

NÉRON MODELS OF 1-MOTIVES AND DUALITY

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Abstract

In this paper, we propose a definition of Néron models of arbitrary Deligne 1-motives over Dedekind schemes, extending Néron models of semi-abelian varieties. The key property of our Néron models is that they satisfy a generalization of Grothendieck’s duality conjecture in SGA 7 when the residue fields of the base scheme at closed points are perfect. The assumption on the residue fields is unnecessary for the class of 1-motives with semistable reduction everywhere. In general, this duality holds after inverting the residual characteristics. The definition of Néron models involves careful treatment of ramification of lattice parts and its interaction with semi-abelian parts. This work is a complement to Grothendieck’s philosophy on Néron models of motives of arbitrary weights.

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1. Introduction

1.1. Aim of the paper. Let X be an irreducible Dedekind scheme with function field K . Let U be either a dense open subscheme of X or equal to

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Spec K . Recall from [12, (10.1.10)] that a smooth 1-motive M over U in the sense of Deligne is a complex of group schemes $[Y \rightarrow G]$ over U whose degree -1 term Y is a lattice (étale locally isomorphic to \mathbf{Z}^n for some n) and degree 0 term G is an extension of an abelian scheme by a torus. Raynaud [33] studied monodromy (i.e. the defect of good reduction around $X \setminus U$) of 1-motives.

In this paper, we define a certain model $\mathcal{N}(M)$ of M over X , which we call the *Néron model* of M , generalizing Néron (lft) models of semi-abelian varieties [8, Chap. 10] (see also [21], [30] for more recent studies). Grothendieck, in [19, Exp. IX, §0.1], imagined a possibility of a theory of Néron models of motives of arbitrary weights. On the other hand, there have been several studies of Néron models of Hodge structures such as [17]. Our study of Néron models of 1-motives is a complement to such studies. We hope that our study sheds some light on possible Néron models of more general motives.

One key property of our Néron model $\mathcal{N}(M)$ is that it satisfies a generalization of *Grothendieck's duality conjecture* [19, IX, Conj. 1.3] when the residue fields of $X \setminus U$ are perfect. This conjecture is originally for $M = A$ an abelian variety, in which case (with perfect residue fields) it is solved by the author [39] after many partial results by other researchers. By Bertapelle-Bosch [3], the conjecture in its original form (for abelian varieties) may fail when a residue field is imperfect. Without the assumption on residue fields, the original conjecture is true if A has semistable reduction everywhere by Werner [42] or after inverting the residual characteristics by Bertapelle [2]. We prove that our Néron models $\mathcal{N}(M)$ of 1-motives M satisfy a duality under the same assumptions as those results (i.e. for the case of semistable M and for the case of residual characteristics being inverted). The duality results we prove strongly suggest that our definition of Néron models is “correct”. If Néron models of more general motives make any sense, then it will be a very interesting problem to try to generalize Grothendieck's duality conjecture to such models.

Our Néron model $\mathcal{N}(M)$ represents, in the derived category of X_{sm} , the truncation $\tau_{\leq 0} Rj_* M$ in degrees ≤ 0 of the derived pushforward of M by the natural morphism $j : U_{\text{sm}} \rightarrow X_{\text{sm}}$. Here X_{sm} is the smooth site of X , i.e. the category of smooth X -schemes with X -scheme morphisms endowed with the étale topology, and U_{sm} similarly. Hence $\mathcal{N}(M)$ encodes $j_* Y$, $R^1 j_* Y$ and the kernel of the morphism $R^1 j_* Y \rightarrow R^1 j_* G$. The sheaf $R^1 j_* Y$ has finite stalks and contains information about (possibly wild) ramification of the lattice Y . If Y is unramified along $X \setminus U$, then $R^1 j_* Y = 0$, and $\mathcal{N}(M)$ is simplified as $[j_* Y \rightarrow j_* G]$.

Bosch-Xarles [9, Def. 4.1] defines the Néron model of a complex of sheaves C on (the local rigid-analytic version of) U_{sm} as $R^0 j_* C$. Including information about the degree -1 term (or $j_* Y$) is a new feature of the present work. Our duality contains the results of Xarles [43] and Bertapelle-González-Avilés [4, Thm. 1.1] as a special case where M is a torus. The result of Xarles mentioned here is essentially about $\tau_{\leq 1} Rj_* Y$. Hence the information of the whole $\tau_{\leq 0} Rj_* M$ is crucial in order to even formulate duality.

According to González-Avilés, Xarles made an (unsuccessful) attempt in 1996 to generalize his result [43] to arbitrary 1-motives. The present work has been done independently of his attempt.

1.2. Main results. Now we state our results. Let $j : U_{\text{sm}} \rightarrow X_{\text{sm}}$ and K as above. Denote the category of 1-motives over U by \mathcal{M}_U , which has a natural additive functor to the bounded derived category $D^b(U_{\text{sm}})$ of sheaves on the site U_{sm} . Let SmGp/X be the category of commutative separated smooth group schemes over X . It has a natural additive functor to the bounded derived category $D^b(X_{\text{sm}})$ of sheaves on the site X_{sm} and hence inherits the notion of quasi-isomorphism of complexes from $D^b(X_{\text{sm}})$. Denote the resulting localization of the category of bounded complexes in SmGp/X by $D^b(\text{SmGp}/X)$. See Def. 2.11 for a more detailed definition and why $D^b(\text{SmGp}/X)$ is triangulated. We have a natural triangulated functor $D^b(\text{SmGp}/X) \rightarrow D^b(X_{\text{sm}})$. The existence of Néron models of semi-abelian varieties (i.e. representability of the sheaf j_*G) is generalized to 1-motives as follows.

THEOREM A. *There exists a canonical additive functor $\mathcal{N} : \mathcal{M}_U \rightarrow D^b(\text{SmGp}/X)$ such that the diagram*

$$\begin{array}{ccc} \mathcal{M}_U & \xrightarrow{\mathcal{N}} & D^b(\text{SmGp}/X) \\ \downarrow & & \downarrow \\ D^b(U_{\text{sm}}) & \xrightarrow{\tau_{\leq 0}Rj_*} & D^b(X_{\text{sm}}) \end{array}$$

is commutative.

This means that the complex of sheaves $\tau_{\leq 0}Rj_*M$ is represented by a complex of separated smooth group schemes over X , which is unique up to quasi-isomorphism and behaves functorially in M in the derived category. The construction of $\mathcal{N}(M)$ for $M = [Y \rightarrow G] \in \mathcal{M}_U$ needs, as auxiliary data, a finite étale covering V of U such that $Y \times_U V$ extends to a lattice over the normalization of X in V (which means that V kills ramification of Y along $X \setminus U$). To each such choice of V , we assign a certain canonical complex $\mathcal{N}(M, V)$ in SmGp/X with terms in degrees -1 and 0 representing $\tau_{\leq 0}Rj_*M$. As an object of $D^b(\text{SmGp}/X)$, this complex does not depend on V .

Actually this canonical complex $\mathcal{N}(M, V)$ is more useful than the object $\mathcal{N}(M)$ of $D^b(\text{SmGp}/X)$ that it represents, since functoriality in triangulated categories is difficult to use for some purposes. For example, the mapping cone of the morphism $\mathcal{N}_0(M) \rightarrow \mathcal{N}(M)$ mentioned below will be constructed using this actual complex representative. Nonetheless, the well-definedness of $\mathcal{N}(M)$ makes sense only in $D^b(\text{SmGp}/X)$.

The representability of the terms of $\mathcal{N}(M, V)$ is important; otherwise we would not have much control of the fiber of $\mathcal{N}(M)$ (and $\mathcal{P}(M)$ mentioned below) over Z (see Prop. 3.1 (2) and Prop. 3.2). Just having a complex of sheaves representing $\tau_{\leq 0}Rj_*M$ is not sufficient in this regard.

Next, to state our duality results, assume that $U \subset X$ is open (so either $U \neq \text{Spec } K$ or X has finitely many points) with reduced complement $i : Z \hookrightarrow X$. For $M = [Y \rightarrow G] \in \mathcal{M}_U$, let \mathcal{Y}_0 be the extension by zero of Y along $j : U \hookrightarrow X$ and \mathcal{G}_0 the maximal open subgroup scheme of the Néron model of G along j with connected fibers. (We do not use the more standard notation \mathcal{G}^0 , in order to avoid confusion with the zeroth term of a complex, in this highly derived categorical paper.) Define the *connected Néron model* of M by $\mathcal{N}_0(M) = [\mathcal{Y}_0 \rightarrow \mathcal{G}_0] \in \text{SmGp}/X$. We will define a canonical morphism $\mathcal{N}_0(M) \rightarrow \mathcal{N}(M)$ in $D^b(\text{SmGp}/X)$. There is a canonical mapping cone of this morphism. This cone is supported on Z (up to quasi-isomorphism). The fiber over Z of this cone is a complex of étale group schemes in degrees -1 and 0 with finitely generated groups of geometric points. Denote this complex of étale group schemes over Z by $\mathcal{P}(M) \in D^b(\mathbf{Z}_{\text{ét}})$ and call it the *Néron component complex* of M . We have a canonical distinguished triangle

$$\mathcal{N}_0(M) \rightarrow \mathcal{N}(M) \rightarrow i_*\mathcal{P}(M)$$

in $D^b(X_{\text{sm}})$. Let $M^\vee \in \mathcal{M}_U$ be the dual 1-motive of M ([12, (10.2.12), (10.2.13)]). Denote the derived tensor product by \otimes^L , shift of complexes by $[1]$ and the derived sheaf-Hom functor by $R \mathbf{Hom}$. We will define canonical morphisms

$$\begin{aligned} \mathcal{N}_0(M^\vee) \otimes^L \mathcal{N}(M) &\rightarrow \mathbf{G}_m[1], \\ \mathcal{P}(M^\vee) \otimes^L \mathcal{P}(M) &\rightarrow \mathbf{Z}[1] \end{aligned}$$

in $D(X_{\text{sm}})$, $D(\mathbf{Z}_{\text{ét}})$, respectively. They induce morphisms

$$\begin{aligned} \zeta_M : \mathcal{N}(M^\vee) &\rightarrow \tau_{\leq 0} R \mathbf{Hom}_{X_{\text{sm}}}(\mathcal{N}_0(M), \mathbf{G}_m[1]), \\ \zeta_{0M} : \mathcal{N}_0(M^\vee) &\rightarrow \tau_{\leq 0} R \mathbf{Hom}_{X_{\text{sm}}}(\mathcal{N}(M), \mathbf{G}_m[1]), \\ \eta_M : \mathcal{P}(M^\vee) &\rightarrow R \mathbf{Hom}_{\mathbf{Z}_{\text{sm}}}(\mathcal{P}(M), \mathbf{Z}[1]). \end{aligned}$$

If the residue field of Z at a point $x \in Z$ has characteristic $p \geq 0$, then by the residual characteristic exponent of Z at x , we mean p if $p > 0$ and 1 if $p = 0$.

THEOREM B.

- (1) ζ_M is an isomorphism.
- (2) ζ_{0M} and ζ_{0M^\vee} are both isomorphisms if and only if η_M is an isomorphism if and only if η_{M^\vee} is an isomorphism.
- (3) η_M is an isomorphism if M is semistable (meaning that Y is unramified and G is semistable along j).
- (4) $\eta_M \otimes \mathbf{Z}[1/n]$ is an isomorphism, where n is the product of the residual characteristic exponents of Z .
- (5) η_M is an isomorphism if the residue fields of Z are perfect.

(1) is more or less trivial (akin to the adjunction $j^* \leftrightarrow j_*$ or $j_! \leftrightarrow j^*$). Therefore the real content of duality is the three equivalent statements in (2), which is a generalization of Grothendieck’s duality conjecture. (3) easily reduces

to Grothendieck’s duality conjecture for semistable abelian varieties proved in [42]. For (4), we define l -adic realizations of $\mathcal{N}(M)$ and $\mathcal{N}_0(M)$ (resp. $\mathcal{P}(M)$) as constructible complexes of sheaves of \mathbf{Z}_l -modules on X (resp. Z), where l is a prime invertible on Z , and use the six operations formalism (in particular, duality) in l -adic derived categories. (5) generalizes the result of [39] for abelian varieties. We will prove (5) using the duality for cohomology of local fields with perfect residue fields with coefficients in M that is established in [39, Thm. (9.1)].

1.3. Remarks and organization. Here are some remarks. If $U = \text{Spec } K$ and X has infinitely many points, then $\mathcal{N}_0(M) = [\mathcal{Y}_0 \rightarrow \mathcal{G}_0]$ still makes sense; see Def. 2.18 for the definition of the extension by zero \mathcal{Y}_0 in this setting. But \mathcal{Y}_0 is not locally of finite type over X since $\text{Spec } K$ is not. If one wants a duality in this case, one should first extend M to a 1-motive over some dense open subscheme V of X and then consider the above duality for the morphism $V \hookrightarrow X$.

The target category $D^b(\text{SmGp}/X)$ of the Néron model functor \mathcal{N} is certainly not the best possible one. In the current form, we cannot consider transitivity of Néron model functors along two dense open subschemes $V \hookrightarrow U \hookrightarrow X$. Also, an arbitrary object of $D^b(\text{SmGp}/X)$ does not seem to have any meaningful notion of dual such that the double dual recovers the original object. For this reason, we do not attempt to lift the morphisms ζ_M and ζ_{0M} to $D^b(\text{SmGp}/X)$. The correct target (resp. source) category might be a suitably defined (non-derived) category of “constructible” or even “perverse” 1-motives over X (resp. U), and the functors \mathcal{N} and \mathcal{N}_0 might be viewed as j_* and $j_!$ between such categories.

Other kinds of realizations of Néron models should be explored. Among such would be the universal one after inverting the residual characteristics, i.e. as mixed étale motives over X in the sense of Cisinski-Déglise [10]. The answers to this and the previous questions might exist along the lines of the work of Pepin Lehalleur [31].

The above duality results are essentially of local nature, reduced to each point of Z . Global duality as studied in [29, III, §3, 9, 11] and [40] should be extended to Néron models of 1-motives.

We will see in Prop. 2.26 an example where the Néron model of a 1-motive arises geometrically from a relative curve over X with an étale local section over U . This suggests that Néron models of 1-motives might have some role in the study of rational points of curves over K valued in ramified extensions of K and the index problem for curves.

Now the organization of the paper is as follows. In §2, after collecting some facts about representability of sheaves on the smooth site, we define Néron models and connected Néron models, thereby proving Thm. A. In §3, we first study some generalities on morphisms of topologies without exact pullback functors, such as the one $i : Z_{\text{sm}} \rightarrow X_{\text{sm}}$ and the change of topologies $X_{\text{fppf}} \rightarrow X_{\text{sm}}$. Then we define Néron component complexes. In §4, we define the duality morphisms ζ_M , ζ_{0M} and η_M . We prove Thm. B (1), (2) and (3). We also prove

a weaker version of (4), namely that $\eta_M \otimes \mathbf{Q}$ is an isomorphism, by some arguments on connected-étale sequences. In §5, we define l -adic realizations and prove Thm. B (4). The weaker version of (4) proved earlier is necessary for this since derived l -adic completions of semi-abelian varieties and lattices are both \mathbf{Z}_l -lattices up to shift and destroy their distinction. In §6, we prove Thm. B (5). We quickly recall the formalism of the ind-rational pro-étale site from [39], [40] and the duality result [39, Thm. (9.1)] on cohomology of local fields with perfect residue field with coefficients in M . From this, we deduce its version for cohomology of the ring of integers of such a local field with coefficients in $\mathcal{N}(M)$, from which (5) follows.

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Notation. The categories of sets and abelian groups are denoted by Set and Ab , respectively. All groups, group schemes and sheaves of groups are assumed commutative. For an additive category \mathcal{A} , the category of complexes in \mathcal{A} in cohomological grading is denoted by $\text{Ch}(\mathcal{A})$. Its full subcategories of bounded below, bounded above and bounded complexes are denoted by $\text{Ch}^+(\mathcal{A})$, $\text{Ch}^-(\mathcal{A})$ and $\text{Ch}^b(\mathcal{A})$, respectively. If $A \rightarrow B$ is a morphism in $\text{Ch}(\mathcal{A})$, then its mapping cone is denoted by $[A \rightarrow B]$. The homotopy category of $\text{Ch}^\bullet(\mathcal{A})$ for $\bullet = +, -, b$ or (blank) is denoted by $K^\bullet(\mathcal{A})$. If \mathcal{A} is abelian, then its derived category is denoted by $D^\bullet(\mathcal{A})$. The canonical truncation functors for $D(\mathcal{A})$ in degrees $\leq n$ and $\geq n$ are denoted by $\tau_{\leq n}$ and $\tau_{\geq n}$, respectively. If we say $A \rightarrow B \rightarrow C$ is a distinguished triangle in a triangulated category, we implicitly assume that a morphism $C \rightarrow A[1]$ to the shift of A is given, and the triangle $A \rightarrow B \rightarrow C \rightarrow A[1]$ is distinguished. If $A \rightarrow B$ is a morphism in a triangulated category together with a certain canonical choice of a mapping cone, then this mapping cone is still denote by $[A \rightarrow B]$ unless confusion may occur. For a site S , the categories of sheaves of sets and abelian groups are denoted by $\text{Set}(S)$ and $\text{Ab}(S)$. We denote $\text{Ch}^\bullet(S) = \text{Ch}^\bullet(\text{Ab}(S))$ and use the notation $K^\bullet(S)$, $D^\bullet(S)$ similarly. The Hom and sheaf- Hom functors for $\text{Ab}(S)$ are denoted by Hom_S and \mathbf{Hom}_S , respectively. Their right derived functors are denoted by Ext_S^n , $R\text{Hom}_S$ and \mathbf{Ext}_S^n , $R\mathbf{Hom}_S$, respectively. The tensor product functor \otimes is over the ring \mathbf{Z} (or, on some site, the sheaf of rings \mathbf{Z}). Its left derived functor is denoted by \otimes^L . For a morphism of sites $f : S' \rightarrow S$, we denote by f^* the pullback functor for sheaves of abelian groups.

2. Definition of Néron models

For a scheme X , we denote the smooth site of X by X_{sm} . It is the category of smooth X -schemes with X -scheme morphisms endowed with the étale (or

equivalently, smooth) topology. We denote the category of separated smooth group schemes (commutative, as assumed throughout the paper) over X by SmGp/X and the category of quasi-separated smooth (commutative!) group algebraic spaces over X by SmGp'/X . They are additive categories. The full subcategory of SmGp/X (resp. SmGp'/X) consisting of objects étale over X are denoted by EtGp/X (resp. EtGp'/X).

By a Dedekind scheme, we mean a noetherian regular scheme of dimension ≤ 1 . A separated smooth group algebraic space over a Dedekind scheme is a scheme by [32, Thm. (3.3.1)]. Hence $\text{SmGp}/X \subset \text{SmGp}'/X$ if X is Dedekind.

PROPOSITION 2.1. *Let X be an irreducible Dedekind scheme with function field K . Let U be either a dense open subscheme of X or equal to $\text{Spec } K$. Then the inclusion morphism $j : U \hookrightarrow X$ induces a morphism of sites $j : U_{\text{sm}} \rightarrow X_{\text{sm}}$ (defined by the functor sending a smooth X -scheme X' to $X' \times_X U$).*

Proof. The only non-trivial part is the exactness of the pullback functor $j^{*\text{set}} : \text{Set}(X_{\text{sm}}) \rightarrow \text{Set}(U_{\text{sm}})$ for sheaves of sets. To show this, we may assume that $X = \text{Spec } A$ is affine. If U is open in X , then $j^{*\text{set}}$ is just the restriction functor, hence exact. Assume $U = \text{Spec } K$. Let $F \in \text{Set}(X_{\text{sm}})$. Then $j^{*\text{set}}F$ is the sheafification of the presheaf that sends a smooth K -algebra B to the direct limit of the sets $F(A')$, where A' runs through smooth A -algebras with fixed A -algebra homomorphisms to B . The index category for this direct limit is filtered since K and hence B are filtered direct limits of smooth A -algebras. Since filtered direct limits and sheafification are exact, we know that $j^{*\text{set}}$ is exact. □

In the rest of this section, assume the following:

Situation 2.2.

- X is an irreducible Dedekind scheme with function field K .
- U is either a dense open subscheme of X or equal to $\text{Spec } K$.
- $j : U \hookrightarrow X$ is the inclusion morphism.
- $j : U_{\text{sm}} \rightarrow X_{\text{sm}}$ is the morphism of sites induced by j as in Prop. 2.1.

As above, we assume that X is irreducible (and, in particular, non-empty), so that its function field K makes sense. A Dedekind scheme is a finite disjoint union of irreducible Dedekind schemes ([8, 1.1]). The arguments in this paper do not involve with descent problems that require careful treatment of reducible Dedekind schemes. Note that $\text{Spec } K \subset X$ is open if and only if $\text{Spec } K$ is (locally) of finite type over X if and only if X has finitely many points if and only if X has a finite open covering by local Dedekind schemes.

PROPOSITION 2.3. *Let Y be a (possibly infinite) disjoint union of finite étale connected schemes over U . Then j_*Y is a separated étale scheme over X .*

Proof. If Z is a smooth X -scheme, then any connected component of $Z \times_X U$ uniquely extends to a connected component of Z . Therefore j_* as a functor $\text{Set}(U_{\text{sm}}) \rightarrow \text{Set}(X_{\text{sm}})$ commutes with disjoint unions. Hence we may assume that Y is connected. Let \bar{Y} be the normalization of X in Y . Let $V \subset \bar{Y}$ be the maximal open subscheme étale over X . If Z is a smooth X -scheme, then any U -morphism $Z \times_X U \rightarrow Y$ uniquely extends to an X -morphism $Z \rightarrow \bar{Y}$ since Z is normal. This morphism factors through V . This means that $j_*Y = V$, which is separated étale. \square

PROPOSITION 2.4. *Let Y be an étale group scheme over X and $F \in \text{Ab}(X_{\text{sm}})$ a sheaf. Let $\varphi : Y \rightarrow F$ be any morphism in $\text{Ab}(X_{\text{sm}})$ and $\text{Ker}(\varphi) \in \text{Ab}(X_{\text{sm}})$ its kernel. Then $\text{Ker}(\varphi)$ is an open subscheme of Y and, in particular, an étale X -scheme.*

Proof. Let N be the union of the open subschemes of Y that map to zero in F . Then N itself maps to zero in F . Any X -morphism $Z \rightarrow Y$ from a quasi-compact smooth X -scheme Z is a faithfully flat smooth morphism followed by an open immersion. Hence if Z maps to zero in F , then it factors through N . Thus $N = \text{ker}(\varphi)$. \square

PROPOSITION 2.5. *Let $G_1, G_2 \in \text{SmGp}'/X$. Then any extension G_3 of G_1 by G_2 in $\text{Ab}(X_{\text{sm}})$ is in SmGp'/X . If $G_1, G_2 \in \text{SmGp}/X$, then $G_3 \in \text{SmGp}/X$.*

Proof. We know that $G_3 \in \text{SmGp}'/X$ by descent. Since X is Dedekind, we know by [32, Thm. (3.3.1)] that a separated group algebraic space over X is a scheme. Hence the second statement follows. \square

If G is an extension of an abelian scheme by a torus over U and if $U = \text{Spec } K$, then j_*G is represented by the Néron (lft) model [8, 10.1/7], which is in SmGp/X . If $U \subset X$ is dense open, we still have $j_*G \in \text{SmGp}/X$ by the arguments in [8, 10.1/9]. In this case, j_*G is the open subgroup scheme of the Néron model of $G \times_X K$ along $\text{Spec } K \rightarrow X$ with connected fibers over U . We still call j_*G the Néron model of G (along $j : U \hookrightarrow X$).

PROPOSITION 2.6. *Let $0 \rightarrow H \rightarrow G \rightarrow Y \rightarrow 0$ be an extension of group schemes over U such that H is an extension of an abelian scheme by a torus and Y is a lattice. Then $j_*G \in \text{SmGp}/X$.*

Proof. We have an exact sequence $0 \rightarrow j_*H \rightarrow j_*G \rightarrow j_*Y \rightarrow R^1j_*H$ in $\text{Ab}(X_{\text{sm}})$. As above, we have $j_*H \in \text{SmGp}/X$. Also $j_*Y \in \text{EtGp}/X$ by Prop. 2.3. By Prop. 2.4, we know that the kernel of $j_*Y \rightarrow R^1j_*H$ is in EtGp/X . Therefore $j_*G \in \text{SmGp}/X$ by Prop. 2.5. \square

Let \mathcal{M}_U be the category of smooth 1-motives over U in the sense of Deligne [12, (10.1.10)]. An object $M = [Y \rightarrow G]$ of \mathcal{M}_U is a complex consisting of a

lattice Y placed in degree -1 , an extension G of an abelian scheme by a torus placed in degree 0 and a morphism $Y \rightarrow G$ of group schemes over U . A morphism in \mathcal{M}_U is a morphism of complexes of group schemes over U . In particular, \mathcal{M}_U is a full subcategory of $\text{Ch}^b(\text{SmGp}/U)$.

For $M = [Y \rightarrow G] \in \mathcal{M}_U$ and a finite étale covering V of U , we denote the Weil restriction $\text{Res}_{V/U}(Y \times_U V)$ of $Y \times_U V$ by $Y_{(V)}$. For references on Weil restrictions, see [8, 7.6] and [11, A.5]. We have a natural injective morphism $Y \hookrightarrow Y_{(V)}$. We set $Y_{(/V)} = Y_{(V)}/Y$. The objects $Y_{(V)}$ and $Y_{(/V)}$ are lattices over U . We denote the cokernel of the diagonal embedding $Y \hookrightarrow Y_{(V)} \oplus G$ in $\text{Ab}(U_{\text{sm}})$ by $G_{(V)}$ (which depends on not only V and G but the whole M despite of the notation). We have a morphism between exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & Y & \longrightarrow & Y_{(V)} & \longrightarrow & Y_{(/V)} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & G & \longrightarrow & G_{(V)} & \longrightarrow & Y_{(/V)} & \longrightarrow & 0 \end{array}$$

in $\text{Ab}(U_{\text{sm}})$. In particular, we have $G_{(V)} \in \text{SmGp}/U$ by Prop. 2.5. Define

$$M_{(V)} = [Y_{(V)} \rightarrow G_{(V)}] \in \text{Ch}^b(\text{SmGp}/U).$$

Then the natural morphism $M \rightarrow M_{(V)}$ is a quasi-isomorphism in $\text{Ch}^b(U_{\text{sm}})$. For a morphism $W \rightarrow V$ of finite étale coverings of U , we have natural morphisms $Y_{(V)} \rightarrow Y_{(W)}$ and $G_{(V)} \rightarrow G_{(W)}$ by functoriality of the Weil restriction and hence a morphism $M_{(V)} \rightarrow M_{(W)}$.

DEFINITION 2.7. Let $M = [Y \rightarrow G] \in \mathcal{M}_U$. A *good covering* of U with respect to M (or Y) and X is a finite étale covering V of U such that $Y \times_U V$ extends to a lattice over the normalization of X in V .

(We do not introduce a piece of notation for the above mentioned extension of $Y \times_U V$ as we do not have to.) The key properties of good coverings are that any finite étale covering that factors through a good covering is good and that the following holds.

PROPOSITION 2.8. *For any lattice Y over U , a good covering exists. If V is a good covering of U with respect to Y and X , then we have $R^1j_* Y_{(V)} = 0$.*

Proof. A lattice can be trivialized by a finite étale covering by [14, X, Prop. 5.11, Thm. 5.16]. Such a covering is good. Let V be a good covering of U with respect to Y . Let \bar{V} be the normalization of X in V . Note that the Weil restriction functor $\text{Ab}(V_{\text{sm}}) \rightarrow \text{Ab}(U_{\text{sm}})$ is nothing but the pushforward functor for the finite étale morphism $V \rightarrow U$. Since the pushforward functor for a finite morphism is exact in the étale topology ([28, II, Cor. 3.6]), we know that the Weil restriction functor $\text{Ab}(V_{\text{sm}}) \rightarrow \text{Ab}(U_{\text{sm}})$ is exact (see also [11, A.5.4]). Hence the sheaf $R^1j_* Y_{(V)} \in \text{Ab}(X_{\text{sm}})$ is the étale sheafification of the presheaf that sends

a smooth X -scheme X' to $H^1(X' \times_X V, Y)$. Hence it is enough to assume that X is strict henselian local and U is the generic point of X , and show that $H^1(X' \times_X V, Y) = 0$ for the strict henselization X' of any smooth X -scheme at any point. By goodness, Y extends to a lattice over the strict henselian scheme $X' \times_X \bar{V}$. Hence Y becomes trivial over $X' \times_X V$. Since $X' \times_X V$ is regular, we have $H^1(X' \times_X V, \mathbf{Z}) = 0$ and so $H^1(X' \times_X V, Y) = 0$. \square

DEFINITION 2.9. Let $M = [Y \rightarrow G] \in \mathcal{M}_U$ and V a good covering of U with respect to M and X . We define

$$\mathcal{N}(M, V) = j_*M_{(V)} = [j_*Y_{(V)} \rightarrow j_*G_{(V)}],$$

which is an object of $\text{Ch}^b(\text{SmGp}/X)$ by Prop. 2.6. We call $\mathcal{N}(M, V)$ the *Néron model of M with respect to V* (along $j : U \rightarrow X$). The assignment $V \mapsto \mathcal{N}(M, V)$ is contravariantly functorial.

PROPOSITION 2.10. Let $M = [Y \rightarrow G] \in \mathcal{M}_U$ and V a good covering of U with respect to M and X . Consider the morphisms

$$\mathcal{N}(M, V) \rightarrow Rj_*M_{(V)} \cong Rj_*M$$

in $D^b(X_{\text{sm}})$. Then the induced morphism

$$\mathcal{N}(M, V) \rightarrow \tau_{\leq 0}Rj_*M$$

is an isomorphism in $D^b(X_{\text{sm}})$.

Proof. By Prop. 2.8, we have an exact sequence

$$0 \rightarrow H^{-1}Rj^*M \rightarrow j_*Y_{(V)} \rightarrow j_*G_{(V)} \rightarrow H^0Rj^*M \rightarrow 0.$$

This proves the proposition. \square

The inclusion functor $\text{SmGp}'/X \hookrightarrow \text{Ab}(X_{\text{sm}})$ induces a triangulated functor $K^b(\text{SmGp}'/X) \rightarrow K^b(X_{\text{sm}})$.

DEFINITION 2.11. We say that a morphism in $K^b(\text{SmGp}/X)$, $K^b(\text{SmGp}'/X)$, $K^b(\text{EtGp}/X)$ or $K^b(\text{EtGp}'/X)$ is a *quasi-isomorphism* if it is so in $K^b(X_{\text{sm}})$. We define $D^b(\text{SmGp}/X)$ to be the localization of $K^b(\text{SmGp}/X)$ by the null system of objects quasi-isomorphic to zero ([24, Thm. 10.2.3]). We define $D^b(\text{SmGp}'/X)$, $D^b(\text{EtGp}/X)$ and $D^b(\text{EtGp}'/X)$ similarly.

A priori, $D^b(\text{SmGp}/X)$ (and $D^b(\text{SmGp}'/X)$) might not be a locally small category and might be as large as $D^b(X_{\text{sm}})$. Smooth group schemes over X with connected fibers are quasi-compact ([13, Exp. VI B, Cor. 3.6]) and hence form a small set. Therefore the problem is the cardinalities of the component groups of the fibers. If one wants to prove the local smallness of $D^b(\text{SmGp}/X)$ using

[24, Rmk. 7.1.14], it suffices to show that for any quasi-isomorphism $F' \rightarrow F$ in $K^b(\text{SmGp}/X)$, there exists a quasi-isomorphism $F'' \rightarrow F'$ in $K^b(\text{SmGp}/X)$ such that the cardinalities of the component groups of the fibers of the terms of F'' are no bigger than those of F . While this seems likely verified by a limit argument on component groups, we content ourselves with not necessarily locally small categories. But notice that the relative component group $G/G_0 = \pi_0(G)$ (the quotient by the relative identity component G_0 ; [13, Exp. VI B, Cor. 3.5], [32, (3.2), (d)]; see also the first paragraph of the proof of Prop. 3.4 below) of any smooth group scheme and group algebraic space G in this paper are all \mathbf{Z} -constructible ([29, II, §0, “Constructible sheaves”]) over X . Hence one may instead use the full subcategory of SmGp/X and SmGp'/X consisting of objects with \mathbf{Z} -constructible relative component groups. This full subcategory is small, and hence any localization of its bounded homotopy category is small.

PROPOSITION 2.12. *The natural functors induce a commutative diagram of triangulated functors*

$$\begin{CD} D^b(\text{EtGp}/X) @>>> D^b(\text{EtGp}'/X) @>>> D^b(X_{\text{et}}) \\ @VVV @VVV @VVV \\ D^b(\text{SmGp}/X) @>>> D^b(\text{SmGp}'/X) @>>> D^b(X_{\text{sm}}). \end{CD}$$

Proof. This follows from [24, Thm. 10.2.3]. □

PROPOSITION 2.13. *Let $0 \rightarrow F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow 0$ be a term-wise exact sequence in $\text{Ch}^b(X_{\text{sm}})$ with $F_i \in \text{Ch}^b(\text{SmGp}/X)$ for any i . Then there exists a canonical morphism $F_3 \rightarrow F_1[1]$ in $D^b(\text{SmGp}/X)$ such that the triangle $F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow F_1[1]$ is distinguished in $D^b(\text{SmGp}/X)$ and maps to the canonical distinguished triangle $F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow F_1[1]$ in $D^b(X_{\text{sm}})$. Similar statements hold for $D^b(\text{SmGp}'/X)$, $D^b(\text{EtGp}/X)$ and $D^b(\text{EtGp}'/X)$.*

Proof. Set $F'_3 = [F_1 \rightarrow F_2]$. We have $F'_3 = F_2 \oplus F_1[1]$ as a graded object forgetting the differentials. The first projection $F_2 \oplus F_1[1] \rightarrow F_2$ followed by the morphism $F_2 \rightarrow F_3$ gives a morphism $F'_3 \rightarrow F_3$ in $\text{Ch}^b(\text{SmGp}/X)$ (i.e. commutative with the differentials). For any n , the diagram with exact rows

$$\begin{CD} H^n F_1 @>>> H^n F_2 @>>> H^n F'_3 @>>> H^{n+1} F_1 @>>> H^{n+1} F_2 \\ @| @| @VVV @| @| \\ H^n F_1 @>>> H^n F_2 @>>> H^n F_3 @>>> H^{n+1} F_1 @>>> H^{n+1} F_2 \end{CD}$$

in $\text{Ab}(X_{\text{sm}})$ is commutative. Hence $F'_3 \rightarrow F_3$ is a quasi-isomorphism. The required morphism is given by the composite $F_3 \xleftarrow{\sim} F'_3 \rightarrow F_1[1]$ in $D^b(\text{SmGp}/X)$. □

PROPOSITION 2.14. For any object $M = [Y \rightarrow G] \in \mathcal{M}_U$, choose a good covering V of U with respect to M and X and consider the object

$$\mathcal{N}(M, V) \in D^b(\mathrm{SmGp}/X).$$

For any morphism $M = [Y \rightarrow G] \rightarrow M' = [Y' \rightarrow G'] \in \mathcal{M}_U$, choose a good covering V of U with respect to both M and M' and X and consider the morphism

$$\mathcal{N}(M, V) \rightarrow \mathcal{N}(M', V) \in D^b(\mathrm{SmGp}/X).$$

These assignments define a well-defined additive functor $\mathcal{M}_U \rightarrow D^b(\mathrm{SmGp}/X)$.

Proof. Let $M = [Y \rightarrow G] \in \mathcal{M}_U$ and V a good covering of U with respect to M and X . If $W \rightarrow V$ is a morphism from another finite étale covering W of U , then the induced morphism $\mathcal{N}(M, V) \rightarrow \mathcal{N}(M, W)$ is a quasi-isomorphism by Prop. 2.10. Let $f, g : W \rightrightarrows V$ be two U -morphisms. Then the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & Y_{(V)} & \longrightarrow & Y_{(V)} & \longrightarrow & 0 \\ & & \downarrow 0 & & \downarrow f-g & & \downarrow f-g & & \\ 0 & \longrightarrow & Y & \longrightarrow & Y_{(W)} & \longrightarrow & Y_{(W)} & \longrightarrow & 0 \end{array}$$

is commutative. Hence the morphism $f - g : Y_{(V)} \rightarrow Y_{(W)}$ factors through the quotient $Y_{(V)}$. The composite $G_{(V)} \rightarrow Y_{(V)} \rightarrow Y_{(W)}$ as a diagonal arrow in the commutative diagram

$$\begin{array}{ccc} Y_{(V)} & \longrightarrow & G_{(V)} \\ \downarrow f-g & & f-g \downarrow \\ Y_{(W)} & \longrightarrow & G_{(W)} \end{array}$$

from the right upper term to the left lower term splits the diagram into two commutative triangles. This means that the two morphisms $f, g : M_{(V)} \rightrightarrows M_{(W)}$ are homotopic to each other. Applying j_* term-wise, we know that the two morphisms $f, g : \mathcal{N}(M, V) \rightrightarrows \mathcal{N}(M, W)$ are also homotopic to each other. Therefore $\mathcal{N}(M, V) \in D^b(\mathrm{SmGp}/X)$ is independent of the choice of V . The rest is an easy consequence of this. \square

DEFINITION 2.15. We denote the functor $\mathcal{M}_U \rightarrow D^b(\mathrm{SmGp}/X)$ defined in Prop. 2.14 by \mathcal{N} . Hence $\mathcal{N}(M) = \mathcal{N}(M, V)$ in $D^b(\mathrm{SmGp}/X)$ for $M \in \mathcal{M}_U$ and a good covering V of U with respect to M and X . We call $\mathcal{N}(M)$ the *Néron model* of M (along $j : U \rightarrow X$).

PROPOSITION 2.16. For any $M \in \mathcal{M}_U$, the image of $\mathcal{N}(M)$ under the functor $D^b(\mathrm{SmGp}/X) \rightarrow D^b(X_{\mathrm{sm}})$ is canonically identified with $\tau_{\leq 0} \mathrm{R}j_* M$.

Proof. This follows from Prop. 2.10. \square

Prop. 2.14 and 2.16 together finish the proof of Thm. A. Next we consider connected Néron models.

PROPOSITION 2.17. *Let $Y \in \text{EtGp}/U$. Identify U with its image in Y as the zero section, which is an open and closed subset of Y . Then the scheme $(Y \setminus U) \sqcup X$ has a unique X -group scheme structure with zero section X compatible with the U -group scheme structure of Y . It is separated over X , and moreover étale over X if $U \subset X$ is open.*

Proof. Obvious. □

DEFINITION 2.18. For $Y \in \text{EtGp}/U$, we denote the group scheme $(Y \setminus U) \sqcup X$ over X in Prop. 2.17 by \mathcal{Y}_0 and call it the *extension by zero* of Y to X .

PROPOSITION 2.19. *Let $Y \in \text{EtGp}/U$ with extension by zero \mathcal{Y}_0 over X . Let G be a group scheme over X . Then the natural homomorphism*

$$\text{Hom}_X(\mathcal{Y}_0, G) \rightarrow \text{Hom}_U(Y, G \times_X U)$$

is an isomorphism. If $U \subset X$ is open, then we have $\mathcal{Y}_0 = j_! Y$, where $j_! : \text{Ab}(U_{\text{sm}}) \rightarrow \text{Ab}(X_{\text{sm}})$ is the left adjoint of j^ .*

Proof. Obvious. □

DEFINITION 2.20. Let $M = [Y \rightarrow G] \in \mathcal{M}_U$. Let \mathcal{Y}_0/X be the extension by zero of Y and \mathcal{G}_0/X be the maximal open subgroup scheme of the Néron model \mathcal{G} of G with connected fibers. We define

$$\mathcal{N}_0(M) = [\mathcal{Y}_0 \rightarrow \mathcal{G}_0],$$

which is a complex of (not necessarily locally finite type) group schemes over X . The assignment $M \mapsto \mathcal{N}_0(M)$ is an additive functor. We call $\mathcal{N}_0(M)$ the *connected Néron model* of M (along $j : U \rightarrow X$).

We have $\mathcal{N}_0(M) \in \text{Ch}^b(\text{SmGp}/X)$ if $U \subset X$ is open.

PROPOSITION 2.21. *Let $M = [Y \rightarrow G] \in \mathcal{M}_U$. For good coverings V of U with respect to M , the natural morphisms*

$$\mathcal{N}_0(M) \rightarrow \mathcal{N}(M, V)$$

of complexes of group schemes over X are contravariantly functorial in V . In particular, if $U \subset X$ is open, then they induce a canonical morphism

$$\mathcal{N}_0(M) \rightarrow \mathcal{N}(M) \text{ in } D^b(\text{SmGp}/X).$$

Proof. Obvious. □

In the next two propositions, consider the following situation:

Situation 2.22.

$$\begin{array}{ccc} U' & \xrightarrow{\quad} & X' \\ \downarrow & \scriptstyle j' & \downarrow \\ U & \xrightarrow{\quad j \quad} & X. \end{array}$$

is a cartesian diagram of schemes such that the both horizontal morphisms are as in Situation 2.2.

In this situation, we say that the formation of Néron models commutes with the base change X'/X if the natural morphism

$$\mathcal{N}(M, V) \times_X X' \rightarrow \mathcal{N}(M \times_U U', V \times_U U')$$

in $\text{Ch}^b(\text{SmGp}/X')$ is an isomorphism for any $M \in \mathcal{M}_U$ and any good covering V of U with respect to M and X . In this case, the natural morphism

$$\mathcal{N}_0(M) \times_X X' \rightarrow \mathcal{N}_0(M \times_U U')$$

of complexes of group schemes over X' is an isomorphism since the base change $(\cdot) \times_X X'$ preserves the maximal subgroup scheme with connected fibers. The natural morphism

$$\mathcal{N}(M) \times_X X' \rightarrow \mathcal{N}(M \times_U U')$$

in $D^b(\text{SmGp}/X')$ is also an isomorphism.

PROPOSITION 2.23. *The formation of Néron models commutes with the base change X'/X if $X' \rightarrow X$ is a regular morphism. This happens, in particular, if $X' \rightarrow X$ is an étale morphism, the localization of X at a closed point, or the (strict) henselization of local X .*

Proof. The statement holds if $X' \rightarrow X$ is an open immersion. The statement is Zariski local on X and X' . Hence we may assume that both X and X' are affine.

By the structure of $\mathcal{N}(M, V)$, it is enough to show that $(j_*T) \times_X X' \xrightarrow{\sim} j'_*(T \times_U U')$ for any smooth U -scheme T such that j_*T (resp. $j'_*(T \times_U U')$) is representable by a smooth X -scheme (resp. smooth X' -scheme). Since $X' \rightarrow X$ is a regular morphism between noetherian affine schemes, we know by Popescu's theorem [41, Thm. 1.1] that X' can be written as a filtered inverse limit $\varinjlim X'_\lambda$ of smooth affine X -schemes. Let X'' be a smooth affine X' -scheme. Then there exist an index λ_0 and a smooth affine X'_{λ_0} -scheme X''_{λ_0} such that $X'' \cong X' \times_{X'_{\lambda_0}} X''_{\lambda_0}$. Set $X''_\lambda = X'_\lambda \times_{X'_{\lambda_0}} X''_{\lambda_0}$ for $\lambda \geq \lambda_0$. Then

$$\begin{aligned} \Gamma(X'', (j_*T) \times_X X') &= \varinjlim_{\lambda \geq \lambda_0} \Gamma(X''_\lambda \times_X U, T) \\ &= \Gamma(X'' \times_X U, T) = \Gamma(X'', j'_*(T \times_U U')). \end{aligned}$$

Hence $(j_*T) \times_X X' \xrightarrow{\sim} j'_*(T \times_U U')$. □

PROPOSITION 2.24. *The formation of Néron models commutes with the base change X'/X if $X = \text{Spec } A$ is local and $X' = \text{Spec } \hat{A}$ its completion.*

Proof. We may assume that X is strictly henselian (local!) and $U = \text{Spec } K$ its generic point. Write $U' = \text{Spec } K'$. We denote the base change $(\cdot) \times_X X'$ of X -schemes by $(\cdot)'$. Let V be a good covering of U with respect to M and X . The proof of Prop. 2.3 shows that $j_* Y$ is the maximal open subscheme of the normalization of X in Y étale over X . This description shows that $(j_* Y)' \xrightarrow{\sim} j'_* Y'$. Similarly, we have $(j_* Y_{(V)})' \xrightarrow{\sim} j'_* Y'_{(V')}$ and $(j_* Y_{(V)})' \xrightarrow{\sim} j'_* Y'_{(V')}$. Also, the formation of Néron (lft) model of semi-abelian varieties commutes with completion by [8, 10.1/3]. Thus $(j_* G)' \xrightarrow{\sim} j'_* G'$. We have exact sequences

$$\begin{aligned} 0 \rightarrow j_* G \rightarrow j_* G_{(V)} \rightarrow j_* Y_{(V)} \rightarrow R^1 j_* G, \\ 0 \rightarrow j'_* G' \rightarrow j'_* G'_{(V')} \rightarrow j'_* Y'_{(V')} \rightarrow R^1 j'_* G' \end{aligned}$$

in $\text{Ab}(X_{\text{sm}})$, $\text{Ab}(X'_{\text{sm}})$, respectively, and a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & (j_* G)' & \longrightarrow & (j_* G_{(V)})' & \longrightarrow & (j_* Y_{(V)})' \\ & & \downarrow \wr & & \downarrow & & \downarrow \wr \\ 0 & \longrightarrow & j'_* G' & \longrightarrow & j'_* G'_{(V')} & \longrightarrow & j'_* Y'_{(V')} \end{array}$$

in SmGp/X' . We want to show that the second vertical morphism is an isomorphism. Let $C \in \text{EtGp}/X$ be the kernel of $j_* Y_{(V)} \rightarrow R^1 j_* G$ and $D \in \text{EtGp}/X'$ the kernel of $j'_* Y'_{(V')} \rightarrow R^1 j'_* G'$. The above diagram induces an injective morphism $C' \hookrightarrow D$ in EtGp/X' , which is an isomorphism after j'^* . It is enough to show that $C' \xrightarrow{\sim} D$, for which it is enough to show that $\Gamma(X', C') \xrightarrow{\sim} \Gamma(X', D)$. We have $\Gamma(X', C') = \Gamma(X, C)$ since $C \in \text{EtGp}/X$. We have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma(X, C) & \longrightarrow & \Gamma(U, Y_{(V)}) & \longrightarrow & H^1(U, G) \\ & & \downarrow & & \downarrow \wr & & \downarrow \\ 0 & \longrightarrow & \Gamma(X', D) & \longrightarrow & \Gamma(U', Y'_{(V')}) & \longrightarrow & H^1(U', G'). \end{array}$$

Since G is smooth and X henselian local, the right vertical homomorphism $H^1(U, G) \rightarrow H^1(U', G')$ (or $H^1(K, G) \rightarrow H^1(K', G')$) is an isomorphism by [16, Prop. 3.5.3 (2)]. Thus $\Gamma(X, C) \xrightarrow{\sim} \Gamma(X', D)$. Hence $(j_* G_{(V)})' \xrightarrow{\sim} j'_* G'_{(V')}$. Therefore $\mathcal{N}(M, V)' \xrightarrow{\sim} \mathcal{N}(M', V')$. \square

There are slightly more flexible representatives of $\mathcal{N}(M)$ in $\text{Ch}^b(\text{SmGp}/U)$ than $\mathcal{N}(M, V)$.

PROPOSITION 2.25. *Let $M = [Y \rightarrow G] \in \mathcal{M}_U$. Let $0 \rightarrow Y \rightarrow Y' \rightarrow Y'' \rightarrow 0$ be an exact sequence of lattices over U such that $R^1 j_* Y' = 0$. Denote the cokernel*

of the diagonal embedding $Y \hookrightarrow Y' \oplus G$ by G' . Then there exists a canonical isomorphism $\mathcal{N}(M) \cong [j_* Y' \rightarrow j_* G']$ in $D^b(\mathrm{SmGp}/X)$.

Proof. Let V be a finite étale covering of U such that Y' (and hence Y and Y'') is trivial over V . Choose a retraction (left-inverse) $Y' \times_U V \twoheadrightarrow Y \times_U V$ to the inclusion $Y \times_U V \hookrightarrow Y' \times_U V$. Such a retraction exists since the cokernel $Y'' \times_U V$ is a lattice. This retraction corresponds to a morphism $Y' \rightarrow Y_{(V)}$ such that the composite $Y \rightarrow Y' \rightarrow Y_{(V)}$ is the natural inclusion. Hence we have a morphism $[Y' \rightarrow G'] \rightarrow [Y_{(V)} \rightarrow G_{(V)}]$ in $\mathrm{Ch}^b(\mathrm{SmGp}/U)$. We have $[j_* Y' \rightarrow j_* G'] \cong \tau_{\leq 0} Rj_* M$ in $D^b(X_{\mathrm{sm}})$ since $R^1 j_* Y' = 0$. Hence $[j_* Y' \rightarrow j_* G'] \rightarrow [j_* Y_{(V)} \rightarrow j_* G_{(V)}]$ in $\mathrm{Ch}^b(\mathrm{SmGp}/X)$ is a quasi-isomorphism. Thus $[j_* Y' \rightarrow j_* G'] \xrightarrow{\sim} \mathcal{N}(M)$ in $D^b(\mathrm{SmGp}/X)$. A different choice of a retraction $Y' \times_U V \twoheadrightarrow Y \times_U V$ gives a morphism $[Y' \rightarrow G'] \rightarrow [Y_{(V)} \rightarrow G_{(V)}]$ homotopic to the previous one by the same argument as the proof of Prop. 2.14. Hence the isomorphism $[j_* Y' \rightarrow j_* G'] \xrightarrow{\sim} \mathcal{N}(M)$ in $D^b(\mathrm{SmGp}/X)$ is canonical. \square

Here is a simple example where the Néron model of a 1-motive arises geometrically from a relative curve over X with an étale local section over U .

PROPOSITION 2.26. *Assume that X is excellent and the residue fields of $X \setminus U$ are perfect. Let $S \rightarrow X$ be a proper flat morphism with 1-dimensional geometrically connected fibers from a regular scheme S such that $S_U = S \times_X U \rightarrow U$ is smooth. Let $T \rightarrow X$ be a finite flat morphism from a regular scheme T such that $T_U = T \times_X U \rightarrow U$ is étale. Let $s : T \rightarrow S$ be an X -morphism.*

Denote by Y the kernel of the norm map $\mathrm{Res}_{T_U/U} \mathbf{Z} \twoheadrightarrow \mathbf{Z}$ and by A the relative Jacobian $\mathrm{Pic}_{S_U/U}^0$. Let $\mathrm{Res}_{T_U/U} \mathbf{Z} \rightarrow \mathrm{Pic}_{S_U/U}$ be the morphism in SmGp/U induced by the restriction $T_U \rightarrow S_U$ of s and $Y \rightarrow A$ its restriction. Set $M = [Y \rightarrow A] \in \mathcal{M}_U$. Denote by $(\mathrm{Pic}_{S/X})_{\mathrm{sep}}$ the maximal separated quotient of $\mathrm{Pic}_{S/X}$ ([32, (8.0.1)]).

Then the morphism $\mathrm{Res}_{T_U/U} \mathbf{Z} \rightarrow \mathrm{Pic}_{S_U/U}$ in SmGp/U uniquely extends to a morphism $\mathrm{Res}_{T/X} \mathbf{Z} \rightarrow (\mathrm{Pic}_{S/X})_{\mathrm{sep}}$ in SmGp/X , and we have a canonical isomorphism

$$\mathcal{N}(M) \cong [\mathrm{Res}_{T/X} \mathbf{Z} \rightarrow (\mathrm{Pic}_{S/X})_{\mathrm{sep}}]$$

in $D^b(\mathrm{SmGp}/X)$.

Proof. We have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & Y & \longrightarrow & \mathrm{Res}_{T_U/U} \mathbf{Z} & \longrightarrow & \mathbf{Z} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & A & \longrightarrow & \mathrm{Pic}_{S_U/U} & \xrightarrow{\mathrm{deg}} & \mathbf{Z} \longrightarrow 0 \end{array}$$

in SmGp/U . Hence the cokernel of the diagonal embedding $Y \hookrightarrow \mathrm{Res}_{T_U/U} \mathbf{Z} \oplus A$ is $\mathrm{Pic}_{S_U/U}$. We have $R^1 j_* \mathrm{Res}_{T_U/U} \mathbf{Z} = 0$ by the same argument as the proof

of Prop. 2.8. Hence we have a canonical isomorphism

$$\mathcal{N}(M) \cong [j_* \operatorname{Res}_{T_U/U} \mathbf{Z} \rightarrow j_* \operatorname{Pic}_{S_U/U}]$$

in $D^b(\operatorname{SmGp}/X)$ by Prop. 2.25. We have $j_* \operatorname{Res}_{T_U/U} \mathbf{Z} \cong \operatorname{Res}_{T/X} \mathbf{Z}$ since $\operatorname{Res}_{T/X} \mathbf{Z}$ is an étale X -scheme and the pushforward of \mathbf{Z} by the morphism $T_{U,\text{ét}} \hookrightarrow T_{\text{ét}}$ is \mathbf{Z} . By the assumption on the residue fields of $X \setminus U$ and [18, Eq. (4.10 bis)], we know that the natural morphism $\operatorname{Pic}_{S/X} \rightarrow j_* \operatorname{Pic}_{S_U/U}$ is surjective in $\operatorname{Ab}(X_{\text{ét}})$ whose kernel is a skyscraper étale sheaf. Hence $(\operatorname{Pic}_{S/X})_{\text{sep}} \xrightarrow{\sim} j_* \operatorname{Pic}_{S_U/U}$ in SmGp/X . This proves the proposition. \square

The assumption “the residue fields of $X \setminus U$ are perfect” is only used to ensure that $\operatorname{Pic}_{S/X} \rightarrow j_* \operatorname{Pic}_{S_U/U}$ is surjective in $\operatorname{Ab}(X_{\text{ét}})$. This latter condition is satisfied also if $S \times_X \mathcal{O}_{X,x}^{\text{sh}} \rightarrow \operatorname{Spec} \mathcal{O}_{X,x}^{\text{sh}}$ has a section (or slightly weaker, has index 1) for any $x \in X \setminus U$, where $\mathcal{O}_{X,x}^{\text{sh}}$ is the strict henselian local ring at x , as stated before [18, Eq. (4.13)].

3. Component complexes

Recall from [40, §2.4] that a premorphism of sites $f : S' \rightarrow S$ between sites defined by pretopologies is a functor f^{-1} from the underlying category of S to the underlying category of S' sending covering families to covering families such that $f^{-1}(T_2 \times_{T_1} T_3) = f^{-1}T_2 \times_{f^{-1}T_1} f^{-1}T_3$ whenever $T_2 \rightarrow T_1$ appears in a covering family. Such a functor f^{-1} is called a morphism of topologies from S to S' in [1, Def. 2.4.2]. By [38, Lem. 3.7.2], the pullback $f^* : \operatorname{Ab}(S) \rightarrow \operatorname{Ab}(S')$ admits a left derived functor $Lf^* : D(S) \rightarrow D(S')$, which is left adjoint to $Rf_* : D(S') \rightarrow D(S)$. Statement (1) in the following proposition is already used in the proof of [40, Lem. 3.2.6].

PROPOSITION 3.1. *Let $f : S' \rightarrow S$ be a premorphism of sites defined by pretopologies. Assume that the underlying category of S has finite products and the underlying functor f^{-1} of f commutes with these products.*

- (1) *There exists a canonical isomorphism*

$$Rf_* R \mathbf{Hom}_{S'}(Lf^* F, F') \cong R \mathbf{Hom}_S(F, Rf_* F')$$

in $D(S)$ functorial in $F \in D(S)$ and $F' \in D(S')$.

- (2) *Denote the sheafification functor for S or S' by $(\cdot)^\sim$. Let F be a bounded above complex of representable presheaves of abelian groups on S . Then we have $Lf^* \tilde{F} = f^* \tilde{F} = (f^{-1} F)^\sim$ in $D(S')$, where f^* and f^{-1} in the middle and right-hand sides are applied term-wise.*

Proof. First note that the pullback $f^{*\text{set}} : \operatorname{Set}(S) \rightarrow \operatorname{Set}(S')$ for sheaves of sets commutes with finite products. Indeed, for $F \in \operatorname{Set}(S)$, the sheaf $f^{*\text{set}} F$ is the sheafification of the presheaf that sends $X' \in S'$ to the direct limit of $F(X)$, where X runs through objects of S together with morphisms $X' \rightarrow f^{-1} X$ in S' .

Using this presentation and the assumption on f^{-1} , it is routine to check that the assignment $F \mapsto f^{*\text{set}}F$ commutes with finite products. (If one wants a reference, see [6, Thm. 1.5, Ex. 3.1].)

Second, if F is a complex of representable presheaves of abelian groups on S , then $f^*\tilde{F} = f^{*\text{set}}\tilde{F} = (f^{-1}F)^\sim$ in $\text{Ch}(S')$.

(1) This is [24, Thm. 18.6.9 (iii)] when f is a morphism of sites. The only part that needs exactness of $f^{*\text{set}}$ is the proof of [24, Prop. 17.6.7 (i)]. It does not need full exactness but only commutativity with finite products.

(2) We have a spectral sequence

$$E_1^{ij} = L_{-j}f^*\tilde{F}^i \Rightarrow H^{i+j}Lf^*\tilde{F},$$

where \tilde{F}^i is the i -th term of the complex \tilde{F} . Hence we may assume that F has a term only in degree zero. By the method of proof of [39, Rmk. (5.1.2)] (i.e. using Mac Lane's resolution of F), the statement to prove reduces to the statement $f^{*\text{set}}(\tilde{F}^m) = ((f^*F)^\sim)^m$ for all $m \geq 0$, where the upper scripts m denote products of m copies. This statement is true by the assumption on f^{-1} and the two remarks above. □

PROPOSITION 3.2. *Let $f : Y \rightarrow X$ be a morphism of schemes and $f : Y_{\text{sm}} \rightarrow X_{\text{sm}}$ the induced premorphism of sites. Then for any bounded above complex F of smooth group algebraic spaces over X , we have $Lf^*F = F \times_X Y$ (term-wise fiber product).*

Proof. Consider the category of smooth algebraic spaces over X with morphisms of algebraic spaces over X endowed with the étale topology. Let $X_{\text{sm}'}$ be the resulting site. The identity functor defines a morphism of sites $X_{\text{sm}'} \rightarrow X_{\text{sm}}$ inducing an equivalence on the topoi. Let $f' : Y_{\text{sm}'} \rightarrow X_{\text{sm}'}$ be the premorphism of sites induced by f . We have $Lf'^* = Lf^*$ since these functors are intrinsic to the topoi. Since the terms of F are now representable in $X_{\text{sm}'}$, Prop. 3.1 shows that $Lf^*F = f^*F = F \times_X Y$. □

In the rest of this paper, we consider the following situation:

Situation 3.3.

- X is an irreducible Dedekind scheme with function field K .
- U is a dense open subscheme of X with complement Z with reduced induced structure.
- $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ are the inclusion morphisms.
- $j : U_{\text{sm}} \rightarrow X_{\text{sm}}$ and $i : Z_{\text{sm}} \rightarrow X_{\text{sm}}$ are the premorphisms of sites induced by j and i , respectively.

The scheme Z is a finite set of closed points of X . Note that we disallow $U = \text{Spec } K$ from now on (if X has infinitely many points). Hence the connected Néron model $\mathcal{N}_0(M)$ of $M \in \mathcal{M}_U$ is in $\text{Ch}^b(\text{SmGp}/X)$. Let $\text{EtGp}^f/Z \subset \text{EtGp}/Z$ be the full subcategory of groups with finitely generated geometric fibers.

For an object P of $D^b(\text{EtGp}^f/Z)$, we denote its linear dual $R\text{Hom}_{Z_{\text{ét}}}(P, \mathbf{Z}) \in D^b(\text{EtGp}^f/Z)$ by P^{LD} . Note that if P is not constant, then $i_*P \in \text{EtGp}'/X$ is only an algebraic space and not a scheme by [32, Prop. (3.3.6.1)] (see also [25, Introduction, Example 2]).

PROPOSITION 3.4. *Let $M = [Y \rightarrow G] \in \mathcal{M}_U$ and V a good covering of U with respect to M and X . Let \mathcal{Y}_0/X be the extension by zero of Y and \mathcal{G}_0/X be the maximal open subgroup scheme of the Néron model \mathcal{G} of G with connected fibers. Define*

$$\begin{aligned} \mathcal{P}'(M, V) &= [j_*Y_{(V)}/\mathcal{Y}_0 \rightarrow j_*G_{(V)}/\mathcal{G}_0] \in \text{Ch}^b(X_{\text{sm}}), \\ \mathcal{P}(M, V) &= i^*\mathcal{P}'(M, V) \in \text{Ch}^b(Z_{\text{sm}}). \end{aligned}$$

Then we have $\mathcal{P}'(M, V) \in \text{Ch}^b(\text{EtGp}'/X)$ and $\mathcal{P}(M, V) \in \text{Ch}^b(\text{EtGp}/Z)$. We have a distinguished triangle

$$\mathcal{N}_0(M) \rightarrow \mathcal{N}(M, V) \rightarrow \mathcal{P}'(M, V)$$

in $D^b(\text{SmGp}'/X)$. The morphism $\mathcal{P}'(M, V) \rightarrow i_*\mathcal{P}(M, V)$ is a quasi-isomorphism. In particular, we have a distinguished triangle

$$\mathcal{N}_0(M) \rightarrow \mathcal{N}(M, V) \rightarrow i_*\mathcal{P}(M, V)$$

in $D^b(\text{SmGp}'/X)$. For any morphism $W \rightarrow V$ of finite étale coverings of U , the morphisms $\mathcal{P}'(M, V) \rightarrow \mathcal{P}'(M, W)$ and $\mathcal{P}(M, V) \rightarrow \mathcal{P}(M, W)$ are both quasi-isomorphisms.

Proof. We have $j_*G = \mathcal{G}$. We show that $\mathcal{G}/\mathcal{G}_0 \in \text{EtGp}'/X$. Write \mathcal{G} as a union of quasi-compact open subschemes $S_i \subset G$ (which might not be group subschemes). For each i , the connected components of fibers of $S_i \rightarrow X$ form a quasi-separated étale algebraic space $\pi_0(S_i/X)$ over X ([26, (6.8.1) (i)], [34, Thm. 2.5.2 (i)]). Since $\varinjlim_i \pi_0(S_i/X) = \mathcal{G}/\mathcal{G}_0$, we know that the sheaf of groups $\mathcal{G}/\mathcal{G}_0$ is also a quasi-separated étale algebraic space, so it is in EtGp'/X .

We have an exact sequence

$$(3.1) \quad 0 \rightarrow j_*G/\mathcal{G}_0 \rightarrow j_*G_{(V)}/\mathcal{G}_0 \rightarrow j_*Y_{(V)} \rightarrow R^1j_*G$$

in $\text{Ab}(X_{\text{sm}})$. The kernel of $j_*Y_{(V)} \rightarrow R^1j_*G$ is in EtGp/X by Prop. 2.3 and 2.4. Hence $j_*G_{(V)}/\mathcal{G}_0 \in \text{EtGp}'/X$. The same argument shows that $j_*Y_{(V)}/\mathcal{Y}_0 \in \text{EtGp}'/X$. Hence $\mathcal{P}'(M, V) \in \text{Ch}^b(\text{EtGp}'/X)$ and consequently $\mathcal{P}(M, V) \in \text{Ch}^b(\text{EtGp}/Z)$.

The sequence

$$0 \rightarrow \mathcal{N}_0(M) \rightarrow \mathcal{N}(M, V) \rightarrow \mathcal{P}'(M, V) \rightarrow 0$$

is a term-wise exact sequence of complexes in $\text{Ab}(X_{\text{sm}})$. Hence by Prop. 2.13, it defines a distinguished triangle in $D^b(\text{SmGp}'/X)$.

We have a distinguished triangle

$$j_!j^*\mathcal{P}'(M, V) \rightarrow \mathcal{P}'(M, V) \rightarrow i_*i^*\mathcal{P}'(M, V)$$

in $D^b(X_{\text{et}})$ and hence in $D^b(X_{\text{sm}})$. We have

$$j^* \mathcal{P}'(M, V) = [Y_{(V)}/Y \rightarrow G_{(V)}/G] = 0$$

in $D^b(U_{\text{sm}})$. Hence $\mathcal{P}'(M, V) \rightarrow i_* i^* \mathcal{P}'(M, V)$ is a quasi-isomorphism.

We have a morphism of distinguished triangles

$$\begin{array}{ccccc} \mathcal{N}_0(M) & \longrightarrow & \mathcal{N}(M, V) & \longrightarrow & \mathcal{P}'(M, V) \\ \parallel & & \downarrow & & \downarrow \\ \mathcal{N}_0(M) & \longrightarrow & \mathcal{N}(M, W) & \longrightarrow & \mathcal{P}'(M, W). \end{array}$$

(Remember that there are hidden shifted terms $\mathcal{N}_0(M)[1]$ in the triangles and a hidden commutative square next to the right square.) The middle vertical morphism is a quasi-isomorphism by Prop. 2.10. Hence so is the right vertical one. \square

DEFINITION 3.5. For $M = [Y \rightarrow V] \in \mathcal{M}_U$ and V a good covering of U with respect to M and X , we call $\mathcal{P}(M, V) \in \text{Ch}^b(\text{EtGp}/Z)$ the *Néron component complex of M with respect to V* . It is contravariantly functorial in V .

PROPOSITION 3.6. For M and V as above, we have $\mathcal{P}(M, V) \in \text{Ch}^b(\text{EtGp}^f/Z)$. Its term in degree -1 is a lattice.

Proof. The Néron model j_*G has finitely generated groups of geometric connected components ([20, Prop. 3.5]). Hence the exact sequence (3.1) shows that $\mathcal{P}'(M, V)$ has finitely generated geometric fibers. Its degree -1 term $j_*Y_{(V)}/\mathcal{Y}_0$ becomes $i^*j_*Y_{(V)}$ after pulling back to Z , which is a lattice. \square

PROPOSITION 3.7. For any object $M = [Y \rightarrow G] \in \mathcal{M}_U$, choose a good covering V of U with respect to M and X and consider the object

$$\mathcal{P}'(M, V) \in D^b(\text{EtGp}'/X).$$

For any morphism $M = [Y \rightarrow G] \rightarrow M' = [Y' \rightarrow G'] \in \mathcal{M}_U$, choose a good covering V of U with respect to both M and M' and X and consider the morphism

$$\mathcal{P}'(M, V) \rightarrow \mathcal{P}'(M', V) \in D^b(\text{EtGp}'/X).$$

These assignments define a well-defined additive functor $\mathcal{M}_U \rightarrow D^b(\text{EtGp}'/X)$. We denote this functor as $M \mapsto \mathcal{P}'(M)$.

Proof. The same proof as Prop. 2.14 works. \square

DEFINITION 3.8. For $M \in \mathcal{M}_U$, we define

$$\mathcal{P}(M) = i^* \mathcal{P}'(M) \in D^b(\text{EtGp}/Z)$$

and call it the *Néron component complex* of M . The assignment $M \mapsto \mathcal{P}(M)$ defines an additive functor $\mathcal{M}_U \rightarrow D^b(\text{EtGp}/Z)$. We have a canonical distinguished triangle

$$\mathcal{N}_0(M) \rightarrow \mathcal{N}(M) \rightarrow i_*\mathcal{P}(M)$$

in $D^b(\text{SmGp}'/X)$ coming from Prop. 3.4.

Note that $Li^*\mathcal{P}'(M) = i^*\mathcal{P}'(M)$ in $D^b(Z_{\text{sm}})$ by Prop. 3.2. Hence the pull-back functor i^* in the above definition can be understood in either the derived or non-derived sense.

PROPOSITION 3.9. *We have $\mathcal{P}(M) \in D^b(\text{EtGp}^f/Z)$, concentrated in degrees -1 and 0 . Its H^{-1} is a lattice.*

Proof. This follows from Prop. 3.6. □

PROPOSITION 3.10. *The formation of Néron component complexes commutes with regular base change and completion. More precisely, in Situation 2.22, assume that $U \rightarrow X$ is open and $X' \rightarrow X$ is regular or the completion of a local X . Let $Z' = Z \times_X X'$. Then we have*

$$\mathcal{P}(M, V) \times_Z Z' \xrightarrow{\sim} \mathcal{P}(M \times_U U', V \times_U U')$$

in $\text{Ch}^b(\text{SmGp}/Z')$ and

$$\mathcal{P}(M) \times_Z Z' \xrightarrow{\sim} \mathcal{P}(M \times_U U')$$

in $D^b(\text{SmGp}/Z')$ for any $M \in \mathcal{M}_U$ and any good covering V of U with respect to M and X .

Proof. This follows from Prop. 2.23 and 2.24. □

4. Duals and duality pairings of Néron models

In the rest of this paper, we work in the following situation with a set of notation:

Situation 4.1.

- $j : U_{\text{sm}} \rightarrow X_{\text{sm}}$, $i : Z_{\text{sm}} \rightarrow X_{\text{sm}}$ and K are as in Situation 3.3.
- $M = [Y \rightarrow G] \in \mathcal{M}_U$ is a 1-motive, where Y is a lattice and G is an extension of an abelian scheme A by a torus T .
- Let \mathcal{T} , \mathcal{G} , \mathcal{A} be the Néron models of T , G , A , respectively, along j . Denote their open subgroup schemes with connected fibers by \mathcal{T}_0 , \mathcal{G}_0 , \mathcal{A}_0 , respectively. Denote the extension by zero of Y by \mathcal{Y}_0 .
- Fibers over Z are generally denoted by putting the subscript $(\cdot)_Z$. The notation $(\cdot)_{Z_0}$ or $(\cdot)_{0Z}$ mean the identity component of $(\cdot)_Z$.

- $M^\vee = [Y' \rightarrow G']$ is the dual of M (see below). The objects above corresponding to M^\vee are denoted by putting primes $(\cdot)'$, such as A' , \mathcal{F}'_0 , \mathcal{G}'_{0Z} , etc.

The dual 1-motive M^\vee [12, (10.2.12), (10.2.13)] of M is equipped with a canonical \mathbf{G}_m -bi-extension $M^\vee \otimes^L M \rightarrow \mathbf{G}_m[1]$ as a morphism in $D(U_{\text{fppf}})$ or $D(U_{\text{sm}})$. The induced morphism

$$(4.1) \quad M^\vee \rightarrow \tau_{\leq 0} R \mathbf{H}om_{U_{\text{fppf}}}(M, \mathbf{G}_m[1])$$

via the derived tensor-Hom adjunction ([24, Thm. 18.6.4 (vii)]) is an isomorphism in $D(U_{\text{fppf}})$ ([23, Thm. 1.11 (2)]).

PROPOSITION 4.2. *For any scheme S , let $\alpha : S_{\text{fppf}} \rightarrow S_{\text{sm}}$ be the premorphism of sites defined by the identity functor. Then for any bounded complex of smooth group algebraic spaces F over S , we have $R\alpha_*F = F$ and $L\alpha^*F = F$. Let F_1, F_2, F_3 be bounded complexes of smooth group algebraic spaces over S . If $F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow F_1[1]$ is a triangle of bounded complexes of smooth group algebraic spaces over S , then it is distinguished in $D(S_{\text{fppf}})$ if and only if it is so in $D(S_{\text{sm}})$. Morphisms $F_1 \otimes^L F_2 \rightarrow F_3$ in $D(S_{\text{fppf}})$ bijectively correspond to morphisms $F_1 \otimes^L F_2 \rightarrow F_3$ in $D(S_{\text{sm}})$.*

Proof. We have $R\alpha_*F = F$ since fppf cohomology with coefficients in smooth group algebraic spaces agrees with étale cohomology ([28, III, Rmk. 3.11 (b)]). We have $L\alpha^*F = F$ by Prop. 3.1 (2) and the proof of Prop. 3.2. These facts imply the statement about the distinguished triangle. We have

$$\begin{aligned} \text{Hom}_{D(S_{\text{fppf}})}(F_1 \otimes^L F_2, F_3) &= \text{Hom}_{D(S_{\text{fppf}})}(L\alpha^*F_1 \otimes^L L\alpha^*F_2, F_3) \\ &= \text{Hom}_{D(S_{\text{sm}})}(F_1 \otimes^L F_2, R\alpha_*F_3) \\ &= \text{Hom}_{D(S_{\text{sm}})}(F_1 \otimes^L F_2, F_3). \end{aligned}$$

This shows the last statement. □

PROPOSITION 4.3. *The isomorphism (4.1) induces an isomorphism*

$$M^\vee \xrightarrow{\sim} \tau_{\leq 0} R \mathbf{H}om_{U_{\text{sm}}}(M, \mathbf{G}_m[1])$$

in $D(U_{\text{sm}})$.

Proof. Let $\alpha : U_{\text{fppf}} \rightarrow U_{\text{sm}}$ be the premorphism of sites defined by the identity functor. We apply $\tau_{\leq 0} R\alpha_*$ to the mentioned isomorphism. By Prop. 4.2, we have $\tau_{\leq 0} R\alpha_* M^\vee = \tau_{\leq 0} M^\vee = M^\vee$ and $L\alpha^*M = M$. Also $\tau_{\leq 0} R\alpha_* \tau_{\leq 0} = \tau_{\leq 0} R\alpha_*$ since $\tau_{\leq 0} R\alpha_* \tau_{\geq 1} = 0$. Hence

$$\tau_{\leq 0} R\alpha_* \tau_{\leq 0} R \mathbf{H}om_{U_{\text{fppf}}}(M, \mathbf{G}_m[1]) = \tau_{\leq 0} R \mathbf{H}om_{U_{\text{sm}}}(M, \mathbf{G}_m[1])$$

using Prop. 3.1. This proves the proposition. □

DEFINITION 4.4. We define

$$\begin{aligned} \mathcal{N}(M)^\vee &= \tau_{\leq 0} R \mathbf{Hom}_{X_{\text{sm}}}(\mathcal{N}(M), \mathbf{G}_m[1]) \in D(X_{\text{sm}}), \\ \mathcal{N}_0(M)^\vee &= \tau_{\leq 0} R \mathbf{Hom}_{X_{\text{sm}}}(\mathcal{N}_0(M), \mathbf{G}_m[1]) \in D(X_{\text{sm}}). \end{aligned}$$

Let \mathcal{G}_m/X be the Néron model of \mathbf{G}_m/U . Recall from [29, III, proof of Lem. C.10] that $R^1 j_* \mathbf{G}_m = 0$. Hence we have $\tau_{\leq 0}(Rj_* \mathbf{G}_m[1]) = \mathcal{G}_m[1]$. We have a canonical exact sequence

$$0 \rightarrow \mathbf{G}_m \rightarrow \mathcal{G}_m \rightarrow i_* \mathbf{Z} \rightarrow 0$$

in SmGp/X .

PROPOSITION 4.5. Consider the morphisms

$$Rj_* M^\vee \otimes^L Rj_* M \rightarrow Rj_*(M^\vee \otimes^L M) \rightarrow Rj_* \mathbf{G}_m[1]$$

and the induced morphism

$$\mathcal{N}(M^\vee) \otimes^L \mathcal{N}(M) \rightarrow \mathcal{G}_m[1]$$

in $D(X_{\text{sm}})$. The two induced morphisms

$$\begin{aligned} \mathcal{N}(M^\vee) &\rightarrow \tau_{\leq 0} R \mathbf{Hom}_{X_{\text{sm}}}(\mathcal{N}(M), \mathcal{G}_m[1]) \\ &\rightarrow \tau_{\leq 0} R \mathbf{Hom}_{X_{\text{sm}}}(\mathcal{N}_0(M), \mathcal{G}_m[1]) \end{aligned}$$

are both isomorphisms in $D(X_{\text{sm}})$.

Proof. Applying $\tau_{\leq 0} Rj_*$ to the isomorphism in Prop. 4.3 and using $\tau_{\leq 0} Rj_* \tau_{\leq 0} = \tau_{\leq 0} Rj_*$, we have

$$\mathcal{N}(M^\vee) \xrightarrow{\sim} \tau_{\leq 0} Rj_* R \mathbf{Hom}_{U_{\text{sm}}}(M, \mathbf{G}_m[1]).$$

We have $j^* \mathcal{N}(M) = j^* \mathcal{N}_0(M) = M$. Hence by Prop. 3.1, we have

$$\begin{aligned} \mathcal{N}(M^\vee) &\xrightarrow{\sim} \tau_{\leq 0} R \mathbf{Hom}_{X_{\text{sm}}}(\mathcal{N}(M), Rj_* \mathbf{G}_m[1]) \\ &\xrightarrow{\sim} \tau_{\leq 0} R \mathbf{Hom}_{X_{\text{sm}}}(\mathcal{N}_0(M), Rj_* \mathbf{G}_m[1]). \end{aligned}$$

Let $F = R \mathbf{Hom}_{X_{\text{sm}}}(\mathcal{N}(M), \cdot)$ or $R \mathbf{Hom}_{X_{\text{sm}}}(\mathcal{N}_0(M), \cdot)$. By the distinguished triangle

$$F \tau_{\leq 0}(Rj_* \mathbf{G}_m[1]) \rightarrow F(Rj_* \mathbf{G}_m[1]) \rightarrow F \tau_{\geq 1}(Rj_* \mathbf{G}_m[1])$$

and the fact $\tau_{\leq 0}(Rj_* \mathbf{G}_m[1]) = \mathcal{G}_m[1]$, we have

$$\tau_{\leq 0} F(\mathcal{G}_m[1]) \xrightarrow{\sim} \tau_{\leq 0} F(Rj_* \mathbf{G}_m[1]).$$

Hence the result follows. □

If M is an abelian scheme, then the isomorphisms in Prop. 4.5 agree with the isomorphism in [29, III, Lem. C.10] by construction.

PROPOSITION 4.6. *The morphism*

$$\mathcal{N}(M^\vee) \otimes^L \mathcal{N}(M) \rightarrow \mathcal{G}_m[1]$$

in Prop. 4.5 and the corresponding morphism

$$\mathcal{N}(M^{\vee\vee}) \otimes^L \mathcal{N}(M^\vee) \rightarrow \mathcal{G}_m[1]$$

for M^\vee are compatible under the biduality isomorphism $M \xrightarrow{\sim} M^{\vee\vee}$ and switching the tensor factors.

Proof. This follows from the fact that the morphism $M^\vee \otimes^L M \rightarrow \mathbf{G}_m[1]$ and the corresponding morphism $M^{\vee\vee} \otimes^L M^\vee \rightarrow \mathbf{G}_m[1]$ for M^\vee are compatible under $M \xrightarrow{\sim} M^{\vee\vee}$ and switching the tensor factors by the symmetric description of the pairings [12, (10.2.12)]. \square

PROPOSITION 4.7. *We have*

$$\tau_{\leq 0} R \mathbf{Hom}_{X_{\text{sm}}}(\mathcal{N}_0(M), i_* \mathbf{Z}[1]) = 0.$$

Hence the isomorphism in Prop. 4.5 induces a canonical isomorphism

$$\mathcal{N}(M^\vee) \cong \tau_{\leq 0} R \mathbf{Hom}_{X_{\text{sm}}}(\mathcal{N}_0(M), \mathbf{G}_m[1]) = \mathcal{N}_0(M)^\vee.$$

Proof. We have

$$R \mathbf{Hom}_{X_{\text{sm}}}(\mathcal{N}_0(M), i_* \mathbf{Z}[1]) = i_* R \mathbf{Hom}_{Z_{\text{sm}}}(i^* \mathcal{N}_0(M), \mathbf{Z}[1])$$

by Prop. 3.1 and Prop. 3.2. We have $i^* \mathcal{N}_0(M) = \mathcal{G}_{Z_0}$ since $i^* \mathcal{Y}_0 = 0$. Hence

$$\tau_{\leq 0} R \mathbf{Hom}_{X_{\text{sm}}}(\mathcal{N}_0(M), i_* \mathbf{Z}[1]) = i_* \tau_{\leq 0} R \mathbf{Hom}_{Z_{\text{sm}}}(\mathcal{G}_{Z_0}, \mathbf{Z}[1]).$$

The fibers of the group \mathcal{G}_{Z_0} at the points of Z are connected smooth algebraic groups. Hence the same argument as [29, III, paragraph after Lem. C.10] shows that

$$\mathbf{Hom}_{Z_{\text{sm}}}(\mathcal{G}_{Z_0}, \mathbf{Z}) = 0,$$

$$\mathbf{Ext}_{Z_{\text{sm}}}^1(\mathcal{G}_{Z_0}, \mathbf{Z}) = \mathbf{Hom}_{Z_{\text{sm}}}(\pi_0(\mathcal{G}_{Z_0}), \mathbf{Q}/\mathbf{Z}) = 0.$$

The result then follows. \square

By the definition of $\mathcal{N}_0(M)^\vee$ and the derived tensor-Hom adjunction, the isomorphism $\mathcal{N}(M^\vee) \xrightarrow{\sim} \mathcal{N}_0(M)^\vee$ in Prop. 4.7 gives a \mathbf{G}_m -bi-extension

$$\mathcal{N}_0(M) \otimes^L \mathcal{N}(M^\vee) \rightarrow \mathbf{G}_m[1]$$

as a morphism in $D(X_{\text{sm}})$. Switching M and M^\vee and applying the derived tensor-Hom adjunction, we have a morphism

$$\mathcal{N}_0(M^\vee) \rightarrow R \mathbf{Hom}_{X_{\text{sm}}}(\mathcal{N}(M), \mathbf{G}_m[1])$$

This morphism factors through the truncation $\tau_{\leq 0}$ of the right-hand side since $\mathcal{N}_0(M^\vee)$ is concentrated in degrees ≤ 0 .

DEFINITION 4.8. We denote the above obtained morphisms in $D(X_{\text{sm}})$ by

$$\begin{aligned} \zeta_M &: \mathcal{N}(M^\vee) \xrightarrow{\sim} \mathcal{N}_0(M)^\vee, \\ \zeta_{0M} &: \mathcal{N}_0(M^\vee) \rightarrow \mathcal{N}(M)^\vee. \end{aligned}$$

Hence Thm. B (1) has been proven in Prop. 4.7. If M is an abelian scheme, then the morphism ζ_{0M} agrees with the morphism in [29, III, Lem. C.11] by construction.

PROPOSITION 4.9. *We have*

$$\begin{aligned} \tau_{\leq 0} R \mathbf{Hom}_{X_{\text{sm}}}(\mathcal{N}(M), i_* \mathbf{Z}[1]) &\xleftarrow{\sim} \tau_{\leq 0} R \mathbf{Hom}_{X_{\text{sm}}}(i_* \mathcal{P}(M), i_* \mathbf{Z}[1]) \\ &= i_* \tau_{\leq 0} R \mathbf{Hom}_{Z_{\text{sm}}}(\mathcal{P}(M), \mathbf{Z}[1]) \\ &\xrightarrow{\sim} i_* \mathcal{P}(M)^{\text{LD}}[1]. \end{aligned}$$

Proof. The first isomorphism follows from Prop. 4.7. The second equality is Prop. 3.1. The third isomorphism follows from the fact that the category of abelian groups has projective dimension one, $\text{Ext}^1(\mathbf{Z}, \cdot) = 0$ and Prop. 3.9. \square

PROPOSITION 4.10. *Let us abbreviate the functor $R \mathbf{Hom}_{X_{\text{sm}}}$ as $[\cdot, \cdot]_X$. Consider the two distinguished triangles*

$$\begin{aligned} \mathcal{N}_0(M^\vee) &\rightarrow \mathcal{N}(M^\vee) \rightarrow i_* \mathcal{P}(M^\vee), \\ [\mathcal{N}(M), \mathbf{G}_m[1]]_X &\rightarrow [\mathcal{N}(M), \mathcal{G}_m[1]]_X \rightarrow [\mathcal{N}(M), i_* \mathbf{Z}[1]]_X. \end{aligned}$$

The morphism

$$\mathcal{N}(M^\vee) \rightarrow [\mathcal{N}(M), \mathcal{G}_m[1]]_X$$

in the middle coming from Prop. 4.5 can uniquely be extended to a morphism of distinguished triangles

$$\begin{array}{ccccc} \mathcal{N}_0(M^\vee) & \longrightarrow & \mathcal{N}(M^\vee) & \longrightarrow & i_* \mathcal{P}(M^\vee), \\ \downarrow & & \downarrow & & \downarrow \\ [\mathcal{N}(M), \mathbf{G}_m[1]]_X & \longrightarrow & [\mathcal{N}(M), \mathcal{G}_m[1]]_X & \longrightarrow & [\mathcal{N}(M), i_* \mathbf{Z}[1]]_X. \end{array}$$

The left vertical morphism agrees with the morphism ζ_{0M} . The middle morphism becomes an isomorphism after truncation $\tau_{\leq 0}$.

Proof. If we show that any morphism from $\mathcal{N}_0(M^\vee)$ or $\mathcal{N}_0(M^\vee)[1]$ to $[\mathcal{N}(M), i_* \mathbf{Z}[1]]_X$ is necessarily zero, then the general lemma on triangulated categories [39, Lem. (4.2.5)] gives the desired unique extension. Let f be such a morphism. Then f factors through the truncation $\tau_{\leq 0}$ of $[\mathcal{N}(M), i_* \mathbf{Z}[1]]_X$,

which is isomorphic to $i_*\mathcal{P}(M)^{\text{LD}}[1]$ by Prop. 4.9. By adjunction, f corresponds to a morphism from \mathcal{G}'_{Z_0} or $\mathcal{G}'_{Z_0}[1]$ to $\mathcal{P}(M)^{\text{LD}}[1]$. We have a spectral sequence

$$E_2^{ij} = \mathbf{Ext}_{Z_{\text{sm}}}^i(H^{-j-1}\mathcal{P}(M), \mathbf{Z}) \Rightarrow H^{i+j}(\mathcal{P}(M)^{\text{LD}}[1]).$$

From this and by Prop. 3.9, we obtain an isomorphism and an exact sequence

$$H^{-1}(\mathcal{P}(M)^{\text{LD}}[1]) = \mathbf{Hom}_{Z_{\text{sm}}}(H^0\mathcal{P}(M), \mathbf{Z}),$$

$$0 \rightarrow \mathbf{Ext}_{Z_{\text{sm}}}^1(H^0\mathcal{P}(M), \mathbf{Z}) \rightarrow H^0(\mathcal{P}(M)^{\text{LD}}[1]) \rightarrow \mathbf{Hom}_{Z_{\text{sm}}}(H^{-1}\mathcal{P}(M), \mathbf{Z}) \rightarrow 0.$$

Hence $\mathcal{P}(M)^{\text{LD}}[1] \in D^b(\text{EtGp}^f/Z)$ is concentrated in degrees -1 and 0 whose H^{-1} is a lattice. Therefore, about cohomology objects H^n of $\mathcal{P}(M)^{\text{LD}}[1]$, we have

$$\mathbf{Hom}_{Z_{\text{sm}}}(\mathcal{G}'_{Z_0}, H^{-1} \text{ or } H^0) = \mathbf{Ext}_{Z_{\text{sm}}}^1(\mathcal{G}_{Z_0}, H^{-1}) = 0,$$

and f has to be zero.

On the other hand, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{N}_0(M^\vee) \otimes^L \mathcal{N}(M) & \longrightarrow & \mathcal{N}(M^\vee) \otimes^L \mathcal{N}(M) \\ \downarrow & & \downarrow \\ \mathbf{G}_m[1] & \longrightarrow & \mathcal{G}_m[1], \end{array}$$

where the right vertical morphism is the one in Prop. 4.5 and the left vertical morphism is the one appearing in the definition of ζ_{0M} (in the paragraph before Def. 4.8). Translating this using the derived tensor-Hom adjunction, we see that the left square in the statement of the proposition is also commutative if the morphism ζ_{0M} is used in the left vertical morphism. Hence the left vertical morphism has to be ζ_{0M} by uniqueness. The middle morphism becomes an isomorphism after truncation $\tau_{\leq 0}$ by Prop. 4.5. \square

The objects in the upper row of the diagram in this proposition are concentrated in degrees ≤ 0 . Hence the third vertical morphisms factor as

$$i_*\mathcal{P}(M^\vee) \rightarrow i_*\mathcal{P}(M)^{\text{LD}}[1],$$

where we used the isomorphism in Prop. 4.9. It is a morphism in $D(X_{\text{sm}})$. Pulling back by i , we have a morphism $\mathcal{P}(M^\vee) \rightarrow \mathcal{P}(M)^{\text{LD}}[1]$ in $D^b(\text{EtGp}^f/Z)$. By the derived tensor-Hom adjunction, this corresponds to a morphism

$$\mathcal{P}(M^\vee) \otimes^L \mathcal{P}(M) \rightarrow \mathbf{Z}[1]$$

in $D(Z_{\text{sm}})$ (or $D(Z_{\text{et}})$).

DEFINITION 4.11. We denote the above obtained morphism in $D^b(\text{EtGp}^f/Z)$ by

$$\eta_M : \mathcal{P}(M^\vee) \rightarrow \mathcal{P}(M)^{\text{LD}}[1].$$

If M is an abelian scheme, then the morphism η_M agrees with Grothendieck's pairing by [29, III, Lem. C.11].

PROPOSITION 4.12. *We have*

$$\tau_{\leq 1} R \mathbf{Hom}_{X_{\text{sm}}} (i_* \mathcal{P}(M), \mathcal{G}_m[1]) = 0.$$

Proof. We have

$$R \mathbf{Hom}_{X_{\text{sm}}} (i_* \mathcal{P}(M), Rj_* \mathbf{G}_m[1]) = Rj_* R \mathbf{Hom}_{U_{\text{sm}}} (j^* i_* \mathcal{P}(M), \mathbf{G}_m[1]) = 0$$

by Prop. 3.1 (1) (or [24, Thm. 18.6.9 (iii)], noting that j is a morphism of sites). Together with $R^1 j_* \mathbf{G}_m = 0$, we have

$$R \mathbf{Hom}_{X_{\text{sm}}} (i_* \mathcal{P}(M), \mathcal{G}_m[1]) = R \mathbf{Hom}_{X_{\text{sm}}} (i_* \mathcal{P}(M), \tau_{\geq 2} Rj_* \mathbf{G}_m).$$

This is concentrated in degrees ≥ 2 . □

PROPOSITION 4.13. *Consider the diagram*

$$\begin{array}{ccccc} [\mathcal{N}(M), \mathcal{G}_m[1]]_X & \longrightarrow & [\mathcal{N}_0(M), \mathcal{G}_m[1]]_X & \longleftarrow & [\mathcal{N}_0(M), \mathbf{G}_m[1]]_X \\ \downarrow & & & & \downarrow \\ [\mathcal{N}(M), i_* \mathbf{Z}[1]]_X & \longleftarrow & [i_* \mathcal{P}(M), i_* \mathbf{Z}[1]]_X & \longrightarrow & [i_* \mathcal{P}(M), \mathbf{G}_m[1]]_X[1] \end{array}$$

of natural morphisms in $D(X_{\text{sm}})$. Note that the mapping cones of the four horizontal morphisms are all concentrated in degrees ≥ 1 by Prop. 4.7 and 4.12 and hence the horizontal morphisms become isomorphisms after truncation $\tau_{\leq 0}$. Then the resulting two morphisms

$$\mathcal{N}_0(M)^\vee \rightrightarrows i_* \mathcal{P}(M)^{\text{LD}}[1]$$

are equal, or equivalently, the above diagram becomes a commutative diagram after $\tau_{\leq 0}$.

Proof. Denote the distinguished triangles

$$\begin{array}{c} \mathcal{N}_0(M) \rightarrow \mathcal{N}(M) \rightarrow i_* \mathcal{P}(M), \\ \mathbf{G}_m[1] \rightarrow \mathcal{G}_m[1] \rightarrow i_* \mathbf{Z}[1] \end{array}$$

by $A \rightarrow B \rightarrow C$, $D \rightarrow E \rightarrow F$, respectively. Then the diagram can be written as

$$\begin{array}{ccccc} [B, E]_X & \longrightarrow & [A, E]_X & \longleftarrow & [A, D]_X \\ \downarrow & & & & \downarrow \\ [B, F]_X & \longleftarrow & [C, F]_X & \longrightarrow & [C, D]_X[1]. \end{array}$$

As noted, $[A, F]_X$ and $[C, E]_X[1]$ are concentrated in degrees ≥ 1 . We want to show that the two morphisms $\tau_{\leq 0}[A, E]_X \rightrightarrows \tau_{\leq 0}[C, F]_X$ are equal. Let

$T \in D(X_{\text{sm}})$ be any object concentrated in degrees ≤ 0 . Applying $\text{Hom}_{X_{\text{sm}}}(T, \cdot)$, we are comparing two homomorphisms

$$\text{Hom}_{X_{\text{sm}}}(T \otimes^L A, E) \rightrightarrows \text{Hom}_{X_{\text{sm}}}(T \otimes^L C, F).$$

Denote $T \otimes^L A, T \otimes^L B, T \otimes^L C$ by A', B', C' , respectively. We want to show that the diagram

$$\begin{array}{ccccc} \text{Hom}_{X_{\text{sm}}}(B', E) & \xrightarrow{\sim} & \text{Hom}_{X_{\text{sm}}}(A', E) & \xleftarrow{\sim} & \text{Hom}_{X_{\text{sm}}}(A', D) \\ \downarrow & & & & \downarrow \\ \text{Hom}_{X_{\text{sm}}}(B', F) & \xleftarrow{\sim} & \text{Hom}_{X_{\text{sm}}}(C', F) & \xrightarrow{\sim} & \text{Hom}_{X_{\text{sm}}}(C', D[1]) \end{array}$$

is commutative. We know that $R \text{Hom}_{X_{\text{sm}}}(A', F)$ and $R \text{Hom}_{X_{\text{sm}}}(C', E[1])$ are concentrated in degrees ≥ 1 . Let $f \in \text{Hom}_{X_{\text{sm}}}(A', E)$. Sending f to $\text{Hom}_{X_{\text{sm}}}(C', F)$ via the left side of the diagram, we have a commutative diagram

$$(4.2) \quad \begin{array}{ccccccc} A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \\ \downarrow f & & \downarrow & & \downarrow & & \downarrow f \\ E & \longrightarrow & F & \longrightarrow & D[1] & \longrightarrow & E[1] \end{array}$$

with diagonal arrows from B' to E and C' to F splitting the left and middle squares into commutative triangles. (Note that the right square is automatically commutative since $\text{Hom}_{X_{\text{sm}}}(C', E[1]) = 0$.) Hence we have a commutative diagram

$$\begin{array}{ccccccc} B' & \longrightarrow & C' & \longrightarrow & A'[1] & \longrightarrow & B'[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ E & \longrightarrow & F & \longrightarrow & D[1] & \longrightarrow & E[1]. \end{array}$$

By an axiom of triangulated categories, there exists a morphism $A'[1] \rightarrow D[1]$ that completes this diagram into a morphism of distinguished triangles. This morphism diagonally splits the right square of the diagram (4.2) into commutative triangles. From this, we see that the two images of f in $\text{Hom}_{X_{\text{sm}}}(C', F)$ are equal. This proves the proposition. \square

PROPOSITION 4.14. *The diagram*

$$\begin{array}{ccccc} \mathcal{N}_0(M^\vee) & \longrightarrow & \mathcal{N}(M^\vee) & \longrightarrow & i_*\mathcal{P}(M^\vee), \\ \downarrow \zeta_{0M} & & \downarrow \zeta_M & & \downarrow d \circ i_*\eta_M \\ [\mathcal{N}(M), \mathbf{G}_m[1]]_X & \longrightarrow & [\mathcal{N}_0(M), \mathbf{G}_m[1]]_X & \longrightarrow & [i_*\mathcal{P}(M), \mathbf{G}_m[1]]_X[1]. \end{array}$$

is a morphism of distinguished triangles, where the d in the right vertical morphism is the connecting morphism $i_*\mathbf{Z}[1] \rightarrow \mathbf{G}_m[2]$ of the triangle $\mathbf{G}_m \rightarrow \mathcal{G}_m \rightarrow i_*\mathbf{Z}$.

Proof. It is enough to show the commutativity of the squares after applying $\tau_{\leq 0}$ to the lower row since the upper row consists of objects concentrated in degrees ≤ 0 . (Note that there are actually three squares whose commutativity has to be checked, one of which is hidden in the diagram.) Prop. 4.6 and 4.13 show that the lower row after $\tau_{\leq 0}$ can be identified with the lower row after $\tau_{\leq 0}$ of the diagram in Prop. 4.10. This implies the result. \square

PROPOSITION 4.15. *Under the identification $M \xrightarrow{\sim} M^{\vee\vee}$, the dual*

$$\eta_M^{\text{LD}} : \mathcal{P}(M) \rightarrow \mathcal{P}(M^{\vee})^{\text{LD}}[1]$$

of η_M agrees with

$$\eta_{M^{\vee}} : \mathcal{P}(M) \rightarrow \mathcal{P}(M^{\vee})^{\text{LD}}[1].$$

In particular, η_M is an isomorphism if and only if $\eta_{M^{\vee}}$ is so.

Proof. Applying the derived tensor-Hom adjunction to the diagram in Prop. 4.14 and interchanging the tensor factors, we have a morphism of distinguished triangles

$$\begin{array}{ccccc} \mathcal{N}_0(M) & \longrightarrow & \mathcal{N}(M) & \longrightarrow & i_*\mathcal{P}(M), \\ \downarrow \zeta_{0M^{\vee}} & & \downarrow \zeta_{M^{\vee}} & & \downarrow d \circ i_*\eta_M^{\text{LD}} \\ [\mathcal{N}(M^{\vee}), \mathbf{G}_m[1]]_X & \longrightarrow & [\mathcal{N}_0(M^{\vee}), \mathbf{G}_m[1]]_X & \longrightarrow & [i_*\mathcal{P}(M^{\vee}), \mathbf{G}_m[1]]_X[1]. \end{array}$$

Using the uniqueness part of Prop. 4.10, we know that $i_*\eta_M^{\text{LD}} = i_*\eta_{M^{\vee}}$. \square

The following together with Prop. 4.15 proves Thm. B (2).

PROPOSITION 4.16. *The morphisms ζ_{0M} and $\zeta_{0M^{\vee}}$ are both isomorphisms if and only if η_M is an isomorphism. If these equivalent conditions are satisfied, then the diagram in Prop. 4.14 induces an isomorphism of distinguished triangles*

$$\begin{array}{ccccc} \mathcal{N}_0(M^{\vee}) & \longrightarrow & \mathcal{N}(M^{\vee}) & \longrightarrow & i_*\mathcal{P}(M^{\vee}), \\ \downarrow \wr \zeta_{0M} & & \downarrow \wr \zeta_M & & \downarrow \wr i_*\eta_M \\ \mathcal{N}(M)^{\vee} & \longrightarrow & \mathcal{N}_0(M)^{\vee} & \longrightarrow & i_*\mathcal{P}(M)^{\text{LD}}[1]. \end{array}$$

Proof. Suppose that η_M is an isomorphism. By Prop. 4.7 and Def. 4.8, the morphism $\zeta_M : \mathcal{N}(M^{\vee}) \rightarrow \mathcal{N}_0(M)^{\vee}$ is an isomorphism. Then the five lemma applied to the diagram in Prop. 4.14 shows that $\zeta_{0M} : \mathcal{N}_0(M^{\vee}) \rightarrow \mathcal{N}(M)^{\vee}$ is an isomorphism. On the other hand, Prop. 4.15 shows that $\eta_{M^{\vee}}$ is also an isomorphism. Hence the above argument applied to M^{\vee} implies that $\zeta_{0M^{\vee}}$ is an isomorphism.

Conversely, suppose that ζ_{0M} and $\zeta_{0M^{\vee}}$ are isomorphisms. Then the same argument as above shows that η_M and $\eta_{M^{\vee}}$ are isomorphisms in H^{-1} and injec-

tive in H^0 . Set $P = \mathcal{P}(M)$ and $P' = \mathcal{P}(M^\vee)$. Denote the torsion part by $(\cdot)_{\text{tor}}$ and torsion-free quotient by $(\cdot)_{/\text{tor}}$. Then we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(P')_{\text{tor}} & \longrightarrow & H^0(P') & \longrightarrow & H^0(P')_{/\text{tor}} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & H^0(P)_{\text{tor}}^{\text{PD}} & \longrightarrow & H^1(P)^{\text{LD}} & \longrightarrow & H^{-1}(P)^{\text{LD}} \longrightarrow 0, \end{array}$$

where $H^0(P)_{\text{tor}}^{\text{PD}}$ is a shorthand for $(H^0(P)_{\text{tor}})^{\text{PD}}$. The middle vertical morphism is injective. The right vertical one is an isomorphism by Prop. 4.15. These imply that the left vertical morphism is injective. Switching M and M^\vee , we know that $H^0(P)_{\text{tor}} \rightarrow H^0(P')_{\text{tor}}$ is also injective. As $H^0(P)_{\text{tor}}$ and $H^0(P')_{\text{tor}}$ are finite étale, we conclude that these injective morphisms are all isomorphisms. Therefore η_M and η_{M^\vee} are isomorphisms.

The last statement about the diagram follows from the isomorphism

$$d : [i_*\mathcal{P}(M), i_*\mathbf{Z}[1]]_X \xrightarrow{\sim} \tau_{\leq 0}[i_*\mathcal{P}(M), \mathbf{G}_m[2]]_X,$$

which is a consequence of Prop. 4.12. □

PROPOSITION 4.17. *The morphism η_M is an isomorphism if and only if the corresponding morphism*

$$\eta_{M \times_U K_x^{\text{sh}}} : \mathcal{P}(M^\vee \times_U K_x^{\text{sh}}) \rightarrow \mathcal{P}(M \times_U K_x^{\text{sh}})^{\text{LD}}[1],$$

for the strict henselian local field K_x^{sh} at any point $x \in Z$ is an isomorphism. We may replace K_x^{sh} by its completion.

Proof. This follows from Prop. 3.10. □

The following proves Thm. B (3).

PROPOSITION 4.18. *The morphism η_M is an isomorphism if M is semistable over X , i.e. if its torus and lattice parts are unramified over X and its abelian scheme part is semistable over X .*

Proof. We may assume that X is strictly henselian local by Prop. 4.17. If $M = \mathbf{G}_m$ or $\mathbf{Z}[1]$, then $\mathcal{P}(M) = \mathbf{Z}$ or $\mathbf{Z}[1]$, respectively, and η_M is an isomorphism. Hence η_M is an isomorphism if M is an unramified torus or an unramified lattice shifted by one. If M is a semistable abelian variety, then η_M is Grothendieck’s pairing, which is an isomorphism by [42].

Now we treat a general semistable M . Set $\mathcal{Y} = j_*Y$. By assumption, X itself is a good covering of X . Hence $\mathcal{N}(M) = [\mathcal{Y} \rightarrow \mathcal{G}]$ and $\mathcal{N}_0(M) = [\mathcal{Y}_0 \rightarrow \mathcal{G}_0]$. We have $R^1j_*T = 0$ since T is a trivial torus and by the proof of [29, III, Lem. C.10]. Hence we have an exact sequence $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{A} \rightarrow 0$. We have $\mathcal{G}_0 \cap \mathcal{F} = \mathcal{F}_0$ since \mathcal{G}_0 is of finite type and $\pi_0(\mathcal{F}_Z)$ is torsion-free.

Therefore we have an exact sequence $0 \rightarrow \mathcal{F}_0 \rightarrow \mathcal{G}_0 \rightarrow \mathcal{A}_0 \rightarrow 0$. We have morphisms of distinguished triangles

$$\begin{array}{ccccc} \mathcal{F}'_0 & \longrightarrow & \mathcal{G}'_0 & \longrightarrow & \mathcal{A}'_0 \\ \downarrow & & \downarrow & & \downarrow \\ [\mathcal{Y}[1], \mathbf{G}_m[1]]_X & \longrightarrow & [[\mathcal{Y} \rightarrow \mathcal{A}], \mathbf{G}_m[1]]_X & \longrightarrow & [\mathcal{A}, \mathbf{G}_m[1]]_X \end{array}$$

and

$$\begin{array}{ccccc} \mathcal{G}'_0 & \longrightarrow & [\mathcal{Y}'_0 \rightarrow \mathcal{G}'_0] & \longrightarrow & \mathcal{Y}'_0[1] \\ \downarrow & & \downarrow & & \downarrow \\ [[\mathcal{Y} \rightarrow \mathcal{A}], \mathbf{G}_m[1]]_X & \longrightarrow & [[\mathcal{Y} \rightarrow \mathcal{G}], \mathbf{G}_m[1]]_X & \longrightarrow & [\mathcal{T}, \mathbf{G}_m[1]]_X. \end{array}$$

In either diagram, the left and right vertical morphisms become isomorphisms after truncation $\tau_{\leq 0}$ by the previously treated cases. Therefore the four lemmas imply that the middle vertical morphisms become isomorphisms after $\tau_{\leq 0}$. \square

The following proves a weaker version of Thm. B (4).

PROPOSITION 4.19. *The morphism η_M becomes an isomorphism after tensoring with \mathbf{Q} .*

Proof. We may assume that X is strictly henselian local and $U = \text{Spec } K$ by Prop. 4.17. First we describe $\mathcal{P}(M) \otimes \mathbf{Q}$. We have $R^1 j_* \mathbf{Q} = 0$ since $H^1(X' \times_X U, \mathbf{Q}) = 0$ for any quasi-compact smooth X -scheme X' by [29, II, Lem. 2.10]. From this, by taking a finite Galois extension of K that trivializes Y and arguing with a Hochschild-Serre spectral sequence, we know that $R^1 j_* Y \otimes \mathbf{Q} = 0$. Hence $\mathcal{N}(M) \otimes \mathbf{Q} = [j_* Y \rightarrow j_* G] \otimes \mathbf{Q}$. Therefore $\mathcal{P}(M) \otimes \mathbf{Q} = [\Gamma(U, Y) \rightarrow \pi_0(\mathcal{G}_Z)] \otimes \mathbf{Q}$, where we are viewing $\mathcal{P}(M)$ as a complex of abstract abelian groups since X is strictly henselian and hence Z is a geometric point. Let F be the cokernel of $\mathcal{G}_Z \rightarrow \mathcal{A}_Z$ in $\text{Ab}(Z_{\text{sm}})$. Then $\Gamma(Z, F)$ is a subgroup of $H^1(U, T)$. If K' is a finite Galois extension of K that trivializes T , then the exact sequence

$$0 \rightarrow H^1(\text{Gal}(K'/K), T(K')) \rightarrow H^1(K, T) \rightarrow \Gamma(\text{Gal}(K'/K), H^1(K', T)),$$

the vanishing $H^1(K', \mathbf{G}_m) = 0$ and [35, VIII, §2, Cor. 1 to Prop. 4] show that $H^1(K, T)$ is killed by $[K' : K]$. Hence so is $\Gamma(Z, F)$. The snake lemma for the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{G}_{Z_0}/\mathcal{F}_{Z_0} & \longrightarrow & \mathcal{G}_Z/\mathcal{F}_{Z_0} & \longrightarrow & \pi_0(\mathcal{G}_Z) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{A}_{Z_0} & \longrightarrow & \mathcal{A}_Z & \longrightarrow & \pi_0(\mathcal{A}_Z) \longrightarrow 0 \end{array}$$

gives an exact sequence

$$\mathcal{T}_Z \cap \mathcal{G}_{Z0} \rightarrow \pi_0(\mathcal{T}_Z) \rightarrow \mathbf{Ker}(\pi_0(\mathcal{G}_Z) \rightarrow \pi_0(\mathcal{A}_Z)) \rightarrow \mathcal{A}_{Z0}/\mathcal{G}_{Z0} \rightarrow F$$

in $\text{Ab}(Z_{\text{sm}})$. The first term is of finite type over Z . Hence the first morphism has finite image. The third morphism is a morphism from a finitely generated abelian group to a smooth algebraic group. The group of Z -valued points of the cokernel of this morphism is killed by $[K' : K]$. Such a morphism has finite image since Z is a geometric point. The group $\pi_0(\mathcal{A}_Z)$ is finite. Thus we have $\pi_0(\mathcal{T}_Z) \otimes \mathbf{Q} = \pi_0(\mathcal{G}_Z) \otimes \mathbf{Q}$. Therefore

$$\mathcal{P}(M) \otimes \mathbf{Q} = [\Gamma(U, Y) \rightarrow \pi_0(\mathcal{T}_Z)] \otimes \mathbf{Q}.$$

Note that Y and T' are Cartier dual to each other. So are Y' and T . The morphism $\eta_M \otimes \mathbf{Q}$ decomposes into two parts

$$\begin{aligned} \Gamma(U, Y') \otimes \mathbf{Q} &\rightarrow \text{Hom}(\pi_0(\mathcal{T}_Z) \otimes \mathbf{Q}, \mathbf{Q}), \\ \pi_0(\mathcal{T}'_Z) \otimes \mathbf{Q} &\rightarrow \text{Hom}(\Gamma(U, Y) \otimes \mathbf{Q}, \mathbf{Q}) \end{aligned}$$

given by

$$\Gamma(U, Y') = \text{Hom}_U(T, \mathbf{G}_m) \rightarrow \text{Hom}_Z(\mathcal{T}_Z, \mathcal{G}_m) \rightarrow \text{Hom}(\pi_0(\mathcal{T}_Z), \mathbf{Z})$$

and the corresponding morphism for Y and T' . It is a classical fact that the two parts above are isomorphisms.¹ Hence $\eta_M \otimes \mathbf{Q}$ is an isomorphism. \square

PROPOSITION 4.20. *The morphism ζ_{0M} induces an isomorphism in cohomologies in degrees $\neq 0$ and an injection in H^0 . The morphism η_M induces an isomorphism in cohomologies in degrees $\neq -1, 0$ and an injection in H^{-1} . We have an exact sequence*

$$0 \rightarrow \text{Coker}(H^{-1}\eta_M) \rightarrow \text{Coker}(H^0\zeta_{0M}) \rightarrow \text{Ker}(H^0\eta_M) \rightarrow 0.$$

*Each of these three terms as well as $\text{Coker}(H^0\eta_M)$ is of the form i_*N for some finite étale group scheme N over Z .*

Proof. The domains and codomains of the morphisms ζ_{0M} and η_M are concentrated in degrees $-1, 0$. The diagram in Prop. 4.10 shows that $H^{-1}\zeta_{0M}$ is injective and induces exact sequences

$$\begin{aligned} 0 \rightarrow \text{Coker}(H^{-1}\zeta_{0M}) &\rightarrow \text{Ker}(H^{-1}\eta_M) \rightarrow \text{Ker}(H^0\zeta_{0M}) \rightarrow 0, \\ 0 \rightarrow \text{Coker}(H^{-1}\eta_M) &\rightarrow \text{Coker}(H^0\zeta_{0M}) \rightarrow \text{Ker}(H^0\eta_M) \rightarrow 0. \end{aligned}$$

¹One way to quickly see this is the following. Let l be a prime invertible on Z . The Kummer sequence gives $H^1(U, V_l T) = \pi_0(\mathcal{T}) \otimes \mathbf{Q}_l$, where V_l is the rational l -adic Tate module of T . The l -adic representation $V_l T$ over U is the Tate twist of the dual of $Y' \otimes \mathbf{Q}_l$. Hence the duality $H^1(U, V_l T) \leftrightarrow \Gamma(U, Y' \otimes \mathbf{Q}_l)$ of l -adic cohomology of strict henselian discrete valuation fields ([22, Exp. I, Thm. 5.1]) gives the result.

All these groups are torsion by Prop. 4.19. The group $\text{Ker}(H^{-1}\eta_M)$ is torsion-free by Prop. 3.9, hence zero. Therefore $\text{Coker}(H^{-1}\zeta_{0M}) = \text{Ker}(H^0\zeta_{0M}) = 0$. The groups $\text{Coker}(H^n\eta_M)$ and $\text{Ker}(H^n\eta_M)$ for any n are of the stated form. Hence so is their extension $\text{Coker}(H^0\zeta_{0M})$. \square

In particular, if ζ_{0M} is an isomorphism, then η_M is an isomorphism in cohomologies of degrees $\neq 0$ and an injection with finite étale cokernel in H^0 . This might not imply that η_M is an isomorphism if no additional assumption is made on ζ_{0M^\vee} . This point seems to exist already for the case that $M = A$ is an abelian scheme; see [7, Prop. 5.1]. An injection of finite étale groups of the same order is an isomorphism, but it is not clear whether $\pi_0(\mathcal{A}_x)$ and $\pi_0(\mathcal{A}'_x)$ for $x \in Z$ have the same order or not. According to Lorenzini [27, Rmk. 7.1], it is not known that the groups of geometric points of $\pi_0(\mathcal{A}_x)$ and $\pi_0(\mathcal{A}'_x)$ are abstractly isomorphic.

5. l -adic realization and perfectness for l -part

We continue working in Situation 4.1. Let l be a prime number invertible on X . Below we use the formalism of derived categories of l -adic sheaves given by [5, §6], [37, Tag 09C0] for technical simplicity. One can also use [15]. We denote the pro-étale site of X by X_{proet} ([5, Def. 4.1.1]). The derived completeness and the derived completion ([5, Lem. 3.4.9, Prop. 3.5.1 (3)])

$$\hat{F} = R \varprojlim_n (F \otimes^L \mathbf{Z}/l^n \mathbf{Z})$$

of an object $F \in D(X_{\text{proet}})$ is always taken with respect to the ideal $l\mathbf{Z} \subset \mathbf{Z}$. See [37, Tag 091J], [40, §2.3] for how we choose derived inverse limits functorially in derived categories. In fact, we have

$$\hat{F} = R \mathbf{Hom}_{X_{\text{proet}}}(\mathbf{Q}_l/l, F)[1]$$

by [37, Tag 099B]. The constructibility of a derived complete $F \in D(X_{\text{proet}})$ is always taken with respect to the ideal $l\mathbf{Z} \subset \mathbf{Z}$ (or $l\mathbf{Z}_l \subset \mathbf{Z}_l$; [5, Lem. 3.5.6]) unless otherwise noted. The same notation applies to the pro-étale sites $U_{\text{proet}}, Z_{\text{proet}}$. For a derived complete $F \in D(X_{\text{proet}})$, we denote

$$F^\vee = R \mathbf{Hom}_{X_{\text{proet}}}(F, \mathbf{Z}_l(1)[2]),$$

where we set $\mathbf{Z}_l = \varprojlim_n \mathbf{Z}/l^n \mathbf{Z} \in \text{Ab}(X_{\text{proet}})$ and the Tate twist $\mathbf{Z}_l(1) = \varprojlim_n \mathbf{Z}/l^n \mathbf{Z}(1) \in \text{Ab}(X_{\text{proet}})$ as sheaves. (This notation F^\vee does not clash with dual 1-motives M^\vee since a non-zero 1-motive is never derived complete. It is also different from the linear dual of \mathbf{Z}_l -lattices due to the twisted shift (1)[2].) The same notation F^\vee applies to a derived complete $F \in D(U_{\text{proet}})$. For a derived complete $F \in D(Z_{\text{proet}})$, we denote

$$F^\vee = R \mathbf{Hom}_{Z_{\text{proet}}}(F, \mathbf{Z}_l).$$

For the six operations formalism, see [5, §6.7]. The l -adic Tate module $T_l(\cdot)$ of a sheaf is the inverse limit of the l^n -torsion parts for $n \geq 0$. Let $v : X_{\text{proet}} \rightarrow X_{\text{et}}$ be the morphism of sites defined by the identity functor ([5, §5]). We naturally regard objects of $D(X_{\text{et}})$ as objects of $D(X_{\text{proet}})$ via pullback v^* (omitting v^* from the notation). A similar convention applies to $v : Z_{\text{proet}} \rightarrow Z_{\text{et}}$ and $v : U_{\text{proet}} \rightarrow U_{\text{et}}$.

PROPOSITION 5.1. *Let $\alpha : X_{\text{sm}} \rightarrow X_{\text{et}}$ be the morphism of sites defined by the identity. Then the objects $\alpha_* \mathcal{N}(M) \otimes^L \mathbf{Z}/l\mathbf{Z}$ and $\alpha_* \mathcal{N}_0(M) \otimes^L \mathbf{Z}/l\mathbf{Z}$ of $D^b(X_{\text{et}})$ are constructible complexes of sheaves of $\mathbf{Z}/l\mathbf{Z}$ -modules.*

Proof. The object $\alpha_* \mathcal{G}_0 \otimes^L \mathbf{Z}/l\mathbf{Z} = \mathcal{G}_0/l\mathcal{G}_0$ is constructible. The object $\alpha_* \mathcal{G}_0 \otimes^L \mathbf{Z}/l\mathbf{Z}$ is the l -torsion part of \mathcal{G}_0 shifted by one, which is constructible. Hence $\alpha_* \mathcal{N}_0(M) \otimes^L \mathbf{Z}/l\mathbf{Z}$ is constructible. Since $\mathcal{P}(M) \in D^b(\text{EtGp}^f/Z)$, it follows that $\alpha_* \mathcal{N}(M) \otimes^L \mathbf{Z}/l\mathbf{Z}$ is also constructible. \square

In the rest of this section, we will omit α_* and simply denote the image of a sheaf or a complex of sheaves F over the smooth site (of U , X or Z) by F .

DEFINITION 5.2. Viewing $\mathcal{N}(M)$ as an object of $D(X_{\text{proet}})$, we call its derived completion $\mathcal{N}(M)^\wedge \in D(X_{\text{proet}})$ the l -adic realization of $\mathcal{N}(M)$ and denote it by $\hat{\mathcal{N}}(M)$. The l -adic realizations $\hat{\mathcal{N}}_0(M) = \mathcal{N}_0(M)^\wedge \in D(X_{\text{proet}})$ and $\hat{\mathcal{P}}(M) = \mathcal{P}(M)^\wedge \in D(Z_{\text{proet}})$ are defined similarly.

Yet another convention: in the rest of this section, we will use the pro-étale topology only and denote the morphisms $U_{\text{proet}} \rightarrow X_{\text{proet}}$ and $Z_{\text{proet}} \rightarrow Z_{\text{proet}}$ induced by $j : U \hookrightarrow X$ and $i : Z \hookrightarrow X$ simply by j and i . This change of notation does not make a difference for relevant groups after derived completion. More precisely:

PROPOSITION 5.3. *Let $i : Z_{\text{proet}} \rightarrow X_{\text{proet}}$ as above. Let $H \in \text{SmGp}'/X$. Then the natural reduction morphism $i^*H \rightarrow H \times_X Z$ induces an isomorphism $(i^*H)^\wedge \xrightarrow{\sim} (H \times_X Z)^\wedge$ in $D(Z_{\text{proet}})$.*

Proof. The reduction morphism $i^*H \rightarrow H \times_X Z$ is surjective by smoothness. We need to show that the multiplication by l on the kernel of $i^*H \rightarrow H \times_X Z$ in $\text{Ab}(Z_{\text{proet}})$ is an isomorphism. We may assume that X is strictly henselian and $U = \text{Spec } K$. Since Z is then a geometric point and H locally of finite type, it is enough to show that $l : \text{Ker}(H(X) \rightarrow H(Z))$ is bijective. By dividing H by the schematic closure of the identity section of $H \times_X U$, we may assume that H is a separated scheme ([32, Prop. 3.3.5]). The multiplication by l is an étale morphism on H by [8, 7.3/2 (b)]. In particular, $\text{Ker}(l) \subset H$ is a separated étale group scheme over X . Hence the map $\text{Ker}(l)(X) \rightarrow \text{Ker}(l)(Z)$ is

bijjective since X is henselian. Therefore $l : \text{Ker}(H(X) \twoheadrightarrow H(Z))$ is injective. Let $a \in \text{Ker}(H(X) \twoheadrightarrow H(Z))$. Then the inverse image $l^{-1}(a) \subset H$ is a separated étale X -scheme whose special fiber contains 0 (the identity element). Hence $l^{-1}(a)(X)$ is non-empty since X is henselian. Thus $l : \text{Ker}(H(X) \twoheadrightarrow H(Z))$ is also surjective. This implies the result. \square

PROPOSITION 5.4. *The objects $\hat{\mathcal{N}}(M)$, $\hat{\mathcal{N}}_0(M)$ of $D(X_{\text{proet}})$ and the object $\hat{\mathcal{P}}(M)$ of $D(Z_{\text{proet}})$ are constructible. We have $\hat{\mathcal{P}}(M) = \mathcal{P}(M) \otimes \mathbf{Z}_l$. We have a canonical distinguished triangle*

$$\hat{\mathcal{N}}_0(M) \rightarrow \hat{\mathcal{N}}(M) \rightarrow i_*\hat{\mathcal{P}}(M)$$

in $D(X_{\text{proet}})$.

Proof. The statements for $\hat{\mathcal{N}}(M)$ and $\hat{\mathcal{N}}_0(M)$ follow from Prop. 5.1 and [5, Prop. 3.5.1 (2)]. Since $\mathcal{P}(M) \in D^b(\text{EtGp}^f/Z)$, it follows that $\hat{\mathcal{P}}(M) = \mathcal{P}(M) \otimes \mathbf{Z}_l$ is constructible. Applying the derived completion to the triangle in Def. 3.8, we get the stated distinguished triangle. \square

PROPOSITION 5.5. *We have canonical isomorphisms*

$$i^*\hat{\mathcal{N}}_0(M) \cong T_l\mathcal{G}_{Z_0}[1], \quad i^!\hat{\mathcal{N}}(M) \cong T_l i^*(R^1j_*M)[-1],$$

which are shifts of \mathbf{Z}_l -lattices over Z .

Proof. By [5, Rmk. 6.5.10], the derived completion commutes with i^* and j^* . By the distinguished triangle $i^! \rightarrow i^* \rightarrow i^*Rj_*j^*$ of functors $D(X_{\text{proet}}) \rightarrow D(Z_{\text{proet}})$ ([5, Lem. 6.1.16]), we know that the derived completion also commutes with $i^!$. We have $i^*\mathcal{Y}_0 = 0$. Also $i^*\mathcal{G}_0^\wedge = \mathcal{G}_{Z_0}^\wedge = T_l\mathcal{G}_{Z_0}[1]$ by Prop. 5.3 and the fact that the smooth group scheme \mathcal{G}_{Z_0} with connected fibers is l -divisible. Hence $i^*\hat{\mathcal{N}}_0(M) = T_l\mathcal{G}_{Z_0}[1]$. We have a distinguished triangle

$$i^!\mathcal{N}(M) \rightarrow i^*\mathcal{N}(M) \rightarrow i^*Rj_*M.$$

We have $i^*\mathcal{N}(M) = \tau_{\leq 0}i^*Rj_*M$. We claim that the l -primary part of $i^*R^nj_*M$ is divisible for $n = 1$ and zero for $n \geq 2$. This implies

$$i^!\hat{\mathcal{N}}(M) = (\tau_{\geq 1}i^*Rj_*M)^\wedge[-1] = (i^*R^1j_*M)^\wedge[-2] = T_l i^*(R^1j_*M)[-1]$$

as desired.

Now we prove the above claim. We may assume that X is strictly henselian and $U = \text{Spec } K$. We need to show that the l -primary part of $H^n(K, M)$ is divisible for $n = 1$ and zero for $n \geq 2$. It is enough to show this for the case $M = G$ and the case $M = Y[1]$. Let C be the l -primary part of G_{tor} or $Y \otimes \mathbf{Q}_l/\mathbf{Z}_l$. We need to show that $H^n(K, C)$ is divisible for $n = 1$ and zero for $n \geq 2$. Since C is l -divisible with finite l -torsion part, this follows from the fact that the l -cohomological dimension of K is 1 ([36, II, §4.3, Prop. 12]). \square

PROPOSITION 5.6. *The objects $i^! \hat{\mathcal{N}}_0(M)$ and $(i^* \hat{\mathcal{N}}(M))^\vee$ are concentrated in degrees 0 and 1. The H^0 of these objects are \mathbf{Z}_l -lattices. The objects $i^* \hat{\mathcal{N}}_0(M)$ and $(i^! \hat{\mathcal{N}}(M))^\vee$ are concentrated in degree -1 .*

Proof. We have a distinguished triangle

$$i^! \hat{\mathcal{N}}_0(M) \rightarrow i^! \hat{\mathcal{N}}(M) \rightarrow \hat{\mathcal{P}}(M).$$

Hence the statement about $i^! \hat{\mathcal{N}}_0(M)$ follows from Prop. 3.9 and 5.5. We also have a distinguished triangle

$$i^* \hat{\mathcal{N}}_0(M) \rightarrow i^* \hat{\mathcal{N}}(M) \rightarrow \hat{\mathcal{P}}(M).$$

Hence by the same propositions, we know that $i^* \hat{\mathcal{N}}(M)$ is concentrated in degrees $-1, 0$, whose H^{-1} is a \mathbf{Z}_l -lattice. This implies the statement about $(i^* \hat{\mathcal{N}}(M))^\vee$. The rest is already in Prop. 5.5. \square

PROPOSITION 5.7. *For $F, G \in D(X_{\text{proet}})$, we have*

$$R \mathbf{Hom}_{X_{\text{proet}}}(F, G)^\wedge = R \mathbf{Hom}_{X_{\text{proet}}}(\hat{F}, \hat{G}).$$

Proof. The derived tensor product $(\cdot) \otimes^L \mathbf{Z}/l^n \mathbf{Z}$ is given by the mapping cone of multiplication by l^n , which commutes with $R \mathbf{Hom}_{X_{\text{proet}}}(F, \cdot)$. The derived inverse limit $R \varprojlim$ also commutes with $R \mathbf{Hom}_{X_{\text{proet}}}(F, \cdot)$. Hence we have

$$\begin{aligned} R \mathbf{Hom}_{X_{\text{proet}}}(F, G)^\wedge &= R \varprojlim_n R \mathbf{Hom}_{X_{\text{proet}}}(F, G \otimes^L \mathbf{Z}/l^n \mathbf{Z}) \\ &= R \varprojlim_n R \mathbf{Hom}_{X_{\text{proet}}}(\hat{F}, G \otimes^L \mathbf{Z}/l^n \mathbf{Z}) \\ &= R \mathbf{Hom}_{X_{\text{proet}}}(\hat{F}, \hat{G}), \end{aligned}$$

where the second equality comes from the fact that the mapping cone of $F \rightarrow \hat{F}$ is uniquely l -divisible. \square

Therefore the morphism ζ_{0M} induces a morphism

$$\hat{\zeta}_{0M} : \hat{\mathcal{N}}_0(M^\vee) \rightarrow R \mathbf{Hom}_{X_{\text{proet}}}(\hat{\mathcal{N}}(M), \mathbf{Z}_l(1)[2]) = \hat{\mathcal{N}}(M)^\vee$$

via derived completion. Its pullback $(M^\vee)^\wedge \rightarrow \hat{M}^\vee$ to U_{proet} is an isomorphism since $(M^\vee)^\wedge[-1] = T_l M^\vee$ and $\hat{M}[-1] = T_l M$ are dual to each other after Tate twist.

PROPOSITION 5.8. *The morphism $\hat{\zeta}_{0M}$ is an isomorphism. In $H^{-1}i^*$, it induces a perfect pairing*

$$T_l \mathcal{G}_{Z_0}^l \leftrightarrow T_l i^* R^1 j_* M$$

of \mathbf{Z}_l -lattices over Z .

Proof. The statement about $H^{-1}i^*$ is a consequence of Prop. 5.5, once we show that $\hat{\zeta}_{0M}$ is an isomorphism. We have

$$i^*(\hat{\mathcal{N}}(M)^\vee) = (i^!\hat{\mathcal{N}}(M))^\vee$$

by Verdier duality. Since

$$j^*\hat{\zeta}_{0M} : j^*(\hat{\mathcal{N}}_0(M^\vee)) \rightarrow j^*(\hat{\mathcal{N}}(M)^\vee)$$

is an isomorphism, it is enough to show that the morphism

$$i^*\hat{\zeta}_{0M} : i^*(\hat{\mathcal{N}}_0(M^\vee)) \rightarrow i^*(\hat{\mathcal{N}}(M)^\vee) = (i^!\hat{\mathcal{N}}(M))^\vee$$

is an isomorphism. The morphism $\hat{\zeta}_{0M}$ induces a canonical morphism of distinguished triangles

$$\begin{array}{ccccc} i^*Rj_*(\hat{\mathcal{N}}_0(M^\vee))[-1] & \longrightarrow & i^!(\hat{\mathcal{N}}_0(M^\vee)) & \longrightarrow & i^*(\hat{\mathcal{N}}_0(M^\vee)) \\ \downarrow \wr & & \downarrow & & \downarrow \\ (i^*Rj_*\hat{\mathcal{N}}(M))^\vee & \longrightarrow & (i^*\hat{\mathcal{N}}(M))^\vee & \longrightarrow & (i^!\hat{\mathcal{N}}(M))^\vee. \end{array}$$

Denote the upper triangle by $E \rightarrow F \rightarrow G$ and lower by $E' \rightarrow F' \rightarrow G'$. Then by Prop. 5.6, we have a commutative diagram

$$\begin{array}{ccc} H^1E & \xrightarrow{\sim} & H^1F \\ \downarrow \wr & & \downarrow \\ H^1E' & \xrightarrow{\sim} & H^1F' \end{array}$$

and a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^{-1}G & \longrightarrow & H^0E & \longrightarrow & H^0F \longrightarrow 0 \\ & & \downarrow & & \downarrow \wr & & \downarrow \\ 0 & \longrightarrow & H^{-1}G' & \longrightarrow & H^0E' & \longrightarrow & H^0F' \longrightarrow 0, \end{array}$$

and the cohomologies of E, F, G, E', F', G' are zero in all other degrees. The first diagram shows that the morphism $H^1F \rightarrow H^1F'$ is an isomorphism. The second diagram shows that $H^0F \rightarrow H^0F'$ is surjective. It is an isomorphism up to torsion by Prop. 4.19. It is also a morphism between \mathbf{Z}_l -lattices by Prop. 5.6. These imply that $H^0F \rightarrow H^0F'$ is an isomorphism. Therefore $H^{-1}G \rightarrow H^{-1}G'$ is also an isomorphism. Hence $F \rightarrow F'$ and $G \rightarrow G'$ are both isomorphisms. This proves that $\hat{\zeta}_{0M}$ is an isomorphism. \square

PROPOSITION 5.9. *The morphism $\mathcal{P}(M^\vee) \rightarrow \mathcal{P}(M)^{\text{LD}}[1]$ becomes an isomorphism after tensoring with \mathbf{Z}_l .*

Proof. The derived completion of the diagram in Prop. 4.14 gives a morphism of distinguished triangles

$$\begin{array}{ccccc}
 \hat{\mathcal{N}}_0(M^\vee) & \longrightarrow & \hat{\mathcal{N}}(M^\vee) & \longrightarrow & i_*\hat{\mathcal{P}}(M^\vee) \\
 \downarrow \hat{\zeta}_{0M} & & \downarrow \hat{\zeta}_M & & \downarrow \hat{\eta}_M \\
 (\hat{\mathcal{N}}(M))^\vee & \longrightarrow & (\hat{\mathcal{N}}_0(M))^\vee & \longrightarrow & i_*\hat{\mathcal{P}}(M)^\vee[1].
 \end{array}$$

The morphism $\hat{\zeta}_{0M}$ is an isomorphism by Prop. 5.8. The morphism $\hat{\zeta}_M$ can be obtained by applying the dual $(\cdot)^\vee$ to $\hat{\zeta}_{0M^\vee}$ by the definition of ζ_{0M} . Hence $\hat{\zeta}_M$ is an isomorphism. (This is not a consequence of the fact that $\zeta_M : \mathcal{N}(M^\vee) \rightarrow \mathcal{N}_0(M)^\vee$ is an isomorphism, since derived completion does not commutes with the truncation $\tau_{\leq 0}$ that appears in Def. 4.4.) Therefore $\hat{\eta}_M$ is an isomorphism. □

The following finishes the proof of Thm. B (4).

PROPOSITION 5.10. *Any of the kernel or cokernel of $H^n\zeta_M$ and $H^n\eta_M$ for any n is of the form i_*N for some finite étale group scheme N over Z whose fiber over any $x \in Z$ has order a power of the residual characteristic exponent at x .*

Proof. This follows from Prop. 5.9 and 4.20. □

6. Duality for cohomology of Néron models and perfectness for p -part

We continue working in Situation 4.1. Assume that $X = \text{Spec } \mathcal{O}_K$ is the spectrum of a complete discrete valuation ring \mathcal{O}_K with *perfect* residue field k of characteristic $p > 0$ and $U = \text{Spec } K$ its generic point. Below we use the same notation as [39, §2.1], [40, §2.3] about the ind-rational pro-étale site of k . We also write $Z = x$, $\mathcal{O}_K = \hat{\mathcal{O}}_x$, $K = \hat{K}_x$ in order to match the notation in [40, §2.5]. Let k^{indrat} be the category of ind-rational k -algebras and $\text{Spec } k_{\text{proet}}^{\text{indrat}}$ the ind-rational pro-étale site of k defined in [39, §2.1], [40, §2.3]. Let

$$\begin{aligned}
 R\Gamma(\mathcal{O}_K, \cdot), R\Gamma_x(\mathcal{O}_K, \cdot) &: D(\mathcal{O}_K, \text{fppf}) \rightarrow D(k_{\text{proet}}^{\text{indrat}}) \\
 R\Gamma(K, \cdot) &: D(K_{\text{fppf}}) \rightarrow D(k_{\text{proet}}^{\text{indrat}})
 \end{aligned}$$

be the functors defined in [40, §2.5]. The composites of them with the n -th cohomology object functor H^n for any n is denoted by $\mathbf{H}^n(\mathcal{O}_K, \cdot)$, $\mathbf{H}_x^n(\mathcal{O}_K, \cdot)$ and $\mathbf{H}^n(K, \cdot)$, respectively, and we set $\Gamma(\mathcal{O}_K, \cdot) = \mathbf{H}^0(\mathcal{O}_K, \cdot)$, $\Gamma(K, \cdot) = \mathbf{H}^0(K, \cdot)$. We denote the Serre dual functor by $(\cdot)^{\text{SD}} = R \mathbf{Hom}_{k_{\text{proet}}^{\text{indrat}}}(\cdot, \mathbf{Z})$ ([39, §2.4]). An object $C \in D(k_{\text{proet}}^{\text{indrat}})$ is said to be Serre reflexive if the canonical morphism $C \rightarrow C^{\text{SDSD}}$ is an isomorphism.

We have the canonical trace morphism

$$R\Gamma(K, \mathbf{G}_m) \rightarrow R\Gamma_x(\mathcal{O}_K, \mathbf{G}_m)[1] = \mathbf{Z}$$

by [40, Eq. (2.5.7)]. The morphism

$$M^\vee \rightarrow R \mathbf{Hom}_{K_{\text{fppf}}}(M, \mathbf{G}_m[1])$$

induces morphisms

$$\begin{aligned} R\Gamma(K, M^\vee) &\rightarrow R \mathbf{Hom}_{k_{\text{proct}}^{\text{indrat}}}(R\Gamma(K, M), R\Gamma(K, \mathbf{G}_m[1])) \\ &\rightarrow R \mathbf{Hom}_{k_{\text{proct}}^{\text{indrat}}}(R\Gamma(K, M), \mathbf{Z}[1]) = R\Gamma(K, M)^{\text{SD}}[1] \end{aligned}$$

in $D(k_{\text{proct}}^{\text{indrat}})$ as in [40, §2.5]. Its Serre dual (when M and M^\vee are switched)

$$R\Gamma(K, M^\vee)^{\text{SDSD}} \rightarrow R\Gamma(K, M)^{\text{SD}}[1]$$

is an isomorphism by [39, Thm. (9.1)] and the comparison results in [40, Appendix A]. This result is the main input for the results of this section.

By Prop. 4.2, the distinguished triangle in Def. 3.8 and the commutative diagram in Prop. 4.10 can be translated in the fppf site.

PROPOSITION 6.1. *The distinguished triangle in Def. 3.8 and the localization triangle in [40, §2.5] (i.e. the definition of $R\Gamma_x$ as a mapping cone) induce a commutative diagram of distinguished triangles*

$$\begin{array}{ccccc} R\Gamma_x(\mathcal{O}_K, \mathcal{N}_0(M)) & \longrightarrow & R\Gamma_x(\mathcal{O}_K, \mathcal{N}(M)) & \longrightarrow & \mathcal{P}(M) \\ \downarrow & & \downarrow & & \parallel \\ R\Gamma(\mathcal{O}_K, \mathcal{N}_0(M)) & \longrightarrow & R\Gamma(\mathcal{O}_K, \mathcal{N}(M)) & \longrightarrow & \mathcal{P}(M) \\ \downarrow & & \downarrow & & \downarrow \\ R\Gamma(K, M) & \xlongequal{\quad\quad\quad} & R\Gamma(K, M) & \longrightarrow & 0. \end{array}$$

Proof. Obvious. □

PROPOSITION 6.2.

- (1) About $R\Gamma(\mathcal{O}_K, \mathcal{N}_0(M))^{\text{SDSD}}$: The H^{-1} is the Tate module $T\Gamma(K, G)_{\text{sAb}}$ of the maximal semi-abelian subgroup $\Gamma(K, G)_{\text{sAb}}$ of the proalgebraic group $\Gamma(K, G)$. The H^0 is the quotient $\Gamma(K, G)_{0/\text{sAb}}$ of the identity component $\Gamma(K, G)_0$ by $\Gamma(K, G)_{\text{sAb}}$.
- (2) About $R\Gamma_x(\mathcal{O}_K, \mathcal{N}(M))^{\text{SD}}$: The H^{-1} is the Pontryagin dual $\pi_0(\mathbf{H}^1(K, M))^{\text{PD}}$ of the component group $\pi_0(\mathbf{H}^1(K, M))$ of the ind-algebraic group $\mathbf{H}^1(K, M)$. The H^0 is the dual

$$\mathbf{H}^1(K, M)_0^{\text{SD}'} := \mathbf{Ext}_{k_{\text{proct}}^{\text{indrat}}}^1(\mathbf{H}^1(K, M)_0, \mathbf{Q}/\mathbf{Z})$$

of the identity component $\mathbf{H}^1(K, M)_0$.

- (3) About $R\Gamma_x(\mathcal{O}_K, \mathcal{N}(M))$: The H^0 is $H^{-1}\mathcal{P}(M)$. The H^1 is $H^0\mathcal{P}(M)$. The H^2 is $\mathbf{H}^1(K, M)$.

- (4) About $R\Gamma(\mathcal{O}_K, \mathcal{N}(M))^{\text{SD}}$: The H^0 is $H^0(\mathcal{P}(M)^{\text{LD}})$. The H^1 is $H^1(\mathcal{P}(M)^{\text{LD}})$. The H^2 is the dual

$$\Gamma(K, G)_0^{\text{SD}'} := \mathbf{Ext}_{k_{\text{proet}}^{\text{indrat}}}^1(\Gamma(K, G)_0, \mathbf{Q}/\mathbf{Z})$$

of the identity component $\Gamma(K, G)_0$.

In both cases (1) and (2), the H^{-1} is a pro-finite-étale group scheme over k and the H^0 is a connected unipotent proalgebraic group over k . In both cases (3) and (4), the H^0 is a lattice, $H^1 \in \text{EtGp}^f/k$ and H^2 an ind-algebraic group with unipotent identity component. All the four complexes above are Serre reflexive. We have $H^n = 0$ for all the complexes for all other degrees.

Proof. (1) Since \mathcal{Y}_0 is the extension by zero of the étale group Y , we have $R\Gamma(\mathcal{O}_K, \mathcal{Y}_0) = 0$ by [39, Prop. (5.2.3.4)]. We have

$$R\Gamma(\mathcal{O}_K, \mathcal{G}_0) = \Gamma(\mathcal{O}_K, \mathcal{G}_0) = \Gamma(\mathcal{O}_K, \mathcal{G}_0)_0 = \Gamma(K, G)_0,$$

where the first equality is [39, Prop. (3.4.1)] (with the smoothness of \mathcal{G}_0), the second [39, Prop. (3.4.2) (a)] and the third [39, Prop. (3.1.3) (c)]. Hence $R\Gamma(\mathcal{O}_K, \mathcal{N}_0(M)) = \Gamma(K, G)_0$, which is a connected proalgebraic group by [39, Prop. (3.4.2) (a)]. Therefore the description of its double dual follows from [39, Prop. (2.4.1) (d) and Footnote 7].

(2) We have

$$R\Gamma(\mathcal{O}_K, \mathcal{N}(M)) = \tau_{\leq 0}R\Gamma(\mathcal{O}_K, Rj_*M) = \tau_{\leq 0}R\Gamma(K, M)$$

by [39, Prop. (3.4.1)] (truncation commutes with exact functors). Hence one of the distinguished triangles in Prop. 6.1 shows that

$$R\Gamma_x(\mathcal{O}_K, \mathcal{N}(M)) = \mathbf{H}^1(K, M)[-2]$$

since $R\Gamma(K, M)$ is concentrated in degrees $-1, 0, 1$ (see [39, first paragraph of §9]). By loc. cit., we know that $\mathbf{H}^1(K, M)$ is an ind-algebraic group with unipotent identity component. Hence [39, Prop. (2.4.1) (b)] gives the required description of its Serre dual.

(3) One of the distinguished triangles in Prop. 6.1 and what we saw right above give a distinguished triangle

$$R\Gamma_x(\mathcal{O}_K, \mathcal{N}_0(M)) \rightarrow \mathbf{H}^1(K, M)[-2] \rightarrow \mathcal{P}(M).$$

The result follows from this.

(4) One of the distinguished triangles in Prop. 6.1 and what we saw in the proof of (1) above give a distinguished triangle

$$\mathcal{P}(M)^{\text{LD}} \rightarrow R\Gamma(\mathcal{O}_K, \mathcal{N}(M))^{\text{SD}} \rightarrow \Gamma(K, G)_0^{\text{SD}}.$$

We have

$$\begin{aligned} \Gamma(K, G)_0^{\text{SD}} &= R \mathbf{Hom}_{k_{\text{proet}}^{\text{indrat}}}(\Gamma(K, G)_0, \mathbf{Q}/\mathbf{Z})[-1] \\ &= \mathbf{Ext}_{k_{\text{proet}}^{\text{indrat}}}^1(\Gamma(K, G)_0, \mathbf{Q}/\mathbf{Z})[-2] \\ &= \Gamma(K, G)_0^{\text{SD}'}[-2] \end{aligned}$$

by [39, Prop. (2.3.3) (d), (2.4.1) (a)]. Hence the statements in (4) follow. The group $\Gamma(K, G)_0^{\text{SD}'}$ is an ind-algebraic group with unipotent identity component by [39, (2.4.1) (d)].

We can check that the cohomology objects of all the four complexes are Serre reflexive using [39, (2.4.1) (b)]. Hence the four complexes themselves are Serre reflexive. \square

By Prop. 4.2, the morphism

$$\mathcal{N}(M^\vee) \rightarrow R \mathbf{Hom}_{\mathcal{O}_{K, \text{sm}}}(\mathcal{N}_0(M), \mathbf{G}_m[1])$$

in $D(\mathcal{O}_{K, \text{sm}})$ given in Def. 4.8 induces a morphism

$$\mathcal{N}(M^\vee) \rightarrow R \mathbf{Hom}_{\mathcal{O}_{K, \text{fppf}}}(\mathcal{N}_0(M), \mathbf{G}_m[1])$$

in $D(\mathcal{O}_{K, \text{fppf}})$. Hence [39, Prop. (3.3.8)] and the trace morphism give a morphism of distinguished triangles

$$\begin{array}{ccccc} R\Gamma(\mathcal{O}_K, \mathcal{N}(M)) & \longrightarrow & R\Gamma(K, M) & \longrightarrow & R\Gamma_x(\mathcal{O}_K, \mathcal{N}(M))[1] \\ \downarrow & & \downarrow & & \downarrow \\ R\Gamma_x(\mathcal{O}_K, \mathcal{N}_0(M^\vee))^{\text{SD}} & \longrightarrow & R\Gamma(K, M^\vee)^{\text{SD}}[1] & \longrightarrow & R\Gamma(\mathcal{O}_K, \mathcal{N}_0(M^\vee))^{\text{SD}}[1] \end{array}$$

Applying SD, shifting by one and using the Serre reflexivity of $R\Gamma_x(\mathcal{O}_K, \mathcal{N}_0(M^\vee))$ (Prop. 6.2), we have a morphism of distinguished triangles

$$(6.1) \quad \begin{array}{ccccc} R\Gamma(\mathcal{O}_K, \mathcal{N}_0(M^\vee))^{\text{SDSD}} & \longrightarrow & R\Gamma(K, M^\vee)^{\text{SDSD}} & \longrightarrow & R\Gamma_x(\mathcal{O}_K, \mathcal{N}_0(M^\vee))[1] \\ \downarrow & & \downarrow & & \downarrow \\ R\Gamma_x(\mathcal{O}_K, \mathcal{N}(M))^{\text{SD}} & \longrightarrow & R\Gamma(K, M)^{\text{SD}}[1] & \longrightarrow & R\Gamma(\mathcal{O}_K, \mathcal{N}(M))^{\text{SD}}[1]. \end{array}$$

To simplify the notation, we denote the upper triangle by $C \rightarrow D \rightarrow E$ and lower by $C' \rightarrow D' \rightarrow E'$. As noted earlier, the middle vertical morphism is an isomorphism by [39, Thm. (9.1)], so $D \xrightarrow{\sim} D'$. The above diagram induces a morphism from the long exact sequence of cohomologies of $C \rightarrow D \rightarrow E$ to the long exact sequence of cohomologies of $C' \rightarrow D' \rightarrow E'$. We can spell it out using Prop. 6.2 as follows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & T\Gamma(K, G')_{\text{sAb}} & \longrightarrow & H^{-1}D & \longrightarrow & H^{-1}\mathcal{P}(M^\vee) \\ & & \downarrow & & \downarrow \wr & & \downarrow \\ 0 & \longrightarrow & (\pi_0 \mathbf{H}^1(K, M))^{\text{PD}} & \longrightarrow & H^{-1}D' & \longrightarrow & H^0(\mathcal{P}(M))^{\text{LD}} \\ \longrightarrow & & \Gamma(K, G')_{0/\text{sAb}} & \longrightarrow & H^0D & \longrightarrow & H^0\mathcal{P}(M) \longrightarrow 0 \\ & & \downarrow & & \downarrow \wr & & \downarrow \\ \longrightarrow & & \mathbf{H}^1(K, M)_0^{\text{SD}' } & \longrightarrow & H^0D' & \longrightarrow & H^1(\mathcal{P}(M))^{\text{LD}} \longrightarrow 0, \end{array}$$

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H^1 D & \longrightarrow & \mathbf{H}^1(K, M^\vee) & \longrightarrow & 0 \\
 & & \downarrow \wr & & \downarrow & & \\
 0 & \longrightarrow & H^1 D' & \longrightarrow & \Gamma(K, G)_0^{\text{SD}'} & \longrightarrow & 0.
 \end{array}$$

The upper, middle and lower diagrams are for the H^{-1} , H^0 and H^1 , respectively.

PROPOSITION 6.3. *The morphism of distinguished triangles (6.1) is an isomorphism of distinguished triangles. The H^1 of the right vertical isomorphism gives an isomorphism*

$$(6.2) \quad \mathbf{H}^1(K, M^\vee) \xrightarrow{\sim} \Gamma(K, G)_0^{\text{SD}'}$$

Proof. The latter statement about H^1 is clear by the paragraph before the proposition. Applying $R\Gamma_x(\mathcal{O}_K, \cdot)$ to the diagram in Prop. 4.14, and using [39, Prop. 3.3.8] and the trace morphism, we have a morphism of distinguished triangles

$$(6.3) \quad \begin{array}{ccccc}
 R\Gamma_x(\mathcal{O}_K, \mathcal{N}_0(M^\vee)) & \longrightarrow & R\Gamma_x(\mathcal{O}_K, \mathcal{N}(M^\vee)) & \longrightarrow & \mathcal{P}(M^\vee) \\
 \downarrow & & \downarrow & & \downarrow \\
 R\Gamma(\mathcal{O}_K, \mathcal{N}(M))^{\text{SD}} & \longrightarrow & R\Gamma(\mathcal{O}_K, \mathcal{N}_0(M))^{\text{SD}} & \longrightarrow & \mathcal{P}(M)^{\text{LD}}[1].
 \end{array}$$

The morphism in H^2 of the left vertical morphism is the isomorphism (6.2). The objects $\mathcal{P}(M)$ and $\mathcal{P}(M^\vee)^{\text{LD}}[1]$ are concentrated in degrees $-1, 0$ by Prop. 3.9 and the proof of Prop. 4.10. Hence the left horizontal two morphisms are both isomorphisms in H^2 . The upper middle term $R\Gamma_x(\mathcal{O}_K, \mathcal{N}(M^\vee))$ is concentrated in degree 2 as we saw in the proof of Prop. 6.2 (2). As we saw in the proof of Prop. 6.2 (1), the object $R\Gamma(\mathcal{O}_K, \mathcal{N}_0(M)) = \Gamma(K, G)_0$ is a connected proalgebraic group. Hence its Serre dual is concentrated in degree 2 by [39, Prop. (2.4.1) (b)]. Therefore the lower middle term $R\Gamma(\mathcal{O}_K, \mathcal{N}_0(M))^{\text{SD}}$ in the above diagram is also concentrated in degree 2. Combining all these, we know that the middle vertical morphism in the above diagram is an isomorphism. Its Serre dual is $C \rightarrow C'$ with M replaced by M^\vee . Therefore $C \rightarrow C'$ is an isomorphism. Hence $E \rightarrow E'$ is an isomorphism. \square

PROPOSITION 6.4. *The morphism η_M is an isomorphism.*

Proof. The right vertical morphism in (6.3) is η_M by Prop. 4.14 and the construction of (6.3). The middle vertical morphism is an isomorphism as seen in the proof of Prop. 6.3. The left vertical morphism is $E \rightarrow E'$ up to shift, which is an isomorphism by Prop. 6.3. Therefore η_M is an isomorphism. \square

The following finishes the proof of Thm. B (5) and hence of Thm. B itself.

PROPOSITION 6.5. *Let X be an irreducible Dedekind scheme and $j : U \hookrightarrow X$ a dense open subscheme with complement Z . Assume the residue fields of Z are perfect. Then for any $M \in \mathcal{M}_U$, the morphism η_M is an isomorphism.*

Proof. This follows from Prop. 4.17, 5.9 (for zero residual characteristics) and 6.4 (for positive residual characteristics). \square

Remark 6.6. The right-hand side of (6.2) depends only on G and not on Y . Hence the left-hand side actually depends only on $[Y' \rightarrow A']$ and not on T' . This can also be checked directly by noting that $\mathbf{H}^n(K, T') = 0$ for $n \geq 1$ by [39, Prop. (3.4.3) (e)], the distinguished triangle $T' \rightarrow M^\vee \rightarrow [Y' \rightarrow A']$ and hence an isomorphism $\mathbf{H}^1(K, M^\vee) \xrightarrow{\sim} \mathbf{H}^1(K, [Y' \rightarrow A'])$. In particular, (6.2) can be written as

$$\mathbf{H}^1(K, [Y' \rightarrow A']) \xrightarrow{\sim} \Gamma(K, G)_0^{\text{SD}'}$$

A similar remark exists for Prop. 5.8.

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