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A p-ANALOGUE OF THE MULTIPLE EULER CONSTANT

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Abstract

We study a p-analogue of the multiple Euler constant. Then we show that it can be described by the congruence zeta function attached to powers of $\mathbf{G}_m$ over $\mathbf{F}_p$. Moreover, we show that it converges to the multiple Euler constant as $p \to 1$.

1. Introduction

Let $r = 1, 2, 3, \ldots$ and $p > 1$. Then we define a p-analogue of the multiple Euler constant $\gamma_r(p)$ by

$$\gamma_r(p) := \left(\frac{\log p}{p-1}\right)^r \sum_{n=1}^{\infty} \frac{n^{r-1} p^{(r-1)n}}{[n]_p^r} - \log\left(\frac{p}{p-1}\right)$$
$$= (\log p)^r \sum_{n=1}^{\infty} \frac{n^{r-1} p^{(r-1)n}}{(p^n - 1)^r} - \log\left(\frac{p}{p-1}\right),$$

where

$$[x]_p := \frac{p^x - 1}{p - 1}.$$

Now we define the congruence zeta function $\zeta_{\mathbf{G}_m^{n-1}/\mathbf{F}_p}(s)$ as

$$\zeta_{\mathbf{G}_m^{n-1}/\mathbf{F}_p}(s) := \exp\left(\sum_{m=1}^{\infty} \frac{|\mathbf{G}_m^{n-1}(\mathbf{F}_{p^m})|}{m} p^{-ms}\right).$$

We prove the following results:

THEOREM 1.

$$\gamma_r(p) = \sum_{n=2}^{\infty} c_r(n) \log \zeta_{\mathbf{G}_m^{n-1}/\mathbf{F}_p}(n),$$

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where $c_r(n)$ is defined by

$$\sum_{n=1}^{\infty} c_r(n)x^{n-1} = \left(\sum_{n=1}^{\infty} \frac{x^{n-1}}{n}\right)^r = \left(\frac{-\log(1-x)}{x}\right)^r, \quad (|x| < 1).$$

We remark that in [5], Kurokawa and Taguchi studied

$$\gamma(p) := \sum_{m=1}^{\infty} \frac{1}{[m]_p},$$

which satisfies the following identity

$$\gamma_1(p) = \frac{\log p}{p-1}\gamma(p) - \log\left(\frac{p}{p-1}\right).$$

Theorem 1 is a generalization of [5, Theorem 1]:

$$\sum_{n=1}^{\infty} \frac{1}{n} \log \zeta_{\mathbf{G}_m^{n-1}/\mathbf{F}_p}(n) = \frac{\log p}{p-1}\gamma(p)$$
$$\Leftrightarrow \sum_{n=2}^{\infty} \frac{1}{n} \log \zeta_{\mathbf{G}_m^{n-1}/\mathbf{F}_p}(n) = \frac{\log p}{p-1}\gamma(p) - \log\left(\frac{p}{p-1}\right)$$
$$\Leftrightarrow \sum_{n=2}^{\infty} \frac{1}{n} \log \zeta_{\mathbf{G}_m^{n-1}/\mathbf{F}_p}(n) = \gamma_1(p).$$

In [8], Kurokawa and Wakayama studied

$$\gamma^{\mathrm{KW}}(p) = \sum_{m=1}^{\infty} \frac{1}{[m]_p} + \frac{(p-1)\log(p-1)}{\log p} - \frac{p-1}{2}$$
$$= \gamma(p) + \frac{(p-1)\log(p-1)}{\log p} - \frac{p-1}{2}$$
$$= \frac{p-1}{\log p}\gamma_1(p) + \frac{p-1}{2}$$

and proved that

$$\lim_{p\to 1} \gamma^{\mathrm{KW}}(p) = \gamma,$$

where $\gamma = 0.5772\ldots$ is the usual Euler constant.

Now we define the multiple zeta function $\zeta_r(s)$ and the multiple Euler constant γ_r as

$$\zeta_r(s) := \sum_{n=1}^{\infty} (n+r-2)\cdots n \cdot n^{-s}$$

and

$$\gamma_r := \lim_{s\to r}\left(\zeta_r(s) - \frac{1}{s-r}\right) + H_{r-1}$$

respectively, where

$$H_{r-1} := \sum_{k=1}^{r-1}\frac{1}{k}.$$

is the Harmonic number. We have

$$\zeta_1(s) = \zeta(s),$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function, and

$$\gamma_1 = \gamma$$

respectively.

We give some numerical examples of the values of $\gamma^{\mathrm{KW}}(p)$ and $\gamma_r(p)$:

p	$\gamma^{\mathrm{KW}}(p)$	$\gamma_1(p)$	$\gamma_2(p)$	$\gamma_3(p)$	$\gamma_4(p)$
1.5	0.8438...	0.4815...	1.3824...	3.4248...	9.3462...
1.1	0.6309...	0.5537...	1.5300...	3.6510...	9.6542...
1.01	0.5826...	0.5747...	1.5722...	3.7146...	9.7395...
1.001	0.5777...	0.5769...	1.5767...	3.7214...	9.7484...
1	γ	$\gamma_1 = \gamma$ $= 0.5772\ldots$	$\gamma_2 = \gamma + 1$ $= 1.5772\ldots$	$\gamma_3 = \gamma + \zeta(2) + \frac{3}{2}$ $= 3.7221\ldots$	$\gamma_4 = \gamma + 2\zeta(3) + 3\zeta(2) + \frac{11}{6}$ $= 9.7494\ldots$

The above values of γ_r are calculated as follows.

THEOREM 2.

(1) *Let* $\begin{bmatrix} n \\ k \end{bmatrix}$ *be the Stirling number of the first kind defined by*

$$x(x+1)\cdots(x+n-1) = \sum_{k=1}^{n}\begin{bmatrix} n \\ k \end{bmatrix} x^k.$$

Then

$$\zeta_r(s) = \sum_{k=1}^{r-1}\begin{bmatrix} r-1 \\ k \end{bmatrix}\zeta(s-k)$$

$$= \zeta(s-r+1) + \sum_{k=1}^{r-2}\begin{bmatrix} r-1 \\ k \end{bmatrix}\zeta(s-k)$$

has an analytic continuation to all $s \in \mathbf{C}$. *It has simple poles at* $s = k+1$ *with residue* $\begin{bmatrix} r-1 \\ k \end{bmatrix}$ *for* $k = 1, \ldots, r-1$.

(2)

$$\gamma_r = \gamma + \sum_{k=1}^{r-2} \begin{bmatrix} r-1 \\ k \end{bmatrix} \zeta(r-k) + H_{r-1}.$$

In Theorem 3 below we need the absolute zeta function $\zeta_{\mathbf{G}_m^n/\mathbf{F}_1}(s)$ of $\mathbf{G}_m^n = \mathrm{GL}(1)^n$ $(n \geq 1)$ defined as

$$\zeta_{\mathbf{G}_m^n/\mathbf{F}_1}(s) := \lim_{p \to 1} \zeta_{\mathbf{G}_m^n/\mathbf{F}_p}(s),$$

where the right hand side is the congruence zeta function; see Soulé [9] and Connes-Consani [1]. We recall another construction of absolute zeta functions using absolute automorphic forms given by [4], [5], [6]. For a function

$$f : \mathbf{R}_{>0} \to \mathbf{R}$$

satisfying the absolute automorphy

$$f\left(\frac{1}{x}\right) = Cx^{-D} f(x)$$

for constants C and D, we define the absolute zeta function $\zeta_f(s)$ of f by

$$\zeta_f(s) := \exp\left(\frac{\partial}{\partial w} Z_f(w,s)\bigg|_{w=0}\right)$$

with

$$Z_f(w,s) := \frac{1}{\Gamma(w)} \int_1^\infty f(x) x^{-s-1} (\log x)^{w-1}\, dx.$$

By this construction, $\zeta_{\mathbf{G}_m^n/\mathbf{F}_1}(s)$ is given as

$$\zeta_{\mathbf{G}_m^n/\mathbf{F}_1}(s) = \zeta_f(s)$$

for $f(x) = (x-1)^n$.

Theorem 3. *For any* $r \geq 1$,

$$\gamma_r = \sum_{n=2}^{\infty} c_r(n) \log \zeta_{\mathbf{G}_m^{n-1}/\mathbf{F}_1}(n).$$

Theorem 4. *For any* $r \geq 1$,

$$\lim_{p \to 1} \gamma_r(p) = \gamma_r.$$

2. Proof of Theorem 1

First, we prove the following formula:

LEMMA 1. *For* $n \geq 2$,

$$\zeta_{\mathbf{G}_m^{n-1}/\mathbf{F}_p}(n) = \prod_{k=1}^{n}(1 - p^{-k})^{(-1)^k \binom{n-1}{k-1}}.$$

Proof of Lemma 1.

$$\begin{aligned}
\zeta_{\mathbf{G}_m^n/\mathbf{F}_p}(s) &= \exp\left(\sum_{m=1}^{\infty} \frac{|\mathbf{G}_m^n(\mathbf{F}_{p^m})|}{m} p^{-ms}\right) \\
&= \exp\left(\sum_{m=1}^{\infty} \frac{(p^m - 1)^n}{m} p^{-ms}\right) \\
&= \exp\left(\sum_{m=1}^{\infty} \frac{1}{m}\left(\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}p^{mk}\right)p^{-ms}\right) \\
&= \prod_{k=0}^{n}(1 - p^{-(s-k)})^{(-1)^{n+1-k}\binom{n}{k}}.
\end{aligned}$$

Then letting $s = n + 1$, and taking $n - 1$ instead of n we have Lemma 1. □

Now, we prove Theorem 1. Using Lemma 1, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} c_r(n) \log \zeta_{\mathbf{G}_m^{n-1}/\mathbf{F}_p}(n) &= \sum_{n=1}^{\infty} c_r(n) \log\left(\prod_{k=1}^{n}(1 - p^{-k})^{(-1)^k\binom{n-1}{k-1}}\right) \\
&= \sum_{n=1}^{\infty} c_r(n) \sum_{k=1}^{n}(-1)^k\binom{n-1}{k-1}\log(1 - p^{-k}) \\
&= \sum_{n=1}^{\infty} c_r(n) \sum_{k=0}^{n-1}(-1)^{k+1}\binom{n-1}{k}\log(1 - p^{-(k+1)}) \\
&= \sum_{n=1}^{\infty} c_r(n) \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\sum_{m=1}^{\infty}\frac{1}{m}p^{-m(k+1)} \\
&= \sum_{n=1}^{\infty} c_r(n) \sum_{m=1}^{\infty}\frac{p^{-m}}{m}\left(\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}p^{-mk}\right) \\
&= \sum_{n=1}^{\infty} c_r(n) \sum_{m=1}^{\infty}\frac{p^{-m}}{m}(1 - p^{-m})^{n-1}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=1}^{\infty} \frac{p^{-m}}{m} \sum_{n=1}^{\infty} c_r(n)(1-p^{-m})^{n-1} \\
&= \sum_{m=1}^{\infty} \frac{p^{-m}}{m} \left(\frac{-\log(1-(1-p^{-m}))}{1-p^{-m}} \right)^r \\
&= \sum_{m=1}^{\infty} \frac{p^{-m}}{m} \left(\frac{m \log p}{1-p^{-m}} \right)^r \\
&= (\log p)^r \sum_{m=1}^{\infty} \frac{m^{r-1} p^{-m}}{(1-p^{-m})^r} \\
&= (\log p)^r \sum_{m=1}^{\infty} \frac{m^{r-1} p^{(r-1)m}}{(p^m-1)^r} \\
&= \left(\frac{\log p}{p-1} \right)^r \sum_{m=1}^{\infty} \frac{m^{r-1} p^{(r-1)m}}{[m]_p^r}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\zeta_{\mathbf{G}_m^0/\mathbf{F}_p}(1) &= (1-p^{-1})^{-1} \\
&= \frac{p}{p-1},
\end{aligned}
$$

we have

$$
\begin{aligned}
\sum_{n=2}^{\infty} c_r(n) \log \zeta_{\mathbf{G}_m^{n-1}/\mathbf{F}_p}(n) &= \sum_{n=1}^{\infty} c_r(n) \log \zeta_{\mathbf{G}_m^{n-1}/\mathbf{F}_p}(n) - c_r(1) \log \zeta_{\mathbf{G}_m^0/\mathbf{F}_p}(1) \\
&= \left(\frac{\log p}{p-1} \right)^r \sum_{m=1}^{\infty} \frac{m^{r-1} p^{(r-1)m}}{[m]_p^r} - \log\left(\frac{p}{p-1} \right) \\
&= \gamma_r(p).
\end{aligned}
$$

Hence, we obtain Theorem 1. Q.E.D.

3. Proof of Theorem 2

First, we prove (1) of Theorem 2. Since

$$
(n+r-2)\cdots n = \sum_{k=1}^{r-1} \begin{bmatrix} r-1 \\ k \end{bmatrix} n^k,
$$

we have

$$\zeta_r(s) = \sum_{n=1}^{\infty}\left(\sum_{k=1}^{r-1}\begin{bmatrix} r-1 \\ k \end{bmatrix} n^k\right) n^{-s}$$
$$= \sum_{k=1}^{r-1}\begin{bmatrix} r-1 \\ k \end{bmatrix}\left(\sum_{n=1}^{\infty} n^{k-s}\right)$$
$$= \sum_{k=1}^{r-1}\begin{bmatrix} r-1 \\ k \end{bmatrix}\zeta(s-k)$$
$$= \zeta(s-r+1) + \sum_{k=1}^{r-2}\begin{bmatrix} r-1 \\ k \end{bmatrix}\zeta(s-k).$$

(2) of Theorem 2 is given by

$$\gamma_r = \lim_{s\to r}\left(\zeta_r(s) - \frac{1}{s-r}\right) + H_{r-1}$$
$$= \lim_{s\to r}\left\{\left(\zeta(s-r+1) - \frac{1}{s-r}\right) + \sum_{k=1}^{r-2}\begin{bmatrix} r-1 \\ k \end{bmatrix}\zeta(s-k)\right\} + H_{r-1}$$
$$= \gamma + \sum_{k=1}^{r-2}\begin{bmatrix} r-1 \\ k \end{bmatrix}\zeta(r-k) + H_{r-1},$$

where we used

$$\lim_{s\to r}\left(\zeta(s-r+1) - \frac{1}{s-r}\right) = \lim_{s\to 1}\left(\zeta(s) - \frac{1}{s-1}\right)$$
$$= \gamma.$$

Thus, we obtain Theorem 2. Q.E.D.

4. Proof of Theorem 3

Since

$$\zeta_r(s) = \sum_{n=1}^{\infty}(n+r-2)\cdots n\cdot n^{-s}$$
$$= \sum_{n=1}^{\infty}(r-1)!\binom{n+r-2}{r-1}n^{-s}$$
$$= (r-1)!\sum_{n=0}^{\infty}\binom{n+r-1}{r-1}(n+1)^{-s}$$
$$= (r-1)!\sum_{n_1,\ldots,n_r\geq 0}(n_1+\cdots+n_r+1)^{-s},$$

we have

$$\zeta_r(s) = (r-1)! \sum_{n_1,\dots,n_r \geq 0} (n_1 + \cdots + n_r + 1)^{-s}.$$

(Here we remark that the multiple Euler constant γ_r is essentially studied in [7] where we proved the limit formula for the multiple Hurwitz zeta function.) Hence

$$\zeta_r(s) = \frac{(r-1)!}{\Gamma(s)} \int_1^\infty \frac{x^{-1}}{(1-x^{-1})^r} (\log x)^{s-1} \frac{dx}{x},$$

where we used

$$\begin{aligned}
\frac{x^{-1}}{(1-x^{-1})^r} &= x^{-1}\left(\sum_{n=0}^{\infty} x^{-n}\right)^r \\
&= x^{-1}\left(\sum_{n_1=0}^{\infty} x^{-n_1}\right) \cdots \left(\sum_{n_r=0}^{\infty} x^{-n_r}\right) \\
&= \sum_{n_1,\dots,n_r \geq 0} x^{-(n_1+\cdots+n_r+1)}.
\end{aligned}$$

This gives

$$\begin{aligned}
\zeta_r(s) &= \frac{(r-1)!}{\Gamma(s)} \int_1^\infty \left(\frac{\log x}{1-x^{-1}}\right)^r x^{-2} (\log x)^{s-r-1}\, dx \\
&= \frac{(r-1)!}{\Gamma(s)} \int_1^\infty \left(\frac{-\log(1-(1-x^{-1}))}{1-x^{-1}}\right)^r x^{-2} (\log x)^{s-r-1}\, dx.
\end{aligned}$$

Putting $u = 1 - x^{-1}$, we have

$$\begin{aligned}
\left(\frac{-\log(1-(1-x^{-1}))}{1-x^{-1}}\right)^r &= \left(\frac{-\log(1-u)}{u}\right)^r \\
&= \left(\sum_{n=1}^{\infty} \frac{1}{n} u^{n-1}\right)^r \\
&= \sum_{n=1}^{\infty} c_r(n) u^{n-1} \\
&= \sum_{n=1}^{\infty} c_r(n)(1-x^{-1})^{n-1}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\zeta_r(s) &= \sum_{n=1}^{\infty} \frac{(r-1)!}{\Gamma(s)} \int_1^{\infty} c_r(n)(1-x^{-1})^{n-1} x^{-2} (\log x)^{s-r-1}\, dx \\
&= \sum_{n=1}^{\infty} c_r(n) \frac{(r-1)!}{\Gamma(s)} \int_1^{\infty} (1-x^{-1})^{n-1} x^{-2} (\log x)^{s-r-1}\, dx \\
&= \sum_{n=1}^{\infty} c_r(n) \frac{(r-1)!}{\Gamma(s)} \int_1^{\infty} \left(\sum_{k=1}^{n} (-1)^{k-1} \binom{n-1}{k-1} x^{-k-1} \right) (\log x)^{s-r-1}\, dx \\
&= \sum_{n=1}^{\infty} c_r(n) \frac{(r-1)!}{\Gamma(s)} \sum_{k=1}^{n} (-1)^{k-1} \binom{n-1}{k-1} \Gamma(s-r) k^{r-s} \\
&= \sum_{n=1}^{\infty} c_r(n) \frac{(r-1)! \sum_{k=1}^{n} (-1)^{k-1} \binom{n-1}{k-1} k^{r-s}}{(s-1)\cdots(s-r)} \\
&= \frac{(r-1)!}{(s-1)\cdots(s-r)} + \sum_{n=2}^{\infty} c_r(n) \frac{(r-1)! Z_{\mathbf{G}_m^{n-1}/\mathbf{F}_1}(s-r, n)}{(s-1)\cdots(s-r)},
\end{aligned}$$

where

$$\begin{aligned}
Z_{\mathbf{G}_m^n/\mathbf{F}_1}(w, s) &= \frac{1}{\Gamma(w)} \int_1^{\infty} \left(\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} x^k \right) x^{-s-1} (\log x)^{w-1}\, dx \\
&= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{1}{\Gamma(w)} \int_1^{\infty} x^{-(s-k)-1} (\log x)^{w-1}\, dx \\
&= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (s-k)^{-w}.
\end{aligned}$$

This expression gives the analytic continuation of $\zeta_r(s)$ as a meromorphic function as in Hasse [3]. Since

$$\lim_{s\to r} \left(\frac{(r-1)!}{(s-1)\cdots(s-r)} - \frac{1}{s-r} \right) = -H_{r-1},$$

$$\lim_{s\to r} \frac{(r-1)! Z_{\mathbf{G}_m^{n-1}/\mathbf{F}_1}(s-r, n)}{(s-1)\cdots(s-r)} = \log \zeta_{\mathbf{G}_m^{n-1}/\mathbf{F}_1}(n),$$

we have

$$\begin{aligned}
\gamma_r &= \lim_{s\to r} \left(\zeta_r(s) - \frac{1}{s-r} \right) + H_{r-1} \\
&= \lim_{s\to r} \left(\frac{(r-1)!}{(s-1)\cdots(s-r)} - \frac{1}{s-r} \right)
\end{aligned}$$

$$+\lim_{s\to r}\sum_{n=2}^{\infty} c_r(n)\frac{(r-1)!Z_{\mathbf{G}_m^{n-1}/\mathbf{F}_1}(s-r,n)}{(s-1)\cdots(s-r)}+H_{r-1}$$

$$=\sum_{n=2}^{\infty} c_r(n)\log\zeta_{\mathbf{G}_m^{n-1}/\mathbf{F}_1}(n). \qquad \text{Q.E.D.}$$

5. Proof of Theorem 4

First, we prove the following formula:

LEMMA 2. *For* $q=p^{-1}$ *let*

$$\gamma_r^\star(q):=\frac{\log q}{q-1}\int_0^1\left(1-\left(\frac{\log x}{x-1}\right)^r\right)\frac{d_qx}{\log x},$$

where we use the Jackson q*-integral defined by*

$$\int_0^1 f(x)\,d_qx=\sum_{m=1}^{\infty} f(q^m)(q^m-q^{m+1}).$$

Then:

$$\gamma_r^\star(q)=\sum_{n=2}^{\infty} c_r(n)\log\zeta_{\mathbf{G}_m^{n-1}/\mathbf{F}_p}(n)$$
$$=\gamma_r(p)$$

when $q^{-1}=p$ *is a prime number.*

Proof of Lemma 2. By the definition

$$\gamma_r^\star(q)=\frac{\log q}{q-1}\sum_{m=1}^{\infty}\left(1-\left(\frac{\log(q^m)}{q^m-1}\right)^r\right)\frac{q^m-q^{m+1}}{\log(q^m)}$$
$$=-\sum_{m=1}^{\infty}\frac{q^m}{m}\left(1-\left(\frac{-\log(1-(1-q^m))}{1-q^m}\right)^r\right)$$
$$=-\sum_{m=1}^{\infty}\frac{q^m}{m}\left(1-\sum_{n=1}^{\infty}c_r(n)(1-q^m)^{n-1}\right)$$
$$=\sum_{n=2}^{\infty}c_r(n)\left(\sum_{m=1}^{\infty}\frac{q^m}{m}(1-q^m)^{n-1}\right)$$
$$=\sum_{n=2}^{\infty}c_r(n)\sum_{m=1}^{\infty}\frac{q^m}{m}\left(\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}q^{mk}\right)$$

$$= \sum_{n=2}^{\infty} c_r(n) \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \left(\sum_{m=1}^{\infty} \frac{q^{m(k+1)}}{m} \right)$$

$$= \sum_{n=2}^{\infty} c_r(n) \sum_{k=0}^{n-1} (-1)^{k+1} \binom{n-1}{k} \log(1-q^{k+1})$$

$$= \sum_{n=2}^{\infty} c_r(n) \log\left(\prod_{k=0}^{n-1} (1-q^{k+1})^{(-1)^{k+1}\binom{n-1}{k}} \right)$$

$$= \sum_{n=2}^{\infty} c_r(n) \log\left(\prod_{k=1}^{n} (1-q^{k})^{(-1)^{k}\binom{n-1}{k-1}} \right)$$

$$= \sum_{n=2}^{\infty} c_r(n) \log \zeta_{\mathbf{G}_m^{n-1}/\mathbf{F}_p}(n).$$

Hence, by Theorem 1

$$\gamma_r^{\star}(q) = \gamma_r(p). \qquad \square$$

Next, we prove Theorem 4. For $0 < x < 1$ let

$$f_r(x) := \left(1 - \left(\frac{\log x}{x-1}\right)^r\right) \frac{1}{\log x}.$$

Then it is not difficult to see that $f(x)$ is a monotone decreasing continuous function with

$$\lim_{x \to 1} f_r(x) = \frac{r}{2}$$

and

$$\lim_{x \to 0} f_r(x) = \begin{cases} 1 & \cdots \quad r = 1, \\ +\infty & \cdots \quad r \geq 2. \end{cases}$$

These are shown as follows:

$$\lim_{x \to 1} f_r(x) = \lim_{x \to 1} \frac{1 - \dfrac{\log x}{x-1}}{\log x} \left(1 + \frac{\log x}{x-1} + \left(\frac{\log x}{x-1}\right)^2 + \cdots + \left(\frac{\log x}{x-1}\right)^{r-1}\right)$$
$$= \frac{r}{2},$$

where we used that

$$\lim_{x \to 1} \frac{1 - \dfrac{\log x}{x-1}}{\log x} = \frac{1}{2},$$

$$\lim_{x \to 1} \frac{\log x}{x-1} = 1.$$

Similarly,

$$\lim_{x\to 0} f_r(x) = \lim_{x\to 0} \frac{1-\dfrac{\log x}{x-1}}{\log x}\left(1+\frac{\log x}{x-1}+\left(\frac{\log x}{x-1}\right)^2+\cdots+\left(\frac{\log x}{x-1}\right)^{r-1}\right)$$
$$= \begin{cases} 1 & \cdots \quad r=1, \\ +\infty & \cdots \quad r\geq 2. \end{cases}$$

where we used

$$\lim_{x\to 0} \frac{1-\dfrac{\log x}{x-1}}{\log x} = 1,$$
$$\lim_{x\to 0} \frac{\log x}{x-1} = +\infty.$$

Let us show the monotone decreasing property. First, look at the case $r=1$. Then,

$$f_1(x) = \frac{1}{1-x}+\frac{1}{\log x} \quad (0<x<1)$$

and

$$f_1'(x) = \frac{1}{(x-1)^2}-\frac{1}{(\log x)^2 x}.$$

Hence,

$$f_1'(x)<0 \ \ (0<x<1) \Leftrightarrow \frac{\log x}{x-1}<\frac{1}{\sqrt{x}} \ \ (0<x<1),$$

and putting $u=x^{-1/2}$ $(u>1)$, the above inequality is equivalent to

$$\log u < \frac{1}{2}\left(u-\frac{1}{u}\right) \quad (u>1).$$

Let

$$g(u) = \frac{1}{2}\left(u-\frac{1}{u}\right)-\log u \quad (u>1).$$

Then

$$g'(u) = \frac{1}{2}+\frac{1}{2u^2}-\frac{1}{u}$$
$$= \frac{1}{2}\left(1-\frac{1}{u}\right)^2 > 0$$

and

$$\lim_{u\to 1} g(u) = 0.$$

Hence, we see that

$$f_1'(x) < 0 \quad \text{for } 0 < x < 1.$$

Next let $r \geq 2$. Then

$$\begin{aligned} f_r(x) &= \left(1 - \left(\frac{\log x}{x-1}\right)^r\right)\frac{1}{\log x} \\ &= \left(1 - \frac{\log x}{x-1}\right)\frac{1}{\log x}\cdot\left(1 + \frac{\log x}{x-1} + \cdots + \left(\frac{\log x}{x-1}\right)^{r-1}\right) \\ &= f_1(x)\cdot(1 + h(x) + \cdots + h(x)^{r-1}), \end{aligned}$$

where

$$h(x) = \frac{\log x}{x-1}.$$

Hence, it is sufficient to show that $h(x) > 0$ is monotone decreasing for $0 < x < 1$. We know that

$$\lim_{x\to 1} h(x) = 1,$$
$$\lim_{x\to 0} h(x) = +\infty$$

and

$$h'(x) = \frac{x - 1 - x\log x}{x(x-1)^2}.$$

Thus,

$$\begin{aligned} h'(x) < 0 \ (0 < x < 1) &\Leftrightarrow \ \log x > 1 - \frac{1}{x} \ \ (0 < x < 1) \\ &\underset{x=1/u}{\Leftrightarrow} \log u < u - 1 \ \ (u > 1), \end{aligned}$$

which is well-known. This proves the monotone decreasing property of $f_r(x)$.

Then we obtain the following inequalities

$$q\int_0^1 f_r(x)\,dx \leq \int_0^1 f_r(x)\,d_qx \leq \int_0^1 f_r(x)\,dx.$$

Actually, from

$$f_r(q^m)(q^m - q^{m+1}) \leq \int_{q^{m+1}}^{q^m} f_r(x)\,dx$$

we see that

$$\begin{aligned}\int_0^1 f_r(x)\,d_qx &= \sum_{m=1}^{\infty} f_r(q^m)(q^m - q^{m+1})\\ &\le \sum_{m=1}^{\infty}\int_{q^{m+1}}^{q^m} f_r(x)\,dx\\ &= \int_0^q f_r(x)\,dx\\ &\le \int_0^1 f_r(x)\,dx.\end{aligned}$$

Next, from

$$f_r(q^m)(q^{m-1} - q^m) \ge \int_{q^m}^{q^{m-1}} f_r(x)\,dx$$

we get

$$f_r(q^m)(q^m - q^{m+1}) \ge q\int_{q^m}^{q^{m-1}} f_r(x)\,dx.$$

Hence,

$$\begin{aligned}\int_0^1 f_r(x)\,d_qx &\ge q\sum_{m=1}^{\infty}\int_{q^m}^{q^{m-1}} f_r(x)\,dx\\ &= q\int_0^1 f_r(x)\,dx.\end{aligned}$$

From the inequalities

$$q\int_0^1 f_r(x)\,dx \le \int_0^1 f_r(x)\,d_qx \le \int_0^1 f_r(x)\,dx$$

we obtain

$$\lim_{q\to 1}\int_0^1 f_r(x)\,d_qx = \int_0^1 f_r(x)\,dx.$$

Thus,

$$\begin{aligned}\lim_{p\to 1}\gamma_r(p) &= \lim_{q\to 1}\gamma_r^{\star}(q)\\ &= \lim_{q\to 1}\int_0^1 f_r(x)\,d_qx\\ &= \int_0^1 f_r(x)\,dx.\end{aligned}$$

Finally, we show that

$$\int_0^1 f_r(x)\,dx = \gamma_r.$$

This is a generalization of Euler's result ([2] 1776)

$$\int_0^1 \left(\frac{1}{1-x} + \frac{1}{\log x}\right) dx = \gamma,$$

which is nothing but the case $r = 1$. Here we know that $\zeta_r(s)$ has a simple pole at $s = r$ with residue 1 (see Theorem 2 (1)). Using the integral expression of $\zeta_r(s)$

$$\zeta_r(s) = \frac{(r-1)!}{\Gamma(s)} \int_0^1 \frac{\left(\log \frac{1}{x}\right)^{s-1}}{(1-x)^r}\,dx$$

we have

$$\begin{aligned}
\lim_{s\to r}\left(\zeta_r(s) - \frac{1}{s-r}\right) &= \lim_{s\to r}\left[\frac{(r-1)!}{\Gamma(s)}\int_0^1 \left\{\left(\frac{\log \frac{1}{x}}{1-x}\right)^r - 1\right\}\left(\log\frac{1}{x}\right)^{s-r-1} dx\right. \\
&\qquad \left. + \left(\frac{(r-1)!}{(s-1)\cdots(s-r)} - \frac{1}{s-r}\right)\right] \\
&= \int_0^1 \left\{\left(\frac{\log \frac{1}{x}}{1-x}\right)^r - 1\right\}\frac{dx}{\log \frac{1}{x}} - H_{r-1} \\
&= \int_0^1 \left(1 - \left(\frac{\log x}{x-1}\right)^r\right)\frac{dx}{\log x} - H_{r-1}.
\end{aligned}$$

Hence, by the definition of γ_r we obtain

$$\int_0^1 f_r(x)\,dx = \gamma_r.$$

This proves Theorem 4. Q.E.D.

Remark. It seems rather difficult to prove Theorem 4 from Theorem 3 directly: termwise convergence is easily seen, but it seems that we must give difficult arguments to show the possibility for the exchange of the summation and the limit. Actually this difficulty made us to use the integral expressions as in the above proof of Theorem 4.

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