y^4 + z^3x = 0$
$(\mathbf{Z}_4 \times \mathbf{Z}_4) \rtimes \mathfrak{S}_3$	96	NH3 : $x^4 + y^4 + z^4 = 0$ (Fermat curve)
$PSL_2(\mathbf{F}_7)$	168	NH4 : $z^3y + y^3x + x^3z = 0$ (Klein curve)

Table 3.2. $A \rtimes B$ is the semidirect product of A and B. For $\mathbf{Z}_4 \circledcirc \mathfrak{A}_4$, see [4] p. 10. NOTE: Both $\mathbf{Z}_4 \circledcirc \mathfrak{A}_4$ and $SL_2(\mathbf{F}_3) \rtimes \mathbf{Z}_2$ have the same identification number of finite group: "GAP Id. [48, 33]" ([4] p. 10, [19] p. 9), so they are isomorphic.

Here:

- NH1 is an $\mathfrak{A}_4$-curve, as $\mathfrak{A}_4 \subset \mathrm{Aut}(\mathrm{NH1})$ $(= \mathfrak{S}_4)$.
- NH2 is excluded, as $\mathfrak{A}_4 \not\subset \mathrm{Aut}(\mathrm{NH2})$ $(= SL_2(\mathbf{F}_3) \rtimes \mathbf{Z}_2)$; see [19] p. 9.
- NH3 and NH4 are special cases of NH1 at the value of $t = 0$ and $t = \dfrac{-3 \pm 3\sqrt{-7}}{2}$ respectively. In fact, NH1 for $t = 0$ is the Fermat curve and NH1 for $t = \dfrac{-3 \pm 3\sqrt{-7}}{2}$ is isomorphic to the Klein curve ([15] p. 121 Theorem 2).

This confirms (ii) of Theorem 3.1.

Supplement: Technical lemmas on groups

LEMMA 3.2. *For* $U_6 := \langle a, b : a^2 = b^{12} = abab^7 = 1 \rangle$, *the following hold*:

(i) $ba = ab^5$.

(ii) *Any element of* U_6 *is written as* b^k *or* ab^k $(k = 0, 1, \ldots, 11)$. *Consequently* $U_6 = \{b^k, ab^k : k = 0, 1, \ldots, 11\}$.

(iii) *Any subgroup* $H \subset U_6$ *of order* 12 *is normal in* U_6 *and* $U_6/H \cong \mathbf{Z}_2$. *Moreover* $b^2 \in H$.

Proof. (i): The relation $abab^7 = 1$ is rewritten as $ba = a^{-1}b^{-7}$. Here $a^{-1} = a$ and $b^{-7} = b^5$ (as $a^2 = b^{12} = 1$), thus $ba = ab^5$.

(ii): Use (i).

(iii): Since $|U_6| = 24$ and $|H| = 12$, H is of index 2 in U_6, so normal. We show that $b^2 \in H$. If $b \in H$, this is trivial. If $b \notin H$, then b determines the generator $\bar{b}$ of $U_6/H \cong \mathbf{Z}_2$, so $\bar{b}^2 = 1$, thus $b^2 \in H$. □

LEMMA 3.3. (1) $\mathfrak{A}_4$ *is "not" a subgroup of* D_{12}.

(2) $\mathfrak{A}_4$ *is "not" a subgroup of* U_6.

Proof. (1): Since $|\mathfrak{A}_4| = |D_{12}|$ $(= 12)$, if $\mathfrak{A}_4 \subset D_{12}$ then $\mathfrak{A}_4 = D_{12}$, which contradicts the fact that D_{12} has an element of order 6 while $\mathfrak{A}_4$ does not (because the order of any element of $\mathfrak{A}_4$ is either 1, 2 or 3).

(2): Since $|\mathfrak{A}_4| = 12$, if $\mathfrak{A}_4 \subset U_6$ then $b^2 \in \mathfrak{A}_4$ (Lemma 3.2 (iii)). The order of this element is 6, which is a contradiction. □

4. Proof of main results

Unless otherwise mentioned, all curves are assumed to be of genus 3. The tetrahedral group $\mathfrak{T}$ naturally acts on the cable surface Σ of the tetrahedron. By Kerckhoff's theorem [13], we may give a complex structure on Σ such that this action is holomorphic. Recall that $\mathfrak{T} \cong \mathfrak{A}_4$, so Σ is an $\mathfrak{A}_4$-curve. Its branch data on Σ is $(0; 2, 2, 3, 3)$ (Proposition 2.5). An $\mathfrak{A}_4$-curve is said to be *of tetra type* if the $\mathfrak{A}_4$-action is topologically equivalent to the standard tetrahedral group action on Σ.

Definition 4.1. An $\mathfrak{A}_4$-curve of tetra type is called a *tetra curve*.

We determine all tetra curves, in fact we show that B and C_t $(t \in \mathbf{C} \backslash \{\pm 2, -1\})$ in Theorem 3.1 exhaust all tetra curves. This is a consequence of a chain of claims:

Claim I. *C_t for some t is a tetra curve.*

Proof. If none of C_t is a tetra curve, then *only* B is a tetra curve, which however cannot occur due to the non-rigidity of a tetra surface (Corollary 2.6). □

From the non-rigidity of a tetra surface (Corollary 2.6), the following holds:

Claim II. *Let C_{t_0} be a tetra curve, then there exists an open neighborhood U of t_0 in $\mathbf{C} \backslash \{\pm 2, -1\}$ such that for any $t \in U$, C_t is a tetra curve.*

In fact every C_t $(t \in \mathbf{C} \backslash \{\pm 2, -1\})$ is a tetra curve. To show this, we need preparation. Let S be the complex surface in $\mathbf{P}^2 \times \mathbf{C}$ defined by

$$S := \{([x : y : z], t) \in \mathbf{P}^2 \times \mathbf{C} : x^4 + y^4 + z^4 + t(x^2y^2 + y^2z^2 + z^2x^2) = 0\},$$

and let $p : S \to \mathbf{C}$ be the projection $([x : y : z], t) \mapsto t$; and $C_t = p^{-1}(t)$.

Lemma 4.2. (1) *$M := S \backslash (C_{\pm 2} \cup C_{-1})$ is non-singular.* (2) *Set $\Omega := \mathbf{C} \backslash \{\pm 2, -1\}$, then the restriction $p : M \to \Omega$ is a fibration.*

Proof. (1) follows from the fact that all of singular points of S lie on C_2 (Lemma 5.1 below). (2) is clear, because no degeneration occurs in $p : M \to \Omega$ (Lemma 5.6 below). □

LEMMA 4.3. *For each point $b \in \Omega$, there exists an open neighborhood V of b in Ω such that the $\mathfrak{A}_4$-actions on all C_t $(t \in V)$ are topologically the same.*

Proof. Since $p : M \to \Omega$ is a fibration, by the Ehresmann fibration theorem there exists a sufficiently small open neighborhood V of b in Ω such that the restriction $p : p^{-1}(V) \to V$ is diffeomorphically isomorphic to a projection $C \times V \to V$ (where $C = C_b$). The fiberwise $\mathfrak{A}_4$-action on $p^{-1}(V)$ corresponds to a fiberwise $\mathfrak{A}_4$-action on $C \times V$. Identifying the fiber $C \times \{t\}$ $(t \in V)$ with C in an obvious way, we regard the $\mathfrak{A}_4$-action on $C \times \{t\}$ $(t \in V)$ as a family of $\mathfrak{A}_4$-actions on a single C. This amounts to a family of injective homomorphisms $\iota_t : \mathfrak{A}_4 \to \mathrm{MCG}(C)$, where $\mathrm{MCG}(C)$ denotes the mapping class group of C. Since $\mathrm{MCG}(C)$ is discrete, ι_t must be constant. Therefore the $\mathfrak{A}_4$-actions on all C_t $(t \in V)$ are topologically the same. □

Now we can show:

CLAIM III. *Every C_t $(t \in \Omega)$ is a tetra curve.*

Proof. It suffices to show that the $\mathfrak{A}_4$-actions on all C_t $(t \in \Omega)$ are topologically the same. Take the open neighborhood U in Claim II as a *maximal* one. We claim that U is the whole of Ω. Otherwise there is a boundary point (say b) of U in Ω. By Lemma 4.3, there exists an open neighborhood V of b in Ω such that the $\mathfrak{A}_4$-actions on all C_t $(t \in V)$ are topologically the same. So $V \subset U$, which contradicts the fact that $b \notin U$. □

Our next task is to show that B is also a tetra curve. Let Δ be a sufficiently small disk centered at $t = 2$ in $\mathbf{C}$ and set $W := p^{-1}(\Delta)$. Consider the restriction $p : W \to \Delta$ of $p : S \to \mathbf{C}$ around the singular fiber $C_2 = p^{-1}(2)$ $(= 2\mathbf{P}^1$; see Lemma 5.4). After showing that B arises as the central fiber of a stable reduction of $p : W \to \Delta$, we will show that B is a tetra curve. Note first that W has eight isolated singularities, which lie on C_2 and exhaust all singularities of S (see Lemma 5.1 below). These eight singularities are A_1-singularities (see Lemma 5.3 below).

Now let $p' : W' \to \Delta$ be the family obtained from $p : W \to \Delta$ by the base change $t - 2 = s^2$, where explicitly

$$W' := \{([x : y : z], s) \in \mathbf{P}^2 \times \Delta : x^4 + y^4 + z^4 + (s^2 + 2)(x^2y^2 + y^2z^2 + z^2x^2) = 0\}.$$

The central fiber $p'^{-1}(0)$ of $p' : W' \to \Delta$ is identical to $p^{-1}(2)$ $(= C_2)$, so $p'^{-1}(0) \cong \mathbf{P}^1$. Here W' is singular in codimension 1 (W' is 'bent' along $p'^{-1}(0)$), and so non-normal. Let $\nu : N \to W'$ be the normalization of W'.

Then $p'' := p' \circ \nu : N \to \Delta$ is a (non-degenerating) family of smooth curves, which is the *stable reduction* of $p : W \to \Delta$.

$$\begin{tikzcd} N \arrow[r, "\nu"] \arrow[dr, dashed, "p''"'] & W' \arrow[d, "p'"] & W \arrow[d, "p"] \\ & \Delta \arrow[r, "\text{base}", "\text{change}"'] & \Delta. \end{tikzcd} \tag{4.1}$$

LEMMA 4.4. *Let $r_1, r_2, \dots, r_8 \in p^{-1}(0)$ be the eight singularities of W and $r'_1, r'_2, \dots, r'_8 \in p'^{-1}(0)$ be the corresponding points of W' under the identification of $p^{-1}(0)$ with $p'^{-1}(0)$. Then the restriction $\nu : p''^{-1}(0) \to p'^{-1}(0)$ $(\cong \mathbf{P}^1)$ of $\nu : N \to W'$ is a double covering with eight branch points $r'_1, r'_2, \dots, r'_8$.*

Proof. For each point $q \in p^{-1}(0)$, let $q' \in p'^{-1}(0)$ denote the corresponding point under the identification of $p^{-1}(0)$ with $p'^{-1}(0)$. To show the assertion, we describe the normalization $\nu : N \to W'$ around each point $q' \in p'^{-1}(0)$. We separate into two cases depending on the position of q' (below, we take a coordinate of the disk Δ so that the center is the origin: $\Delta = \{T \in \mathbf{C} : |T| < 1\}$):

CASE 1. $q' \in \{r'_1, r'_2, \dots, r'_8\}$: In this case W is defined by $TX = Y^2$ around q (and $p : W \to \Delta$ is given by $(X, Y, T) \mapsto T$ around q). Here q corresponds to the origin $(X, Y, T) = (0, 0, 0)$. The base change $T \mapsto T^2$ turns $TX = Y^2$ to $T^2X = Y^2$, which is the defining equation of W' around q' (and $p' : W' \to \Delta$ is given by $(X, Y, T) \mapsto T$ around q'). Here q' corresponds to the origin $(X, Y, T) = (0, 0, 0)$. Let $W_{q'}$ be a sufficiently small neighborhood of q'. Then $W_{q'} \cap p'^{-1}(0)$, given by $T = Y = 0$ (the X-axis), is the non-normal locus of $W_{q'}$. The normalization of $W_{q'}$ is given by $(u, v) \in \mathbf{C}^2 \mapsto (v^2, uv, u) \in W'_{q'}$; note that $(X, Y, T) = (v^2, uv, u)$ satisfies $T^2X = Y^2$, as $u^2v^2 = (uv)^2$. (Precisely speaking, we need to shrink $\mathbf{C}^2$ around the origin.) On the u-axis in $\mathbf{C}^2$, this normalization is given by $v \mapsto v^2$, which is a double covering over the origin q'. See Figure 4.1.

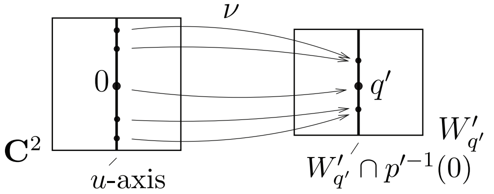

FIGURE 4.1. The restriction of $\nu : \mathbf{C}^2 \to W'_{q'}$ to the u-axis is two-to-one outside $0 \in \mathbf{C}^2$ while ramified at 0.

CASE 2. $q' \in p'^{-1}(0)\backslash\{r'_1, r'_2, \dots, r'_8\}$: In this case W is defined by $T = Y^2$ around q (and $p : W \to \Delta$ is given by $(X, Y, T) \mapsto T$ around q). Accordingly W' is defined by $T^2 = Y^2$ around q' (and $p' : W' \to \Delta$ is given by $(X, Y, T) \mapsto T$ around q'). Let $W_{q'}$ be a sufficiently small neighborhood of q'. Then $W_{q'} \cap p'^{-1}(0)$, given by $Y = T = 0$ (the X-axis), is the non-normal locus of $W_{q'}$. The normalization of $W_{q'} : T^2 = Y^2$ is explicitly given by $V_+ \amalg V_- \to W_{q'}$, where $V_+ := \{(u, v, w) \in \mathbf{C}^3 : w = v\}$ and $V_- := \{(u, v, w) \in \mathbf{C}^3 : w = -v\}$, and $v|_{V_+}$ and $v|_{V_-}$ are the 'identity' maps. (Precisely speaking, we need to shrink V_+ and V_- around the origins.) This normalization isomorphically maps the u-axis in V_+ and the u-axis in V_- to the X-axis in $W_{q'}$, which is an unramified double covering.

The descriptions in Case 1 and Case 2 together imply the assertion.

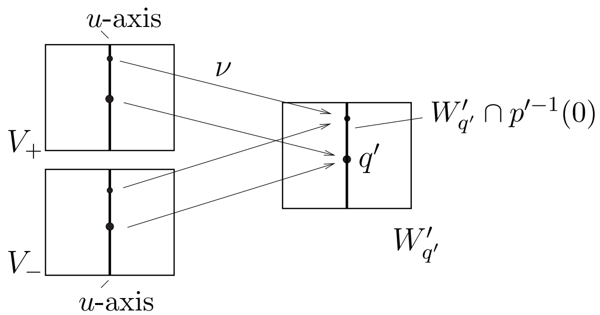

FIGURE 4.2. The restriction of $\nu : V_+ \amalg V_- \to W_{q'}$ to the u-axis in V_+ and the u-axis in V_- is two-to-one and unramified.

A genus 3 curve branched over $\mathbf{P}^1$ at eight points is necessarily hyperelliptic, and the double covering is the quotient under the hyperelliptic involution—the ramification points are the fixed points of hyperelliptic involution. Thus the following holds:

COROLLARY 4.5. *In Lemma* 4.4, $p''^{-1}(0)$ *is a hyperelliptic curve*, *and the eight ramification points of* ν *are the fixed points of its hyperelliptic involution* ι, *so that* $p''^{-1}(0)/\iota = p'^{-1}(0)$ $(= p^{-1}(0))$.

We next show that the hyperelliptic curve $A := p''^{-1}(0)$ admits an $\mathfrak{A}_4$-action. Consider the commutative diagram:

$$
\begin{array}{ccc}
N & \longrightarrow & W \\
{\scriptstyle p''}\big\downarrow & & \big\downarrow{\scriptstyle p} \\
\Delta & \longrightarrow & \Delta.
\end{array}
\tag{4.2}
$$

Then the $\mathfrak{A}_4$-action on $W \backslash p^{-1}(0)$ lifts to an $\mathfrak{A}_4$-action on $N \backslash A$ such that it maps each fiber $p''^{-1}(u)$ $(u \neq 0)$ to itself (Remark 4.7 below). The commutativity of (4.2) implies that each fiber $p''^{-1}(u)$ $(u \neq 0)$ is isomorphic to C_s $(s = u^2)$, so from Claim III it is a tetra curve (equipped with the $\mathfrak{A}_4$-action). We thus obtain the following:

Lemma 4.6. (1) *$A := p''^{-1}(0)$ is a hyperelliptic curve.* (2) *$p'' : N \backslash A \to \Delta \backslash \{0\}$ is a family of smooth $\mathfrak{A}_4$-curves—the $\mathfrak{A}_4$-action on $N \backslash A$ maps each fiber $p''^{-1}(u)$ $(u \neq 0)$ to itself, and $p''^{-1}(u)$ $(u \neq 0)$ is a tetra curve.*

Remark 4.7. Any finite group action on a plane curve in $\mathbf{P}^2$ is the restriction of a projective linear action on $\mathbf{P}^2$, that is, the finite group acts as a subgroup of $PGL_3(\mathbf{C})$ ([16] Corollary 5.3.19 p. 382). So in our context, the $\mathfrak{A}_4$-action on W is of the form $([x : y : z], s) \mapsto (g([x : y : z]), s)$, where $g \in PGL_3(\mathbf{C})$ (and $s := t - 2$). This action naturally defines an $\mathfrak{A}_4$-action on $N \backslash A$. Indeed as W is defined by $f([x : y : z], s) := x^4 + y^4 + z^4 + (s + 2)(x^2y^2 + y^2z^2 + z^2x^2) = 0$ in $\mathbf{P}^2 \times \Delta$, $N \backslash A$ is defined by $f([x : y : z], s^2) = 0$ in $\mathbf{P}^2 \times (\Delta \backslash \{0\})$, thus the $\mathfrak{A}_4$-action on W defines an $\mathfrak{A}_4$-action on $N \backslash A$. (Caution: N itself is not simply defined by $f([x : y : z], s^2) = 0$.)

The $\mathfrak{A}_4$-action on $N \backslash A$ uniquely extends to an $\mathfrak{A}_4$-action on N that maps $A = p''^{-1}(0)$ to itself (see Remark 4.8 below). In particular A is an $\mathfrak{A}_4$-curve. With Lemma 4.6 (1), *A is a hyperelliptic $\mathfrak{A}_4$-curve.* Such a curve is unique—it is B (see Theorem 3.1), thus $A = B$.

Remark 4.8. Let $\pi : M \to \Delta$ be a family of smooth curves and set $X := \pi^{-1}(0)$. Suppose that the restriction $\pi : M \backslash X \to \Delta \backslash \{0\}$ is a family of smooth G-curves (G: a finite group). Then the G-action on $M \backslash X$ uniquely extends to a G-action on M that maps X to itself (see [1] p. 115).

We show that the $\mathfrak{A}_4$-actions on all fibers of $p'' : N \to \Delta$ are topologically equivalent. First by the Ehresmann fibration theorem, $p'' : N \to \Delta$ may be topologically considered as the projection $A \times \Delta \to \Delta$ (recall that Δ is a sufficiently small disk). Then applying the argument in the proof of Lemma 4.3 shows that the $\mathfrak{A}_4$-actions on all fibers of $p'' : N \to \Delta$ are topologically the same. Since the $\mathfrak{A}_4$-actions on all fibers of $p'' : N \backslash A \to \Delta \backslash \{0\}$ are of tetra type (Lemma 4.6 (2)), the $\mathfrak{A}_4$-action on A is also of tetra type. Thus A $(= B)$ is also a tetra curve.

We summarize the results so far obtained as follows:

Theorem 4.9. (1) *The tetra curves are exhausted by B and C_t $(t \in \mathbf{C} \backslash \{\pm 2, -1\})$.*

(2) *W (and S) has eight singularities and they lie on C_2 and all are A_1-singularities.*

(3) *Let $p'' : N \to \Delta$ be the stable reduction of $p : W \to \Delta$ via a base change $\Delta \to \Delta$, $t - 2 \mapsto (t - 2)^2$. Then the central fiber of p'' is B.*

Remark 4.10. The $\mathfrak{A}_4$-action on B (and C_t) corresponds to an embedding of $\mathfrak{A}_4$ (as a subgroup) into the mapping class group MCG_3 of a genus 3 curve. Then in MCG_3, $\mathfrak{A}_4$ and the hyperelliptic involution ι commute, which follows from the commutativity of the $\mathfrak{A}_4$-action and the $\mathbf{Z}_2$-action ($\mathbf{Z}_2 = \langle \iota \rangle$) on $N \backslash A$; see the paragraph above Lemma 4.6.

Now let $\mathfrak{r} : M \to W$ be the minimal resolution of the eight A_1-singularities $r_1, r_2, \ldots, r_8$ (in Lemma 4.4), where each $E_i := \mathfrak{r}^{-1}(r_i)$ $(i = 1, 2, \ldots, 8)$ is $\mathbf{P}^1$ with self-intersection number -2, that is, E_i is a (-2)-curve. The composition of $\mathfrak{r}$ with $p : W \to \Delta$ is a degeneration $\pi := p \circ \mathfrak{r} : M \to \Delta$ of smooth curves of genus 3, whose singular fiber is $2\mathbf{P}^1 + \sum_{i=1}^{8} E_i$ (Figure 4.3), where each E_i intersects $2\mathbf{P}^1$ transversally. The monodromy of $\pi : M \to \Delta$ is the hyperelliptic involution in Corollary 4.5.

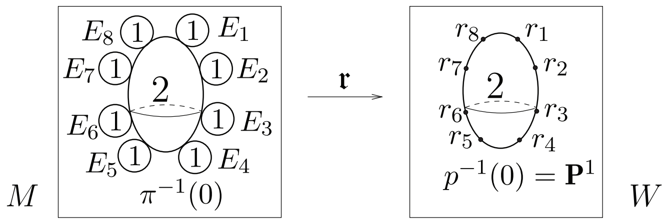

FIGURE 4.3. The minimal resolution $\mathfrak{r} : M \to W$.

We formalize the above as follows:

PROPOSITION 4.11. *Let $\mathfrak{r} : M \to W$ be the minimal resolution of the eight A_1-singularities $r_1, r_2, \ldots, r_8$ of W; each $E_i := \mathfrak{r}^{-1}(r_i)$ is a (-2)-curve. Then the composition of $\mathfrak{r}$ with $p : W \to \Delta$ is a degeneration $\pi := p \circ \mathfrak{r} : M \to \Delta$ of smooth curves of genus* 3, *whose singular fiber is* $2\mathbf{P}^1 + \sum_{i=1}^{8} E_i$, *and the monodromy of $\pi : M \to \Delta$ is the hyperelliptic involution in Corollary* 4.5.

5. The singularities and singular fibers of the $\mathfrak{A}_4$-family

Let S be the complex surface in $\mathbf{P}^2 \times \mathbf{C}$ defined by

$$S := \{([x : y : z], t) \in \mathbf{P}^2 \times \mathbf{C} : x^4 + y^4 + z^4 + t(x^2y^2 + y^2z^2 + z^2x^2) = 0\},$$

and let $p : S \to \mathbf{C}$ be the projection $([x : y : z], t) \mapsto t$ and $C_t := p^{-1}(t)$. The restriction of $p : S \to \mathbf{C}$ to $\mathbf{C} \backslash \{\pm 2, -1\}$ is the family of $\mathfrak{A}_4$-curves appearing in Theorem 3.1 (ii).

LEMMA 5.1. *S has eight isolated singularities* $([\pm\omega : \pm\omega^2 : 1], 2)$ *and* $([\pm\omega^2 : \pm\omega : 1], 2)$, *where* $\omega := e^{2\pi i/3}$, *which lie on the fiber* $C_2 = p^{-1}(2)$.

Proof. Take an open covering $\mathbf{P}^2 = U \cup V \cup W$, where $U = \{z = 1\}$, $V = \{x = 1\}$, and $W = \{y = 1\}$. We show that the singularities of S lie on $(U \cap V \cap W) \times \mathbf{C}$ and they are $([\pm\omega : \pm\omega^2 : 1], 2)$ and $([\pm\omega^2 : \pm\omega : 1], 2)$.

We first determine the singularities of S on $U \times \mathbf{C}$. The defining equation of S on $U \times \mathbf{C}$ is given by $f(x, y, t) = x^4 + y^4 + 1 + t(x^2y^2 + x^2 + y^2)$. Let $q = (x, y, t) \in S|_{U\times\mathbf{C}}$, then

$$\text{(a)} \quad x^4 + y^4 + 1 + t(x^2y^2 + x^2 + y^2) = 0.$$

Suppose that q is a singularity, equivalently

$$\partial_x f(q) = \partial_y f(q) = \partial_t f(q) = 0 \quad \text{(Jacobi criterion)},$$

or explicitly

$$\begin{cases} \text{(b)} \quad x(4x^2 + 2t(y^2 + 1)) = 0, \\ \text{(c)} \quad y(4y^2 + 2t(x^2 + 1)) = 0, \\ \text{(d)} \quad x^2y^2 + x^2 + y^2 = 0. \end{cases}$$

We claim that $x \neq 0$ and $y \neq 0$. Indeed if $x = 0$ then (a) and (d) become (a)$'$ $y^4 + ty^2 + 1 = 0$ and (d)$'$ $y^2 = 0$, so $1 = 0$ (absurd!). Similarly if $y = 0$ then (a) and (d) yield a contradiction, so $y \neq 0$. Dividing now (b) by x and (c) by y yields

$$\begin{cases} \text{(b)}' \quad 4x^2 + 2t(y^2 + 1) = 0, \\ \text{(c)}' \quad 4y^2 + 2t(x^2 + 1) = 0. \end{cases}$$

We next claim that $x^2 \neq -1$ and $y^2 \neq -1$. If $x^2 = -1$ then (c)$'$ implies $y = 0$ (contradiction). Similarly if $y^2 = -1$ then (b)$'$ implies $x = 0$ (contradiction). Now eliminating t from (b)$'$ and (c)$'$ yields (e) $\dfrac{2x^2}{y^2+1} = \dfrac{2y^2}{x^2+1}$. From (d), $x^2 = \dfrac{-y^2}{y^2+1}$. Substituting this into (e) yields $y^2 = -2, \omega, \omega^2$, so $(x^2, y^2) = (-2, -2), (\omega, \omega^2), (\omega^2, \omega)$. Here the first one is excluded, as it does not satisfy (a). The others indeed satisfy all of (a), (b), (c), (d) for $t = 2$. Therefore the singularities of S on $U \times \mathbf{C}$ are $([\pm\omega : \pm\omega^2 : 1], 2)$ and $([\pm\omega^2 : \pm\omega : 1], 2)$.

Similarly the singularities of S on $V \times \mathbf{C}$ are $([1 : \pm\omega : \pm\omega^2], 2)$ and $([1 : \pm\omega^2 : \pm\omega], 2)$. They are 'equal' to $([\pm\omega : \pm\omega^2 : 1], 2)$ and $([\pm\omega^2 : \pm\omega : 1], 2)$ (projective coordinates!). Similarly the singularities of S on $W \times \mathbf{C}$ are $([\pm\omega^2 : 1 : \pm\omega], 2)$ and $([\pm\omega : 1 : \pm\omega^2], 2)$, and they are also 'equal' to $([\pm\omega : \pm\omega^2 : 1], 2)$ and $([\pm\omega^2 : \pm\omega : 1], 2)$. □

Remark 5.2. The eight points $[\pm\omega : \pm\omega^2 : 1]$, $[\pm\omega^2 : \pm\omega : 1]$ on $\mathbf{P}^2$ are the base points of the pencil $\{C_t\}_{t\in\mathbf{P}^1}$.

LEMMA 5.3. *All eight singularities* $([\pm\omega : \pm\omega^2 : 1], 2)$ *and* $([\pm\omega^2 : \pm\omega : 1], 2)$ *of* S *are an* A_1*-singularities.*

Proof. It suffices to check that the Hessian of $f(x,y,t) = x^4 + y^4 + 1 + t(x^2y^2 + x^2 + y^2)$ at each singularity is nonzero, that is, nondegenerate (as this is equivalent to the singularity being A_1; see, e.g. [17]). The Hessian matrix of f is

$$H = \begin{pmatrix} 12x^2 + 2t(y^2+1) & 4txy & 2x(y^2+1) \\ 4txy & 12y^2 + 2t(x^2+1) & 2y(x^2+1) \\ 2x(y^2+1) & 2y(x^2+1) & 0 \end{pmatrix}.$$

At a singularity $(x,y,t) = (\omega, \omega^2, 2)$, H is given by

$$\begin{pmatrix} 12\omega^2 & 8 & -2 \\ 8 & 12\omega & -2 \\ -2 & -2 & 0 \end{pmatrix},$$

whose determinant is nonzero (indeed 16). Similarly for the other singularities, the Hessian is nonzero. □

We next determine the singular fibers of $p : S \to \mathbf{C}$.

LEMMA 5.4. *The curves C_t for $t = \pm 2, -1$ are reducible. In fact*:
(i) C_2 *is* $\mathbf{P}^1$ *of multiplicity* 2.
(ii) C_{-2} *consists of four* $\mathbf{P}^1$*'s and any two of them intersect at one point.*
(iii) C_{-1} *consists of two* $\mathbf{P}^1$*'s intersecting at four points.*

Proof. The defining equations (DEs) of C_t for $t = \pm 2, -1$ factorize as follows (so C_t for $t = \pm 2, -1$ are reducible):
DE for $t = 2$:

$$x^4 + y^4 + z^4 + 2(x^2y^2 + y^2z^2 + z^2x^2) = (x^2 + y^2 + z^2)^2.$$

DE for $t = -2$:

$$\begin{aligned} &x^4 + y^4 + z^4 - 2(x^2y^2 + y^2z^2 + z^2x^2) \\ &\quad = (x+y+z)(x+y-z)(x-y+z)(x-y-z). \end{aligned}$$

DE for $t = -1$: Where $\omega := e^{2\pi i/3}$,

$$x^4 + y^4 + z^4 - (x^2y^2 + y^2z^2 + z^2x^2) = (x^2 + \omega y^2 + \omega^2 z^2)(x^2 + \omega^2 y^2 + \omega z^2).$$

Note that any factor of the above factorizations is linear or quadratic, so it defines $\mathbf{P}^1$. Thus each irreducible component of C_t for $t = \pm 2, -1$ is $\mathbf{P}^1$. The other assertions are immediate from the above factorizations. □

LEMMA 5.5. *Any curve C_t for $t \neq \pm 2, -1$ is smooth.*

Proof. Set $F(x,y,z) := x^4 + y^4 + z^4 + t(x^2y^2 + y^2z^2 + z^2x^2)$. Then $[x:y:z] \in C_t$ is a singularity if and only if $\partial_x F = \partial_y F = \partial_z F = 0$, or explicitly

$$x(2x^2 + t(y^2+z^2)) = y(2y^2 + t(z^2+x^2)) = z(2z^2 + t(x^2+y^2)) = 0. \tag{5.1}$$

We separate into two cases:

CASE 1. $xyz \neq 0$: Then (5.1) is simplified into

$$2x^2 + t(y^2 + z^2) = 2y^2 + t(z^2 + x^2) = 2z^2 + t(x^2 + y^2) = 0.$$

Thus $\begin{pmatrix} 2 & t & t \\ t & 2 & t \\ t & t & 2 \end{pmatrix}\begin{pmatrix} x^2 \\ y^2 \\ z^2 \end{pmatrix} = 0$. This has a nontrivial solution precisely when $\begin{vmatrix} 2 & t & t \\ t & 2 & t \\ t & t & 2 \end{vmatrix} = 0$, that is, $2t^3 - 6t^2 + 8 = 0$, so $t = 2, -1$.

CASE 2. $xyz = 0$: Then *no* two of x, y, z can be 0 (for instance if $x = y = 0$, then from (5.1), $z = 0$, so $x = y = z = 0$, which contradicts $[x : y : z] \in \mathbf{P}^2$). We may thus assume that $x = 0$ and $yz \neq 0$. Then $2y^2 + tz^2 = 2z^2 + ty^2 = 0$, so $\begin{pmatrix} 2 & t \\ t & 2 \end{pmatrix}\begin{pmatrix} y^2 \\ z^2 \end{pmatrix} = 0$. This has a nontrivial solution precisely when $\begin{vmatrix} 2 & t \\ t & 2 \end{vmatrix} = 0$, that is, $-t^2 + 4 = 0$, so $t = \pm 2$.

We thus conclude that C_t is singular if and only if $t = \pm 2, -1$. □

By Lemmas 5.4 and 5.5, the following is obtained:

LEMMA 5.6. *A curve C_t $(t \in \mathbf{C})$ is singular precisely when $t = \pm 2, -1$: the singular curves C_2, C_{-2}, C_{-1} are explicitly described in Lemma* 5.4.

REFERENCES

[1] E. ARBARELLO, M. CORNALBA AND P. GRIFFITHS, Geometry of algebraic curves: Volume II, Grundlehren der mathematischen Wissenschaften **268**, Springer-Verlag, Berlin, 2011.

[2] K. ALWALEED AND F. SAKAI, Geometry and computation of 2-Weierstrass points on Kuribayashi quartic curves, Saitama Math. J. **26** (2009), 67–82.

[3] F. BARDELLI, Lectures on stable curves, Lectures on Riemann surfaces (M. Cornalba, X. Gómez-Mont and A. Verjovsky, eds.), World Scientific, Singapore, 1989, 648–704.

[4] F. BARS, On the automorphisms groups of genus 3 curves, Surveys in Math. and Math. Sciences **2** (2012), 83–124, https://www.researchgate.net/publication/228866630.

[5] E. BUJALANCE, J. M. GAMBOA AND G. GROMADZKI, The full automorphism groups of hyperelliptic Riemann surfaces, Manuscripta Math. **79** (1993), 267–282.

[6] T. BREUER, Characters and automorphism groups of compact Riemann surfaces, London Math. Soc. lecture note series **280**, Cambridge Univ. Press, Cambridge, 2000.

[7] I. V. DOLGACHEV, Classical algebraic geometry: A modern view, Cambridge Univ. Press, Cambridge, 2012.

[8] B. FARB AND D. MARGALIT, A primer on mapping class groups, Princeton math. series **49**, Princeton Univ. Press, New Jersey, 2012.

[9] P. GRIFFITHS AND J. HARRIS, Principles of algebraic geometry, Wiley Classics Library, John Wiley and Sons, New York, 1994.

[10] J. Gutierrez, D. Sevilla and T. Shaska, Hyperelliptic curves of genus 3 with prescribed automorphism group, Computational aspects of algebraic curves (T. Shaska, eds.), Lecture notes series on computing **13**, World Scientific, Singapore, 2005, 109–123.
[11] R. Hartshorne, Algebraic geometry, Graduate texts in math. **52**, Springer-Verlag, Berlin, 1977.
[12] R. Hirakawa and S. Takamura, Degenerations and fibrations of Riemann surfaces associated with regular polyhedra and soccer ball, J. Math. Soc. Japan **69** (2017), 1213–1233.
[13] S. Kerckhoff, The Nielsen realization problem, Ann. of Math. **117** (1983), 235–265.
[14] F. Klein, Lectures on the icosahedron and the solution of equation of the fifth degree, Cosimo Inc., New York, 2007.
[15] A. Kuribayashi and E. Sekita, On a Family of Riemann surfaces I, Bull. Facul. Sci. Eng. Chuo Univ. **22** (1979), 107–129.
[16] M. Namba, Geometry of projective algebraic curves, Marcel Dekker, New York, 1984.
[17] M. Oka, On the bifurcation of the multiplicity and topology of the Newton boundary, J. Math. Soc. Japan **31** (1979), 435–450.
[18] M. Oka, Non-degenerate complete intersection singularity, Hermann, Paris, 1997.
[19] K. Parattu and A. Wingerter, Finite groups of order less than or equal to 100, https://www.mimuw.edu.pl/~zbimar/small_groups.pdf.
[20] K. Sasaki and S. Takamura, Quotient families associated with the Klein curve, Preprint (2017).
[21] S. Takamura, Towards the classification of atoms of degenerations, II, (Cyclic quotient construction of degenerations of complex curves), RIMS Preprint **1344** (2001).
[22] S. Takamura, Towards the classification of atoms of degenerations, III, (Splitting Deformations of Degenerations of Complex Curves), Lecture notes in math. **1886**, Springer-Verlag, Berlin, 2006.

Ryota Hirakawa
Department of Mathematics
Kyoto University
Kitashirakawa Oiwake-cho, Sakyo-ku
Kyoto 606-8502
Japan
E-mail: r-hirakawa@cr.math.sci.osaka-u.ac.jp