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RELATIONSHIPS AMONG NON-FLAT TOTALLY GEODESIC SURFACES IN SYMMETRIC SPACES OF TYPE A AND THEIR POLYNOMIAL REPRESENTATIONS

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Abstract

We give computational systems of polynomial representations of the composition maps of non-flat totally geodesic surfaces of the symmetric spaces of type A which are obtained by K. Mashimo, and the Cartan imbeddings of symmetric spaces of type A to $SU(n)$. We obtain the relationships among the non-flat totally geodesic surfaces in symmetric spaces of types AI, AII and AIII by this methods.

1. Introduction

In [6], K. Mashimo classified non-flat totally geodesic surfaces in symmetric spaces of classical type. Since the induced metric on the symmetric space by Cartan imbedding coincides with the normal metric (which comes from Killing form on the symmetric space) up to positive constant, the composition map of Mashimo's totally geodesic immersion and the Cartan imbedding is a totally geodesic immersion of two dimensional sphere S^2 to $SU(n)$.

We will show that this totally geodesic immersion is a some restriction of the irreducible representation of the 3-dimensional simple Lie group $SU(2)$ to $SU(n)$. By using this, we give methods of the computation of this composition map by taking account of polynomials very quickly (Theorems 5.1–5.3). From these explicit representations of totally geodesic immersions of S^2 of symmetric spaces of type A, we obtain the relationships among non-flat totally geodesic immersions of 3-types. Also we can compute the Gauss curvature of the totally geodesic surface which is corresponding to the irreducible representation of $SU(2)$ to $SU(n)$, in the unified way.

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2. Irreducible representation of $SU(2)$

We give the complex irreducible representations of special unitary group $SU(2)$. Let $V(d)$ be the complex vector space of homogeneous polynomials $P(z,w)$ of degree d in two variables (z,w) of $\mathbf{C}^2$. That is

$$V(d) = \mathrm{span}_{\mathbf{C}}\{z^k w^{d-k} \mid k \in \{0,\dots,d\}\}.$$

The Hermitian inner product $\langle\,,\rangle$ on $V(d)$ is given by

$$\langle f_1, f_2\rangle = \sum_{k=0}^{d} k!(d-k)!a_k\overline{b_k}, \tag{2.1}$$

for $f_1(z,w) = \sum_{k=0}^{d} a_k z^k w^{d-k}$ and $f_2(z,w) = \sum_{k=0}^{d} b_k z^k w^{d-k} \in V(d)$. We set

$$P_k(z,w) = \frac{z^k w^{d-k}}{\sqrt{k!(d-k)!}}, \tag{2.2}$$

for any $k \in \{0,\dots,d\}$. Then, $(P_0 \quad P_1 \quad \cdots \quad P_d)$ is an orthonormal basis of $V(d)$, with respect to (2.1).

Let $\rho_d : SU(2) \to End\{V(d)\}$ be the representation which is defined by

$$(\rho_d(g)P)(z,w) = P\left({}^t\!\left(g^{-1}\begin{pmatrix} z \\ w\end{pmatrix}\right)\right) = P((z,w)\bar{g}), \tag{2.3}$$

for any function $P \in V(d)$, and $g \in SU(2)$. Then the representation matrix $\mu_{d+1}(g)$ of the representation ρ_d with respect to the orthonormal basis is given by

$$(\rho_d(g)P_0 \quad \rho_d(g)P_1 \quad \cdots \quad \rho_d(g)P_d) = (P_0 \quad P_1 \quad \cdots \quad P_d)\mu_{d+1}(g),$$

then $\mu_{d+1}(g) \in SU(d+1) \subset M_{(d+1)\times(d+1)}$. We put $n = d+1$, then for any $g = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a}\end{pmatrix} \in SU(2)$ and $i,j \in \{1,\dots,n\}$, the (i,j)th entry of the matrix $\mu_n(g)$ is given by

$$\begin{aligned}(\mu_n(g))_{ij} &= \langle \rho_{n-1}(g)P_{j-1}, P_{i-1}\rangle \\ &= \sqrt{\frac{(n-i)!(i-1)!}{(n-j)!(j-1)!}}\sum_{(s,t)}(-1)^s({}_{n-j}\mathrm{C}_s \cdot {}_{j-1}\mathrm{C}_t)a^{(n-j-s)}\bar{a}^{(j-1-t)}b^s\bar{b}^t,\end{aligned} \tag{2.4}$$

where s runs in the set $\{0,\dots,n-j\}$ and t runs in $\{0,\dots,j-1\}$ with the relation $t = j-i+s$.

3. Symmetric spaces of type A and Cartan imbeddings

In this section, we give the Cartan involutions of type A, and the decomposition of the Lie algebra $\mathfrak{su}(n)$ of $SU(n)$ by these involutions. Let $SU(n)$ be the special unitary group of degree n defined by

$$SU(n) = \{g \in M_{n\times n}(\mathbf{C}) \mid \langle gu, gv\rangle = \langle u, v\rangle \text{ for any } u, v \in \mathbf{C}^n, \det(g) = 1\}.$$

We denote by σ the Cartan involution of $SU(n)$ and $K = \{g \in SU(n) \mid \sigma(g) = g\}$ is the isotropy subgroup of σ.

We write down the Cartan decomposition of the Lie algebra $\mathfrak{su}(n)$ by σ. Since σ is the Cartan involution, the differential $\sigma_*|_e$ at the identity element e

$$\sigma_*|_e : \mathfrak{su}(n) \to \mathfrak{su}(n)$$

has two eigenvalues ± 1. Let $\mathfrak{p}$ and $\mathfrak{k}$ be the eigenvector space corresponding to the eigenvalue -1 and $+1$, respectively. Then the subspace $\mathfrak{p}$ can be identified with the tangent space $T_{eK}(SU(n)/K)$ at the origin $eK \in SU(n)/K$. We recall Cartan involutions of type A and the Cartan decomposition of $\mathfrak{su}(n)$ by each involution. To represent this, we put

$$J = \begin{pmatrix} O_{n\times n} & -I_n \\ I_n & O_{n\times n} \end{pmatrix}, \quad I_{p,q} = \begin{pmatrix} I_p & O_{p\times q} \\ O_{q\times p} & -I_q \end{pmatrix}.$$

Then, the Cartan involutions and decompositions are given by

Type	Cartan involution	$\mathfrak{k}$	$\mathfrak{p}$
AI	$\sigma_{\mathrm{I},n}(g) = \bar{g}$ (outer)	$\mathfrak{so}(n)$	$\sqrt{-1}U$
AII	$\sigma_{\mathrm{II},2n}(g) = J\bar{g}J^{-1}$ (outer)	$\mathfrak{sp}(n)$	$\begin{pmatrix} Z_1 & Z_2 \\ \bar{Z}_2 & -\bar{Z}_1 \end{pmatrix}$
AIII	$\sigma_{\mathrm{III},(p,q)}(g) = I_{p,q}gI_{p,q}$ (inner)	$\mathfrak{s}(\mathfrak{u}(p) \oplus \mathfrak{u}(q))$	$\begin{pmatrix} O_{p\times p} & -{}^t\bar{Z} \\ Z & O_{q\times q} \end{pmatrix}$

where $U \in M_{n\times n}(\mathbf{R})$ satisfies ${}^tU = U$, $\mathrm{tr}\, U = 0$, $Z_1 \in \mathfrak{su}(n)$, $Z_2 \in \mathfrak{so}(n, \mathbf{C})$ and $Z \in M_{q\times p}(\mathbf{C})$. We give the definition of Cartan imbeddings as follows. For any $g \in SU(n)$, the map $\tilde{\mathrm{Car}}_\sigma : SU(n) \to SU(n)$ is defined by

$$\tilde{\mathrm{Car}}_\sigma(g) = g\sigma(g^{-1}).$$

The map $\tilde{\mathrm{Car}}_\sigma$ induces the imbedding

$$\mathrm{Car}_\sigma : SU(n)/K \to SU(n),$$

which is called Cartan imbedding. The image of this imbedding is a totally geodesic submanifold in $SU(n)$. The table of Cartan imbeddings of type A is given by

Type	Source	Target	Cartan imbedding
AI	$SU(n)/SO(n)$	$SU(n)$	$\mathrm{Car}_{\sigma_{\mathrm{I},n}}(g\cdot SO(n)) = g\ {}^t g$
AII	$SU(2n)/Sp(n)$	$SU(2n)$	$\mathrm{Car}_{\sigma_{\mathrm{II},2n}}(g\cdot Sp(n)) = gJ\ {}^t gJ^{-1}$
AIII	$SU(p+q)/S(U(p)\times U(q))$	$SU(p+q)$	$\mathrm{Car}_{\sigma_{\mathrm{III},(p,q)}}(g\cdot S(U(p)\times U(q))) = gI_{p,q}\ {}^t\bar{g}I_{p,q}$

4. Lie triple system and the totally geodesic surfaces

In [6], K. Mashimo classified non-flat totally geodesic surfaces in symmetric spaces of classical type. In this section, we briefly recall his result. For any $l \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$, we define the map $\mathscr{M}_l : \mathbf{R}^2 \to SU(n)/K_l$ as

$$\mathscr{M}_l(t,s) = \exp tX_2^l \exp sX_3^l K_l,$$

where K_l is the isotropy group with respect to the symmetric space of type Al. Here $X_2^l, X_3^l \in \mathfrak{p}_l$ defined in the below. Then we obtain non-flat totally geodesic immersions $\mathscr{M}_l$ (surfaces) in symmetric spaces of type Al.

For Type AI, we set the two tangent vectors $X_2^{\mathrm{I}}, X_3^{\mathrm{I}} \in \mathfrak{p}_{\mathrm{I}}$ as follows

$$X_2^{\mathrm{I}} = \sqrt{-1}\sum_{i=1}^{n}(n+1-2i)E_{i,i}, \tag{4.1}$$

$$X_{3,\varepsilon}^{\mathrm{I}} = -\sqrt{-1}\left[\sum_{i=1}^{n-2}\sqrt{i(n-i)}S_{i,i+1} + \varepsilon\sqrt{n-1}S_{n-1,n}\right], \tag{4.2}$$

for

$$\varepsilon = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 2, \\ \pm 1 & \text{if } n \equiv 0 \pmod 2, \end{cases}$$

where $E_{i,j}$ is the $n \times n$ matrix whose (i,j)-th entry is 1 and all of whose other entries are 0, and $S_{i,j} = E_{i,j} + E_{j,i}$ is the symmetric matrix. We put the subspace $\mathfrak{m} = \mathrm{span}_{\mathbf{R}}\{X_2^{\mathrm{I}}, X_3^{\mathrm{I}}\}$ in $\mathfrak{p}_{\mathrm{I}}$. In fact, $[X_2^{\mathrm{I}}, X_{3,\varepsilon}^{\mathrm{I}}] \in \mathfrak{k} = \mathfrak{so}(n)$, and the 2-dimensional non-abelian subspace $\mathfrak{m}$ in $\mathfrak{p}$ is a Lie triple system. We obtain the relationship between the two vectors X_2^{I}, $X_{3,\varepsilon}^{\mathrm{I}}$ and the matrix of irreducible representation μ_n. If we set the base of $\mathfrak{su}(2)$ by

$$E_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \tag{4.3}$$

then we can easily check that, for $\varepsilon = +1$,

$$\frac{d}{dt}\mu_n(\exp(tE_i))\bigg|_{t=0} = X_i^{\mathrm{I}},$$

for any $i \in \{1,2\}$, and

$$\frac{d}{dt}\mu_n(\exp(tE_3))\bigg|_{t=0} = X^{\mathrm{I}}_{3,+1},$$

where $X^{\mathrm{I}}_1 = \frac{1}{2}[X^{\mathrm{I}}_2, X^{\mathrm{I}}_3] \in \mathfrak{k}_{\mathrm{I}}$. For $\varepsilon = -1$, we set

$$Ad(I_{n-1,1})\left(\frac{d}{dt}\mu_n(\exp(tE_i))\bigg|_{t=0}\right) = X^{\mathrm{I}}_i,$$

for any $i \in \{1,2\}$, and

$$Ad(I_{n-1,1})\left(\frac{d}{dt}\mu_n(\exp(tE_3))\bigg|_{t=0}\right) = X^{\mathrm{I}}_{3,-1}.$$

In the same way, for type AII set the two tangent vectors $X^{\mathrm{II}}_2, X^{\mathrm{II}}_3 \in \mathfrak{p}_{\mathrm{II}}$ by

$$X^{\mathrm{II}}_2 = \sum_{i=1}^{n-1}\sqrt{i(n-i)}(G_{i,i+1} - G_{n+i,n+i+1}), \tag{4.4}$$

$$X^{\mathrm{II}}_3 = \sqrt{-1}\sum_{i=1}^{n-1}\sqrt{i(n-i)}(S_{i,i+1} + S_{n+i,n+i+1}), \tag{4.5}$$

where $G_{i,j} = E_{i,j} - E_{j,i}$ is the skew-symmetric matrix. The 2-dimensional non-abelian subspace $\mathfrak{m} = \mathrm{span}_{\mathbf{R}}\{X^{\mathrm{II}}_2, X^{\mathrm{II}}_3\} \subset \mathfrak{p}_{\mathrm{II}}$ is a Lie triple system.

Next, set the matrix $Q_n \in O(n)$ as

$$Q_n = \begin{pmatrix} & & & 1 \\ & 0 & 1 & \\ & & \cdot^{\cdot^{\cdot}} & \\ & \cdot^{\cdot^{\cdot}} & 0 & \\ 1 & & & \end{pmatrix} \in O(n), \tag{4.6}$$

that is $(Q_n)_{ij} = \delta_j^{n+1-i}$. We define the matrix of reducible representation $\mu_{\mathrm{II},2n}(g)$ of $SU(2)$ by

$$\begin{aligned} \mu_{\mathrm{II},2n}(g) &= \mathbf{Ad}(I_n \oplus Q_n)\Delta\mu_n(g) \\ &= \begin{pmatrix} I_n & O_{n\times n} \\ O_{n\times n} & Q_n \end{pmatrix}\begin{pmatrix} \mu_n(g) & O_{n\times n} \\ O_{n\times n} & \mu_n(g) \end{pmatrix}\begin{pmatrix} I_n & O_{n\times n} \\ O_{n\times n} & Q_n \end{pmatrix}^{-1}. \end{aligned} \tag{4.7}$$

By (4.3) and (4.6), we have

$$\left.\frac{d}{dt}\mu_{\mathrm{II},2n}(\exp(tE_1))\right|_{t=0} = X_2^{\mathrm{II}}, \tag{4.8}$$

$$\left.\frac{d}{dt}\mu_{\mathrm{II},2n}(\exp(tE_3))\right|_{t=0} = -X_3^{\mathrm{II}}, \tag{4.9}$$

$$\left.\frac{d}{dt}\mu_{\mathrm{II},2n}(\exp(tE_2))\right|_{t=0} = X_1^{\mathrm{II}}, \tag{4.10}$$

where $X_1^{\mathrm{II}} = \frac{1}{2}[X_2^{\mathrm{II}}, X_3^{\mathrm{II}}] \in \mathfrak{k}_{\mathrm{II}}$.

Lastly, for the symmetric space $SU(p+q)/S(U(p)\times U(q))$ of Type AIII, where $p = q$ or $p = q+1$, set the two tangent vectors $X_2^{\mathrm{III}}, X_3^{\mathrm{III}} \in \mathfrak{p}_{\mathrm{III}}$ by

$$\begin{aligned} X_2^{\mathrm{III}} = {} & \sum_{i=1}^{q} \sqrt{(2i-1)(p+q+1-2i)}\,G_{i,q+1+i} \\ & + \sum_{i=1}^{p-1} \sqrt{2i(p+q-2i)}\,G_{p+i,i+1}, \end{aligned} \tag{4.11}$$

$$\begin{aligned} X_3^{\mathrm{III}} = \sqrt{-1}\Bigg[& \sum_{i=1}^{q} \sqrt{(2i-1)(p+q+1-2i)}\,S_{p+i,i} \\ & + \sum_{i=1}^{p-1} \sqrt{2i(p+q-2i)}\,S_{i+1,p+i}\Bigg]. \end{aligned} \tag{4.12}$$

Then the non-abelian 2-dimensional subspace $\mathfrak{m} = \mathrm{span}_{\mathbf{R}}\{X_2^{\mathrm{III}}, X_3^{\mathrm{III}}\} \subset \mathfrak{p}_{\mathrm{III}}$ is a Lie triple system. We give the relationship between the basis X_2^{III}, X_3^{III} and the matrix of irreducible representation $\mu_{\mathrm{III},(p,q)}(g)$. We set the matrix $Q' \in O(p+q)$ by

$$(Q')_{ij} = \begin{cases} \delta_j^{2i-1} & (1 \le i \le p), \\ \delta_j^{2(i-p)} & (p+1 \le i \le p+q). \end{cases} \tag{4.13}$$

Let $\mu_{\mathrm{III},(p,q)}(g)$ be the matrix of irreducible representation of $SU(2)$ defined by

$$\mu_{\mathrm{III},(p,q)}(g) = \mathbf{Ad}(Q')\mu_{p+q}(g) = Q'\mu_{p+q}(g)(Q')^{-1}. \tag{4.14}$$

By (4.3), we have

$$\left.\frac{d}{dt}\mu_{\mathrm{III},2n}(\exp(tE_1))\right|_{t=0} = X_2^{\mathrm{III}}, \tag{4.15}$$

$$\left.\frac{d}{dt}\mu_{\mathrm{III},2n}(\exp(tE_3))\right|_{t=0} = -X_3^{\mathrm{III}}, \tag{4.16}$$

$$\left.\frac{d}{dt}\mu_{\mathrm{III},2n}(\exp(tE_2))\right|_{t=0} = X_1^{\mathrm{III}}, \tag{4.17}$$

where $X_1^{\mathrm{III}} = \frac{1}{2}[X_2^{\mathrm{III}}, X_3^{\mathrm{III}}] \in \mathfrak{k}_{\mathrm{III}}$.

5. Composition of Cartan imbedding and μ_n

In this section, we give the new methods of construction of non-flat totally geodesic surfaces in symmetric spaces of type A. Since the images of the Cartan imbeddings of type A are totally geodesic submanifolds and the irreducible representation are the totally geodesic immersions of S^2 into symmetric spaces of type A, the composition map of these two maps is the immersion of S^2 to $SU(n)$. In this case, we can show that the image of the composition map is a non-flat totally geodesic surface (which is diffeomorphic to S^2 or RP^2) in $SU(n)$. The purpose of this paper is to give the system of computation of the representations of these maps by polynomials. In order to show our result (Theorem 5.1), we start by showing the following Lemma.

LEMMA 5.1. *Let $\mu_n : SU(2) \to SU(n)$ be the irreducible matrix representation of $SU(2)$ and $\sigma_{\mathrm{I},n} : SU(n) \to SU(n)$ be the Cartan involution of type* AI. *Then*

$$\sigma_{\mathrm{I},n} \circ \mu_n(g) = \mu_n \circ \sigma_{\mathrm{I},2}(g),$$

for any $g \in SU(2)$.

Proof. For any $g = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \in SU(2)$, the (i,j)th entry of the matrix $\mu_n(g)$ is given by

$$\begin{aligned}(\sigma_{\mathrm{I},n} \circ \mu_n(g))_{ij} &= (\overline{\mu_n(g)})_{ij} \\ &= \overline{\sqrt{\frac{(n-i)!(i-1)!}{(n-j)!(j-1)!}} \sum_{(s,t)} (-1)^s ({}_{n-j}\mathrm{C}_s \cdot {}_{j-1}\mathrm{C}_t) a^{(n-j-s)} \bar{a}^{(j-1-t)} b^s \bar{b}^t} \\ &= \sqrt{\frac{(n-i)!(i-1)!}{(n-j)!(j-1)!}} \sum_{(s,t)} (-1)^s ({}_{n-j}\mathrm{C}_s \cdot {}_{j-1}\mathrm{C}_t) \bar{a}^{(n-j-s)} a^{(j-1-t)} \bar{b}^s b^t \\ &= (\mu_n(\bar{g}))_{ij} = (\mu_n \circ \sigma_{\mathrm{I},2}(g))_{ij},\end{aligned}$$

for any $i, j \in \{1, \dots, n\}$. Hence we obtain the desired result. □

THEOREM 5.1. *Let $\tilde{\mathrm{C}}\mathrm{ar}_{\sigma_{\mathrm{I},n}} : SU(n) \to SU(n)$ be the map defined by*

$$\tilde{\mathrm{C}}\mathrm{ar}_{\sigma_{\mathrm{I},n}}(h) = h\sigma_{\mathrm{I},n}(h^{-1}) = h\, {}^t h,$$

for any $h \in SU(n)$. Then the composition map $\tilde{\mathrm{C}}\mathrm{ar}_{\sigma_{\mathrm{I},n}} \circ \mu_n : SU(2) \to SU(n)$ satisfies

$$\tilde{\mathrm{C}}\mathrm{ar}_{\sigma_{\mathrm{I},n}} \circ \mu_n(g) = \mu_n \circ \tilde{\mathrm{C}}\mathrm{ar}_{\sigma_{\mathrm{I},2}}(g)$$

for any $g \in SU(2)$ where $\mu_n : SU(2) \to SU(n)$ is the irreducible matrix representation of $SU(2)$ of $SU(n)$.

Proof. We see that

$$\tilde{\mathrm{Car}}_{\sigma_{\mathrm{I},n}} \circ \mu_n(g) = \mu_n(g)\sigma_{\mathrm{I},n}((\mu_n(g))^{-1}) = \mu_n(g)\sigma_{\mathrm{I},n}(\mu_n(g^{-1})). \tag{5.1}$$

By Lemma 5.1 and μ_n is a homomorphism, we get

$$\mu_n(g)\sigma_{\mathrm{I},n}(\mu_n(g^{-1})) = \mu_n(g)\mu_n(\sigma_{\mathrm{I},2}(g^{-1})) = \mu_n(g\sigma_{\mathrm{I},2}(g^{-1})) = \mu_n \circ \tilde{\mathrm{Car}}_{\sigma_{\mathrm{I},2}}(g). \tag{5.2}$$

By (5.1),(5.2), we obtain the desired result. □

COROLLARY 5.1. *Let* $\tilde{\mathrm{Car}}_{\sigma_{\mathrm{I},n}} \circ \mu_n$ *be the map defined as above* (*in Theorem* 5.1), *then we can define the immersion* $\varphi_{\mathrm{I}} : SU(2)/SO(2)(\simeq S^2) \to SU(n)$ *as*

$$\varphi_{\mathrm{I}} \circ \pi_{\mathrm{I}}(g) := \tilde{\mathrm{Car}}_{\sigma_{\mathrm{I},n}} \circ \mu_n(g),$$

for any $g = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \in SU(2)$ *where* $\pi_{\mathrm{I}} : SU(2) \to SU(2)/SO(2)$ *in the natural projection. By Theorem* 5.1, *the immersion is given by*

$$\varphi_{\mathrm{I}} \circ \pi_{\mathrm{I}}(g) = \varphi_{\mathrm{I}}(g \cdot SO(2)) = \mu_n\left(\begin{pmatrix} \alpha & \sqrt{-1}u \\ \sqrt{-1}u & \bar{\alpha} \end{pmatrix}\right),$$

where

$$\begin{pmatrix} \alpha & \sqrt{-1}u \\ \sqrt{-1}u & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} a^2 + \bar{b}^2 & ab - \bar{a}\bar{b} \\ ab - \bar{a}\bar{b} & \bar{a}^2 + b^2 \end{pmatrix} = \tilde{\mathrm{Car}}_{\sigma_{\mathrm{I},2}}\left(\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}\right).$$

Here $\alpha = a^2 + \bar{b}^2 \in \boldsymbol{C}$ *and* $u = -\sqrt{-1}(ab - \bar{a}\bar{b}) \in \boldsymbol{R}$ *satisfy* $u^2 + \alpha\bar{\alpha} = 1$.

Next, to prove Theorem 5.2, by (4.6) we prepare the following two Lemmas.

LEMMA 5.2. *For any* $g \in SU(2)$, *we have*

$$\mathbf{Ad}(Q_n)(\mu_n \circ \sigma_{\mathrm{I},2}(g)) = \mu_n \circ \sigma_{\mathrm{III},(1,1)}(g).$$

Proof. By (2.2), we have

$$\rho_{n-1}(Q_2)(P_0 \quad \cdots \quad P_{n-1}) = (P_0 \quad \cdots \quad P_{n-1})Q_n. \tag{5.3}$$

Therefore, we obtain

$$\mu_n(Q_2) = Q_n. \tag{5.4}$$

We can easily see that

$$\mathbf{Ad}(Q_2)(\sigma_{\mathrm{I},2}(g)) = \sigma_{\mathrm{III},(1,1)}(g). \tag{5.5}$$

By (5.5), we get

$$
\begin{aligned}
(5.6) \qquad \mathbf{Ad}(Q_n)(\mu_n(\sigma_{\mathrm{I},2}(g))) &= \mathbf{Ad}(\mu_n(Q_2))(\mu_n(\sigma_{\mathrm{I},2}(g))) \\
&= \mu_n(\mathbf{Ad}(Q_2)(\sigma_{\mathrm{I},2}(g))) \\
&= \mu_n(\sigma_{\mathrm{III},(1,1)}(g))
\end{aligned}
$$

By (5.6), we get the desired result. □

LEMMA 5.3. *Let* $\mu_{\mathrm{II},2n} : SU(2) \to SU(2n)$ *be the reducible matrix representation of* $SU(2)$ *in* (4.7) *and* $\sigma_{\mathrm{II},2n} : SU(2n) \to SU(2n)$ *be the Cartan involution of type* AII. *Then*

$$\sigma_{\mathrm{II},2n} \circ \mu_{\mathrm{II},2n}(g) = \mu_{\mathrm{II},2n} \circ \sigma_{\mathrm{III},(1,1)}(g),$$

for any $g \in SU(2)$, *where* $\sigma_{\mathrm{III},(1,1)} : SU(2) \to SU(2)$ *is the Cartan involution of type* AIII.

Proof. By Lemma 5.2, we have

$$
\begin{aligned}
\sigma_{\mathrm{II},2n} \circ \mu_{\mathrm{II},2n}(g) &= \mathbf{Ad}\left(\begin{pmatrix} O_{n\times n} & -I_n \\ I_n & O_{n\times n} \end{pmatrix}\right) \overline{\mathbf{Ad}\left(\begin{pmatrix} I_n & O_{n\times n} \\ O_{n\times n} & Q_n \end{pmatrix}\right)\begin{pmatrix} \mu_n(g) & O_{n\times n} \\ O_{n\times n} & \mu_n(g) \end{pmatrix}} \\
&= \mathbf{Ad}\left(\begin{pmatrix} I_n & O_{n\times n} \\ O_{n\times n} & Q_n \end{pmatrix}\right) \mathbf{Ad}\left(\begin{pmatrix} O_{n\times n} & -Q_n \\ Q_n^{-1} & O_{n\times n} \end{pmatrix}\right)\begin{pmatrix} \mu_n(\bar{g}) & O_{n\times n} \\ O_{n\times n} & \mu_n(\bar{g}) \end{pmatrix} \\
&= \mathbf{Ad}\left(\begin{pmatrix} I_n & O_{n\times n} \\ O_{n\times n} & Q_n \end{pmatrix}\right)\begin{pmatrix} \mathbf{Ad}(Q_n)\mu_n(\sigma_{I,2}(g)) & O_{n\times n} \\ O_{n\times n} & \mathbf{Ad}(Q_n)\mu_n(\sigma_{I,2}(g)) \end{pmatrix} \\
&= \mathbf{Ad}\left(\begin{pmatrix} I_n & O_{n\times n} \\ O_{n\times n} & Q_n \end{pmatrix}\right)\begin{pmatrix} \mu_n \circ \sigma_{\mathrm{III},(1,1)}(g) & O_{n\times n} \\ O_{n\times n} & \mu_n \circ \sigma_{\mathrm{III},(1,1)}(g) \end{pmatrix} \\
&= \mu_{\mathrm{II},2n} \circ \sigma_{\mathrm{III},(1,1)}(g).
\end{aligned}
$$

Hence we obtain the desired result. □

THEOREM 5.2. *Let* $\tilde{\mathrm{Car}}_{\sigma_{\mathrm{II},2n}} : SU(2n) \to SU(2n)$ *be the map given by*

$$\tilde{\mathrm{Car}}_{\sigma_{\mathrm{II},2n}}(g) = g\sigma_{\mathrm{II},2n}(g^{-1}) = gJ\bar{g}J^{-1}.$$

Then the composition of two maps $\tilde{\mathrm{Car}}_{\sigma_{\mathrm{II},2n}} \circ \mu_{\mathrm{II},2n} : SU(2) \to SU(2n)$ *satisfies*

$$\tilde{\mathrm{Car}}_{\sigma_{\mathrm{II},2n}} \circ \mu_{\mathrm{II},2n}(g) = \mu_{\mathrm{II},2n} \circ \tilde{\mathrm{Car}}_{\sigma_{\mathrm{III},(1,1)}}(g),$$

for any $g \in SU(2)$, *where* $\mu_{\mathrm{II},2n} : SU(2) \to SU(2n)$ *be the reducible matrix representation of* $SU(2)$.

Proof. We see that

$$(5.7) \quad \tilde{\mathrm{Car}}_{\sigma_{\mathrm{II},2n}} \circ \mu_{\mathrm{II},2n}(g) = \mu_{\mathrm{II},2n}(g)\sigma_{\mathrm{II},2n}((\mu_{\mathrm{II},2n}(g))^{-1}) = \mu_{\mathrm{II},2n}(g)\sigma_{\mathrm{II},2n}(\mu_{\mathrm{II},2n}(g^{-1})).$$

By Lemma 5.3 and $\mu_{\mathrm{II},2n}$ is a homomorphism, we get

$$\mu_{\mathrm{II},2n}(g)\sigma_{\mathrm{II},2n}(\mu_{\mathrm{II},2n}(g^{-1})) = \mu_{\mathrm{II},2n}(g)\mu_{\mathrm{II},2n}(\sigma_{\mathrm{III},(1,1)}(g^{-1})) \tag{5.8}$$
$$= \mu_{\mathrm{II},2n}(g\sigma_{\mathrm{III},(1,1)}(g^{-1}))$$
$$= \mu_{\mathrm{II},2n} \circ \tilde{\mathrm{Car}}_{\sigma_{\mathrm{III},(1,1)}}(g).$$

By (5.7),(5.8), we obtain the desired result. □

Corollary 5.2. *Let* $\tilde{\mathrm{Car}}_{\sigma_{\mathrm{II},2n}} \circ \mu_{\mathrm{II},2n}$ *be the map defined as above* (*in Theorem* 5.2), *then we can define the immersion* $\varphi_{\mathrm{II},2n} : SU(2)/S(U(1) \times U(1)) \to SU(2n)$ *as*

$$\varphi_{\mathrm{II},2n} \circ \pi_{\mathrm{III}}(g) := \tilde{\mathrm{Car}}_{\sigma_{\mathrm{II},2n}} \circ \mu_{\mathrm{II},2n}(g),$$

for any $g = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \in SU(2)$ *where* $\pi_{\mathrm{III}} : SU(2) \to SU(2)/S(U(1) \times U(1))$ *is the natural projection. By Theorem* 5.2, *the immersion is given by*

$$\varphi_{\mathrm{II},2n} \circ \pi_{\mathrm{III}}(g) = \varphi_{\mathrm{II},2n}(g \cdot S(U(1) \times U(1))) = \mu_{\mathrm{II},2n}\left(\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \alpha \end{pmatrix}\right),$$

where

$$\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \alpha \end{pmatrix} = \begin{pmatrix} |a|^2 - |b|^2 & -2a\bar{b} \\ 2\bar{a}b & |a|^2 - |b|^2 \end{pmatrix} = \tilde{\mathrm{Car}}_{\sigma_{\mathrm{III},(1,1)}}\left(\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}\right).$$

Here $\alpha = |a|^2 - |b|^2 \in \boldsymbol{R}$ *and* $\beta = 2\bar{a}b \in \boldsymbol{C}$, *satisfy* $\alpha^2 + \beta\bar{\beta} = 1$.

Lastly, to prove Theorem 5.3, we prepare the following Lemma.

Lemma 5.4. *Let* $\mu_{\mathrm{III},(p,q)} : SU(2) \to SU(p+q)$ *be the irreducible matrix representation of* $SU(2)$ *and* $\sigma_{\mathrm{III},(p,q)} : SU(p+q) \to SU(p+q)$ *be the Cartan involution of type* AIII, *where* $p = q$ *or* $p = q+1$. *Then*

$$\sigma_{\mathrm{III},(p,q)} \circ \mu_{\mathrm{III},(p,q)}(g) = \mu_{\mathrm{III},(p,q)} \circ \sigma_{\mathrm{III},(1,1)}(g),$$

for any $g \in SU(2)$.

Proof. Let $(P'_0 \ \cdots \ P'_{p+q-1})$ be the basis of $V(p+q-1)$ defined by,

$$(P'_0 \ \cdots \ P'_{p+q-1}) = (P_0 \ \cdots \ P_{p+q-1})(Q')^{-1} \tag{5.9}$$

where Q' is defined in (4.13). Then we have

$$\rho(-I_{1,1})(P'_0 \ \cdots \ P'_{p+q-1}) = (P'_0 \ \cdots \ P'_{p+q-1})(I_{p,q}). \tag{5.10}$$

By (5.9),(5.10), we get

$$\mathbf{Ad}(Q')\mu_{p+q}(I_{1,1}) = I_{p,q}. \tag{5.11}$$

Therefore, by (4.14) and (5.11),

$$\begin{aligned}(5.12)\qquad \sigma_{\mathrm{III},(p,q)} \circ \mu_{\mathrm{III},(p,q)}(g) &= \mathbf{Ad}(I_{p,q})\, \mathbf{Ad}(Q')\mu_{p+q}(g)\\ &= \mathbf{Ad}(Q'\mu_{p+q}(I_{1,1})(Q')^{-1})\, \mathbf{Ad}(Q')\mu_{p+q}(g)\\ &= \mathbf{Ad}(Q')\, \mathbf{Ad}(\mu_{p+q}(I_{1,1}))\mu_{p+q}(g)\\ &= \mathbf{Ad}(Q')\mu_{p+q}(\mathbf{Ad}(I_{1,1})(g))\\ &= \mathbf{Ad}(Q')\mu_{p+q}(\sigma_{\mathrm{III},(1,1)}(g))\\ &= \mu_{\mathrm{III},(p,q)} \circ \sigma_{\mathrm{III},(1,1)}(g).\end{aligned}$$

By (5.12), we obtain the desired result. □

THEOREM 5.3. *Let* $\tilde{\mathrm{Car}}_{\sigma_{\mathrm{III},(p,q)}} : SU(p+q) \to SU(p+q)$ *be the map given by*

$$\tilde{\mathrm{Car}}_{\sigma_{\mathrm{III},(p,q)}}(h) = h\sigma_{\mathrm{III},(p,q)}(h^{-1}) = hI_{p,q}\ {}^t\bar{h}I_{p,q},$$

for any $h \in SU(p+q)$. *Then the composition map* $\tilde{\mathrm{Car}}_{\sigma_{\mathrm{III},(p,q)}} \circ \mu_{\mathrm{III},(p,q)} : SU(2) \to SU(p+q)$ *satisfies*

$$\tilde{\mathrm{Car}}_{\sigma_{\mathrm{III},(p,q)}} \circ \mu_{\mathrm{III},(p,q)}(g) = \mu_{\mathrm{III},(p,q)} \circ \tilde{\mathrm{Car}}_{\sigma_{\mathrm{III},(1,1)}}(g),$$

for any $g \in SU(2)$ *where* $\mu_{\mathrm{III},(p,q)} : SU(2) \to SU(p+q)$ *be the matrix of irreducible representation of* $SU(2)$.

Proof. We see that

$$\begin{aligned}(5.13)\qquad \tilde{\mathrm{Car}}_{\sigma_{\mathrm{III},(p,q)}} \circ \mu_{\mathrm{III},(p,q)}(g) &= \mu_{\mathrm{III},(p,q)}(g)\sigma_{\mathrm{III},(p,q)}((\mu_{\mathrm{III},(p,q)}(g))^{-1})\\ &= \mu_{\mathrm{III},(p,q)}(g)\sigma_{\mathrm{III},(p,q)}(\mu_{\mathrm{III},(p,q)}(g^{-1})).\end{aligned}$$

By Lemma 5.4 and $\mu_{\mathrm{III},(p,q)}$ is a homomorphism, we get

$$\begin{aligned}(5.14)\qquad \mu_{\mathrm{III},(p,q)}(g)\sigma_{\mathrm{III},(p,q)}(\mu_{\mathrm{III},(p,q)}(g^{-1})) &= \mu_{\mathrm{III},(p,q)}(g)\mu_{\mathrm{III},(p,q)}(\sigma_{\mathrm{III},(1,1)}(g^{-1}))\\ &= \mu_{\mathrm{III},(p,q)}(g\sigma_{\mathrm{III},(1,1)}(g^{-1}))\\ &= \mu_{\mathrm{III},(p,q)} \circ \tilde{\mathrm{Car}}_{\sigma_{\mathrm{III},(1,1)}}(g).\end{aligned}$$

By (5.13),(5.14), we obtain the desired result. □

COROLLARY 5.3. *Let* $\tilde{\mathrm{Car}}_{\sigma_{\mathrm{III},(p,q)}} \circ \mu_{\mathrm{III},(p,q)}$ *be the map defined as above (in Theorem* 5.3*), then we can define the immersion* $\varphi_{\mathrm{III}} : S^2 = SU(2)/S(U(1) \times U(1)) \to SU(p+q)$ *as*

$$\varphi_{\mathrm{III}} \circ \pi_{\mathrm{III}}(g) := \tilde{\mathrm{Car}}_{\sigma_{\mathrm{III},(p,q)}} \circ \mu_{\mathrm{III},(p,q)}(g),$$

for any $g = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \in SU(2)$ *and* $\pi_{\mathrm{III}} : SU(2) \to SU(2)/S(U(1) \times U(1))$ *is a natural projection. By Theorem* 5.3, *the immersion is given by*

$$\varphi_{\mathrm{III}} \circ \pi_{\mathrm{III}}(g) = \varphi_{\mathrm{III}}(g \cdot S(U(1) \times U(1))) = \mu_{\mathrm{III},(p,q)}\left(\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}\right),$$

where

$$\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \alpha \end{pmatrix} = \begin{pmatrix} |a|^2 - |b|^2 & -2a\bar{b} \\ 2\bar{a}b & |a|^2 - |b|^2 \end{pmatrix} = \widetilde{\mathrm{Car}}_{\sigma_{\mathrm{III},(1,1)}}\left(\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}\right).$$

Here $\alpha = |a|^2 - |b|^2 \in \boldsymbol{R}$ *and* $\beta = 2\bar{a}b \in \boldsymbol{C}$, *satisfy* $\alpha^2 + \beta\bar{\beta} = 1$.

By Corollaries 5.1 and 5.3, we obtain the topological structure of the image of the totally geodesic surfaces.

Proposition 5.1. *If n is odd, then the non-flat totally geodesic surface which is corresponding to the irreducible representation of type* AI *or* AIII *in* $SU(n)$ *is diffeomorphic to a* 2-*dimensional real projective planes. If n is even, the non-flat totally geodesic surface of the same type in* $SU(n)$ *is diffeomorphic to a* 2-*dimensional sphere.*

6. Isomorphism from the totally geodesic 2-sphere S^2 in $SU(2)$ to the complex projective space $\mathrm{P}^1(\mathbf{C})$

Let $\Psi : SU(2) \to SU(2)$ be the map defined by

$$\Psi(g) = \mathbf{Ad}(K_0)g := K_0 g K_0^{-1} = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} g \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

for $g \in SU(2)$, and $K_0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ (see [3]). Then the map Ψ is an inner automorphism of $SU(2)$, and satisfies

$$\Psi\left(g\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\right) = \Psi(g)\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}. \tag{6.1}$$

The map Ψ induce the map

$$\psi : SU(2)/SO(2) \to SU(2)/S(U(1) \times U(1)) (\simeq \mathrm{P}^1(\mathbf{C})). \tag{6.2}$$

Therefore we get the following diagram;

$$\begin{array}{ccc} SU(2) & \xrightarrow{\quad\Psi\quad} & SU(2) \\ \pi_{\mathrm{I}}\big\downarrow & \circlearrowleft & \big\downarrow\pi_{\mathrm{III}} \\ SU(2)/SO(2) & \xrightarrow[\psi]{} & SU(2)/S(U(1) \times U(1)) \end{array}$$

that is, they satisfy

(6.3) $$\pi_{\mathrm{III}} \circ \Psi = \psi \circ \pi_{\mathrm{I}}.$$

By (6.1), we also obtain

(6.4) $$\Psi \circ \tilde{\mathrm{C}}\mathrm{ar}_{\sigma_{\mathrm{I},2}} = \tilde{\mathrm{C}}\mathrm{ar}_{\sigma_{\mathrm{III},(1,1)}} \circ \Psi.$$

6.1. Relationship between totally geodesic surfaces of type AI and type AIII

By (6.3), we note that the map $\psi : SU(2)/SO(2) \to SU(2)/S(U(1) \times U(1))$ is a diffeomorphism and satisfies

$$\psi(\pi_{\mathrm{I}}(g)) = \mathbf{Ad}(K_0)\pi_{\mathrm{I}}(g)(= \pi_{\mathrm{III}} \circ \Psi(g))$$

for $g \in SU(2)$. If $n = p + q$ $(p = q$ or $p = q + 1)$, then we define the map $F_{\mathrm{III}} : SU(n) \to SU(n) = SU(p + q)$ as

$$F_{\mathrm{III}}(h) = \mathbf{Ad}(Q'\mu_n(K_0))(h).$$

for $h \in SU(n)$. We note that the map F_{III} is a diffeomorphism. Then we have

Theorem 6.1. *Let $F_{\mathrm{III}} : SU(n) \to SU(n) = SU(p + q)$, for $n = p + q$ $(p = q$ or $p = q + 1)$, and $\psi : SU(2)/SO(2) \to SU(2)/S(U(1) \times U(1))$ be maps defined as above, then*

$$F_{\mathrm{III}} \circ \varphi_{\mathrm{I}} = \varphi_{\mathrm{III}} \circ \psi.$$

Or equivalently,

$$\varphi_{\mathrm{III}} = F_{\mathrm{III}} \circ \varphi_{\mathrm{I}} \circ \psi^{-1}.$$

where two maps φ_{I} and φ_{III} are defined in §5.

Proof. In fact, for any $g \in SU(2)$, we have

$$\begin{aligned}
F_{\mathrm{III}} \circ \varphi_{\mathrm{I}} \circ \pi_{\mathrm{I}}(g) &= F_{\mathrm{III}} \circ \mu_n \circ \tilde{\mathrm{C}}\mathrm{ar}_{\sigma_{\mathrm{I},2}}(g) \\
&= (Q'\mu_n(K_0))\mu_n(\tilde{\mathrm{C}}\mathrm{ar}_{\sigma_{\mathrm{I},2}}(g))(Q'\mu_n(K_0))^{-1} \\
&= Q'\mu_n(K_0\,\tilde{\mathrm{C}}\mathrm{ar}_{\sigma_{\mathrm{I},2}}(g)K_0^{-1})Q'^{-1} \\
&= \mu_{\mathrm{III},(\mathrm{p},\mathrm{q})} \circ \Psi \circ \tilde{\mathrm{C}}\mathrm{ar}_{\sigma_{\mathrm{I},2}}(g) \\
&= \mu_{\mathrm{III},(\mathrm{p},\mathrm{q})} \circ \tilde{\mathrm{C}}\mathrm{ar}_{\sigma_{\mathrm{III},(1,1)}} \circ \Psi(g) \\
&= \varphi_{\mathrm{III}} \circ \pi_{\mathrm{III}} \circ \Psi(g) \\
&= \varphi_{\mathrm{III}} \circ \psi \circ \pi_{\mathrm{I}}(g).
\end{aligned}$$

□

We note that if $p = q + 1$, therefore $n = 2q + 1$, then

$$F_{\mathrm{III}}(g) = \mathbf{Ad}(Q'\mu_n(K_0))(g)$$

for $(g \in SU(n))$.

6.2. Relationship between totally geodesic surfaces of type AI and type AII

We define the map $F_{\mathrm{II}} : SU(n) \to SU(n) \times SU(n)\ (\subset SU(2n))$ as

$$\begin{aligned} F_{\mathrm{II}}(h) &= \mathbf{Ad}(\mu_n(K_0) \oplus Q_n\mu_n(K_0))\Delta h \\ &= \mathbf{Ad}(\mu_n(K_0) \oplus \mu_n(Q_2K_0))\Delta h \end{aligned}$$

for $h \in SU(n)$. Then F_{II} is a diffeomorphism from $SU(n)$ to its image of F_{II}. We have

THEOREM 6.2. *Let* $F_{\mathrm{II}} : SU(n) \to SU(n) \times SU(n)\ (\subset SU(2n))$ *and* $\psi : SU(2)/SO(2) \to SU(2)/S(U(1) \times U(1))$ *be maps defined as above, then*

$$F_{\mathrm{II}} \circ \varphi_{\mathrm{I}} = \varphi_{\mathrm{II}} \circ \psi.$$

Or eqivalently

$$\varphi_{\mathrm{II}} = F_{\mathrm{II}} \circ \varphi_{\mathrm{I}} \circ \psi^{-1}.$$

where two maps φ_{I} *and* φ_{II} *are defined in* §5.

Proof. In fact, for any $g \in SU(2)$, we have

$$\begin{aligned} F_{\mathrm{II}} \circ \varphi_{\mathrm{I}} \circ \pi_{\mathrm{I}}(g) &= F_{\mathrm{II}} \circ \mu_n \circ \tilde{\mathrm{C}}\mathrm{ar}_{\sigma_{\mathrm{I},2}}(g) \\ &= \mathbf{Ad}(I_n \oplus Q_n)\Delta(\mathbf{Ad}\mu_n(K_0)\mu_n(\tilde{\mathrm{C}}\mathrm{ar}_{\sigma_{\mathrm{I},2}}(g))) \\ &= \mathbf{Ad}(I_n \oplus Q_n)\Delta\mu_n(\mathbf{Ad}(K_0)(\tilde{\mathrm{C}}\mathrm{ar}_{\sigma_{\mathrm{I},2}}(g))) \\ &= \mathbf{Ad}(I_n \oplus Q_n)\Delta(\mu_n \circ \Psi \circ \tilde{\mathrm{C}}\mathrm{ar}_{\sigma_{\mathrm{I},2}}(g)) \\ &= \mathbf{Ad}(I_n \oplus Q_n)\Delta(\mu_n \circ \tilde{\mathrm{C}}\mathrm{ar}_{\sigma_{\mathrm{III},(1,1)}} \circ \Psi(g)) \\ &= \varphi_{\mathrm{II}} \circ \pi_{\mathrm{III}} \circ \Psi(g) \\ &= \varphi_{\mathrm{II}} \circ \psi \circ \pi_{\mathrm{I}}(g). \end{aligned}$$

□

7. Gauss curvature of the non-flat totally geodesic surface in $SU(n)/K$

In order to define a bi-invariant Riemannian metric $\langle\, ,\rangle$ on $SU(n)$, we set the metric on $T_eSU(n)$ as

$$\langle X, Y\rangle_e = \mathrm{Re}(\mathrm{tr}({}^t\overline{X}Y)), \tag{7.1}$$

for any $X, Y \in T_eSU(n)$. We extends this metric whole on $SU(n)$ by the left-translation of $SU(n)$. By (4.1),(4.2), we put

$$X_2^{\mathrm{I}} = \frac{d}{dt}\mu_n(\exp(tE_2))|_{t=0} = \sqrt{-1}\sum_{i=1}^{n}(n+1-2i)E_{i,i},$$

$$X_{3,+1}^{\mathrm{I}} = \frac{d}{ds}\mu_n(\exp(sE_3))|_{s=0} = -\sqrt{-1}\left[\sum_{i=1}^{n-2}\sqrt{i(n-i)}S_{i,i+1} + \sqrt{n-1}S_{n-1,n}\right].$$

Then $\langle X_2^{\mathrm{I}}, X_3^{\mathrm{I}}\rangle_0 = 0$. Define the map $\varphi_{\mathrm{I}}^0 : \mathbf{R}^2 \to SU(n)$ by

$$\varphi_{\mathrm{I}}^0(t,s) = \exp(tX_2^{\mathrm{I}})\exp(2sX_{3,+1}^{\mathrm{I}})\exp(tX_2^{\mathrm{I}}).$$

The map φ_{I}^0 is the local parametrization of the immersion φ_{I}. In order to compute the induced metric, the local tangent vector fields along the immersion φ_{I}^0 are given by

$$\varphi_{\mathrm{I}*}^0\left(\frac{\partial}{\partial t}\right) = X_2^{\mathrm{I}}\varphi_{\mathrm{I}}^0(t,s) + \varphi_{\mathrm{I}}^0(t,s)X_2^{\mathrm{I}}, \tag{7.2}$$

and

$$\begin{aligned} \varphi_{\mathrm{I}*}^0\left(\frac{\partial}{\partial s}\right) &= 2\,Ad(\exp tX_2^{\mathrm{I}})X_{3,+1}^{\mathrm{I}}\varphi_{\mathrm{I}}^0(t,s) \\ &= 2\varphi_{\mathrm{I}}^0(t,s)\,Ad(\exp(-tX_2^{\mathrm{I}}))X_{3,+1}^{\mathrm{I}}. \end{aligned} \tag{7.3}$$

PROPOSITION 7.1. *Let $\varphi_{\mathrm{I}}^0 : \boldsymbol{R}^2 \to SU(n)$ be the immersion defined as above. Then the induced metric $\varphi_{\mathrm{I}}^{0*}\langle\,,\rangle$ is given by*

$$\varphi_{\mathrm{I}}^{0*}\langle\,,\rangle = 2\{\|X_2^{\mathrm{I}}\|^2 + \langle X_2^{\mathrm{I}}, Ad(\exp(-2sX_{3,+1}^{\mathrm{I}}))X_2^{\mathrm{I}}\rangle\}\,dt^2 + 4\|X_{3,+1}^{\mathrm{I}}\|^2\,ds^2. \tag{7.4}$$

The Gauss curvature K of image of φ_{I}^0 with respect to the induced metric is given by

$$K = \frac{\|[X_2^{\mathrm{I}}, X_{3,+1}^{\mathrm{I}}]\|^2}{4\|X_2^{\mathrm{I}}\|^2\|X_{3,+1}^{\mathrm{I}}\|^2} = \frac{3}{n(n-1)(n+1)}. \tag{7.5}$$

where $[X_2^{\mathrm{I}}, X_{3,+1}^{\mathrm{I}}] = X_2^{\mathrm{I}}X_{3,+1}^{\mathrm{I}} - X_{3,+1}^{\mathrm{I}}X_2^{\mathrm{I}}$ is a usual Lie bracket.

Proof. Since the immersion φ_{I}^0 is a totally geodesic, the Gauss curvature K of this surface coincides with the sectional curvature of the 2-dimensional sub-space spanned by X_2^{I} and $X_{3,+1}^{\mathrm{I}}$ of $SU(n)$ (with respect to the bi-invariant metric $\langle\,,\rangle$ defined as above). That is, K is given by

$$K = \frac{\langle R(X_2^{\mathrm{I}}, X_{3,+1}^{\mathrm{I}})X_{3,+1}^{\mathrm{I}}, X_2^{\mathrm{I}}\rangle}{\|X_2^{\mathrm{I}}\|^2\|X_{3,+1}^{\mathrm{I}}\|^2}, \tag{7.6}$$

where $R(X,Y)Z = ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})Z = -\frac{1}{4}[[X,Y]Z]$ is the curvature tensor of $SU(n)$ where ∇ is a Levi-Civita connection of the bi-invariant metric which satisfies $\nabla_X Y = \frac{1}{2}[X,Y]$ for left-invariants vector fields X, Y, Z on $SU(n)$ (for details see [1]). By (7.6) and Theorem (3.9) in [1], we obtain

$$\frac{\langle R(X_2^{\mathrm{I}}, X_{3,+1}^{\mathrm{I}})X_{3,+1}^{\mathrm{I}}, X_2^{\mathrm{I}}\rangle}{\|X_2^{\mathrm{I}}\|^2\|X_{3,+1}^{\mathrm{I}}\|^2} = \frac{\|[X_2^{\mathrm{I}}, X_{3,+1}^{\mathrm{I}}]\|^2}{4\|X_2^{\mathrm{I}}\|^2\|X_{3,+1}^{\mathrm{I}}\|^2}. \tag{7.7}$$

By (4.1),(4.2), we have

$$\|X_2^{\mathrm{I}}\|^2 = \|X_{3,+1}^{\mathrm{I}}\|^2 = \frac{n(n-1)(n+1)}{3}, \tag{7.8}$$

and

$$\|[X_2^{\mathrm{I}}, X_{3,+1}^{\mathrm{I}}]\|^2 = \frac{4n(n-1)(n+1)}{3}. \tag{7.9}$$

Therefore, we obtain

$$K = \frac{\|[X_2^{\mathrm{I}}, X_{3,+1}^{\mathrm{I}}]\|^2}{4\|X_2^{\mathrm{I}}\|^2\|X_{3,+1}^{\mathrm{I}}\|^2} = \frac{3}{n(n-1)(n+1)}. \qquad \square$$

By Proposition 7.1, we obtain

COROLLARY 7.1. *The Gauss curvature of non-flat totally geodesic surfaces which are corresponding to the irreducible representation of* $SU(2)$ *to* $SU(n)$, (*with the same codimension in* $SU(n)$) *in symmetric space of type* AI *and type* AIII *is the same value*

$$K = \frac{3}{n(n-1)(n+1)}.$$

COROLLARY 7.2. *The Gauss curvature* K *of non-flat totally geodesic surface of* $SU(2n)$ (*which is corresponding to the irreducible representation of* $SU(2)$ *of* $SU(n)$) *in symmetric space of type* AII *is given by*

$$K = \frac{1}{2}\cdot\frac{3}{n(n-1)(n+1)}.$$

8. Examples

By Corollaries 5.1, 5.2 and 5.3, we obtain the following example.

8.1. Type AI

The subspace $\mathfrak{m}$ which is spanned by the following matrices X_2^{I}, $X_{3,+1}^{\mathrm{I}}$ is a Lie triple system of $\mathfrak{p}_{\mathrm{I}}$;

$$X_2^{\mathrm{I}} = \sqrt{-1}\begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -4 \end{pmatrix}, \quad X_{3,+1}^{\mathrm{I}} = -\sqrt{-1}\begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 2 & 0 & \sqrt{6} & 0 & 0 \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ 0 & 0 & \sqrt{6} & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 \end{pmatrix}.$$

The matrix of irreducible representation of $SU(2)$ in $SU(5)$ is given by $\mu_5(g) =$

$$\begin{pmatrix} a^4 & 2a^3\bar{b} & \sqrt{6}a^2\bar{b}^2 & 2a\bar{b}^3 & \bar{b}^4 \\ -2a^3b & a^2(|a|^2 - 3|b|^2) & \sqrt{6}a\bar{b}(|a|^2 - |b|^2) & \bar{b}^2(3|a|^2 - |b|^2) & 2\bar{a}\bar{b}^3 \\ \sqrt{6}a^2b^2 & \sqrt{6}ab(|b|^2 - |a|^2) & |a|^4 - 4|a|^2|b|^2 + |b|^4 & \sqrt{6}\bar{a}\bar{b}(|a|^2 - |b|^2) & \sqrt{6}\bar{a}^2\bar{b}^2 \\ -2ab^3 & b^2(3|a|^2 - |b|^2) & \sqrt{6}\bar{a}b(|b|^2 - |a|^2) & \bar{a}^2(|a|^2 - 3|b|^2) & 2\bar{a}^3\bar{b} \\ b^4 & -2\bar{a}b^3 & \sqrt{6}\bar{a}^2b^2 & -2\bar{a}^3b & \bar{a}^4 \end{pmatrix},$$

where $g = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \in SU(2)$. Then the map from $SU(2)/SO(2)$ to $SU(5)$ is given by

$$\varphi_{\mathrm{I}}(\pi_{\mathrm{I}}(g))$$
$$= \mu_5\left(\begin{pmatrix} \alpha & \sqrt{-1}u \\ \sqrt{-1}u & \bar{\alpha} \end{pmatrix}\right)$$
$$= \begin{pmatrix} \alpha^4 & -2\sqrt{-1}\alpha^3u & -\sqrt{6}\alpha^2u^2 & -2\sqrt{-1}u^3\alpha & u^4 \\ -2\sqrt{-1}\alpha^3u & \alpha^2(|\alpha|^2 - 3u^2) & -\sqrt{-1}\sqrt{6}\alpha u(|\alpha|^2 - |u|^2) & -u^2(3|\alpha|^2 - u^2) & -2\sqrt{-1}\bar{\alpha}u^3 \\ -\sqrt{6}\alpha^2u^2 & \sqrt{-1}\sqrt{6}\alpha u(u^2 - |\alpha|^2) & |\alpha|^4 - 4|\alpha|^2|u^2 + u^4 & -\sqrt{-1}\sqrt{6}\bar{\alpha}u(|\alpha|^2 - u^2) & -\sqrt{6}\bar{\alpha}^2u^2 \\ 2\sqrt{-1}\alpha u^3 & -u^2(3|\alpha|^2 - u^2) & \sqrt{-1}\sqrt{6}\bar{\alpha}u(u^2 - |\alpha|^2) & \bar{\alpha}^2(|\alpha|^2 - 3u^2) & -2\sqrt{-1}\bar{\alpha}^3u \\ u^4 & 2\sqrt{-1}\bar{\alpha}u^3 & -\sqrt{6}\bar{\alpha}^2u^2 & -2\sqrt{-1}\bar{\alpha}^3u & \bar{\alpha}^4 \end{pmatrix},$$

where $\alpha = a^2 + \bar{b}^2$ and $\sqrt{-1}u = ab - \bar{a}\bar{b}$ as above $\pi_{\mathrm{I}}(g) \in SU(2)/SO(2)$. The Gauss curvature K is given by

$$K = \frac{1}{40}.$$

8.2. Type AIII

The subspace $\mathfrak{m}$ which is spanned by the following matrices X_2^{III}, X_3^{III} is a Lie triple system of $\mathfrak{p}_{\mathrm{III}}$;

$$X_2^{\mathrm{III}} = \begin{pmatrix} 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -\sqrt{6} & \sqrt{6} \\ 0 & 0 & 0 & 0 & -2 \\ -2 & \sqrt{6} & 0 & 0 & 0 \\ 0 & -\sqrt{6} & 2 & 0 & 0 \end{pmatrix}, \quad X_3^{\mathrm{III}} = \sqrt{-1}\begin{pmatrix} 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{6} & \sqrt{6} \\ 0 & 0 & 0 & 0 & 2 \\ 2 & \sqrt{6} & 0 & 0 & 0 \\ 0 & \sqrt{6} & 2 & 0 & 0 \end{pmatrix}.$$

The matrix of irreducible representation of $SU(2)$ in $SU(5)$ is given by

$$\mu_{\mathrm{III},(3,2)}(g) = \mu_{\mathrm{III},(3,2)}\left(\begin{pmatrix} a & -\bar{b} \\ b & a \end{pmatrix}\right)$$

$$= \begin{pmatrix} a^4 & \sqrt{6}a^2\bar{b}^2 & \bar{b}^4 & 2a^3\bar{b} & 2a\bar{b}^3 \\ \sqrt{6}a^2b^2 & |a|^4 - 4|a|^2|b|^2 + |b|^4 & \sqrt{6}\bar{a}^2\bar{b}^2 & \sqrt{6}ab(|b|^2 - |a|^2) & \sqrt{6}\bar{a}\bar{b}(|a|^2 - |b|^2) \\ b^4 & \sqrt{6}\bar{a}^2b^2 & \bar{a}^4 & -2\bar{a}b^3 & -2\bar{a}^3b \\ -2a^3b & \sqrt{6}a\bar{b}(|a|^2 - |b|^2) & 2\bar{a}\bar{b}^3 & a^2(|a|^2 - 3|b|^2) & \bar{b}^2(3|a|^2 - |b|^2) \\ -2ab^3 & \sqrt{6}\bar{a}b(|b|^2 - |a|^2) & 2\bar{a}^3\bar{b} & b^2(3|a|^2 - |b|^2) & \bar{a}^2(|a|^2 - 3|b|^2) \end{pmatrix}.$$

The map of $SU(2)/S(U(1) \times U(1))$ to $SU(5)$ is given by

$$\varphi_{\mathrm{III}}(\pi_{\mathrm{III}}(g)) = \mu_{\mathrm{III},(3,2)}\left(\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \alpha \end{pmatrix}\right)$$

$$= \begin{pmatrix} \alpha^4 & \sqrt{6}\alpha^2\bar{\beta}^2 & \bar{\beta}^4 & 2\alpha^3\bar{\beta} & 2\alpha\bar{\beta}^3 \\ \sqrt{6}\alpha^2\beta^2 & \alpha^4 - 4\alpha^2|\beta|^2 + |\beta|^4 & \sqrt{6}\alpha^2\bar{\beta}^2 & \sqrt{6}\alpha\beta(|\beta|^2 - \alpha^2) & \sqrt{6}\alpha\bar{\beta}(\alpha^2 - |\beta|^2) \\ \beta^4 & \sqrt{6}\alpha^2\beta^2 & \alpha^4 & -2\alpha\beta^3 & -2\alpha^3\beta \\ -2\alpha^3\beta & \sqrt{6}\alpha\bar{\beta}(\alpha^2 - |\beta|^2) & 2\alpha\bar{\beta}^3 & \alpha^2(\alpha^2 - 3|\beta|^2) & \bar{\beta}^2(3\alpha^2 - |\beta|^2) \\ -2\alpha\beta^3 & \sqrt{6}\alpha\beta(|\beta|^2 - \alpha^2) & 2\alpha^3\bar{\beta} & \beta^2(3\alpha^2 - |\beta|^2) & \alpha^2(\alpha^2 - 3|\beta|^2) \end{pmatrix},$$

where $\alpha = |a|^2 + |b|^2$ and $\beta = 2\bar{a}b$. Then the Gauss curvature is constant and its value is

$$K = \frac{1}{40}.$$

From these two examples, we see that

$$F_{\mathrm{III}} \circ \varphi_{\mathrm{I}} \circ \pi_{\mathrm{I}}(g) = \varphi_{\mathrm{III}} \circ \pi_{\mathrm{III}} \circ \Psi(g).$$

This relation is a prototype of Theorem 6.1. We note that the following relation holds

$$\mathbf{Ad}(Q')\mu_5(g) = \mu_{\mathrm{III},(3,2)}(g)$$

where $Q' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$.

8.3. Type AII

The subspace $\mathfrak{m}$ which is spanned by the following matrices X_2^{II}, X_3^{II} is a Lie triple system of $\mathfrak{p}_{\mathrm{II}}$.

$$X_2^{\mathrm{II}} = \left(\begin{array}{ccccc:ccccc} 0 & 2 & 0 & 0 & 0 & & & & & \\ -2 & 0 & \sqrt{6} & 0 & 0 & & & & & \\ 0 & -\sqrt{6} & 0 & \sqrt{6} & 0 & & & \mathrm{O}_{5\times 5} & & \\ 0 & 0 & -\sqrt{6} & 0 & 2 & & & & & \\ 0 & 0 & 0 & -2 & 0 & & & & & \\ \hdashline & & & & & 0 & -2 & 0 & 0 & 0 \\ & & & & & 2 & 0 & -\sqrt{6} & 0 & 0 \\ & & \mathrm{O}_{5\times 5} & & & 0 & \sqrt{6} & 0 & -\sqrt{6} & 0 \\ & & & & & 0 & 0 & \sqrt{6} & 0 & -2 \\ & & & & & 0 & 0 & 0 & 2 & 0 \end{array}\right),$$

$$X_3^{\mathrm{II}} = \sqrt{-1}\left(\begin{array}{ccccc:ccccc} 0 & 2 & 0 & 0 & 0 & & & & & \\ 2 & 0 & \sqrt{6} & 0 & 0 & & & & & \\ 0 & \sqrt{6} & 0 & \sqrt{6} & 0 & & & \mathrm{O}_{5\times 5} & & \\ 0 & 0 & \sqrt{6} & 0 & 2 & & & & & \\ 0 & 0 & 0 & 2 & 0 & & & & & \\ \hdashline & & & & & 0 & 2 & 0 & 0 & 0 \\ & & & & & 2 & 0 & \sqrt{6} & 0 & 0 \\ & & \mathrm{O}_{5\times 5} & & & 0 & \sqrt{6} & 0 & \sqrt{6} & 0 \\ & & & & & 0 & 0 & \sqrt{6} & 0 & 2 \\ & & & & & 0 & 0 & 0 & 2 & 0 \end{array}\right).$$

The following is the corresponding matrix of reducible representation of $SU(2)$ in $SU(10)$.

(8.1)
$$\mu_{\mathrm{II},10}(g) = \left(\begin{array}{c:c} \mu_5(g) & O_{5\times 5} \\ \hdashline O_{5\times 5} & \mathbf{A}d(Q_5)\mu_5(g) \end{array}\right).$$

By (8.1), we obtain the map from $SU(2)/S(U(1)\times U(1))$ to $SU(10)$ as

$$\varphi_{\mathrm{II},10}(\pi_{\mathrm{III}}(g)) = \left(\begin{array}{c:c} \mu_5\circ\pi_{\mathrm{III}}(g) & O_{5\times 5} \\ \hdashline O_{5\times 5} & \mathbf{A}d(Q_5)\mu_5\circ\pi_{\mathrm{III}}(g) \end{array}\right),$$

where

(8.2)
$\mathbf{A}d(Q_5)\varphi_{\mathrm{I}}\circ\pi_{\mathrm{I}}(g)$

$$= \begin{pmatrix} \bar{\alpha}^4 & -2\bar{\alpha}^3\beta & \sqrt{6}\bar{\alpha}^2\beta^2 & -2\bar{\alpha}\beta^3 & \beta^4 \\ 2\bar{\alpha}^3\bar{\beta} & \bar{\alpha}^2(|\alpha|^2-3|\beta|^2) & \sqrt{6}\bar{\alpha}\beta(|\beta|^2-|\alpha|^2) & \beta^2(3|\alpha|^2-|\beta|^2) & -2\alpha\beta^3 \\ \sqrt{6}\bar{\alpha}^2\bar{\beta}^2 & \sqrt{6}\bar{\alpha}\bar{\beta}(|\alpha|^2-|\beta|^2) & (|\alpha|^4-4|\alpha|^2|\beta|^2+|\beta|^4) & \sqrt{6}\alpha\beta(|\beta|^2-|\alpha|^2) & \sqrt{6}\alpha^2\beta^2 \\ 2\bar{\alpha}\bar{\beta}^3 & \bar{\beta}^2(3|\alpha|^2-|\beta|^2) & \sqrt{6}\alpha\bar{\beta}(|\alpha|^2-|\beta|^2) & \alpha^2(|\alpha|^2-3|\beta|^2) & -2\alpha^3\beta \\ \bar{\beta}^4 & 2\alpha\bar{\beta}^3 & \sqrt{6}\alpha^2\bar{\beta}^2 & 2\alpha^3\bar{\beta} & \alpha^4 \end{pmatrix}.$$

Here $\alpha = |a|^2 + |b|^2$ and $\beta = 2\bar{a}b$ as above $\pi_{III}(g) \in SU(2)/SU(U(1) \times U(1))$. By (8.2), we get the Gauss curvature as

$$K = \frac{1}{80}.$$

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