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KODAI MATH. J.
42 (2019), 358–375

A CESÀRO AVERAGE OF GENERALISED HARDY-LITTLEWOOD NUMBERS

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Abstract

We continue our recent work on additive problems with prime summands: we already studied the *average* number of representations of an integer as a sum of two primes, and also considered individual integers. Furthermore, we dealt with representations of integers as sums of powers of prime numbers. In this paper, we study a Cesàro weighted partial *explicit* formula for generalised Hardy-Littlewood numbers (integers that can be written as a sum of a prime power and a square) thus extending and improving our earlier results.

1. Introduction

The problem of counting the number of representations of an integer as a sum of some fixed powers of primes, and its variants where some primes are replaced by powers of integers, has received much attention in the last decades. The goal, that has been attained only in part, is to obtain an asymptotic formula for the number of such representations, which is valid for large integers that satisfy some necessary congruence conditions, as in the binary and ternary Goldbach problems. In some cases, conditional results are obtained, that is, it is necessary to assume the truth of some hitherto unproved conjecture like the Riemann Hypothesis. In our previous paper [12] we considered the problem of representing a large integer of suitable parity as a sum of $j \geq 5$ primes, assuming the truth of the Generalised Riemann Hypothesis, and we obtained an individual asymptotic formula with a main term of the expected order of magnitude, and a lower order term which depends explicitly on non-trivial zeros of relevant Dirichlet L-functions. The corresponding problem with a smaller number of summands is harder, and it is convenient to study the *average* number of representations; in fact, assuming the Riemann Hypothesis, in our paper [11] we gave one such result for the standard Goldbach problem where $j = 2$. In this case, averaging has the effect of making the zeros of L-functions irrelevant,

2010 *Mathematics Subject Classification*. Primary 11P32; Secondary 44A10, 33C10.
Key words and phrases. Goldbach-type theorems, Hardy-Littlewood numbers, Laplace transforms, Cesàro averages.
Received June 20, 2018; revised November 19, 2018.

except for the Riemann ζ-function itself, and in fact the development contains a main term and a smaller term which depends on the non-trivial zeros of the ζ-function.

In the paper [14] we introduced a Cesàro-Riesz weight in the summation: the presence of a smooth weight in place, essentially, of the characteristic function of the interval where we are averaging, leads to the possibility of giving a development into several terms, of decreasing order of magnitude, depending on the zeros of the Riemann ζ-function, sometimes in pairs. This weight also enabled us to remove the necessity of assuming the Riemann Hypothesis. The results of this paper have been generalised and improved in [15], where we treated the average number of representations of an integer in the form $p_1^{\ell_1} + p_2^{\ell_2}$, where ℓ_1 and ℓ_2 are fixed positive integers and p_1 and p_2 are prime numbers. The results in [14] have been recently extended by Goldston and Yang [7] and by Brüdern, Kaczorowski and Perelli [2].

As remarked above, we also considered a mixed binary problem with a prime and the square of an integer, the so-called Hardy-Littlewood numbers, in [13]. Similar problems have been studied by Cantarini in [3] and [4]. Our task here is to extend and improve our earlier results on weighted averages. We let $\ell \geq 1$ be an integer and set

$$r_{\ell,2}(n) = \sum_{m_1^\ell + m_2^2 = n} \Lambda(m_1), \tag{1}$$

where Λ is the usual von Mangoldt-function. Our main goal is to give a multi-term development for

$$R_k(N) = R_k(N;\ell) = \sum_{n \leq N} r_{\ell,2}(n) \frac{(1 - n/N)^k}{\Gamma(k+1)}, \tag{2}$$

where $k > 0$. We introduce the following abbreviations for the terms of the development:

$$\mathscr{M}_{1,\ell,k}(N) = \pi^{1/2} \frac{\Gamma(1/\ell)}{2\ell} \frac{N^{1/2+1/\ell}}{\Gamma(k+3/2+1/\ell)} - \frac{\Gamma(1/\ell)}{2\ell} \frac{N^{1/\ell}}{\Gamma(k+1+1/\ell)},$$

$$\mathscr{M}_{2,\ell,k}(N) = -\frac{\pi^{1/2}}{2\ell} \sum_\rho \frac{\Gamma(\rho/\ell)}{\Gamma(k+3/2+\rho/\ell)} N^{\rho/\ell+1/2},$$

$$\mathscr{M}_{3,\ell,k}(N) = \frac{1}{2\ell} \sum_\rho \frac{\Gamma(\rho/\ell)}{\Gamma(k+1+\rho/\ell)} N^{\rho/\ell},$$

$$\mathscr{M}_{4,\ell,k}(N) = -\frac{\pi^{1/2} \log(2\pi)}{2\Gamma(k+3/2)} N^{1/2},$$

$$\mathscr{M}_{5,\ell,k}(N) = \frac{N^{1/4-k/2+1/(2\ell)}}{\pi^{k+1/\ell}} \frac{\Gamma(1/\ell)}{\ell} \sum_{j \geq 1} \frac{J_{k+1/2+1/\ell}(2\pi j N^{1/2})}{j^{k+1/2+1/\ell}},$$

$$\mathscr{M}_{6,\ell,k}(N) = -\frac{N^{1/4-k/2}}{\pi^k}\sum_{\rho}\Gamma\left(\frac{\rho}{\ell}\right)\frac{N^{\rho/(2\ell)}}{\pi^{\rho/\ell}}\sum_{j\geq 1}\frac{J_{k+1/2+\rho/\ell}(2\pi jN^{1/2})}{j^{k+1/2+\rho/\ell}},$$

$$\mathscr{M}_{7,\ell,k}(N) = -\frac{\log(2\pi)}{\pi^k}N^{1/4-k/2}\sum_{j\geq 1}\frac{J_{k+1/2}(2\pi jN^{1/2})}{j^{k+1/2}}.$$

Here ρ runs over the non-trivial zeros of the Riemann zeta-function $\zeta(s)$, Γ is Euler's function and $J_\nu(u)$ denotes the Bessel function of complex order ν and real argument u. The main result of the paper is the following theorem.

THEOREM 1.1. *Let $\ell \geq 1$ be an integer and N be a sufficiently large integer. For $k > 1$ we have*

$$\sum_{n\leq N} r_{\ell,2}(n)\frac{(1-n/N)^k}{\Gamma(k+1)} = \sum_{j=1}^{7}\mathscr{M}_{j,\ell,k}(N) + \mathcal{O}_{k,\ell}(1).$$

Clearly, depending on the size of ℓ, some of the previously listed terms can be included in the error term. Theorem 1.1 generalises and improves our Theorem 1 in [13], which corresponds to the case $\ell = 1$, where the error term there should be read as $\mathcal{O}_k(N^{1/2})$; see Theorem 2.3 of [10]. In fact, in this case we are now able to detect the terms $\mathscr{M}_{4,1,k}$ and $\mathscr{M}_{7,1,k}$.

The basic strategy of the proof depends on the modern version of a classical formula due to Laplace [16], namely

$$\frac{1}{2\pi i}\int_{(a)} v^{-s}e^{v}\,\mathrm{d}v = \frac{1}{\Gamma(s)}, \tag{3}$$

where $\Re(s) > 0$ and $a > 0$; see Formula 5.4(1) on page 238 of Erdélyi et al, [5]. Using a suitable form of this transform, which we describe in §2, we obtain the fundamental relation for the method, viz.

$$N^k R_k(N) = \sum_{n\leq N} r_{\ell,2}(n)\frac{(N-n)^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_{(a)} e^{Nz}z^{-k-1}\tilde{S}_\ell(z)\omega_2(z)\,\mathrm{d}z, \tag{4}$$

where

$$\tilde{S}_\ell(z) = \sum_{m\geq 1}\Lambda(m)e^{-m^\ell z} \quad\text{and}\quad \omega_2(z) = \sum_{m\geq 1} e^{-m^2 z} \tag{5}$$

are the exponential sums that embody the properties of the ℓ-th powers of primes, and of the perfect squares, respectively. Here we need $k > 0$, and consider the complex variable $z = a + iy$ with $a > 0$.

The basic facts that we need are the "explicit formula" for $\tilde{S}_\ell(z)$, that is, its development as a main term and a secondary term which is a sum over non-trivial zeros of the Riemann ζ-function with a very small error, as in (16) below, and the simple connection of $\omega_2(z)$ with $\theta(z) = \sum_{m=-\infty}^{+\infty} e^{-m^2 z}$, since $\theta(z) =$

$1 + 2\omega_2(z)$. Now, we recall that θ satisfies the functional equation (11). We plug these relations into the right-hand side of (4), and exchange summation over zeros with vertical integration, obtaining formally the development in Theorem 1.1. The Bessel functions in $\mathcal{M}_{5,\ell,k}$, $\mathcal{M}_{6,\ell,k}$ and $\mathcal{M}_{7,\ell,k}$ arise from the "modular" terms in the functional equation of θ.

Of course, we need to prove that the exchange referred to above is legitimate, and that the error term arising from the approximation of the exponential sum $\tilde{S}_\ell(z)$ in (16) is small.

Summing up, as in [15] we combine the approach with line integrals with the classical methods dealing with infinite sums over primes, exploited by Hardy and Littlewood (see [8] and [9]) and by Linnik [17]. The main difficulty here is, as in [13], that the problem naturally involves the modular relation for the complex theta function (11). The presence of the Bessel functions in our statement strictly depends on such modularity relation. It is worth mentioning that it is not clear how to get such "modular" terms using the finite sums approach for the function $r_{\ell,2}(n)$. The previously mentioned improvement we get in Theorem 1.1 follows using Lemma 6.1 below, which is proved in [15].

2. Settings

As we mentioned in the previous section, we will need the general case of (3), which can be found in de Azevedo Pribitkin [1], formulae (8) and (9). More precisely, we have

$$\frac{1}{2\pi}\int_{\mathbf{R}} \frac{e^{iDu}}{(a+iu)^s}\,\mathrm{d}u = \begin{cases} \dfrac{D^{s-1}e^{-aD}}{\Gamma(s)} & \text{if } D > 0, \\ 0 & \text{if } D < 0, \end{cases} \tag{6}$$

which is valid for $\sigma = \Re(s) > 0$ and $a \in \mathbf{C}$ with $\Re(a) > 0$, and

$$\frac{1}{2\pi}\int_{\mathbf{R}} \frac{1}{(a+iu)^s}\,\mathrm{d}u = \begin{cases} 0 & \text{if } \Re(s) > 1, \\ 1/2 & \text{if } s = 1, \end{cases} \tag{7}$$

for $a \in \mathbf{C}$ with $\Re(a) > 0$. Formulae (6)–(7) actually enable us to write averages of arithmetical functions by means of line integrals as we will see below.

We will also need Bessel functions of complex order ν and real argument u. For their definition and main properties we refer to Watson [19]. In particular, equation (8) on page 177 gives the Sonine representation:

$$J_\nu(u) := \frac{(u/2)^\nu}{2\pi i}\int_{(a)} s^{-\nu-1}e^s e^{-u^2/4s}\,\mathrm{d}s, \tag{8}$$

where $a > 0$ and $u, \nu \in \mathbf{C}$ with $\Re(\nu) > -1$. We will also use the Poisson integral formula

$$J_\nu(u) := \frac{2(u/2)^\nu}{\pi^{1/2}\Gamma(\nu + 1/2)}\int_0^1 (1 - t^2)^{\nu-1/2}\cos(ut)\,\mathrm{d}t, \tag{9}$$

which holds for $\Re(\nu) > -1/2$ and $u \in \mathbf{C}$. (See eq. (3) on page 48 of [19].) An asymptotic estimate we will need is

$$J_\nu(u) = \left(\frac{2}{\pi u}\right)^{1/2} \cos\left(u - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) + \mathcal{O}_{|\nu|}(u^{-5/2}), \tag{10}$$

which follows from eq. (1) on page 199 of Watson [19].

From now on we assume that $k > 0$. We recall the definitions (5), where $z = a + iy$ with $a > 0$. We also recall that $\theta(z) = \sum_{m=-\infty}^{+\infty} e^{-m^2 z}$ satisfies the functional equation

$$\theta(z) = \left(\frac{\pi}{z}\right)^{1/2} \theta\left(\frac{\pi^2}{z}\right) \quad \text{for } \Re(z) > 0, \tag{11}$$

see, e.g., Proposition VI.4.3 of Freitag and Busam [6, page 340]. Since $\theta(z) = 1 + 2\omega_2(z)$, we immediately get

$$\omega_2(z) = \frac{1}{2}\left(\frac{\pi}{z}\right)^{1/2} - \frac{1}{2} + \left(\frac{\pi}{z}\right)^{1/2} \omega_2\left(\frac{\pi^2}{z}\right) \quad \text{for } \Re(z) > 0. \tag{12}$$

Recalling (1), we can write

$$\tilde{S}_\ell(z)\omega_2(z) = \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} \Lambda(m_1) e^{-(m_1^\ell + m_2^2)z} = \sum_{n \geq 1} r_{\ell,2}(n) e^{-nz}$$

and, by (6)–(7), we see that

$$\sum_{n \leq N} r_{\ell,2}(n) \frac{(N-n)^k}{\Gamma(k+1)} = \sum_{n \geq 1} r_{\ell,2}(n) \left(\frac{1}{2\pi i} \int_{(a)} e^{(N-n)z} z^{-k-1} \, \mathrm{d}z\right). \tag{13}$$

Our first goal is to exchange the series with the line integral in (13). To do so we have to recall that the Prime Number Theorem (PNT) implies the statement

$$\tilde{S}_\ell(a) \sim \frac{\Gamma(1/\ell)}{\ell a^{1/\ell}} \quad \text{for } a \to 0+.$$

In fact, by a straightforward application of the partial summation formula we see that

$$\tilde{S}_\ell(a) = \ell a \int_0^{+\infty} \psi(t) t^{\ell-1} e^{-at^\ell} \, \mathrm{d}t,$$

where ψ is the standard Chebyshev function. It is now convenient to split the integration range at $t_0 = a^{-1/(2\ell)}$. We use the weak upper bound $\psi(t) \ll t$ on $[0, t_0]$, recalling that $\psi(t) = 0$ for $t < 2$. Since $t^\ell e^{-at^\ell} \leq (ae)^{-1}$ for all $t \geq 0$, we immediately see that the contribution of this range to $\tilde{S}_\ell$ is $\ll \ell a^{-1/(2\ell)}$. According to a weak form of the PNT, we have $\psi(t) = t + \mathcal{O}_A(t(\log t)^{-A})$ for any fixed $A > 0$. Hence, completing the missing range, performing the obvious change of variables and using the same bounds as above when needed, we have

$$\int_{t_0}^{+\infty} \psi(t)t^{\ell-1}e^{-at^\ell}\,\mathrm{d}t = \int_0^{+\infty} t^\ell e^{-at^\ell}\,\mathrm{d}t + \mathcal{O}\biggl(\int_0^{t_0} t^\ell e^{-at^\ell}\,\mathrm{d}t\biggr) + \mathcal{O}\biggl(\int_{t_0}^{+\infty} \frac{t^\ell e^{-at^\ell}}{(\log t)^A}\,\mathrm{d}t\biggr)$$
$$= \frac{\Gamma(1+1/\ell)}{\ell a^{1+1/\ell}} + \mathcal{O}(t_0 a^{-1}) + \mathcal{O}\biggl((\log t_0)^{-A}\int_0^{+\infty} t^\ell e^{-at^\ell}\,\mathrm{d}t\biggr)$$
$$= \frac{\Gamma(1+1/\ell)}{\ell a^{1+1/\ell}} + \mathcal{O}_\ell(a^{-1-1/\ell}(\log(1/a))^{-A}).$$

The final result then follows by recalling that $\Gamma(s+1) = s\Gamma(s)$.

We will also use the inequality

$$|\omega_2(z)| \le \omega_2(a) \le \int_0^\infty e^{-at^2}\,\mathrm{d}t \le a^{-1/2}\int_0^\infty e^{-v^2}\,\mathrm{d}v \ll a^{-1/2}, \tag{14}$$

from which we immediately get

$$\sum_{n\ge 1} |r_{\ell,2}(n)e^{-nz}| = \sum_{n\ge 2} r_{\ell,2}(n)e^{-na} = \tilde{S}_\ell(a)\omega_2(a) \ll_\ell a^{-1/\ell-1/2}.$$

Taking into account the estimates

$$|z|^{-1} \asymp \begin{cases} a^{-1} & \text{if } |y| \le a, \\ |y|^{-1} & \text{if } |y| \ge a, \end{cases} \tag{15}$$

where $f \asymp g$ means $g \ll f \ll g$, and

$$|e^{Nz}z^{-k-1}| \asymp e^{Na}\begin{cases} a^{-k-1} & \text{if } |y| \le a, \\ |y|^{-k-1} & \text{if } |y| \ge a, \end{cases}$$

we have

$$\int_{(a)} |e^{Nz}z^{-k-1}|\,|\tilde{S}_\ell(z)\omega_2(z)|\,|\mathrm{d}z| \ll_\ell a^{-1/\ell-1/2}e^{Na}\biggl(\int_{-a}^{a} a^{-k-1}\,\mathrm{d}y + 2\int_a^{+\infty} y^{-k-1}\,\mathrm{d}y\biggr)$$
$$\ll_\ell a^{-1/\ell-1/2}e^{Na}\biggl(a^{-k} + \frac{a^{-k}}{k}\biggr).$$

The last estimate is valid only if $k > 0$. So, for $k > 0$, we can exchange the line integral with the sum over n in (13), thus proving (4).

3. Inserting zeros and modularity

We need $k > 1/2$ in this section. The treatment of the integral on the right-hand side of (4) requires Lemma 6.1. We split $\tilde{S}_\ell(z)$ according to its statement as $\mathcal{S}_\ell(z) + E(a,z,\ell)$ where E satisfies the bound in (31) and

$$\mathcal{S}_\ell(z) := \frac{\Gamma(1/\ell)}{\ell z^{1/\ell}} - \frac{1}{\ell}\sum_\rho z^{-\rho/\ell}\Gamma\biggl(\frac{\rho}{\ell}\biggr) - \log(2\pi), \tag{16}$$

where $\rho = \beta + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$. Formula (4) becomes

$$\sum_{n\le N} r_{\ell,2}(n)\frac{(N-n)^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_{(a)} \mathscr{S}_\ell(z)\omega_2(z)e^{Nz}z^{-k-1}\,\mathrm{d}z + \mathscr{O}\biggl(\int_{(a)} |E(a,z,\ell)|\,|e^{Nz}|\,|z|^{-k-1}|\omega_2(z)|\,|\mathrm{d}z|\biggr).$$

Using (14)–(15) and (31), we see that the error term is

$$\ll_\ell a^{-1/2}e^{Na}\biggl(\int_{-a}^{a} a^{-k-1/2}\,\mathrm{d}y + \int_a^{+\infty} y^{-k-1/2}\log^2(y/a)\,\mathrm{d}y\biggr) \ll_{k,\ell} e^{Na}a^{-k}\biggl(1+\int_1^{+\infty} v^{-k-1/2}\log^2 v\,\mathrm{d}v\biggr) \ll_{k,\ell} e^{Na}a^{-k},$$

provided that $k > 1/2$. Choosing $a = 1/N$, the previous estimate becomes $\ll_{k,\ell} N^k$. Summing up, for $k > 1/2$, we can write

$$\sum_{n\le N} r_{\ell,2}(n)\frac{(N-n)^k}{\Gamma(k+1)} = \frac{1}{2\pi i}\int_{(1/N)} \mathscr{S}_\ell(z)\omega_2(z)e^{Nz}z^{-k-1}\,\mathrm{d}z + \mathscr{O}_{k,\ell}(N^k). \tag{17}$$

We now insert (12) into (17), so that the integral on the right-hand side of (17) becomes

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{(1/N)} \mathscr{S}_\ell(z)\biggl(\frac12\Bigl(\frac{\pi}{z}\Bigr)^{1/2} - \frac12\biggr)e^{Nz}z^{-k-1}\,\mathrm{d}z \\
&\quad + \frac{1}{2\pi i}\int_{(1/N)} \Bigl(\frac{\pi}{z}\Bigr)^{1/2}\mathscr{S}_\ell(z)\omega_2\Bigl(\frac{\pi^2}{z}\Bigr)e^{Nz}z^{-k-1}\,\mathrm{d}z \\
&= \mathscr{I}_1 + \mathscr{I}_2,
\end{aligned} \tag{18}$$

say. We now proceed to evaluate $\mathscr{I}_1$ and $\mathscr{I}_2$. In the next two sections we will use (16) and obtain that I_1 and I_2 split into a number of summands; in later sections we will prove that we can exchange all summations and integrations, in suitable ranges for k, using some properties of the non-trivial zeros of the Riemann ζ-function: see §6. Finally, we perform a change of variables that yields all the summands in the statement of Theorem 1.1.

4. Evaluation of $\mathscr{I}_1$

We need $k > 1/2$ in this section. By a direct computation we can write that

$$\begin{aligned}\mathscr{I}_1 &= \frac{1}{4\pi i}\frac{\Gamma(1/\ell)}{\ell}\int_{(1/N)}\Big(\frac{\pi^{1/2}}{z^{1/2}}-1\Big)e^{Nz}z^{-k-1-1/\ell}\,\mathrm{d}z\\
&\quad-\frac{\pi^{1/2}}{4\ell\pi i}\int_{(1/N)}\sum_{\rho}\Gamma\Big(\frac{\rho}{\ell}\Big)e^{Nz}z^{-k-\rho/\ell-3/2}\,\mathrm{d}z\\
&\quad+\frac{1}{4\ell\pi i}\int_{(1/N)}\sum_{\rho}\Gamma\Big(\frac{\rho}{\ell}\Big)e^{Nz}z^{-k-\rho/\ell-1}\,\mathrm{d}z\\
&\quad-\frac{\log(2\pi)}{4\pi i}\int_{(1/N)}\Big(\frac{\pi^{1/2}}{z^{1/2}}-1\Big)e^{Nz}z^{-k-1}\,\mathrm{d}z\\
&= \mathscr{J}_1+\mathscr{J}_2+\mathscr{J}_3+\mathscr{J}_4,\end{aligned}$$

say. We see now how to evaluate $\mathscr{J}_1$, $\mathscr{J}_2$, $\mathscr{J}_3$ and $\mathscr{J}_4$. The delicate point is the justification of the exchanges required to deal with $\mathscr{J}_2$ and $\mathscr{J}_3$ (see §7 for the details), whereas the computations needed for $\mathscr{J}_1$ and $\mathscr{J}_4$ are straightforward and immediately follow by using the substitution $s = Nz$, by (3). This way we get

$$\mathscr{J}_1 = \frac{\pi^{1/2}}{2}\frac{\Gamma(1/\ell)}{\ell}\frac{N^{k+1/2+1/\ell}}{\Gamma(k+3/2+1/\ell)} - \frac{\Gamma(1/\ell)}{2\ell}\frac{N^{k+1/\ell}}{\Gamma(k+1+1/\ell)} \tag{19}$$

and

$$\mathscr{J}_4 = -\frac{\pi^{1/2}\log(2\pi)}{2\Gamma(k+3/2)}N^{k+1/2} + \frac{\log(2\pi)}{2\Gamma(k+1)}N^{k}. \tag{20}$$

4.1. Evaluation of $\mathscr{J}_2$. Exchanging the sum over ρ with the integral (this can be done for $k > 0$; see §7) and using the substitution $s = Nz$, we have

$$\begin{aligned}\mathscr{J}_2 &= -\frac{\pi^{1/2}}{2\ell}\sum_{\rho}\Gamma\Big(\frac{\rho}{\ell}\Big)\frac{1}{2\pi i}\int_{(1/N)}e^{Nz}z^{-k-\rho/\ell-3/2}\,\mathrm{d}z\\
&= -\frac{\pi^{1/2}}{2\ell}\sum_{\rho}\Gamma\Big(\frac{\rho}{\ell}\Big)N^{k+\rho/\ell+1/2}\frac{1}{2\pi i}\int_{(1)}e^{s}s^{-k-\rho/\ell-3/2}\,\mathrm{d}s\\
&= -\frac{\pi^{1/2}}{2\ell}\sum_{\rho}\frac{\Gamma(\rho/\ell)}{\Gamma(k+3/2+\rho/\ell)}N^{k+\rho/\ell+1/2},\end{aligned} \tag{21}$$

again by (3). By the Stirling formula (30), we remark that the series in $\mathscr{J}_2$ converges absolutely for $k > -1/2$.

4.2. Evaluation of $\mathscr{J}_3$. Arguing as in §7 with $-k-1$ which plays the role of $-k-3/2$ there, we see that we can exchange the sum with the integral provided that $k > 1/2$. Hence, performing again the usual substitution $s = Nz$, we can write

$$\mathscr{I}_3 = \frac{1}{2\ell}\sum_\rho \Gamma\Big(\frac{\rho}{\ell}\Big) N^{k+\rho/\ell}\frac{1}{2\pi i}\int_{(1)} e^s s^{-k-1-\rho/\ell}\,\mathrm{d}s \tag{22}$$
$$= \frac{1}{2\ell}\sum_\rho \frac{\Gamma(\rho/\ell)}{\Gamma(k+1+\rho/\ell)} N^{k+\rho/\ell}.$$

By the Stirling formula (30), we remark that the series in $\mathscr{I}_3$ converges absolutely for $k > 0$.

5. Evaluation of $\mathscr{I}_2$ and conclusion of the proof of Theorem 1.1

We need $k > 1$ in this section. Using (16) and the definition of $\omega_2(\pi^2/z)$ (see (5)) we have

$$\mathscr{I}_2 = \frac{1}{2\pi i}\frac{\Gamma(1/\ell)}{\ell}\int_{(1/N)}\Big(\frac{\pi}{z}\Big)^{1/2}\Big(\sum_{j\ge1} e^{-j^2\pi^2/z}\Big) e^{Nz} z^{-k-1-1/\ell}\,\mathrm{d}z \tag{23}$$
$$-\frac{1}{2\ell\pi i}\int_{(1/N)}\Big(\frac{\pi}{z}\Big)^{1/2}\Big(\sum_{j\ge1} e^{-j^2\pi^2/z}\Big)\Big(\sum_\rho z^{-\rho/\ell}\Gamma\Big(\frac{\rho}{\ell}\Big)\Big) e^{Nz} z^{-k-1}\,\mathrm{d}z$$
$$-\frac{\log(2\pi)}{2\pi i}\int_{(1/N)}\Big(\frac{\pi}{z}\Big)^{1/2}\Big(\sum_{j\ge1} e^{-j^2\pi^2/z}\Big) e^{Nz} z^{-k-1}\,\mathrm{d}z = \mathscr{I}_5 + \mathscr{I}_6 + \mathscr{I}_7,$$

say. We see now how to evaluate $\mathscr{I}_5$, $\mathscr{I}_6$ and $\mathscr{I}_7$. The proof in this section is more delicate than in the previous one. We have to justify inversion as before, but we are then faced with the problem of dealing with series containing values of the Bessel functions, arising from the "modular" terms. We refer to §10 for a detailed discussion of the problem.

5.1. Evaluation of $\mathscr{I}_5$. By means of the substitution $s = Nz$, since the exchange is justified in §8 for $k > 1/2 - 1/\ell$, we get

$$\mathscr{I}_5 = \pi^{1/2}\frac{\Gamma(1/\ell)}{\ell} N^{k+1/2+1/\ell}\sum_{j\ge1}\frac{1}{2\pi i}\int_{(1)} e^s e^{-j^2\pi^2 N/s} s^{-k-3/2-1/\ell}\,\mathrm{d}s.$$

Setting $u = 2\pi j N^{1/2}$ in (8), we obtain

$$J_\nu(2\pi j N^{1/2}) = \frac{(\pi j N^{1/2})^\nu}{2\pi i}\int_{(1)} e^s e^{-j^2\pi^2 N/s} s^{-\nu-1}\,\mathrm{d}s, \tag{24}$$

and hence we have

$$\mathscr{I}_5 = \frac{N^{k/2+1/4+1/(2\ell)}}{\pi^{k+1/\ell}}\frac{\Gamma(1/\ell)}{\ell}\sum_{j\ge1}\frac{J_{k+1/2+1/\ell}(2\pi j N^{1/2})}{j^{k+1/2+1/\ell}}. \tag{25}$$

The absolute convergence of the series in $\mathscr{I}_5$ is studied in §10.

5.2. Evaluation of $\mathcal{I}_6$. With the same substitution used before, since the double exchange between sums and the line integral is justified in §9 for $k > 1$, we see that

$$\mathcal{I}_6 = -\frac{\pi^{1/2}}{\ell} \sum_{\rho} \Gamma\left(\frac{\rho}{\ell}\right) N^{k+1/2+\rho/\ell} \sum_{j \geq 1} \left(\frac{1}{2\pi i} \int_{(1)} e^s e^{-j^2\pi^2 N/s} s^{-k-3/2-\rho/\ell} \, \mathrm{d}s \right).$$

Using (24), we get

$$\mathcal{I}_6 = -\frac{N^{k/2+1/4}}{\pi^k} \sum_{\rho} \Gamma\left(\frac{\rho}{\ell}\right) \frac{N^{\rho/(2\ell)}}{\pi^{\rho/\ell}} \sum_{j \geq 1} \frac{J_{k+1/2+\rho/\ell}(2\pi j N^{1/2})}{j^{k+1/2+\rho/\ell}}. \tag{26}$$

In this case, the absolute convergence of the series in $\mathcal{I}_6$ is more delicate; such a treatment is again described in §10.

5.3. Evaluation of $\mathcal{I}_7$. With the same substitution used before, since the exchange between sum and the line integral is justified in §8 for $k > 1/2$, we see that

$$\mathcal{I}_7 = -\pi^{1/2} \log(2\pi) N^{k+1/2} \sum_{j \geq 1} \frac{1}{2\pi i} \int_{(1)} e^s e^{-j^2\pi^2 N/s} s^{-k-3/2} \, \mathrm{d}s.$$

Using (24), we get

$$\mathcal{I}_7 = -\frac{\log(2\pi)}{\pi^k} N^{k/2+1/4} \sum_{j \geq 1} \frac{J_{k+1/2}(2\pi j N^{1/2})}{j^{k+1/2}}. \tag{27}$$

The absolute convergence of the series in $\mathcal{I}_7$ is studied in §10.

Finally, inserting (19)–(27) into (18) and (17), we obtain

$$\sum_{n \leq N} r_{\ell,2}(n) \frac{(N-n)^k}{\Gamma(k+1)} = N^k \sum_{j=1}^{7} \mathcal{M}_{j,\ell,k}(N) + \mathcal{O}_{k,\ell}(N^k), \tag{28}$$

for $k > 1$. Theorem 1.1 follows dividing (28) by N^k.

6. Lemmas

We recall some basic facts in complex analysis. First, if $z = a + iy$ with $a > 0$, we see that for complex w we have

$$\begin{aligned} z^{-w} &= |z|^{-w} \exp(-iw \arctan(y/a)) \\ &= |z|^{-\Re(w) - i\Im(w)} \exp((-i\Re(w) + \Im(w)) \arctan(y/a)), \end{aligned}$$

so that

$$|z^{-w}| = |z|^{-\Re(w)} \exp(\Im(w) \arctan(y/a)). \tag{29}$$

We also recall that, uniformly for $x \in [x_1, x_2]$, with x_1 and x_2 fixed, and for $|y| \to +\infty$, by the Stirling formula we have

$$|\Gamma(x+iy)| \sim \sqrt{2\pi} e^{-\pi|y|/2} |y|^{x-1/2}; \tag{30}$$

see, e.g., Titchmarsh [18, §4.42].

We will need the following lemmas from Languasco and Zaccagnini [15].

Lemma 6.1 (See Lemma 1 of [15]). *Let $\ell \geq 1$ be an integer, $z = a + iy$, where $a > 0$ and $y \in \mathbf{R}$ and let $\mathscr{S}_\ell(z)$ be defined as in* (16). *Then $\tilde{S}_\ell(z) = \mathscr{S}_\ell(z) + E(a, z, \ell)$ where*

$$E(a, z, \ell) \ll_\ell |z|^{1/2} \begin{cases} 1 & \text{if } |y| \leq a \\ 1 + \log^2(|y|/a) & \text{if } |y| > a. \end{cases} \tag{31}$$

Lemma 6.2 (See Lemma 2 of [15]). *Let $\ell \geq 1$ be an integer, let $\rho = \beta + i\gamma$ run over the non-trivial zeros of the Riemann zeta-function and $\alpha > 1$ be a parameter. The series*

$$\sum_{\rho:\gamma>0} \left(\frac{\gamma}{\ell}\right)^{\beta/\ell - 1/2} \int_1^{+\infty} \exp\left(-\frac{\gamma}{\ell} \arctan \frac{1}{u}\right) \frac{\mathrm{d}u}{u^{\alpha+\beta/\ell}}$$

converges provided that $\alpha > 3/2$. For $\alpha \leq 3/2$ the series does not converge. The result remains true if we insert in the integral a factor $(\log u)^c$, for any fixed $c \geq 0$.

Lemma 6.3 (See Lemma 3 of [15]). *Let $\ell \geq 1$ be an integer, $\alpha > 1$, $z = a + iy$, $a \in (0, 1)$ and $y \in \mathbf{R}$. Let further $\rho = \beta + i\gamma$ run over the non-trivial zeros of the Riemann zeta-function. We have*

$$\sum_\rho \left|\frac{\gamma}{\ell}\right|^{\beta/\ell - 1/2} \int_{\mathbf{Y}_1 \cup \mathbf{Y}_2} \exp\left(\frac{\gamma}{\ell} \arctan \frac{y}{a} - \frac{\pi}{2}\left|\frac{\gamma}{\ell}\right|\right) \frac{\mathrm{d}y}{|z|^{\alpha+\beta/\ell}} \ll_{\alpha,\ell} a^{1-\alpha-1/\ell},$$

where $\mathbf{Y}_1 = \{y \in \mathbf{R} : y\gamma \leq 0\}$ and $\mathbf{Y}_2 = \{y \in [-a, a] : y\gamma > 0\}$. The result remains true if we insert in the integral a factor $(\log(|y|/a))^c$, for any fixed $c \geq 0$.

7. Interchange of the series over zeros with the line integral in $\mathscr{J}_2$, $\mathscr{J}_3$

We need $k > 1/2$ in this section. For $\mathscr{J}_2$ we have to establish the convergence of

$$\sum_\rho \left|\Gamma\left(\frac{\rho}{\ell}\right)\right| \int_{(1/N)} |e^{Nz}| \, |z|^{-k-3/2} |z^{-\rho/\ell}| \, |\mathrm{d}z|, \tag{32}$$

where, as usual, $\rho = \beta + i\gamma$ runs over the non-trivial zeros of the Riemann zeta-function. By (29) and the Stirling formula (30), we are left with estimating

$$\sum_{\rho} \left|\frac{\gamma}{\ell}\right|^{\beta/\ell - 1/2} \int_{\mathbf{R}} \exp\left(\frac{\gamma}{\ell}\arctan(Ny) - \frac{\pi}{2}\left|\frac{\gamma}{\ell}\right|\right) \frac{\mathrm{d}y}{|z|^{k+3/2+\beta/\ell}}. \tag{33}$$

We have just to consider the case $\gamma y > 0$, $|y| > 1/N$ since in the other cases the total contribution is $\ll_k N^{k+1/2+1/\ell}$ by Lemma 6.3 with $\alpha = k + 3/2$ and $a = 1/N$. By symmetry, we may assume that $\gamma > 0$. We have that the integral in (33) is

$$\begin{aligned} &\ll_\ell \sum_{\rho} \left|\frac{\gamma}{\ell}\right|^{\beta/\ell - 1/2} \int_{1/N}^{+\infty} \exp\left(-\frac{\gamma}{\ell}\arctan\frac{1}{Ny}\right) \frac{\mathrm{d}y}{y^{k+3/2+\beta/\ell}} \\ &= N^{k+1/2} \sum_{\rho:\gamma>0} N^{\beta/\ell}\left(\frac{\gamma}{\ell}\right)^{\beta/\ell-1/2} \int_{1}^{+\infty} \exp\left(-\frac{\gamma}{\ell}\arctan\frac{1}{u}\right)\frac{\mathrm{d}u}{u^{k+3/2+\beta/\ell}}. \end{aligned}$$

For $k > 0$, this is $\ll_{k,\ell} N^{k+1/2+1/\ell}$ by Lemma 6.2. This implies that the integrals in (33) and in (32) are both $\ll_{k,\ell} N^{k+1/2+1/\ell}$, and hence this exchange step for $\mathscr{I}_2$ is fully justified.

For $\mathscr{I}_3$, we have to consider

$$\sum_{\rho} \left|\Gamma\left(\frac{\rho}{\ell}\right)\right| \int_{(1/N)} |e^{Nz}|\,|z|^{-k-1}|z^{-\rho/\ell}|\,|\mathrm{d}z|. \tag{34}$$

We can repeat the same reasoning we used for $\mathscr{I}_2$ just replacing $k + 3/2$ with $k + 1$. This means that we need $k > 1/2$ here to get that the integral in (34) is $\ll_{k,\ell} N^{k+1/\ell}$, and that this exchange step for $\mathscr{I}_3$ is fully justified too.

8. Interchange of the series over j with the line integral in $\mathscr{I}_5$, $\mathscr{I}_7$

We need $k > 1/2$ in this section. For $\mathscr{I}_5$ we have to establish the convergence of

$$\sum_{j\ge1} \int_{(1/N)} |e^{Nz}|\,|z|^{-k-3/2-1/\ell} e^{-\pi^2 j^2 \Re(1/z)}|\mathrm{d}z|. \tag{35}$$

A trivial computation gives

$$\Re(1/z) = \frac{N}{1+N^2y^2} \gg \begin{cases} N & \text{if } |y| \le 1/N, \\ 1/(Ny^2) & \text{if } |y| > 1/N. \end{cases} \tag{36}$$

By (36), we can write that the quantity in (35) is

$$\ll_\ell \sum_{j\ge1}\int_0^{1/N} \frac{e^{-j^2N}}{|z|^{k+3/2+1/\ell}}\,\mathrm{d}y + \sum_{j\ge1}\int_{1/N}^{+\infty}\frac{e^{-j^2/(Ny^2)}}{|z|^{k+3/2+1/\ell}}\,\mathrm{d}y = U_1 + U_2, \tag{37}$$

say, since the π^2 factor in the exponential function is negligible. Using (14)–(15), we have

$$U_1 \ll_\ell N^{k+1/2+1/\ell}\omega_2(N) \ll_\ell N^{k+1/\ell} \tag{38}$$

and

$$\begin{aligned} U_2 &\ll_\ell \sum_{j\ge 1}\int_{1/N}^{+\infty} \frac{e^{-j^2/(Ny^2)}}{y^{k+3/2+1/\ell}}\,\mathrm{d}y \\ &\ll_\ell N^{k/2+1/4+1/(2\ell)}\sum_{j\ge 1}\frac{1}{j^{k+1/2+1/\ell}}\int_0^{j^2N} u^{k/2-3/4+1/(2\ell)}e^{-u}\,\mathrm{d}u \\ &\le \Gamma\Bigl(\frac{2k+1+2\ell}{4}\Bigr)N^{k/2+1/4+1/(2\ell)}\sum_{j\ge 1}\frac{1}{j^{k+1/2+1/\ell}} \ll_{k,\ell} N^{k/2+1/4+1/(2\ell)}, \end{aligned} \tag{39}$$

provided that $k > 1/2 - 1/\ell$, where we used the substitution $u = j^2/(Ny^2)$. Inserting (38)–(39) into (37) we get, for $k > 1/2 - 1/\ell$, that the quantity in (35) is $\ll N^{k+1/\ell}$ and so it is for $\mathcal{J}_5$.

For $\mathcal{J}_7$ we have to establish the convergence of

$$\sum_{j\ge 1}\int_{(1/N)} |e^{Nz}|\,|z|^{-k-3/2}e^{-\pi^2 j^2\Re(1/z)}|\mathrm{d}z|. \tag{40}$$

We can repeat the same reasoning we used for $\mathcal{J}_5$ just replacing $k+3/2+1/\ell$ with $k+3/2$. This means that we need $k > 1/2$ here to get that the integral in (40) is $\ll_{k,\ell} N^k$, and that this exchange step for $\mathcal{J}_7$ is fully justified too.

9. Interchange of series with the line integral in $\mathcal{J}_6$

We need $k > 1$ in this section. We first have to establish the convergence of

$$\sum_{j\ge 1}\int_{(1/N)}\Bigl|\sum_\rho \Gamma\Bigl(\frac{\rho}{\ell}\Bigr)z^{-\rho/\ell}\Bigr|\,|e^{Nz}|\,|z|^{-k-3/2}e^{-\pi^2 j^2\Re(1/z)}|\mathrm{d}z|. \tag{41}$$

Using the Prime Number Theorem and (31), we first remark that

$$\Bigl|\sum_\rho \Gamma\Bigl(\frac{\rho}{\ell}\Bigr)z^{-\rho/\ell}\Bigr| \ll_\ell N^{1/\ell} + |z|^{1/2}\log^2(2N|y|). \tag{42}$$

By (36) and (42), we can write that the quantity in (41) is

$$\begin{aligned} &\ll_\ell N^{1/\ell}\sum_{j\ge 1}\int_0^{1/N}\frac{e^{-j^2N}}{|z|^{k+3/2}}\,\mathrm{d}y + N^{1/\ell}\sum_{j\ge 1}\int_{1/N}^{+\infty}\frac{e^{-j^2/(Ny^2)}}{|z|^{k+3/2}}\,\mathrm{d}y \\ &\quad + \sum_{j\ge 1}\int_{1/N}^{+\infty}\log^2(2Ny)\frac{e^{-j^2/(Ny^2)}}{|z|^{k+1}}\,\mathrm{d}y = V_1+V_2+V_3, \end{aligned} \tag{43}$$

say. V_1 can be estimated exactly as U_1 in Section 8 and we get $V_1 \ll_{k,\ell} N^{k+1/\ell}$. For V_2 we can work analogously to U_2 thus obtaining

$$V_2 \ll_{k,\ell} N^{1/\ell} \sum_{j\geq 1} \int_{1/N}^{+\infty} \frac{e^{-j^2/(Ny^2)}}{y^{k+3/2}} \,\mathrm{d}y \ll_{k,\ell} N^{k/2+1/4+1/\ell} \sum_{j\geq 1} \frac{1}{j^{k+1/2}} \int_0^{j^2N} u^{k/2-3/4} e^{-u} \,\mathrm{d}u$$

$$\ll_{k,\ell} \Gamma\left(\frac{2k+1}{4}\right) N^{k/2+1/4+1/\ell} \sum_{j\geq 1} \frac{1}{j^{k+1/2}} \ll_{k,\ell} N^{k/2+1/4+1/\ell},$$

provided that $k > 1/2$, where we used the substitution $u = j^2/(Ny^2)$. Hence, we have

$$V_1 + V_2 \ll_{k,\ell} N^{k+1/\ell}, \tag{44}$$

provided that $k > 1/2$.

Using the substitution $u = j^2/(Ny^2)$, we obtain

$$V_3 \ll_{k,\ell} \sum_{j\geq 1} \int_{1/N}^{+\infty} \log^2(2Ny) \frac{e^{-j^2/(Ny^2)}}{y^{k+1}} \,\mathrm{d}y$$

$$= \frac{N^{k/2}}{8} \sum_{j\geq 1} \frac{1}{j^k} \int_0^{j^2N} u^{k/2-1} \log^2\left(\frac{4j^2N}{u}\right) e^{-u} \,\mathrm{d}u.$$

Hence, a direct computation shows that

$$V_3 \ll_{k,\ell} N^{k/2} \sum_{j\geq 1} \frac{\log^2(jN)}{j^k} \int_0^{j^2N} u^{k/2-1} e^{-u} \,\mathrm{d}u \tag{45}$$

$$+ N^{k/2} \sum_{j\geq 1} \frac{1}{j^k} \int_0^{j^2N} u^{k/2-1} \log^2(u) e^{-u} \,\mathrm{d}u$$

$$\ll_{k,\ell} \Gamma(k/2) N^{k/2} \sum_{j\geq 1} \frac{\log^2(jN)}{j^k} + N^{k/2} \ll_{k,\ell} N^{k/2} \log^2 N$$

provided that $k > 1$. Inserting (44)–(45) into (43) we get, for $k > 1$, that the quantity in (41) is $\ll_{k,\ell} N^{k+1/\ell}$.

Now we have to establish the convergence of

$$\sum_{j\geq 1} \sum_{\rho} \left| \Gamma\left(\frac{\rho}{\ell}\right) \right| \int_{(1/N)} |e^{Nz}|\, |z|^{-k-3/2} |z^{-\rho/\ell}| e^{-\pi^2 j^2 \Re(1/z)} |\mathrm{d}z|. \tag{46}$$

By symmetry, we may assume that $\gamma > 0$. For $y \in (-\infty, 0]$ we have $\gamma \arctan(y/a) - \frac{\pi}{2}\gamma \leq -\frac{\pi}{2}\gamma$. Using (36), (15) and the Stirling formula (30), the quantity we are estimating becomes

$$
\begin{aligned}
(47) \qquad & \ll \sum_{j\geq 1}\sum_{\rho:\gamma>0}\Big(\frac{\gamma}{\ell}\Big)^{\beta/\ell-1/2}\exp\Big(-\frac{\pi}{2}\frac{\gamma}{\ell}\Big) \\
& \times\Big(\int_{-1/N}^{0} N^{k+3/2+\beta/\ell}e^{-j^2N}\,\mathrm{d}y+\int_{-\infty}^{-1/N}\frac{e^{-j^2/(Ny^2)}}{|y|^{k+3/2+\beta/\ell}}\,\mathrm{d}y\Big) \\
& \ll_{k,\ell} N^{k+1/2+1/\ell}\sum_{j\geq 1}e^{-j^2N}\sum_{\rho:\gamma>0}\Big(\frac{\gamma}{\ell}\Big)^{\beta/\ell-1/2}\exp\Big(-\frac{\pi}{2}\frac{\gamma}{\ell}\Big) \\
& \quad + N^{k/2+1/4}\sum_{j\geq 1}\frac{1}{j^{k+1/2}}\sum_{\rho:\gamma>0}\frac{N^{\beta/(2\ell)}}{j^{\beta/\ell}}\Big(\frac{\gamma}{\ell}\Big)^{\beta/\ell-1/2}\exp\Big(-\frac{\pi}{2}\frac{\gamma}{\ell}\Big) \\
& \quad \times\int_0^{j^2N}u^{k/2-3/4+\beta/(2\ell)}e^{-u}\,\mathrm{d}u \\
& \ll_{k,\ell} N^{k+1/\ell}+\Big(\max_{0\leq b\leq 1}\Gamma\Big(\frac{b}{2\ell}+\frac{k}{2}+\frac{1}{4}\Big)\Big)N^{k/2+1/4+1/2\ell} \\
& \quad \times\sum_{j\geq 1}\frac{1}{j^{k+1/2}}\sum_{\rho:\gamma>0}\Big(\frac{\gamma}{\ell}\Big)^{\beta/\ell-1/2}\exp\Big(-\frac{\pi}{2}\frac{\gamma}{\ell}\Big) \\
& \ll_{k,\ell} N^{k+1/\ell},
\end{aligned}
$$

provided that $k>1/2$, where we used the substitution $u=-j^2/(Ny^2)$, (14) and standard density estimates.

Let now $y>0$. Using the Stirling formula (30) and (36), we can write that the quantity in (46) is

$$
\begin{aligned}
(48) \qquad & \ll_{k,\ell} \sum_{j\geq 1}\sum_{\rho:\gamma>0}\Big(\frac{\gamma}{\ell}\Big)^{\beta/\ell-1/2}\exp\Big(-\frac{\pi}{4}\frac{\gamma}{\ell}\Big)\int_0^{1/N}\frac{e^{-j^2N}}{|z|^{k+3/2+\beta/\ell}}\,\mathrm{d}y \\
& + \sum_{j\geq 1}\sum_{\rho:\gamma>0}\Big(\frac{\gamma}{\ell}\Big)^{\beta/\ell-1/2}\int_{1/N}^{+\infty}\exp\Big(\frac{\gamma}{\ell}\Big(\arctan(Ny)-\frac{\pi}{2}\Big)\Big)\frac{e^{-j^2/(Ny^2)}}{|z|^{k+3/2+\beta/\ell}}\,\mathrm{d}y \\
& = W_1+W_2,
\end{aligned}
$$

say. Using (15) and (14), we have that

$$
(49) \qquad W_1 \ll_{k,\ell} N^{k+1/2+1/\ell}\sum_{j\geq 1}e^{-j^2N}\sum_{\rho:\gamma>0}\Big(\frac{\gamma}{\ell}\Big)^{\beta/\ell-1/2}\exp\Big(-\frac{\pi}{4}\frac{\gamma}{\ell}\Big)\ll_{k,\ell} N^{k+1/\ell},
$$

by standard density estimates. Moreover, we get

$$
\begin{aligned}
W_2 &\ll_{k,\ell} \sum_{j\geq 1}\sum_{\rho:\gamma>0}\Big(\frac{\gamma}{\ell}\Big)^{\beta/\ell-1/2}\int_{1/N}^{+\infty} y^{-k-3/2-\beta/\ell}\exp\Big(-\frac{\gamma}{\ell N y}-\frac{j^2}{Ny^2}\Big)\,\mathrm{d}y\\
&\ll_{k,\ell} N^{k/2+1/4}\sum_{j\geq 1}\frac{1}{j^{k+1/2}}\sum_{\rho:\gamma>0}\frac{N^{\beta/(2\ell)}\gamma^{\beta/\ell-1/2}}{j^{\beta/\ell}}\\
&\quad\times\int_0^{j\sqrt{N}} v^{k-1/2+\beta/\ell}\exp\Big(-\frac{\gamma v}{\ell j\sqrt{N}}-v^2\Big)\,\mathrm{d}v,
\end{aligned}
$$

in which we used the substitution $v^2=j^2/(Ny^2)$. We remark that, for $k>1$, we can set $\varepsilon=\varepsilon(k)=(k-1)/2>0$ and that $k-\varepsilon=(k+1)/2>1$. We further remark that $\max_v(v^{k-\varepsilon}e^{-v^2})$ is attained at $v_0=((k-\varepsilon)/2)^{1/2}$, and hence we obtain, for N sufficiently large, that

$$
W_2 \ll_{k,\ell} N^{k/2+1/4}\sum_{j\geq 1}\frac{1}{j^{k+1/2}}\sum_{\rho:\gamma>0}\frac{N^{\beta/(2\ell)}\gamma^{\beta/\ell-1/2}}{j^{\beta/\ell}}\int_0^{j\sqrt{N}} v^{\beta/\ell-1/2+\varepsilon}\exp\Big(-\frac{\gamma v}{\ell j\sqrt{N}}\Big)\,\mathrm{d}v.
$$

Making the substitution $u=\gamma v/(j\sqrt{N})$, we have

$$
\begin{aligned}
W_2 &\ll_{k,\ell} N^{k/2+1/2+\varepsilon/2}\sum_{j\geq 1}\frac{1}{j^{k-\varepsilon}}\sum_{\rho:\gamma>0}\frac{N^{\beta/\ell}}{\gamma^{1+\varepsilon}}\int_0^{\gamma} u^{\beta/\ell-1/2+\varepsilon}e^{-u}\,\mathrm{d}u \qquad (50)\\
&\ll_{k,\ell} N^{k/2+1/2+1/\ell+\varepsilon/2}\sum_{j\geq 1}\frac{1}{j^{k-\varepsilon}}\sum_{\rho:\gamma>0}\frac{1}{\gamma^{1+\varepsilon}}\Big(\max_{0\leq b\leq 1}\Gamma\Big(\frac{b}{\ell}+\frac{1}{2}+\varepsilon\Big)\Big)\\
&\ll_{k,\ell} N^{(3/4)k+1/4+1/\ell},
\end{aligned}
$$

by standard density estimates and the definition of ε. Inserting (49)–(50) into (48) and recalling (47), we get, for $k>1$, that the quantity in (46) is $\ll_{k,\ell} N^{k+1/\ell}$.

10. Absolute convergence of $\mathscr{J}_5$, $\mathscr{J}_6$ and $\mathscr{J}_7$

Using, for $v>0$ fixed, $u\in\mathbf{R}$ and $u\to+\infty$, the estimate

$$
|J_v(u)| \ll_v u^{-1/2} \qquad (51)
$$

which immediately follows from (10), and performing a direct computation, we obtain that $\mathscr{J}_5$ converges absolutely for $k>-1/\ell$ (and for N sufficiently large) and that $\mathscr{J}_5\ll_{k,\ell} N^{k/2+1/(2\ell)}$.

Again using (51) and performing a direct computation as in the previous case, we obtain that $\mathscr{J}_7$ converges absolutely for $k>0$ (and for N sufficiently large) and that $\mathscr{J}_7\ll_{k,\ell} N^{k/2}$.

For the study of the absolute convergence of the series in $\mathscr{J}_6$ we have a different situation. In this case it is better to come back to the Sonine representation of the Bessel functions (8) on the line $\Re(s)=1$. Using the usual substitution $s=Nz$, we are led to consider the quantity

$$\sum_{\rho}\left|\Gamma\left(\frac{\rho}{\ell}\right)\frac{N^{\rho/(2\ell)}}{\pi^{\rho/\ell}}\right|\sum_{j\geq 1}\left|\frac{J_{k+1/2+\rho/\ell}(2\pi jN^{1/2})}{j^{k+1/2+\rho/\ell}}\right|$$
$$\ll_k N^{-k/2-1/4}\sum_{\rho}\left|\Gamma\left(\frac{\rho}{\ell}\right)\right|\sum_{j\geq 1}\int_{(1/N)}|e^{Nz}|\,|z|^{-k-3/2}|z^{-\rho/\ell}|e^{-\pi^2 j^2\Re(1/z)}|\mathrm{d}z|,$$

which is very similar to the one in (46); the only difference is that the sums are interchanged. The argument used in (46)–(50) can be applied in this case too thus showing that the double series in $\mathscr{J}_6$ converges absolutely for $k > 1$.

We thank the Referee for a very careful reading of the first version of this paper.

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