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KODAI MATH. J.
41 (2018), 620–651

ŁOJASIEWICZ EXPONENTS OF NON-DEGENERATE HOLOMOROHIC AND MIXED FUNCTIONS

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Abstract

We consider Łojasiewicz inequalities for a non-degenerate holomorphic function with an isolated singularity at the origin. We give an explicit estimation of the Łojasiewicz exponent in a slightly weaker form than the assertion in Fukui [10]. We also introduce Łojasiewicz inequality for strongly non-degenerate mixed functions and generalize this estimation for mixed functions.

1. Holomorphic functions and Łojasiewicz exponent

Consider an analytic function $f(\mathbf{z})$ with an isolated singularity at the origin. We consider the inequality

$$\exists c > 0, \forall \mathbf{z} \in U, \quad \|\partial f(\mathbf{z})\| \geq c\|\mathbf{z}\|^{\theta} \tag{1}$$

where U is a sufficiently small neighborhood of the origin and $\partial f(z)$ is the gradient vector $\left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right)$. The Łojasiewicz exponent $\ell_0(f)$ of $f(\mathbf{z})$ at the origin is the smallest positive number among θ's which satisfy the inequality (1). It is known that there exists such number $\ell_0(f)$ and it is a rational number [16, 27]. We assume that the Newton boundary of f is non-degenerate hereafter. The purpose of this paper is to give an explicit upper bound of the Łojasiewicz exponent in term of the combinatorics of the Newton boundary. There is a similar estimation proposed by Fukui [10] but he uses some incorrect equality (2.4), [10] in his proof. Thus the proof has a gap and the assertion must be proved in a different way, even if it is true. There is also an estimation by Abderrahmane [2] using Newton number. In this paper, we give an estimation along Fukui's way. Our estimation is apparently a little weaker than that of Fukui but it is enough for our purpose. In the last section of this paper (§4), we will generalize the notion of Łojasiewicz exponent for mixed functions.

2000 *Mathematics Subject Classification*. 32S05.
Key words and phrases. Łojasiewicz inequality, vanishing coordinate, non-convenient.
Received April 26, 2017; revised February 7, 2018.

1.1. Newton boundary and the dual Newton diagram. Let $f(\mathbf{z}) = \sum_{\nu} c_\nu \mathbf{z}^\nu$ be an analytic function with $f(\mathbf{0}) = 0$. Recall that the Newton diagram $\Gamma_+(f)$ is the minimal convex set in the first quadrant of $\mathbf{R}_+^n$ containing $\bigcup_{\nu, c_\nu \neq 0}(\nu + \mathbf{R}_+^n)$. The Newton boundary $\Gamma(f)$ is the union of compact faces of $\Gamma_+(f)$. Let N_+ be the space of non-negative weight vectors. Using the canonical basis, it can be identified with the first quadrant of $\mathbf{R}_+^n$. Let $P = (p_1, \ldots, p_n) \in N_+$ be a non-zero weight. It defines a canonical linear mapping on $\Gamma_+(f)$ by $P(\nu) := \sum_{i=1}^n p_i \nu_i$. We denote the minimal value of P on $\Gamma_+(f)$ by $d(P, f)$ and put $\Delta(P, f) = \{\nu \in \Gamma_+(f) \mid P(\nu) = d(P, f)\}$. This is a face of $\Gamma_+(f)$. We simply write as $d(P)$ or $\Delta(P)$ if no confusion is likely. The dimension of $\Delta(P)$ can be $0, 1, \ldots, n-1$. Recall that $P, Q \in N_+$ are *equivalent* if $\Delta(P) = \Delta(Q)$. This equivalence classes gives N_+ a rational polyhedral cone subdivision $\Gamma^*(f)$ and we call $\Gamma^*(f)$ *the dual Newton diagram*. We also define a partial order in $\Gamma^*(f)$ by

$$P \le Q \Leftrightarrow \Delta(P) \subset \Delta(Q).$$

Denote the set of weights which are equivalent to Q by $[Q]$. Note that the closure $\overline{[P]}$ is equal to the union $\bigcup_{Q \ge P}[Q]$ and

$$\dim \overline{[P]} = \dim [P] = n - \dim \Delta(P).$$

The generators of the 1-dimensional cones of $\Gamma^*(f)$ are called *vertices*. A weight $P \in N_+$ is a vertex if and only if $\dim \Delta(P) = n - 1$. We denote the set of vertices by $\mathscr{V}$. $P = (p_1, \ldots, p_n)$ is called *strictly positive* if $p_i > 0$ for any $i = 1, \ldots, n$. Note that $\Delta(P)$ is compact if and only if P is strictly positive. A vertex which is not strictly positive is either the canonical basis $e_i = (0, \ldots, \underset{\smile}{\overset{i}{1}}, \ldots, 0)$, $1 \le i \le n$ or corresponds to a vanishing coordinate subspace (see §3.1 for the definition).

1.1.1. *Face function.* For $\Delta \subset \Gamma(f)$, put $f_\Delta(\mathbf{z}) := \sum_{\nu \in \Delta} c_\nu \mathbf{z}^\nu$ and we call f_Δ *the face function of* Δ. For a weight vector $P \in N_+$, the face function associated with P is defined by $f_P(\mathbf{z}) = f_{\Delta(P)}(\mathbf{z})$. The monomial $\mathbf{z}^\nu$ and the integer point $\nu \in \Gamma_+(f)$ correspond each other. If Δ is a compact face, f_Δ is a weighted homogeneous polynomial.

1.2. Normalized weight vector. Take a weight vector $P = (p_1, \ldots, p_n)$. Let $d = d(P)$ and assume that $d > 0$. The hyperplane Π in $\mathbf{R}^n$, defined by $p_1\nu_1 + \cdots + p_n\nu_n = d$, contains the face $\Delta(P)$ and all other points $\nu \in \Gamma_+(f) \setminus \Delta(P)$ are above Π. Namely $p_1\nu_1 + \cdots + p_n\nu_n > d$. We call Π *the supporting hyperplane* of the weight vector P. For a weight vector P with $d(P) > 0$, we define *the normalized weight vector* $\hat{P}$ of P (with respect to f) by

$$\hat{P} := (\hat{p}_1, \ldots, \hat{p}_n), \quad \hat{p}_i = p_i / d(P).$$

Hereafter we use this notation $\hat{P}$ throughout this paper. Using the normalized weight vector, $d(\hat{P}) = 1$ and the supporting hyperplane Π is written as

$$\Pi: \quad \hat{p}_1\nu_1 + \cdots + \hat{p}_n\nu_n = 1.$$

Note that the v_j coordinate of the intersection of Π and v_j-axis is $1/\hat{p}_j$.

Assume that $\Delta(P) \cap \Delta(Q) \neq \emptyset$ and consider the line segment $P_t := tP + (1-t)Q$, $0 \le t \le 1$. Note that $\Delta(P_t) = \Delta(P) \cap \Delta(Q)$ for any $0 < t < 1$ and the normalized weight vector $\hat{P}_t$ is simply given by $\hat{P}_t = t\hat{P} + (1-t)\hat{Q}$, provided $d(P) > 0$ and $d(Q) > 0$.

The purpose of this paper is to give an upper bound explicitly for the Łojasiewicz exponent using the combinatorial data of the Newton boundary. Then we give an application for the characterization of the monomials which do not change the topology by adding to f. For a weighted homogeneous non-degenerate polynomial, we prove the estimation conjectured in [6] under the Łojasiewicz non-degeneracy. In §4, we generalize these results for mixed functions.

2. Łojasiewicz exponent for convenient functions

2.1. Preliminary consideration. We first consider the estimation of Łojasiewicz exponent along an analytic curve $C(t)$ which is parametrized as follows. Put $I := \{i \mid z_i(t) \not\equiv 0\}$ and I^c be the complement of I.

$$C(t) : \begin{cases} \mathbf{z}(t) = (z_1(t), \dots, z_n(t)), & \mathbf{z}(0) = 0,\ \mathbf{z}(t) \in \mathbf{C}^{*I} \\ z_i(t) = a_i t^{p_i} + (\text{higher terms}), & p_i \in \mathbf{N},\ i \in I. \end{cases} \tag{2}$$

Here we use the following notations:

$$\mathbf{C}^I := \{\mathbf{z} = (z_1, \dots, z_n) \mid z_j = 0,\ j \notin I\}$$

$$\mathbf{C}^{*I} := \{\mathbf{z} = (z_1, \dots, z_n) \mid z_i \neq 0, \Leftrightarrow i \in I\}$$

$$N_+^I := \{P = (p_1, \dots, p_n) \in N_+ \mid p_j = 0,\ j \notin I\}$$

$$N_+^{*I} := \{P = (p_1, \dots, p_n) \in N_+ \mid p_i \neq 0 \Leftrightarrow i \in I\}$$

$$f^I := f|_{\mathbf{C}^I}.$$

Put $P = (p_i)_{i \in I}$ and $d = d(P, f^I)$. We define

$$M(P) := \max\{p_j \mid j \in I\}, \quad m(P) := \min\{p_j \mid j \in I\}.$$

Note that $\operatorname{ord} \mathbf{z}(t) = m(P)$. Put $q := \operatorname{ord} \partial f^I(\mathbf{z}(t))$. Under the non-degeneracy assumption, we have the inequalities:

$$d - M(P) \le q \le d - m(P) \quad \text{or} \tag{3}$$

$$\frac{d - M(P)}{m(P)} \le \frac{q}{m(P)} \le \frac{d}{m(P)} - 1 = \frac{1}{m(\hat{P})} - 1. \tag{4}$$

Put $Vari(P) = \left\{ z_j \,\middle|\, \dfrac{\partial f_P}{\partial z_j} \not\equiv 0 \right\}$. Namely $Vari(P)$ is the set of variables which appear in f_P. Then we have the obvious estimations:

$$\frac{\partial f}{\partial z_j}(\mathbf{z}(t)) = \left(\frac{\partial f}{\partial z_j}\right)_P(\mathbf{a})t^{d_j} + (\text{higher terms}), \quad d_j = d\left(P, \frac{\partial f}{\partial z_j}\right), \tag{5}$$

$$\text{ord}\,\frac{\partial f}{\partial z_j}(\mathbf{z}(t)) \geq d_j \geq d - p_j. \tag{6}$$

If $z_j \in Vari(P)$, $d_j = d(P, f) - p_j$ and otherwise $d_j > d - p_j$. If $m(P) = p_j$, $d/m(P) = 1/\hat{p}_j$ and this is equal to the j-th coordinate of the intersection of $\Pi(P)$ and v_j axis. We define *the Łojasiewicz exponent of f along $C(t)$* by

$$\ell_0(C(t)) := \frac{\text{ord}\,\partial f(\mathbf{z}(t))}{\text{ord}\,\mathbf{z}(t)}.$$

By (3) and by the non-degeneracy assumption, we have

$$\text{ord}\,\partial f(\mathbf{z}(t)) \leq d - m(P) \tag{7}$$

$$\text{ord}\,f_j(\mathbf{z}(t)) \geq d(P, f_j) \geq d - p_j \tag{8}$$

$$\ell_0(C(t)) \leq \frac{d - m(P)}{m(P)}. \tag{9}$$

For a strictly positive weight vector $P = (p_1, \dots, p_n)$, we define positive invariants

$$\begin{cases} \eta_{i,j}(P) := \dfrac{d - p_j}{p_i} = \dfrac{1 - \hat{p}_j}{\hat{p}_i}, \\ \eta'_{i,j}(P) := \dfrac{d_j}{p_i} = \dfrac{\hat{d}_j}{\hat{p}_i} \\ \eta(P) := \dfrac{d - m(P)}{m(P)} = \dfrac{1}{m(\hat{P})} - 1. \end{cases} \tag{10}$$

where $d_j = d(P, f_j)$ and $\hat{d}_j = d_j/d$.

2.2. Łojasiewicz exceptional monomial. We say that $f(\mathbf{z})$ is *convenient* if for any $1 \leq j \leq n$, Newton boundary $\Gamma(f)$ intersects with v_j-axis at a point $B_j = (0, \dots, \underset{j}{\breve{b_j}}, \dots, 0)$. Recall that a non-degenerate function $f(\mathbf{z})$ has an isolated singularity at the origin, if it is convenient (Corollary (2.3), [19]). Assume that $f(\mathbf{z})$ is convenient as above. Define an integer $B := \max\{b_j \mid j = 1, \dots, n\}$ and let $\mathscr{L} = \{i \mid b_i = B\}$. We call z_i^B, $i \in \mathscr{L}$ a *Łojasiewicz monomial* of f. We say that a Łojasiewicz monomial z_i^B is *Łojasiewicz exceptional* if there exists j, $j \neq i$ and a monomial of the form $z_j z_i^{B'}$, with $B' < B - 1$ which has a non-zero coefficient in f.

Consider the curve parametrized as (2) and assume that $I = \{1, \dots, n\}$. Then we have seen

$$\frac{\text{ord}\,\partial f(\mathbf{z}(t))}{\text{ord}\,\mathbf{z}(t)} \leq \frac{d}{m(P)} - 1 \leq B - 1.$$

This implies the following inequality holds in a small neighborhood of the origin.

$$\|\partial f(\mathbf{z}(t))\| \geq c\|\mathbf{z}(t)\|^{B-1}, \quad c \neq 0. \tag{11}$$

If z_i^B is Łojasiewicz exceptional and let $z_j z_i^{B'}$, $B' < B-1$ be as above. Then $\dfrac{\partial f}{\partial z_j}$ has the monomial $z_i^{B'}$ with non-zero coefficient and ord $\dfrac{\partial f}{\partial z_j}(\mathbf{z}(t))$ can be $p_i B'$ which is smaller than $p_i(B-1)$. In fact, this is the case for the i-axis curve $\mathbf{z}(t)$ where $z_i(t) = t$ and $z_j(t) \equiv 0$ for $j \neq i$. We assert

Assertion 1. *The inequality $\ell_0(f) \leq B-1$ holds for any analytic curve $\mathbf{z}(t)$.*

Proof. As we have shown the assertion for the case $I = \{1, \ldots, n\}$, we need only consider the case where some of $z_i(t)$ is identically zero. In this case, put $I := \{i \mid z_i(t) \not\equiv 0\}$. Then $f^I := f|_{\mathbf{C}^I}$ is a non-degenerate convenient function. Thus by the above argument applied for f^I, we have

$$\text{ord}\, \partial f^I(\mathbf{z}(t)) \leq (\text{ord}\, \mathbf{z}(t)) \times (B_I - 1).$$

Here B_I is defined similarly for f^I. By the obvious inequality ord $\partial f(\mathbf{z}(t)) \leq$ ord $\partial f^I(\mathbf{z}(t))$ and $B_I \leq B$, we get the inequality (11). □

For the practical calculation of the Łojasiewicz exponent, we use the following criterion. This can be proved by the Curve Selection Lemma ([18, 11]).

Proposition 2. *A positive number θ satisfies the Łojasiewicz inequality* (1) *if the inequality*

$$\text{ord}\, \partial f(\mathbf{z}(t)) \leq \theta \times \text{ord}\, \mathbf{z}(t)$$

is satisfied along any non-constant analytic curve $C(t)$ parametrized by an analytic path $\mathbf{z}(t)$ with $\mathbf{z}(0) = \mathbf{0}$. That is $\ell_0(f) = \sup \ell_0(C(t))$ where $C(t)$ moves every possible analytic curves starting from the origin.

Now we have the following result for convenient non-degenerate functions.

Theorem 3. *Let $f(\mathbf{z})$ be a non-degenerate convenient analytic function. Then Łojasiewicz exponent $\ell_0(f)$ satisfies the inequality*: $\ell_0(f) \leq B-1$.

Furthermore if f has a Łojasiewicz non-exceptional monomial, $\ell_0(f) = B-1$.

Proof. We have shown that $\ell_0(f) \leq B-1$. We only need to show the existence of a curve $C(t)$ which takes the equality $q = B-1$, assuming that f has a Łojasiewicz non-exceptional monomial. For this purpose, we assume for simplicity $B = b_1$ and z_1^B is non-exceptional. Note that the Newton boundary of $\dfrac{\partial f}{\partial z_i}$ does not touch the v_1 axis under $B-1$ for any $i > 1$ by the assumption. Thus we can take a sufficiently large integer N and put $P = (1, N, \ldots, N)$. Note that

$d\left(P,\dfrac{\partial f}{\partial z_1}\right)=B-1$ and $d\left(P,\dfrac{\partial f}{\partial z_i}\right)\geq B-1$ for any $i\geq 2$. Consider the curve $C(t)$ defined by $\mathbf{z}(t)=(t,t^N,\dots,t^N)$. Then the above observation tells us that

$$\partial f(\mathbf{z}(t))=(B,*,\dots,*)t^{B-1}+(\text{higher terms}).$$

Thus $\|\partial f(\mathbf{z}(t))\|\approx\|\mathbf{z}(t)\|^{B-1}$. □

In the above proof, if there is a monomial $z_1^{B'}z_j$ with $j\neq 1$, $B'<B-1$, we see that $\operatorname{ord} f_j(\mathbf{z}(t))=B'$. Thus we have $\operatorname{ord} f_1(\mathbf{z}(t))>\operatorname{ord} f_j(\mathbf{z}(t))$. The importance of Łojasiewicz exceptional monomial is observed by Lenarcik [17]. For plane curves ($n=2$), we have also observed that it gives a fake effect to computation of the complexity of plane curve singularity but exceptional monomials can be eliminated without changing the non-degeneracy (Le-Oka [15]). Suppose that $cz_1^B+c'z_1^{B'}z_2$ with $B'\leq B-2$, $c,c'\neq 0$ is in a face function of f. Then take the coordinate change $(z_1,z_2'):=(z_1,z_2+(c/c')z_1^{B-B'})$ to kill the monomial z_1^B. This operation does not work for mixed polynomials.

3. Łojasiewicz exponents for non-convenient functions

In this section, we consider again a non-degenerate function $f(\mathbf{z})$ with isolated singularity at the origin without assuming the convenience of the Newton boundary. It turns out that the estimation of Łojasiewicz exponent is much more complicated without the convenience assumption. We assume that $\Gamma(f)$ has dimension $n-1$ hereafter. If the multiplicity at the origin is greater than 2, this condition is always satisfied.

3.1. Łojasiewicz non-degeneracy along a vanishing coordinate subspace. Let I be a subset of $\{1,\dots,n\}$. We say that $\mathbf{C}^I$ is *a vanishing coordinate subspace* ([21, 22, 9]) if $f^I(\mathbf{z}_I)\equiv 0$. Here f^I is the restriction of f to $\mathbf{C}^I$. We use the notation $\mathbf{C}^I=\{\mathbf{z}\,|\,z_j=0,\ j\notin I\}$ and $\mathbf{z}_I=(z_i)_{i\in I}$. If further $I=\{i\}$ is a vanishing coordinate subspace, we say $\mathbf{C}^{\{i\}}$ *a vanishing axis*. We say a face $\Xi\subset\Gamma_+(f)$ is *essentially non-compact* if there exists a non-strictly positive weight vector $Q=(q_1,\dots,q_n)$ such that $d(Q,f)>0$ and $\Delta(Q)=\Xi$. Let $I(Q)=\{i\,|\,q_i=0\}$. We say also $I(Q)$ *the vanishing direction of* Ξ and write also as $I(\Xi)=I(Q)$. Then the assumption $d(Q,f)>0$ implies $\mathbf{C}^{I(Q)}$ is a vanishing coordinate subspace.

Put $I=I(Q)$ and we assume that $I=\{1,\dots,m\}$. Take an $i\in I$. By the assumption, the gradient vector ∂f does non vanish in a neighborhood of the origin of i-axis except at the origin. This is possible only if there exists a monomial $z_i^{n_i}z_j$ with a non-zero coefficient for some $j\neq i$ in the expansion of f. Then we observe that $j\notin I$, because $\mathbf{C}^I$ is a vanishing coordinate subspace. Let J_i be the set of $j\in I^c$ for which such a monomial $z_i^{n_i}z_j$ exists with a non-zero coefficient in the expansion of $f(\mathbf{z})$. $J_i\neq\emptyset$ for any $i\in I$. We define an integer n_{ij} by

$$n_{ij}:=\min\{n_i\,|\,z_i^{n_i}z_j \text{ has a non-zero coefficient}\}$$

for a fixed $i \in I$ and $j \in J_i$. For brevity, we put $n_{ij} = \infty$ if $j \notin J_i$. Put $B_{ij} := (0, \dots, \overset{i}{\breve{n_{ij}}}, \dots, \overset{j}{\breve{1}}, \dots, 0)$. Note that $B_{ij} \in \Gamma(f)$. Put $J(I) = \bigcup_{i \in I} J_i$. We say that f is *Łojasiewicz non-degengerate* if for any strictly positive weight vector $P \in N^{*I}$ (namely $p_i > 0$, $\forall i \in I$), the following condition is satisfied. Put $I' := \{i \mid p_i = m(P)\}$ and $J(P) := \bigcup_{i \in I'} J_i \subset J(I)$. Then the variety

$$\{\mathbf{z} \in \mathbf{C}^{*I} \mid ((f_j)^I)_P(\mathbf{z}_I) = 0, \ \forall j \in J(P)\}.$$

is empty. In other word, for any $\mathbf{a} \in \mathbf{C}^{*I}$, there exists $j \in J(P)$ such that $((f_j)^I)_P(\mathbf{a}) \neq 0$. Here and hereafter we use the simplified notation for the derivative function: $f_i(\mathbf{z}) := \dfrac{\partial f}{\partial z_i}(\mathbf{z})$, $i = 1, \dots, n$.

3.2. Jacobian dual Newton diagram. We consider the derivatives $f_i(\mathbf{z})$, $i = 1, \dots, n$. We consider their Newton boundary $\Gamma(f_i)$, $i = 1, \dots, n$. As we consider $n+1$ Newton boundaries, we denote by $\Delta(P, f_i)$ the face of $\Gamma(f_i)$ where P takes minimal value, $d(P, f_i)$. We consider the following stronger equivalence relation in the space of non-negative weight vectors. Two weight vectors P, Q are *Jacobian equivalent* if $\Delta(P, f_i) = \Delta(Q, f_i)$ for any $i = 1, \dots, n$ and $\Delta(P, f) = \Delta(Q, f)$. We denote it by $P \underset{J}{\sim} Q$. This gives a polyhedral cone subdivision of N_+ and we denote this as $\Gamma_J^*(f)$ and we call it *the Jacobian dual Newton diagram of* f. $\Gamma_J^*(f)$ is a polyhedral cone subdivision of N_+ which is finer than $\Gamma^*(f)$.

The Jacobian dual Newton diagram can be understood alternatively as follows. Let us consider the function $F(\mathbf{z}) = f(\mathbf{z}) f_1(\mathbf{z}) \cdots f_n(\mathbf{z})$. Then $\Gamma_J^*(f)$ is essentially equivalent to the dual Newton diagram $\Gamma^*(F)$ of F. For any weight vector P, we have $\Delta(P, F) = \Delta(P, f) + \Delta(P, f_1) + \cdots + \Delta(P, f_n)$ where the sum is Minkowski sum. See [3] for the definition. For a weight vector P, the set of equivalent weight vectors in $\Gamma^*(f)$ and $\Gamma_J^*(f)$ is denoted as $[P]$ and $[P]_J$ respectively. We consider the vertices of this subdivision. We denote the set of strictly positive vertices of $\Gamma^*(f)$ and $\Gamma_J^*(f)$ by $\mathscr{V}^+$, $\mathscr{V}_J^+$ respectively. Recall that $e_i = (0, \dots, \overset{i}{\breve{1}}, \dots, 0)$.

Proposition 4. (1) *$P \underset{J}{\sim} Q$ implies $P \sim Q$ in $\Gamma^*(f)$. Conversely if $P \sim Q$ and $f_P(\mathbf{z})$ contains all n-variables, $P \underset{J}{\sim} Q$.*

(2) *A strictly positive weight vector P is in $\mathscr{V}^+$ or $\mathscr{V}_J^+$ if and only if $\dim \Delta(P, f) = n - 1$ or $\dim(\Delta(P, f) + \sum_i \Delta(P, f_i)) = n - 1$ respectively where the summation is Minkowski sum.*

Let $\mathscr{V}_0$ be the set of vertices of $\Gamma^*(f)$ which are not strictly positive.

Proposition 5. *Assume that $P \in \mathscr{V}_0$ and $\mathbf{C}^{I(P)}$ is a non-vanishing subspace. Then P is one of $e_1, \dots, e_n$.*

A vertex $P \in \mathscr{V}_0$ is called *a vanishing vertex* if $\mathbf{C}^{I(P)}$ is a vanishing subspace.

LEMMA 6. *Let $P = (p_1, p_2, \dots, p_n)$ be a vanishing vertex of $\Gamma^*(f)$ in $\mathcal{V}_0$ and put $I = I(P)$. Then the following holds.*

(1) *f_P contains every variable $z_1, \dots, z_n$. In particular, $\hat{p}_i \le 1$ for any i.*
(2) *Any monomial $z_i^a z_j$, $i \in I$ in $f(\mathbf{z})$ must be contained in $f_P(\mathbf{z})$, as $\hat{p}_i = 0$ and $\deg_{\hat{P}} z_i^a z_j = \hat{p}_j \le 1$.*
(3) *There are no monomials $\mathbf{z}^\nu \in \mathbf{C}[\mathbf{z}_I]$ in $f_P(\mathbf{z})$.*

Proof. Suppose that f_P does not contain the variable z_i for some i. Then $\Delta(P) \subset \{\nu_i = 0\}$. Then $e_i \in \overline{[P]}$ and thus a contradiction $d(P, f) = d(e_i, f) = 0$. This proves the first assertion. Consider $z_i^a z_j$, $i \in I$ in f. Consider the normalized vector $\hat{P}$. Then $\hat{p}_i \le 1$ by the assertion (1) and as $\deg_{\hat{P}} z_i^a z_j \ge 1$, this implies $\hat{p}_j = 1$ and $z_i^a z_j$ must be in f_P. If there is a monomial $\mathbf{z}^\nu$ as in the assertion, $\deg_{\hat{P}} \mathbf{z}^\nu = 0$ and an obvious contradiction. □

Example 7. Consider $f(\mathbf{z}) = (z_1^9 + z_2^3 + z_3^6)z_2 + z_3^7 + z_4^7$. $\mathcal{V}_J^+$ has vertices e_1, e_2, e_3, e_4 and R, P, S where

$$R = \left(\frac{1}{12}, \frac{1}{4}, \frac{1}{7}, \frac{1}{7}\right), \quad f_R(\mathbf{z}) = z_1^9 z_2 + z_2^4 + z_3^7 + z_4^7$$

$$P = \left(\frac{2}{21}, \frac{2}{7}, \frac{1}{7}, \frac{1}{7}\right), \quad f_P(\mathbf{z}) = z_3^7 + z_4^7$$

$$S = \left(0, 1, \frac{1}{7}, \frac{1}{7}\right), \quad f_S(\mathbf{z}) = z_1^9 z_2 + z_3^7 + z_4^7.$$

Note that $P \in \mathcal{V}_J^+ \backslash \mathcal{V}^+$ as $f_{2P} = z_1^9 + z_2^3 + z_3^6$ and $\deg_P f_{2P} + \frac{2}{7} = \frac{8}{7} > 1$. $S \in \mathcal{V}_0$ corresponds to the vanishing coordinate subspace $\mathbf{C}^{\{1\}}$. The vertex P is in the simplicial cone $\mathrm{Cone}(\mathrm{R}, \mathrm{e}_1, \mathrm{e}_2)$ as $P = R + \frac{1}{84}e_1 + \frac{1}{28}e_2$. Note that $\mathrm{Cone}(\mathrm{R}, \mathrm{e}_1, \mathrm{e}_2)$ is a regular boundary region. See the definition below.

3.3. Boundary region. We consider equivalence classes $[P]$ and $[P]_J$ in $\Gamma^*(f)$ and $\Gamma_J^*(f)$ respectively. There exist three different cases.

1. An equivalent class $[P]$ (respectively $[P]_J$) is called *an inner region* if the closure $\overline{[P]}$ (resp. $\overline{[P]_J}$) does not contains any vertex of $\mathcal{V}_0$ on the boundary.

2. $[P]$ (respectively $[P]_J$) is called *a regular boundary region* if the closure $\overline{[P]}$ (resp. $\overline{[P]_J}$) contains some vertex e_i but contains no vanishing vertex on the boundary.

3. $[P]$ (resp. $[P]_J$) is called *a vanishing boundary region*, if $\overline{[P]}$ (resp. $\overline{[P]_J}$) contains a vanishing vertex $Q \in \Gamma^*(f)$ (resp. $Q \in \Gamma_J^*(f)$) on the boundary.

3.4. Special admissible paths. Two weight vectors P, Q are called *admissible*, (respectively *J-admissible*) if $\Delta(P) \cap \Delta(Q) \ne \emptyset$ (resp. $\Delta(P) \cap \Delta(Q) \ne \emptyset$ and $\Delta(P, f_i) \cap \Delta(Q, f_i) \ne \emptyset$ for any i). Any weight R in the interior of an admissible lene segment $\overline{PQ}$ satisfies $\Delta(R) = \Delta(P) \cap \Delta(Q)$ (resp. $\Delta(R) = \Delta(P) \cap \Delta(Q)$ and $\Delta(R, f_i) = \Delta(P, f_i) \cap \Delta(Q, f_i)$, $i = 1, \dots, n$). Take a weight vector $P = (p_1, \dots,$

p_n) which is not strictly positive. Put $I = \{i \mid p_i = 0\}$. We say P is a *vanishing weight* (respectively *non-vanishing weight*) if $f^I \equiv 0$ (resp. $f^I \not\equiv 0$).

PROPOSITION 8 (A path in a regular boundary region). *Suppose that two weight P, Q are admissible, Q is strictly positive weight and P is a non-vanishing weight vector with $I = \{i \mid p_i = 0\}$. Then weight vector $R \in \overline{PQ}$ on this line segment (except P) is given in the normalized form as $\hat{R}_t = \hat{Q} + tP$ with $0 \le t < \infty$. In this expression, $\hat{R}_t \to P$ when $t \to \infty$ and there exists a sufficiently large $\delta > 0$ so that $m(\hat{R}_t) \equiv m(\hat{Q}_I)$ and $\eta(R_t) \equiv \eta(\hat{Q}_I)$ for $t \ge \delta$ and $\eta(\hat{Q}_I) \le \eta(\hat{Q})$.*

Proof. For $j \notin I$, $\hat{q}_j + tp_j \to \infty$ and the assertion follows immediately. Here $m(\hat{Q}_I) = \min\{\hat{q}_j \mid j \in I\}$. □

PROPOSITION 9 (A path in a vanishing boundary region). *Suppose that Q is strictly positive and P is a vanishing weight vector. Put $I = \{i \mid p_i = 0\}$. Then $\hat{Q}_t = (1-t)\hat{Q} + t\hat{P}$, $(0 \le s \le 1)$ parametrize the weights on the line segment $\overline{QP}$ and we have the following.*

(1) *Suppose that $\hat{q}_j \ge \hat{q}_i$ for some $i \in I$, $j \notin I$. Then $(1-t)\hat{q}_j + t\hat{p}_j \ge (1-t)\hat{q}_i$ for $0 \le t \le 1$.*
(2) *If there is a $j \notin I$ such that $\hat{q}_j < \hat{q}_i$ for some $i \in I$, there exists $0 < t_0 < 1$ which satisfies $(1-t_0)\hat{q}_j + t_0\hat{p}_j = (1-t_0)\hat{q}_i$.*

Proof. Second assertion follows from the following property.

$$(1-t)\hat{q}_j + t\hat{p}_j \xrightarrow[t\to 1]{} \hat{p}_j > 0, \quad (1-t)\hat{q}_i \xrightarrow[t\to 1]{} 0. \qquad \square$$

3.5. Key lemma. First we prepare an elementary lemma.

LEMMA 10. *Consider a linear fractional function $\varphi(s) = \dfrac{as+b}{cs+d}$ where a, b, c, d are real numbers such that $(c,d) \ne (0,0)$ and $cs+d \ne 0$ for $0 \le s \le 1$. Then if $\varphi'(s) \not\equiv 0$, the sign of $\varphi'(s)$ does not change i.e., $\varphi'(s) > 0$ or $\varphi'(s) < 0$ for any s, $0 \le s \le 1$. Thus $\varphi(s)$ is a monotone function on $[0,1]$.*

Proof. Assertion follows from

$$\varphi'(s) = \frac{ad-bc}{(cs+d)^2}. \qquad \square$$

Assume that P, Q are strictly positive weight vectors. Then the weights on this line segment $\overline{PQ}$ can be parametrized normally as $\hat{R}_s$, $0 < s < 1$:

$$\hat{R}_s = s\hat{P} + (1-s)\hat{Q}.$$

Putting $\hat{P} = (\hat{p}_1, \dots, \hat{p}_n)$ and $\hat{Q} = (\hat{q}_1, \dots, \hat{q}_n)$, we can write $\hat{R}_s = (s\hat{p}_1 + (1-s)\hat{q}_1, \dots, s\hat{p}_n + (1-s)\hat{q}_n)$. We consider the quantities defined in (10):

$$\eta_{ij}(\hat{R}_s) = \frac{1 - \hat{r}_{s,j}}{\hat{r}_{s,i}} = \frac{1 - (s\hat{p}_j + (1-s)\hat{q}_j)}{s\hat{p}_i + (1-s)\hat{q}_i},$$

$$\eta(R_s) = \frac{1 - m(\hat{R}_s)}{m(\hat{R}_s)}$$

$$\eta'_{ij}(R_s) = \frac{\deg(\hat{R}_s, f_j)}{\hat{r}_{s,i}} = \frac{\deg(\hat{R}_s, f_j)}{s\hat{p}_i + (1-s)\hat{q}_i}$$

Applying Lemma 10, we have

Lemma 11. *Assume that P, Q are strictly positive weight vectors.*
(1) *Assume that P, Q are admissible. Then we have*

$$\eta_{ij}(\hat{R}_s) \le \max\{\eta_{ij}(\hat{P}), \eta_{ij}(\hat{Q})\}, \quad 0 < s < 1.$$

In particular, we have

$$\eta(\hat{R}_s) \le \max\{\eta(\hat{P}), \eta(\hat{Q})\}$$

(2) *Assume that P, Q are J-admissible. Then we have*

$$\eta'_{ij}(\hat{R}_s) \le \max\{\eta'_{ij}(\hat{P}), \eta'_{ij}(\hat{Q})\}, \quad 0 < s < 1.$$

3.5.1. *Invariants to be used for the estimation.* Let $\mathscr{V}_J^+$ be the set of strictly positive vertices of $\Gamma_J^*(f)$ and consider the subset $\mathscr{V}_J^{++} \subset \mathscr{V}_J^+$ which are in a vanishing boundary region $[Q]$ of $\Gamma^*(f)$ for some Q. The numbers of $\mathscr{V}^+$, $\mathscr{V}_J^+$, $\mathscr{V}_J^{++}$ are finite. We define the following invariants.

$$\eta_{max}(f) := \max\{\eta(P) \mid P \in \mathscr{V}^+\}$$
$$\eta_{J,max}(f) := \max\{\eta(P) \mid P \in \mathscr{V}^+ \cup \mathscr{V}_J^{++}\},$$
$$\eta'_{J,max}(f) := \max\{\eta'_{k,i}(R) \mid R \in \mathscr{V}_J^{++},\ k, i = 1, \dots, n\}$$
$$\eta''_{J,max}(f) := \max\{\eta_{J,max}(f), \eta'_{J,max}(f)\}.$$

Here $\eta'_{k,i}(R) = d(R, f_i)/r_k$.

3.6. Main theorem. The following estimation is our main result which is a modified weaker version of the assertion in [10]. Recall that we assume that $\dim \Gamma(f) = n - 1$.

Theorem 12. *Let $f(\mathbf{z})$ be a non-degenerate Łojasiewicz non-degenerate function with an isolated singularity at the origin. Then Łojasiewicz exponent $\ell_0(f)$ has the estimation*

$$\ell_0(f) \le \eta''_{J,max}(f).$$

If $\mathscr{V}_J^{++} = \emptyset$, the estimation can be replaced by a better one

$$\ell_0(f) \le \eta_{max}(f).$$

The proof of Theorem 3.6 will be given in §3.7.

3.6.1. *Test Curve*. We consider an analytic curve $C(t)$, $-\varepsilon < t \le \varepsilon$ parametrized as before (2),

$$C(t) : \begin{cases} \mathbf{z}(t) = (z_1(t), \dots, z_n(t)), & \mathbf{z}(0) = 0,\ \mathbf{z}(t) \in \mathbf{C}^{*I} \\ z_i(t) = a_i t^{p_i} + (\text{higher terms}), & i \in I \\ z_j(t) \equiv 0, & j \notin I \end{cases} \tag{12}$$

For simplicity, we assume that $I = \{1, \dots, m\}$. Put $P = (p_1, \dots, p_m) \in N_+^{*I}$ and $\mathbf{a} = (a_1, \dots, a_m) \in \mathbf{C}^{*I}$. We are interested in a best possible upper bound for the positive quantity $\ell_0(C(t)) = \operatorname{ord} \partial f(\mathbf{z}(t))/\operatorname{ord} \mathbf{z}(t)$. Using the notation $f_j = \partial f/\partial z_j$, we have the expansion

$$f_j(\mathbf{z}(t)) = (f_j)_P(\mathbf{a}) t^{d(P, f_j)} + (\text{higher terms}). \tag{13}$$

Note that if $f_P(\mathbf{z})$ contains the variable z_j,

$$(f_j)_P(\mathbf{a}) = (f_P)_j(\mathbf{a}), \quad d(P, f_j) = d(P, f) - p_j. \tag{14}$$

3.6.2. *Curves corresponding to a strictly positive weight vector*. In the previous section, we have seen that a test curve $C(t)$ gives a pair $(P, \mathbf{a}) \in N_+^I \times \mathbf{C}^{*I}$. We consider the converse in the case $I = \{1, \dots, n\}$. Assume we have a strictly positive integer weight vector $P = (p_1, \dots, p_n) \in N_+^n \cap \mathbf{Z}^n$. Taking a coefficient vector $\mathbf{a} = (a_1, \dots, a_n) \in \mathbf{C}^{*n}$, we associate an analytic curve

$$C_P(t, \mathbf{a}) : \mathbf{z}(t) := (a_1 t^{p_1}, \dots, a_n t^{p_n}).$$

The test curve (12) gives the data $(P, \mathbf{a})$ and if $I = \{1, \dots, n\}$, $C_P(t, \mathbf{a})$ and $C(t)$ differs only higher terms. In this case, we also use the notation as $\ell_0(P)$ instead of $\ell_0(C_P(t, \mathbf{a}))$ or $\ell_0(C(t))$ by an abuse of notation. Then by the above discussion and by the non-degeneracy assumption, we have

$$\operatorname{ord} \partial f(\mathbf{z}(t)) \le d(P, f) - m(P) \tag{15}$$

$$\frac{\operatorname{ord} \partial f(\mathbf{z}(t))}{\operatorname{ord} \mathbf{z}(t)} \le \frac{d(P, f)}{m(P)} - 1 \tag{16}$$

$$= \frac{1}{m(\hat{P})} - 1. \tag{17}$$

Note that $\eta(\hat{P}) = \eta(P)$. This estimation does not depend on the choice of representative of the equivalence class of P and the choice of $\mathbf{a}$. The weakness of the above estimation is that $m(\hat{P})$ can be arbitrary small in the vanishing boundary region which makes $\eta(\hat{P}) = \eta(P)$ unbounded.

3.7. Proof of Theorem 12. Take a test curve as (2) and we consider the weight vector P. To prove the theorem, it is enough to prove that $\ell_0(f)(C(t)) \le \eta''_{J,max}(f)$ by Proposition 2. We first consider the case that P is strictly positive which is most essential.

3.7.1. *Strictly positive case.* We first assume that $I = \{1, \dots, n\}$ and P is strictly positive. We divide the situation into three cases.

C-1 $[P]$ is an inner region. That is, $\overline{[P]}$ has only strictly positive weight vectors in the boundary.

C-2 $[P]$ is a regular boundary region.

C-3 $[P]$ is a vanishing boundary region. In this case, we need to consider the subdivision by Jacobian dual Newton diagram. There are three subcases.

C-3-1. $[P]_J$ is an inner region.

C-3-2. $[P]_J$ is a regular boundary region.

C-3-3. $[P]_J$ is also a vanishing boundary region.

We need consider the Jacobian dual diagram only in vanishing boundary regions of $\Gamma^*(f)$.

3.7.2. *Cases C-*1 *and C-*3-1. We start from the inequality $\ell_0(P) \le \eta(P)$ and then estimate $\eta(P)$ by the strictly positive vertices. We use an induction on $\dim[P]$ or $\dim[P]_J$ in Case C-3-1 to show that $\ell_0(P) \le \eta_{max}(f)$ (resp. $\ell_0(P) \le \eta_{J,max}(f)$). The induction starts from the case $\dim[P] = 1$ (respectively $\dim[P]_J = 1$). In this case, the assertion is obvious. If $\dim[P] = r > 1$, we take a line segment $\overline{RS}$ with $R, S \in \partial\overline{[P]}$ (resp. in $\partial\overline{[P]_J}$) passing through P we apply Lemma 11 to get the estimation

$$\ell_0(C(t)) \le \eta(P) \le \max\{\eta(R), \eta(S)\} \le \eta_{max}(f), \quad R, S : \text{admissible}$$
$$\ell_0(C(t)) \le \eta(P) \le \max\{\eta(R), \eta(S)\} \le \eta_{J,max}(f), \quad S : J\text{-admissible}.$$

As $\dim[R], \dim[S] < r$ (resp. $\dim[R]_J, \dim[S]_J < r$), the induction works. For the other cases, we prepare a simple lemma.

Lemma 13. *Assume that* $\dim \Gamma(f) = n - 1$. *Then for any face* $\Delta \subset \Gamma(f)$, *there exists an* $(n-1)$ *dimensional face* Ξ *such that* $\Xi \supset \Delta$. *Equivalently for any weight* P, *the closure* $\overline{[P]}$ *contains a strictly positive vertex* Q. *Respectively* $\overline{[P]_J}$ *contains a strictly positive vertex* $Q \in \mathscr{V}_J^+$.

The assertion is immediate from the assumption that $\dim \Gamma(f) = n - 1$ as $\Gamma_+(f)$ is a n-dimensional convex polyhedral region and $\Gamma(f)$ is the union of compact boundary faces. The assertion for $\Gamma_J^*(f)$, as we can use $F = ff_1 \cdot f_n$ instead of f.

3.7.3. *Case C-*2 *and C-*3-2. We start again from the inequality $\ell_0(P) \le \eta(P)$. Assume that $\overline{[P]}$ (respectively $\overline{[P]_J}$) is a regular boundary region. We prove that $\eta(P) \le \eta_{max}(f)$ (respectively $\eta(P) \le \eta_{J,max}(f)$) by the induction of $\dim[P]$ (resp. $\dim[P]_J$). The argument is completely same in the case $[P]_J$. Take a strictly positive vertex R in $\overline{[P]} \cap \mathscr{V}^+$ (resp. in $R \in \overline{[P]_J} \cap \mathscr{V}_J^+$), using Lemma 13. Take the segment $\overline{RP}$ and extend it further to the right so that it arrives to a boundary point of the region, say Q. Then $[Q]$ is either an inner region or a regular boundary region.

If Q is strictly positive and $[Q]$ is an inner region, we can apply the argument of Case C-1 or C-3-1 and we consider the estimation in $[Q]$ by the inductive argument.

Similarly if Q is strictly positive and $[Q]$ is a regular boundary region, we apply the induction's assumption, as $\dim[P] > \dim[Q]$.

So we assume that Q is not strictly positive. Then Q is a non-vanishing weight vector i.e., $d(Q) = 0$. The normalized form of weight vectors on this segment is given as $\hat{R}_s = \hat{R} + sQ$ with $0 \le s < \infty$ and $\hat{P} = \hat{R}_{s_0}$ for some $s_0 > 0$. Put $I = I(Q)$ and $m_I(\hat{R}) = \min\{\hat{r}_i \mid i \in I\}$, $I' := \{i \mid \hat{r}_i = m_I(\hat{R})\}$ and $I_R := \{j \mid \hat{r}_j = m(\hat{R})\}$.

– If $I' \cap I_R \neq \emptyset$, $m(\hat{R}) = m_I(\hat{R})$ and $m(\hat{R}_s) \equiv \hat{r}_{i_0}$ for any s and $i_0 \in I'$. Thus

$$\ell_0(C(t)) \le \eta(\hat{P}) = \eta(\hat{R}_{s_0}) = \eta(\hat{R}) \le \eta_{max}(f) \quad (\text{resp.} \le \eta_{J,max}(f)).$$

– If $I' \cap I_R = \emptyset$, i.e., $m(\hat{R}) < m_I(\hat{R})$, take $j_0 \in I_R$ such that there exists a small positive number ε and $m(\hat{R}_s) = \hat{r}_{j_0} + sq_{j_0}$ for $s \le \varepsilon$. As $m_I(\hat{R}) > \hat{r}_{j_0}$ but $\hat{r}_{j_0} + sq_{j_0}$ is monotone increasing in s, there exists some s_1 such that $m_I(\hat{R}) = \hat{r}_{j_0} + s_1 q_{j_0}$ and for $s \ge s_1$, $m(\hat{R}_s) = m_I(\hat{R})$. Thus $\eta(\hat{R}_s)$ is monotone decreasing for $0 \le s \le s_1$ and constant for $s \ge s_1$. Thus in any case we get

$$\ell_0(C(t)) \le \eta(P) = \eta(\hat{R}_{s_0}) \le \eta(\hat{R}_0) = \eta(\hat{R}) \le \eta_{max}(f) \quad (\text{resp.} \le \eta_{J,max}(f)).$$

3.7.4. *Case C*-3-3. This case requires careful choice of the line segment for the estimation. For this purpose, we prepare the following Proposition 14 and Lemma 15. A strictly positive weight vector P is *simplicially positive* (respectively J-*simplicially positive*) if there exist strictly positive linearly independent vertices $P_1, \dots, P_s$ of $\Gamma^*(f)$ such that $P_i \in \overline{[P]}$, $i = 1, \dots, s$ (resp. vertices $P_1, \dots, P_s$ of $\Gamma_J^*(f)$ such that $P_i \in \overline{[P]_J}$, $i = 1, \dots, s$) and P is in the interior of the simplex $(P_1, \dots, P_s)$. Here by a simplex $(P_1, \dots, P_s)$, we mean the simplicial cone

$$\mathrm{Cone}(P_1, \dots, P_s) = \left\{ \sum_{i=1}^{s} \lambda_i P_i \,\middle|\, \lambda_i \ge 0 \right\}.$$

Thus a line segment $\overline{PQ}$ is equal to the simplex (P, Q).

Proposition 14. *Let P be a strictly positive weight vector. Then there are two possibilities.*

(1) *P is simplicially positive* (*respectively J-simplicially positive*).

(2) *There are linearly independent vertices $P_1, \dots, P_{q-1}$, $q \le \dim[P]$ of $\overline{[P]}$* (*resp. linearly independent vertices $P_1, \dots, P_{q-1}$, $q \le \dim[P]_J$ of $\overline{[P]_J}$*) *and a weight vector P_q which is not strictly positive so that P is in the interior of the simplex* $(P_1, \dots, P_q)$.

Proof. The assertion follows easily from the fact that $\overline{[P]}$ (resp. $\overline{[P]_J}$) is a polyhedral convex cone. We use induction on $r = \dim[P]$ (resp. $r = \dim[P]_J$). As the proof is completely parallel, we show the assertion in the case of $\Gamma^*(f)$.

Take a strictly positive vertex $P_1 \in \overline{[P]}$ using Lemma 13 and take the line segment $\overline{P_1 P}$ and extending to the right, put Q_1 be the weight on the boundary of $\overline{[P]}$. Thus P is contained in the interior of $\overline{P_1 Q_1}$. Consider $[Q_1]$. Then $\dim[Q_1] < r$. If Q_1 is not strictly positive, we stop the operation. Then $q = 2$ and this case corresponds to Case (2). If Q_1 is still strictly positive but not a vertex, we repeat the argument on $\overline{[Q_1]}$. Take a strictly positive vertex $P_2 \in \overline{[Q_1]}$ and so on. Apply an inductive argument. The operation stops if we arrive at a weight vector which is not strictly positive (then case (2)) or a strictly positive vertex P_q (Case (1)). □

Using this proposition, we have the following choice of a nice line segment.

Lemma 15. *Assume that P is a strictly positive weight. If P is not simplicially positive (respectively not J-simplicially positive), there is a line segment $\overline{RQ}$ such that R is simplicially positive (resp. J-simplicially positive) and Q is not strictly positive.*

Proof. We give the proof for $\Gamma^*(f)$ as the proof is completely parallel for $\Gamma^*_J(f)$. Assume that P is not simplicially positive. Using Proposition 14, we suppose that P is in the interior of the simplex $(P_1, \dots, P_q)$ where $P_1, \dots, P_{q-1}$ are strictly positive vertices and P_q is not strictly positive. Write P by a barycentric coordinates as $P = \sum_{i=1}^{q} \lambda_i P_i$ with $\lambda_i > 0$ and we may assume $\sum_{i=1}^{q} \lambda_i = 1$. Put $\lambda := \sum_{i=1}^{q-1} \lambda_i$, $\mu := 1 - \lambda$ and define $R := (\sum_{i=1}^{q-1} \lambda_i P_i)/\lambda$ and $Q := P_q/\mu$. Then $P = \lambda R + \mu Q$ and $R \in (P_1, \dots, P_{q-1})$. Thus R is simplicially positive. □

Remark 16. For the proof of Case 3-3 below, the Jacobian dual Newton diagram is essential. So we use Lemma 15 for $\Gamma^*_J(f)$.

Proof of Case 3-3. Now we are ready to have an estimation for the Łojasiewicz exponent of our test curve $C(t)$. Suppose that $[P]$ and $[P]_J$ are not vanishing boundary regions. We apply Lemma 15. If P is J-simplicially positive, we have the estimation $\ell_0(C(t)) \le \eta_{J,max}(f)$ by the same argument as in Case 3-1. Thus using Lemma 15, we may assume that P is in the line segment $\overline{RQ}$ where R is J-simplicially positive and Q is not strictly positive. If Q is a non-vanishing weight, we proceed as the case C-3-2 to get the estimation $\ell_0(C(t)) \le \eta_{J,max}(f)$. Thus we assume that Q is a vanishing weight vector and $d(Q, f) > 0$. Assume that $Q = (q_1, \dots, q_n)$ and put $I := \{i \mid q_i = 0\}$. We assume $I = \{1, \dots, m\}$ for simplicity. Note that $\mathbf{C}^I$ is a vanishing coordinate subspace. For each $i \in I$, there exists some $j \notin I$ and a monomial $z_i^{n_{i,j}} z_j$ with a non-zero coefficient, as f has an isolated singularity at the origin. Put J_i be the set of such j for a fixed $i \in I$ and put $J(I) = \bigcup_{i \in I} J_i$. Here $n_{i,j}$ is assumed to be the smallest when j is fixed. Put $\xi_I := \max\{n_{i,j} \mid i \in I, j \in J_i\}$ and $\xi(f)$ be the maximum of ξ_I where I corresponds to a vanishing coordinate subspace.

Put $\eta'_{J,max}(f) := \max\{\eta'_{k,i}(R) \mid R \in \mathcal{V}^{++},\ 1 \le k, i \le n\}$ where $\eta'_{j,i}(R) = d(R, f_i)/r_j$. Under the above situation, we will prove that

$$(\star) \quad \ell_0(C(t)) \le \max\{\xi(f), \eta_{J,max}(f), \eta'_{J,max}(f)\}.$$

Consider the normalized weight vector $\hat{R}_s := (1-s)\hat{R} + s\hat{Q}$, $0 \le s \le 1$. Note that $\hat{R}_0 = \hat{R}$, $\hat{R}_1 = \hat{Q}$ and putting $\hat{R}_s = (\hat{r}_{s,1}, \dots, \hat{r}_{s,n})$,

$$\hat{r}_{s,i} = \begin{cases} (1-s)\hat{r}_i, & 1 \le i \le m \\ (1-s)\hat{r}_i + s\hat{q}_i, & m < i \le n. \end{cases}$$

Put $I' = \{i \in I \mid \hat{r}_i = m(\hat{R}_I)\}$ and $J' = \bigcup_{i \in I'} J_i$. Thus for $i \le m$, the normalized weight $\hat{r}_{sj}$ goes to 0, when s approaches to 1. On the other hand, for $j > m$, $\hat{r}_{sj} \ge \delta$, $0 \le \forall s \le 1$ for some $\delta > 0$. Thus there exists an ε, $1 > \varepsilon > 0$ so that for $1 - \varepsilon \le s \le 1$, $m(\hat{R}_s)$ is taken by $i \in I'$. Note that for $j \in J'$, there exists a small enough ε_2, $\varepsilon_2 \le \varepsilon_1$ so that $(f_j)_{\hat{R}_s} = ((f_j)^I)_{\hat{R}_s}$ as $\hat{r}_{sj} \ge \delta$ for $j > m$ for $1 - \varepsilon_2 \le s \le 1$. Here $(f_j)^I$ is the restriction of f_j to $\mathbf{C}^I$ and $(\hat{R}_s)_I$ is the I projection of $\hat{R}_s$ to N_+^I. That is, $(f_j)_{\hat{R}_s}$ contains only variable $z_1, \dots, z_m$. By the Łojasiewicz non-degeneracy, there exists $i_0 \in I'$ and $j_0 \in J_{i_0}$ such that

$$(f_{j_0})_{\hat{R}_s}(\mathbf{a}) \ne 0, \quad s \ge 1 - \varepsilon_2.$$

By the definition of Jacobian dual Newton diagram, for any $0 < s < 1$, $(f_{j_0})_{\hat{R}_s}$ does not depend on s and thus $(f_{j_0})_{\hat{R}_s}(\mathbf{a}) \ne 0$ for any $0 < s < 1$. Then $d(\hat{R}_s, f_{j_0}) \le (1-s)\hat{r}_{i_0} \times n_{i_0,j_0}$ for any $0 < s \le 1$. The equality takes place if the monomial $z_{i_0}^{n_{i_0,j_0}} z_{j_0}$ is on the face function $(f_{j_0})_{\hat{R}_s}(\mathbf{z})$. Assume that $\hat{P} = \hat{R}_{s_0}$, $0 < s_0 < 1$. Thus we start from the estimation

$$\ell_0(C(t)) \le d(\hat{R}_{s_0}, f_{j_0})/m(\hat{R}_{s_0}). \tag{18}$$

First for $s \ge 1 - \varepsilon_2$, we see that

$$\ell_0(\hat{R}_s) \le \frac{(1-s)\hat{r}_{i_0} n_{i_0,j_0}}{(1-s)\hat{r}_{i_0}} = n_{i_0,j_0}.$$

($s_0 = \mu$ in the proof of Lemma 15.) Put $I_R := \{k \mid \hat{r}_k = m(\hat{R})\}$.

(a) If there exists a $k \in I' \cap I_R$ i.e. $m(\hat{R}) = m(\hat{R}_I)$, we have the estimation $\ell_0(\hat{R}_s) \le n_{i_0,j_0}$ for any s. In particular, $\ell_0(C(t)) \le n_{i_0,j_0}$.

(b) Assume that $I' \cap I_R = \emptyset$ i.e. $m(\hat{R}) < m(\hat{R}_I)$. Choose $k \in I_R$. Then there exists a small number $\varepsilon > 0$ so that $m(\hat{R}_s) = \hat{r}_{s,k}$ for $0 \le s \le \varepsilon$. By the definition of I', this implies that $k \notin I$. There exists $0 < s_1 < 1$ such that $m(\hat{R}_{s_1}) = (1-s_1)\hat{r}_k + s_1\hat{q}_k = (1-s_1)\hat{r}_{i_0}$. We divide this case into two subcases.

(b-1) Assume that $(1-s)\hat{r}_k + s\hat{q}_k$ is monotone increasing in s. Then

$$\ell_0(P) = \ell_0(\hat{R}_{s_0}) \le \eta(\hat{R}_0) = \eta(\hat{R}) \le \eta_{J,max}(f), \quad \text{if } s_0 \le s_1.$$

$$\ell_0(P) = \ell_0(\hat{R}_{s_0}) \le \frac{(1-s_0)\hat{r}_{i_0} n_{i_0,j_0}}{(1-s_0)\hat{r}_{i_0}} = n_{i_0,j_0}, \quad \text{if } s_0 > s_1.$$

(b-2) Assume that $(1-s)\hat{r}_k + s\hat{q}_k$ is monotone decreasing in s. Then

$$\ell_0(\hat{R}_{s_0}) \le \frac{(1-s_0)\hat{r}_{i_0} n_{i_0,j_0}}{(1-s_0)\hat{r}_k + s_0\hat{q}_k} \le \frac{\hat{r}_{i_0} n_{i_0,j_0}}{\hat{r}_k} \le \eta'_{k,j_0}(R), \quad \text{if } s_0 \le s_1 \tag{19}$$

$$\ell_0(\hat{R}_{s_0}) \le \frac{(1-s_0)\hat{r}_{i_0} n_{i_0,j_0}}{(1-s_0)\hat{r}_{i_0}} = n_{i_0,j_0}, \qquad \text{if } s_0 \ge s_1. \tag{20}$$

where the first estimation (19) for $s_0 \le s_1$ follows from the fact that

$$s \mapsto \eta'_{k,j_0}(\hat{R}_s) = \frac{(1-s)\hat{r}_{i_0} n_{i_0,j_0}}{(1-s)\hat{r}_k + s\hat{q}_k}, \quad 0 \le s \le 1$$

is monotone decreasing on s. Then we apply Lemma 11 to get the estimation

$$\ell_0(C(t)) \le \eta'_{k,j_0}(R) \le \eta'_{J,max}(f).$$

Thus we have proved the estimation $(\star)$. Assuming the next lemma for a moment, we can ignore the first term $\xi(f)$ in $(\star)$ and the estimation reduces to

$$\ell_0(C(t)) \le \eta''_{J,max}(f), \quad \text{if } \mathbf{z}(t) \in \mathbf{C}^{*n}.$$

Furthermore if $\mathscr{V}_J^{++} = \emptyset$, $\eta'_{i,j}(P) = \eta_{i,j}(P)$ for $P \in \mathscr{V}^+$, as $f_P(\mathbf{z})$ contains all the variables and therefore $\eta'_{i,j}(P) = \eta_{i,j}(P) \le \eta(P)$ or $\eta'_{J,max}(f) \le \eta_{J,max}(f) = \eta_{max}(f)$.

Lemma 17. *The following inequality holds.*

$$n_{i,j} \le \eta_{max}(f), \quad \forall i \in I, \forall j \in J_i.$$

Proof. Let Ξ be a maximal face of $\Gamma(f)$ for which the face function $f_\Xi(\mathbf{z})$ contains the monomial $z_i^{n_{i,j}} z_j$. The corresponding weight vector P (i.e., $\Delta(P) = \Xi$) is in $\mathscr{V}^+ \subset \Gamma^*(f)$ such that Ξ is subset of the hyperplane

$$p_1 v_1 + \cdots + p_n v_n = d(P)$$

and $f_P(\mathbf{z})$ contains the monomial $z_i^{n_{i,j}} z_j$. Thus $n_{i,j} p_i + p_j = d(P)$. Consider an analytic curve $\mathbf{z}(t)$ corresponding to the weight P. Then we have

$$n_{i,j} = \frac{d(P) - p_j}{p_i} = \eta_{i,j}(P) \le \eta(P) \le \eta_{max}(f). \qquad \square$$

To complete the proof, we have to consider the case where the test curve is in a proper coordinate subspace.

3.7.5. *Test curves in a proper subspace*. We consider the situation of the test curve $\mathbf{z}(t)$ defined in (12) for which $I^c \ne \emptyset$. Recall that $I = \{i \mid z_i(t) \not\equiv 0\}$ and we assume $I = \{1, \dots, m\}$, $m < n$ for simplicity.

Case 1. Assume that $f^I \not\equiv 0$. Recall that $P = (p_1, \dots, p_m)$ and $\mathbf{a} = (a_1, \dots, a_m) \in \mathbf{C}^{*I}$. Let $\Delta_1 = \Delta(P) \subset \Gamma^*(f^I)$. Consider the weight vector $\tilde{P} =$

$(p_1, \dots, p_m, K, \dots, K) \in N_+$ where K is sufficiently large so that $\Delta(\tilde{P}, f) = \Delta_1$, $d := d(P, f^I) = d(\tilde{P}, f)$ and $m(\tilde{P}) = m(P)$. Put $\tilde{\mathbf{a}} = (\mathbf{a}, 1, \dots, 1)$. Consider $\theta_k :=$ ord $\partial f/\partial k(\mathbf{z}(t))$ and put $\theta := \min\{\theta_k \mid \theta_k \neq \infty\}$. Here $\theta_k = \infty$ if $\partial f/\partial k(\mathbf{z}(t)) \equiv 0$ by definition. Then $\ell_0(C(t)) = \theta/m(P)$. Consider the modified curve

$$\tilde{C}(t) : \tilde{\mathbf{z}}(t) = (z_1(t), \dots, z_m(t), t^N, \dots, t^N).$$

Taking N sufficiently large, say $N > \max\{\theta_k \mid \theta_k \neq \infty\}$, it is easy to see that ord $\partial f/\partial k(\tilde{\mathbf{z}}(t)) = \theta_k$ for any k with $\theta_k \neq \infty$. Thus for such an N, $\ell_0(C(t)) = \ell_0(\tilde{C}(t))$. Combining with the previous argument, we get $\ell_0(C(t)) \leq \eta''_{J,max}(f)$.

Case 2. Assume that $f^I \equiv 0$. This implies $\mathbf{C}^I$ is a vanishing coordinate subspace. Thus each monomial in the expansion of f must contain one of $\{z_j \mid j \in I^c\}$. Thus $f_i(\mathbf{z}(t)) \equiv 0$ for $i \in I$. Put $m(P) := \min\{p_i \mid i \in I\}$ and $I' = \{i \in I \mid p_i = m(P)\}$ and J_i be the set of $j \in I^c$ such that a monomial $z_i^{n_{i,j}} z_j$ exists. Then by the Łojasiewicz non-degeneracy, there exists $i_0 \in I'$ and $j_0 \in J_{i_0}$ such that $f_{j_0}(\mathbf{a}) \neq 0$ and thus

$$\frac{\text{ord } \partial f(\mathbf{z}(t))}{\text{ord } \mathbf{z}(t)} \leq \max\{n_{i,j} \mid i \in I', j \in I_i\} \leq \eta_{J,max}(f).$$

This completes the proof of Theorem 12.

Remark 18. It is possible to have $\eta_{J,max} = \eta_{\max}$ or $\eta''_{J,max} = \eta_{J,max}$ in some cases. For example, see the next section. In Example 7, we have the equality $\eta_{J,max} = \eta_{max}$ as the simplex (R, e_1, e_2) is a regular boundary region. In fact, we have $\eta(R) = 11$, $\eta(P) = 9$.

3.8. Weighted homogeneous polynomials. Suppose that $f(\mathbf{z})$ is a weighted homogeneous polynomial with isolated singularity at the origin. There are already a better result by Brzostowski [5, 4]. I thank to Tadeusz Krasiński for informing me these papers.

Theorem 19 (Brzostowski [5, 4]). *Let $f(\mathbf{z})$ be a non-degenerate weighted homogeneous polynomial with an isolated singularity at the origin and* $\dim \Gamma(f) = n - 1$. *Let R be the weight vector of f. Then $\ell_0(f) \leq \eta(R)$.*

Remark 20. For a non-degenerate weighted homogeneous polynomial $f(\mathbf{z})$ with isolated singularity at the origin, Brzostowski does not need the Łojasiewicz non-degeneracy.

3.9. Łojasiewicz Join Theorem. Consider a join type function $f(\mathbf{z}, \mathbf{w}) = g(\mathbf{z}) + h(\mathbf{w})$ where $\mathbf{z} \in \mathbf{C}^n$ and $\mathbf{w} \in \mathbf{C}^m$. Assume that both $g(\mathbf{z})$ and $h(\mathbf{w})$ have isolated singularities at the respective origin. Then

Łojasiewicz Join Theorem 21 ([27], Corollary 2, §2). We have the equality:

$$\ell_0(f) = \max\{\ell_0(g), \ell_0(h)\}.$$

Proof. Put $\mathbf{u} = (\mathbf{z}, \mathbf{w}) \in \mathbf{C}^{n+m}$. Take any analytic curve $C(t) : \mathbf{u}(t) = (\mathbf{z}(t), \mathbf{w}(t))$, $0 \le t \le 1$.

Case 1. If $\mathbf{z}(t) \equiv 0$ (respectively $\mathbf{w}(t) \equiv 0$), $\dfrac{\operatorname{ord} \partial f(\mathbf{u}(t))}{\operatorname{ord} \mathbf{u}(t)} \le \ell_0(h)$ (resp. $\le \ell_0(g)$).

Case 2. Assume that $\mathbf{z}(t) \not\equiv 0$ and $\mathbf{w}(t) \not\equiv 0$. If $\operatorname{ord} \mathbf{z}(t) \le \operatorname{ord} \mathbf{w}(t)$, we have

$$\begin{aligned}\frac{\operatorname{ord} \partial f(\mathbf{u}(t))}{\operatorname{ord} \mathbf{u}(t)} &= \min\left\{\frac{\operatorname{ord} \partial g(\mathbf{z}(t))}{\operatorname{ord} \mathbf{z}(t)}, \frac{\operatorname{ord} \partial h(\mathbf{w}(t))}{\operatorname{ord} \mathbf{z}(t)}\right\} \\ &\le \min\left\{\ell_0(g), \frac{\ell_0(h) \operatorname{ord} \mathbf{w}(t)}{\operatorname{ord} \mathbf{z}(t)}\right\} \\ &\le \ell_0(g).\end{aligned}$$

If $\operatorname{ord} \mathbf{z}(t) > \operatorname{ord} \mathbf{w}(t)$, by the same argument,

$$\frac{\operatorname{ord} \partial f(\mathbf{u}(t))}{\operatorname{ord} \mathbf{u}(t)} \le \ell_0(h).$$

Thus we have $\ell_0(f) \le \max\{\ell_0(g), \ell_0(h)\}$. The equality can be taken by a curve $\mathbf{u}(t) = (\mathbf{z}(t), \mathbf{0})$ or $\mathbf{u}(t) = (0, \mathbf{w}(t))$ which takes $\ell_0(g)$ for $g(\mathbf{z})$ or $\ell_0(h)$ for $h(\mathbf{w})$ respectively. □

Remark 22. As we see in the proof of Theorem 12, it is not necessary to take the Jacobian dual Newton diagram everywhere. We only need consider $\Gamma_J^*(f)$ in the vanishing regions of $\Gamma^*(f)$. Namely if P is a vertex of $\Gamma_J^*(f)$ which is in an inner or a regular boundary region of $\Gamma^*(f)$, we have an estimation $\eta(P) \le \eta_{max}(f)$.

3.10. Application. We consider the hypersurface $V = f^{-1}(0)$ and the transversality problem with the sphere $S_r := \{\mathbf{z} \mid \|\mathbf{z}\| = r\}$. Certainly transversality has been shown by Milnor [18]. However we want to see this property from a slightly different view point. Recall first that S_r and V does not intersect transversely at $\mathbf{z} = \mathbf{a}$ if and only if

(i) $\mathbf{a}$ and $\overline{\partial f(\mathbf{a})}$ are linear dependent over $\mathbf{C}$, or

Using hermitian inner product and the Schwartz inequality, this condition is equivalent to

(ii) $|(\mathbf{a}, \overline{\partial f(\mathbf{a})})_{norm}| = 1$.

3.10.1. *Orthogonality at the limits* (*Whitney* (b)-*regularity*).

Lemma 23. *Assume that f is a non-degenerate and Łojasiewicz non-degenerate holomorphic function with an isolated singularity at the origin. Consider a non-constant analytic curve $\mathbf{z}(t)$ with $\mathbf{z}(0) = \mathbf{0}$ defined as* (2) *and assume that $f_P(\mathbf{a}) = 0$. Then we have*

(1) $\lim_{t\to 0}(\mathbf{z}(t), \overline{\partial f(\mathbf{z}(t))})_{norm} = 0$. *Geometrically this implies the limit direction of* $\mathbf{z}(t)$ *is contained in the limit of the tangent space* $T_{\mathbf{z}(t)}V$ *of* V.

(2) *In particular, there exists a positive number* r_0 *so that the sphere* S_r *intersects* V *transversely for any* $r \le r_0$.

Proof. Let $J = \{j \mid z_j(t) \equiv 0\}$ and let I be the complement of J. We assume that $I = \{1, \dots, m\}$. Consider the Taylor expansion

$$z_i(t) = a_i t^{p_i} + (\text{higher terms}), \quad i \in I.$$

Put $P = (p_i)_{i\in I} \in N_+^I$ as before.

Case 1. Assume that $f^I \not\equiv 0$ and assume for simplicity $p_1 = \cdots = p_\ell < p_{\ell+1} \le \cdots \le p_m$ and put $d = d(P, f^I)$. Note that $\operatorname{ord} \mathbf{z}(t) = p_1$. Put $q = \operatorname{ord} \partial f(\mathbf{z}(t))$. By the assumption, $\lim_{t\to 0} t^{-p_1}\mathbf{z}(t) = \mathbf{a}_\infty$ where $\mathbf{a}_\infty := (a_1, \dots, a_\ell, 0, \dots, 0)$. Put $\mathbf{v}_\infty := \lim_{t\to 0} t^{-q}\partial f(\mathbf{z}(t))$. By the non-degeneracy, we have $q \le d - p_1$. If $q < d - p_1$, the assertion (1) is immediate, as $\mathbf{v}_\infty \in \mathbf{C}^K$ where $K = \{j \mid j > \ell\}$. Assume that $q = d - p_1$. This implies

$$\frac{\partial f^I}{\partial z_j}(\mathbf{a}) = 0, \quad j \ge \ell + 1.$$

By the assumption $f_P^I(\mathbf{a}) = 0$ and the Euler equality of $f_P^I(\mathbf{z})$, we get

$$f_P^I(\mathbf{a}) = \sum_{i=1}^m p_i a_i \frac{\partial f_P^I}{\partial z_i}(\mathbf{a}) = p_1 \sum_{i=1}^\ell a_i \frac{\partial f_P^I}{\partial z_i}(\mathbf{a}) = 0.$$

In this case, note that $\mathbf{v}_\infty = \left(\frac{\partial f_P^I}{\partial z_1}(\mathbf{a}), \dots, \frac{\partial f_P^I}{\partial z_\ell}(\mathbf{a}), 0, \dots, 0\right)$. This implies also

$$\begin{aligned}\lim_{t\to 0}(\mathbf{z}(t), \overline{\partial f}(\mathbf{z}(t)))_{norm} &= (\mathbf{a}_\infty, \mathbf{v}_\infty)_{norm} \\ &= c\sum_{i=1}^\ell a_i \frac{\partial f_P^I}{\partial z_i}(\mathbf{a}) = 0\end{aligned}$$

where c is a non-zero scalar.

Case 2. Assume that $f^I \equiv 0$. Then $\mathbf{z}(t) \in \mathbf{C}^I$ and $\mathbf{C}^I$ is a vanishing coordinate subspace, and thus $\partial f(\mathbf{z}(t)) \in \mathbf{C}^{I^c}$. Thus the assertion is obvious. The assertion (2) follows from (1). Of course, (2) is nothing but the existence of a stable radius which is well known by [18] for a general holomorphic function with an isolated singularity at the origin. □

Remark 24. The assertion of the lemma says that the stratification $\{V\backslash\{\mathbf{0}\}, \{\mathbf{0}\}\}$ is a Whitney b-regular stratification.

3.10.2. *Making* f *convenient without changing the topology*. Let f be a non-degenerate, Łojasiewicz non-degenerate function with isolated singularity at the origin. Choose integer N_i with

$$N_i > \eta''_{J,max}(f) + 1, \quad i = 1, \dots, n \tag{21}$$

and consider a polynomial

$$R(\mathbf{z}) = c_1 z_1^{N_1} + \cdots + c_n z_n^{N_n}.$$

Consider the family of functions:

$$f_s(\mathbf{z}) = f(\mathbf{z}) + sR(\mathbf{z}), \quad 0 \le s \le 1.$$

The coefficients are chosen generically so that f_s is non-degenerate. Note that $f_0 = f$ and f_1 is a convenient and non-degenerate

Theorem 25. *Consider the family of hypersurface $V_s := f_s^{-1}(0)$, $0 \le s \le 1$. There exists a positive number r_0 such that $V_s \cap B_{r_0}$ has a unique singular point at the origin and for any $r \le r_0$ the sphere S_r and V_s intersect transversely for any $0 \le s \le 1$. In particular, the links of f and f_1 are isotopic and their Milnor fibrations are isomorphic.*

Proof. Take r_0 so that there exists a positive number c and the following inequality is satisfied.

$$\|\partial f(\mathbf{z})\| \ge c\|\mathbf{z}\|^{\ell_0(f)}, \quad 0 < \|\mathbf{z}\| \le r_0.$$

Assume that the assertion does not hold. Then we can find an analytic curve $(\mathbf{z}(t), s(t))$, $0 \le t \le 1$ and Laurent series $\lambda(t)$ such that $\mathbf{z}(0) = \mathbf{0}$ and $s(0) = s_0 \in [0,1]$ and

$$\overline{\partial f_{s(t)}(\mathbf{z}(t))} = \lambda(t)\mathbf{z}(t), \quad f_{s(t)}(\mathbf{z}(t)) \equiv 0. \tag{22}$$

Expand $\mathbf{z}(t)$ and $s(t)$ in Taylor expansions

$$z_i(t) = a_i t^{p_i} + (\text{higher terms}), \quad i = 1, \dots, n$$

$$s(t) = s_0 + \beta t^b + (\text{higher terms}), \quad 0 \le s_0 \le 1$$

and let $I = \{i \mid z_i(t) \not\equiv 0\}$, $P = (p_i) \in N_+^{*I}$, and $\mathbf{a} = (a_i) \in \mathbf{C}^{*I}$. Then by the definition of $R(\mathbf{z})$,

$$\operatorname{ord} s(t)\partial R(\mathbf{z}(t)) \ge m(P)(N-1) > m(P)\eta''_{J,max} \tag{23}$$

where $N = \min\{N_1, \dots, N_n\}$. By Theorem 32,

$$\operatorname{ord} \partial f(\mathbf{z}(t)) \le m(P)\eta''_{J,max}. \tag{24}$$

Thus

$$\operatorname{ord} \partial f_{s_0}(\mathbf{z}(t)) = \operatorname{ord} \partial f(\mathbf{z}(t))$$

and

$$\lim_{t\to 0}(\partial f_{s(t)}(\mathbf{z}(t)))_{norm} = \lim_{t\to 0}(\partial f_{s(0)}(\mathbf{z}(t)))_{norm}.$$

This already implies the family of hypersurfaces V_s have isolated singularities at the origin. Note that the equality $f_P(\mathbf{a}) = 0$ follows from the equality $f_{s(t)}(\mathbf{z}(t)) \equiv 0$. The assumption gives us the contradicting equalities

$$|(\mathbf{z}(t), \overline{\partial f_{s(t)}(\mathbf{z}(t))})_{norm}| \equiv 1, \quad \text{and}$$
$$\lim_{t\to 0}(\mathbf{z}(t), \overline{\partial f_{s(t)}(\mathbf{z}(t))})_{norm} = 0$$

where the first equality follows from the assumption (22) and the second convergence follows from (24) and Lemma 23. Thus the family f_s, $0 \le s \le 1$ has a uniform stable radius and the isomorphisms of the Milnor fibrations are easily obtained using tubular Milnor fibration and Ehresman's fibration theorem ([28]). □

Remark 26. By the same argument, it is easy to see that $f_t(\mathbf{z}) = f(\mathbf{z}) + t\mathbf{z}^\nu$ with $\sum_{i=1}^n \nu_i \ge \eta''_{J,max}(f) + 1$ does not change the topology at the origin. A similar result for C^0-sufficiency is proved in Kuo [14].

4. Łojasiewicz inequality for mixed functions

In this section, we will introduce the notion of Łojasiewicz exponent for a mixed function and we generalize the estimation obtained for non-degenerate holomorphic functions in previous sections for $f(\mathbf{z}, \bar{\mathbf{z}})$ which are strongly non-degenerate.

4.1. Newton boundaries and various gradients. Let f be a mixed function expanded as

$$f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{\nu,\mu} c_{\nu\mu}\mathbf{z}^\nu\bar{\mathbf{z}}^\mu.$$

The Newton diagram $\Gamma_+(f)$ is defined as the convex hull of

$$\bigcup_{c_{\nu\mu}\neq 0} ((\nu + \mu) + (\mathbf{R}_+)^n)$$

and the Newton boundary $\Gamma(f)$ is defined by the union of compact faces of $\Gamma_+(f)$ as in the holomorphic case, using the radial weighted degree of the monomial $\mathbf{z}^\nu\bar{\mathbf{z}}^\mu$. Here the radial weight degree with respect to the weight vector $P = (p_1, \dots, p_n)$, is defined by

$$\operatorname{rdeg}_P \mathbf{z}^\nu\bar{\mathbf{z}}^\mu := \sum_{i=1}^n p_i(\nu_i + \mu_i).$$

The notion of *strong non-degeneracy* is introduced in [21] to study Milnor fibration defined by f. Let us recall the definition. Take an arbitrary face Δ of $\Gamma(f)$ of any dimension. The face function is defined by $f_\Delta(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{\nu+\mu\in\Delta} c_{\nu\mu}\mathbf{z}^\nu\bar{\mathbf{z}}^\mu$. Let P be the weight vector. Then f_P is defined as $f_{\Delta(P)}$ as in the holomorphic case. $f_P(\mathbf{z}, \bar{\mathbf{z}})$ is a radially weighted homogeneous polynomial with weight P. A mixed function f is called *strongly non-degenerate* if $f_\Delta : \mathbf{C}^{*n} \to \mathbf{C}$ has no critical point for any face function f_Δ. It is known that such a mixed function admits a Milnor fibration at the origin ([21]).

We assume that $\Gamma(f)$ has dimension $n-1$. We consider two gradient vectors:

$$\partial f := \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right) \tag{25}$$

$$\bar{\partial} f := \left(\frac{\partial f}{\partial \bar{z}_1}, \dots, \frac{\partial f}{\partial \bar{z}_n}\right) \tag{26}$$

To study the behavior of the tangent spaces, it is more useful to use the real and imaginary part of f. Put $f = g + ih$ where g and h are real and imaginary parts of f. Putting $z_j = x_j + iy_j$, g, h are real analytic functions of $2n$ variables $x_1, y_1, \dots, x_n, y_n$. Substituting $x_j = (z_j + \bar{z}_j)/2$ and $y_j = (z_j - \bar{z}_j)/2i$, we consider g, h as mixed functions. As they are real valued mixed functions, we have

$$T_{\mathbf{a}} g^{-1}(0) = \{\mathbf{v} \mid \Re(\mathbf{v}, \bar{\partial} g) = 0\}, \quad T_{\mathbf{a}} h^{-1}(0) = \{\mathbf{v} \mid \Re(\mathbf{v}, \bar{\partial} h) = 0\}.$$

Here $(\mathbf{v}, \mathbf{w})$ is the hermitian inner product. As we have $\overline{\partial g} = \bar{\partial} g$ and $\overline{\partial h} = \bar{\partial} h$ ([22]), various gradients are related by

$$\bar{\partial} f = \bar{\partial} g + i\bar{\partial} h, \quad \overline{\partial f} = \bar{\partial} g - i\bar{\partial} h \tag{27}$$

$$\bar{\partial} g = (\bar{\partial} f + \overline{\partial f})/2, \quad \bar{\partial} h = (\bar{f} - \overline{\partial f})/2i. \tag{28}$$

For a weight vector $P = (p_1, \dots, p_n)$, the real part and imaginary part $g_P = \Re f_P$, $h_P = \Im f_P$ of f_P are real-valued radially weighted homogeneous polynomials with weight P and the Euler equality take the form ([20]):

$$g_P(\mathbf{z}, \bar{\mathbf{z}}) = 2\sum_{i=1}^{n} p_i z_i \frac{\partial g_P}{\partial z_i}, \quad h_P(\mathbf{z}, \bar{\mathbf{z}}) = 2\sum_{i=1}^{n} p_i z_i \frac{\partial h_P}{\partial z_i}. \tag{29}$$

The strong non-degeneracy implies that $\{\bar{\partial} g_P(\mathbf{a}, \bar{\mathbf{a}}), \bar{\partial} h_P(\mathbf{a}, \bar{\mathbf{a}})\}$ are linearly independent over $\mathbf{R}$ for any $\mathbf{a} = (a_1, \dots, a_n) \in \mathbf{C}^{*n}$. We consider the Łojasiewicz inequalities of real-valued mixed functions g and h. For a non-zero vector $\mathbf{w} \in \mathbf{C}^n$, we denote the real hyperplane orthogonal to $\mathbf{w}$ by $\mathbf{w}^\perp := \{\mathbf{v} \in \mathbf{C}^n \mid \Re(\mathbf{v}, \mathbf{w}) = 0\}$. We denote the normalized vector $\mathbf{w}/\|\mathbf{w}\|$ by $\mathbf{w}_{norm}$. The problem (which do not happen for holomorphic functions) is that along a given analytic curve $\mathbf{z}(t)$, $0 \le t \le 1$ with $\mathbf{z}(0) = \mathbf{0}$, the limit directions $\lim_{t\to 0}(\bar{\partial} g(\mathbf{z}(t)))_{norm}$ and $\lim_{t\to 0}(\bar{\partial} h(\mathbf{z}(t)))_{norm}$ can be linearly dependent over $\mathbf{R}$. In this case, we have a proper inclusion:

$$\begin{aligned} \lim_{t\to 0} T_{\mathbf{z}(t)} V_t &= \lim_{t\to 0}((\bar{\partial} g(\mathbf{z}(t)))^\perp \cap (\bar{\partial} h(\mathbf{z}(t)))^\perp) \\ &\subsetneq \lim_{t\to 0}(\bar{\partial} g(\mathbf{z}(t))_{norm})^\perp \cap \lim_{t\to 0}(\bar{\partial} h(\mathbf{z}(t))_{norm})^\perp \end{aligned} \tag{30}$$

The following lemma plays a key role to solve this problem. Let $I = \{1 \le j \le n \mid z_j(t) \not\equiv 0\}$ and I^c be the complement of I.

$$(31) \qquad C(t): \begin{cases} \mathbf{z}(t) = (z_1(t), \dots, z_n(t)), & \mathbf{z}(0) = 0,\ \mathbf{z}(t) \in \mathbf{C}^{*I},\ t > 0 \\ z_i(t) = a_i t^{p_i} + (\text{higher terms}), & a_i \ne 0,\ i \in I \end{cases}$$

Consider the weight vector $P = (p_i) \in N^{*I}$, $\mathbf{a} = (a_i)_{i \in I} \in \mathbf{C}^{*I}$ and we assume that $f^I \ne 0$. For any real valued analytic function $b(t)$ defined on an open neighborhood of $t = 0$, we consider the modified gradient vectors, defined as follows.

$$(\bar\partial g(\mathbf{z}(t)))_{b(t)} := \bar\partial g(\mathbf{z}(t)) + b(t)\bar\partial h(\mathbf{z}(t))$$
$$(\bar\partial h(\mathbf{z}(t)))_{b(t)} := \bar\partial h(\mathbf{z}(t)) + b(t)\bar\partial g(\mathbf{z}(t)).$$

Note that any of the following three pairs generate the same real dimension 2 subspace over $\mathbf{R}$ at $T_{\mathbf{z}(t)}\mathbf{C}^n$.

$$\{\bar\partial g(\mathbf{z}(t)), \bar\partial h(\mathbf{z}(t))\}, \quad \{\bar\partial g(\mathbf{z}(t)), (\bar\partial h(\mathbf{z}(t)))_{b(t)}\}, \quad \{(\bar\partial g(\mathbf{z}(t)))_{b(t)}, \bar\partial h(\mathbf{z}(t))\}$$

The following is a key lemma to generalize the assertions obtained in previous sections for mixed functions. Consider the family of hypersurface $V_t := \{\mathbf{z} \in \mathbf{C}^n \mid f(\mathbf{z}, \bar{\mathbf{z}}) = f(\mathbf{z}(t), \bar{\mathbf{z}}(t))\}$ for $-\varepsilon \le t \le \varepsilon$ where V_t passes through $\mathbf{z}(t)$.

Lemma 27. *Assume that* $\operatorname{ord} \bar\partial g(\mathbf{z}(t)) \le \operatorname{ord} \bar\partial h(\mathbf{z}(t))$ *for simplicity.*

(i) *There exists a real valued analytic function $b(t)$ so that two analytic curves $\bar\partial g(\mathbf{z}(t))$, $(\bar\partial h(\mathbf{z}(t)))_{b(t)}$ have normalized limits*

$$v_{g,\infty} := \lim_{t\to 0} (\bar\partial g(\mathbf{z}(t)))_{norm} \quad \textit{and}$$
$$v'_{h,\infty} := \lim_{t\to 0} ((\bar\partial h(\mathbf{z}(t)))_{b(t)})_{norm}$$

which are linearly independent over $\mathbf{R}$. *The limit of the tangent space $T_{\mathbf{z}(t)} V_t$ is equal to the intersection of the hyperplanes $v_{g,\infty}^{\perp} \cap (v'_{h,\infty})^{\perp}$.*

(ii) *The orders of the vectors $\bar\partial g(\mathbf{z}(t))$, $(\bar\partial h(\mathbf{z}(t)))_{b(t)}$ satisfy the inequality*:

$$\operatorname{ord} \bar\partial g(\mathbf{z}(t)),\ \operatorname{ord}(\bar\partial h(\mathbf{z}(t)))_{b(t)} \le d(P, f^I) - m(P).$$

(iii) *If further $f_P(\mathbf{a}) = 0$, the limit vector $\lim_{t\to 0} \mathbf{z}(t)_{norm}$ is real orthgonal to $v_{g,\infty}$ and $v'_{h,\infty}$ i.e., $\lim_{t\to 0} \mathbf{z}(t)_{norm} \in v_{g,\infty}^{\perp} \cap (v'_{h,\infty})^{\perp}$.*

(iv) *For any analytic functions $b(t)$, $c(t)$,*

$$\operatorname{ord} \bar\partial g(\mathbf{z}(t))_{c(t)},\ \operatorname{ord} \bar\partial h(\mathbf{z}(t))_{b(t)} \le d(P, f^I) - m(P).$$

Here two vectors $\mathbf{v}, \mathbf{w} \in \mathbf{C}^n$ are called to be real orthogonal if $\Re(\mathbf{v}, \mathbf{w}) = 0$.

Proof. The proof is essentially the same as the proof of Theorem 3.14, [9]. For the convenience, we repeat the proof briefly. For further related discussion, see [22, 9]. Put $I = \{i \mid z_i(t) \not\equiv 0\}$ and $J = I^c$. We may assume that $I = \{1, \dots, m\}$, $\mathbf{a} = (a_1, \dots, a_m)$ and

$$p_1 = p_2 = \cdots = p_k < p_{k+1} \le \cdots \le p_m.$$

Put $d = d(P, f^I)$. Under the above assumption, $\operatorname{ord} \mathbf{z}(t) = p_1$ and

$$\lim_{t \to 0} (\mathbf{z}(t))_{norm} = \mathbf{a}(\infty)_{norm} \quad \text{where } \mathbf{a}(\infty) := (a_1, \dots, a_k, 0, \dots, 0).$$

By the definition of P and the assumption $f^I \ne 0$,

$$\frac{\partial g^I}{\partial \bar{z}_i}(\mathbf{z}(t)) = \frac{\partial g_P^I}{\partial \bar{z}_i}(\mathbf{a}) t^{d-p_i} + (\text{higher terms})$$
$$\frac{\partial h^I}{\partial \bar{z}_i}(\mathbf{z}(t)) = \frac{\partial h_P^I}{\partial \bar{z}_i}(\mathbf{a}) t^{d-p_i} + (\text{higher terms})$$

Thus by the strong non-degeneracy assumption, we have the inequality:

$$\operatorname{ord} \bar{\partial} g^I(\mathbf{z}(t)) \le d - p_1, \quad \operatorname{ord} \bar{\partial} h^I(\mathbf{z}(t)) \le d - p_1.$$

For an analytic curve $\mathbf{v}(t)$ with $\mathbf{v}(0) = 0$, we associate scalar vector

$$\beta(\mathbf{v}(t)) := (\beta_1, \dots, \beta_m), \quad \text{where } \beta_i = \operatorname{Coeff}(v_i(t), t^{d-p_i})$$

and integers

$$d(\mathbf{v}(t)) := \min\{\mathrm{ord\ v_i(t)} \mid \mathrm{i} = 1, \dots, \mathrm{m}\},$$
$$\gamma_{\mathbf{v}} := \max\{i \mid \mathrm{ord\ v_i(t) = d(\mathbf{v}(t))}\}.$$

Note that $\gamma_{\mathbf{v}}$ is the largest index for which $\lim_{t \to 0} \mathbf{v}(t)_{norm}$ has non-zero coefficient. We call $\gamma_{\mathbf{v}}$ *the leading index* of $\mathbf{v}(t)$.

We start from two analytic curves $\bar{\partial} g^I(\mathbf{z}(t))$ and $\bar{\partial} h^I(\mathbf{z}(t))$. Put $d_g = \operatorname{ord}(\bar{\partial} g^I(\mathbf{z}(t)))$, $d_h = \operatorname{ord}(\bar{\partial} h^I(\mathbf{z}(t)))$ and $\gamma_g := \gamma_{\bar{\partial} g^I(\mathbf{z}(t))}$, $\gamma_h := \gamma_{\bar{\partial} h^I(\mathbf{z}(t))}$. We assume that $d_g \le d_h$. First we associate $2 \times m$-matrix with complex coefficients by

$$A(\bar{\partial} g^I, \bar{\partial} h^I) := \begin{pmatrix} \beta(\bar{\partial} g^I(\mathbf{z}(t))) \\ \beta(\bar{\partial} h^I(\mathbf{z}(t))) \end{pmatrix} = \begin{pmatrix} \frac{\partial g_P^I}{\partial \bar{z}_1}(\mathbf{a}) & \cdots & \frac{\partial g_P^I}{\partial \bar{z}_m}(\mathbf{a}) \\ \frac{\partial h_P^I}{\partial \bar{z}_1}(\mathbf{a}) & \cdots & \frac{\partial h_P^I}{\partial \bar{z}_m}(\mathbf{a}) \end{pmatrix}$$

By the strong non-degeneracy assumption, two raw complex vectors are linearly independent over $\mathbf{R}$. The normalized limit, $\lim_{t \to 0} (\bar{\partial} g^I(\mathbf{z}(t)))_{norm}$, has non-zero j-th coefficient if and only if $\operatorname{ord} \frac{\partial g^I}{\partial \bar{z}_j}(\mathbf{z}(t)) = d_g$. The following three cases are possible.

(1) $\gamma_g \ne \gamma_h$.
(2) $\gamma_g = \gamma_h$ and $\left\{ \operatorname{Coeff}\left(\frac{\partial g^I}{\partial \bar{z}_{\gamma_g}}(\mathbf{z}(t)), t^{d_g} \right), \operatorname{Coeff}\left(\frac{\partial h^I}{\partial \bar{z}_{\gamma_h}}(\mathbf{z}(t)), t^{d_h} \right) \right\}$ are linearly independent over $\mathbf{R}$.

(3) $\gamma_g = \gamma_h$ and $\left\{\mathrm{Coeff}\left(\frac{\partial g^I}{\partial \bar{z}_{\gamma_g}}(\mathbf{z}(t)), t^{d_g}\right), \mathrm{Coeff}\left(\frac{\partial h^I}{\partial \bar{z}_{\gamma_h}}(\mathbf{z}(t)), t^{d_h}\right)\right\}$ are linearly dependent over $\mathbf{R}$.

In the cases of (1), (2), their normalized limits are linearly independent over $\mathbf{R}$ and there is no operation necessary. In the case of (3), we put $b_1(t) = \rho_1 t^{d_h - d_g}$ and put $(\bar{\partial}h)'(t) := \bar{\partial}h^I(\mathbf{z}(t)) - b_1(t)\bar{\partial}g^I(\mathbf{z}(t))$. Here ρ_1 is the unique real number such that

$$\rho_1 \,\mathrm{Coeff}\left(\frac{\partial g^I}{\bar{z}_{\gamma_g}}(\mathbf{z}(t)), t^{d_g}\right) - \mathrm{Coeff}\left(\frac{\partial h^I}{\bar{z}_{\gamma_h}}(\mathbf{z}(t)), t^{d_h}\right) = 0.$$

After this operation, we have three possible cases.

(1)′ $\gamma_{(\bar{\partial}h)'(t)} \neq \gamma_{\bar{\partial}g^I(\mathbf{z}(t))}$

(2)′ $\gamma_{(\bar{\partial}h)'(t)} = \gamma_{\bar{\partial}g^I(\mathbf{z}(t))}$ but the leading coefficients of $(\bar{\partial}h)'(t)$ and $\bar{\partial}g^I(\mathbf{z}(t))$ are linearly independent over $\mathbf{R}$.

(3)′ $\gamma_{(\bar{\partial}h)'(t)} = \gamma_{\bar{\partial}g^I(\mathbf{z}(t))}$ and the leading coefficients of $(\bar{\partial}h)'(t)$ and $\bar{\partial}g^I(\mathbf{z}(t))$ are still linearly dependent over $\mathbf{R}$.

In the case of $(1)'$ and $(2)'$, we stop the operation. Otherwise we have $(3)'$ and we continue this operation till we get a modified gradient vector

$$(\bar{\partial}h)^{(j)}(t) = \bar{\partial}h^I(\mathbf{z}(t)) - \rho(t)\bar{\partial}g^I(\mathbf{z}(t)), \quad \rho(t) := \sum_{i=1}^{j} b_i(t)$$

for which either its leading index is different from γ_g (case (1)) or the coefficients of the leading index are linearly independent over $\mathbf{R}$ (case (2)). Note that in this operation, the order of $(\bar{\partial}h)^{(j)}(t)$ is strictly increasing in j while the matrix $A(\bar{\partial}g, (\bar{\partial}h)^{(j)})$ is simply changed in the second raw vector by $\beta(\bar{\partial}h^I) - \rho(0)\beta(\bar{\partial}g^I)$. Therefore $(\bar{\partial}h)^{(j)}(t) := \bar{\partial}h^I(\mathbf{z}(t)) - \rho(t)\bar{\partial}g^I(\mathbf{z}(t))$ satisfies

$$\mathrm{ord}(\bar{\partial}h)^{(j)}(t) \leq d - p_1$$

and therefore the operation should stop after finite steps, say k. After the operation is finished, the normalized vector $((\bar{\partial}h)^{(k)}(t))_{norm}$ has linearly independent limit with that of $\bar{\partial}g^I(\mathbf{z}(t))$.

Suppose further $f_P^I(\mathbf{a}) = 0$. This implies $g_P^I(\mathbf{a}) = h_P^I(\mathbf{a}) = 0$. We will show now $\mathbf{a}_\infty = (a_1, \dots, a_k, 0, \dots, 0)$ is orthogonal to the limits of the normalized vectors

$$v_\infty^g := \lim_{t\to 0}(\bar{\partial}g^I(\mathbf{z}(t)))_{norm} \quad \text{and} \quad v_\infty^{(\bar{\partial}h)^{(k)}} := \lim_{t\to 0}((\bar{\partial}h)^{(k)}(t))_{norm}.$$

First we consider v_∞^g. If $d_g < d - p_1$, j-coefficient of v_∞^g is zero for $j \leq k$ and $\Re(v_\infty^g, \mathbf{a}_\infty) = 0$ is obvious. If $d_g = d - p_1$, we must have

$$\frac{\partial g_P^I}{\partial \bar{z}_j}(\mathbf{a}) = 0, \quad k+1 \leq j \leq m$$

and the i-th coefficient of v_∞^g is $\dfrac{\partial g_P^I}{\partial \bar z_i}(\mathbf{a})$ for $1 \le i \le k$ up to a scalar multiplication. Thus the assertion follows from the Euler equality

$$g_P^I(\mathbf{a}) = 0 = \sum_{i=1}^m p_i a_i \frac{\partial g_P^I}{\partial \bar z_i}(\mathbf{a}) = p_1 \sum_{i=1}^k a_i \frac{\partial g_P^I}{\partial \bar z_i}(\mathbf{a}).$$

Now we consider $v_\infty^{(\bar\partial h)^{(k)}}$. We start from the equality $h_P^I(\mathbf{a}) - \rho(0) g_P^I(\mathbf{a}) = 0$. If $d_{(\bar\partial h)^{(k)}(t)} < d - p_1$, $v_\infty^{(\bar\partial h)^{(k)}}$ and $\mathbf{a}_\infty$ are orthogonal by the same reason. Suppose that $d_{(\bar\partial h)^{(k)}(t)} = d - p_1$. Then we must have

$$\frac{\partial h_P^I}{\partial \bar z_j}(\mathbf{a}) - \rho(0)\frac{\partial g_P^I}{\partial \bar z_j}(\mathbf{a}) = 0, \quad k+1 \le j \le m.$$

Thus the assertion follows from the Euler equality of the real valued radially weighted homogeneous polynomial $h_P^I(\mathbf{z}, \bar{\mathbf{z}}) - \rho(0) g_P^I(\mathbf{z}, \bar{\mathbf{z}})$. The assertion (iv) can be shown in a similar way looking at the matrix A before and after. □

Definition 28. Let $\mathbf{z}(t)$ be a analytic curve starting at the origin. Assume that $\bar\partial g(\mathbf{z}(t)) \le \operatorname{ord} \bar\partial h(\mathbf{z}(t))$ and $\{\bar\partial g(\mathbf{z}(t)), \bar\partial h(\mathbf{z}(t))_{c(t)}\}$ (respectively $\operatorname{ord} \bar\partial g(\mathbf{z}(t)) > \bar\partial h(\mathbf{z}(t)))$ and $\{\bar\partial g(\mathbf{z}(t))_{c(t)}, \bar\partial h(\mathbf{z}(t))\}$ is a good modified gradient pair if they have linearly independent normalized limits over $\mathbf{R}$.

Take an arbitrary analytic curve $C(t) : \mathbf{z} = \mathbf{z}(t)$ with $\mathbf{z}(0) = 0$ and a good modified gradient pair, say $\{\bar\partial g(\mathbf{z}(t)), \bar\partial h(\mathbf{z}(t))_{c(t)}\}$ assuming $\bar\partial g(\mathbf{z}(t)) \le \operatorname{ord} \bar\partial h(\mathbf{z}(t))$ for simplicity and suppose that the following inequality is satisfied for sufficiently small t, $0 \le t \le \varepsilon$,

$$\operatorname{ord}(\bar\partial g(\mathbf{z}(t), \bar{\mathbf{z}}(t)))/\operatorname{ord} \mathbf{z}(t) \le \theta \tag{32}$$

$$\operatorname{ord}(\bar\partial h(\mathbf{z}(t), \bar{\mathbf{z}}(t))_{c(t)})/\operatorname{ord} \mathbf{z}(t) \le \theta. \tag{33}$$

(If $\bar\partial g(\mathbf{z}(t)) > \operatorname{ord} \bar\partial h(\mathbf{z}(t))$, we exchange g and h in the above inequalities so that $\bar\partial g$ is to be modified.) We define the Łojasiewicz exponent $\ell_0(C(t))$ along an analytic curve $C(t)$ as the infinimum of such θ satisfying the above inequality. The Łojasiewicz exponent of f is defined by the supremum of $\ell_0(C(t))$ for all analytic curves $C(t)$.

These inequalities are equivalent to the inequality

$$\|\bar\partial g(\mathbf{z}(t), \bar{\mathbf{z}}(t))\|, \ \|\bar\partial h(\mathbf{z}(t), \bar{\mathbf{z}}(t))_{c(t)}\| \ge C\|\mathbf{z}(t)\|^\theta, \quad \exists C > 0$$

for $0 \le t \le \varepsilon$.

Taking $c(t) \equiv 0$, such θ satisfies the usual Łojasiewicz inequalities:

$$\|\bar\partial g(\mathbf{z}, \bar{\mathbf{z}})\| \ge C\|\mathbf{z}\|^\theta \tag{34}$$

$$\|\bar\partial h(\mathbf{z}, \bar{\mathbf{z}})\| \ge C\|\mathbf{z}\|^\theta, \quad \exists C > 0. \tag{35}$$

in a sufficiently small neighborhood of the origin.

Now we are ready to generalize the results which are obtained in previous sections for holomorphic functions.

4.2. Convenient case. We consider a strongly non-degenerate mixed function $f(\mathbf{z}, \bar{\mathbf{z}})$ with an isolated singularity at the origin. A mixed function $f(\mathbf{z}, \bar{\mathbf{z}})$ is called *convenient* if for each $i = 1, \dots, n$, there is a point $B_i = (0, \dots, \underset{i}{\smile}{b_i}, \dots, 0)$ on the Newton boundary $\Gamma(f)$. In the mixed function case, there might exist several corresponding mixed monomial $z_i^{\nu_i} \bar{z}_i^{\mu_i}$ with $\nu_i + \mu_i = b_i$ in the expansion of $f(\mathbf{z}, \bar{\mathbf{z}})$. Such a monomial is called *an i-axis monomial.* Let $B := \max\{b_i \mid i = 1, \dots, n\}$. An i axis monomial $z_i^{\nu_i} \bar{z}_i^{\mu_i}$ is called *Łojasiewicz monomial* if $\nu_i + \mu_i = B$. A Łojasiewicz monomial $z_i^{\nu_i} \bar{z}_i^{\mu_i}$ is *exceptional* if there exists a monomial $z_i^{\nu_i'} \bar{z}_i^{\mu_i'} w_j$ where $w_j = z_j$ or $\bar{z}_j$ in the expansion of $f(\mathbf{z}, \bar{\mathbf{z}})$ such that $\nu_i' + \mu_i' < B - 1$.

Theorem 29. *Let $f(\mathbf{z}, \bar{\mathbf{z}})$ be a strongly non-degenerate convenient mixed function. Then Łojasiewicz exponent $\ell_0(f)$ satisfies the inequality: $\ell_0(f) \le B - 1$. Furthermore if f has a Łojasiewicz non-exceptional monomial, we have the equality $\ell_0(f) = B - 1$.*

Using Lemma 27, the proof is completely parallel to that of Theorem 3.

4.3. Non-convenient mixed polynomials. We consider the case of non-degenerate mixed polynomials with an isolated singularity at the origin. One point is how to define "*Łojasiewicz non-degeneracy*" for mixed functions.

Let $\mathbf{C}^I$ be a vanishing coordinate subspace and let J be the complement of I. For each i, there must exist a mixed monomial $z_i^{n_{ij}} \bar{z}_i^{m_{ij}} w_j$ with $j \in J$ as we have assumed that the origin is an isolated singularity. Hereafter we use variable w_j for either $w_j = z_j$ or $w_j = \bar{z}_j$ for simplicity. We take ℓ_{ij} to be the minimum of $\{n_{ij} + m_{ij}\}$ for fixed i, j. The point $B_{ij} := (0, \dots, \underset{i}{\smile}{\ell_{ij}}, \dots, \underset{j}{\smile}{1} \dots, 0)$ is a point of $\Gamma(f)$. We call a monomial $z_i^{n_{ij}} \bar{z}_i^{m_{ij}} w_j$ *an almost i-axis monomial* if $n_{ij} + m_{ij} = \ell_{ij}$. Let J_i be the set of j for which such almost i-axis monomial exists and put $J(I) = \bigcup_{i \in I} J_i$ as in Part I.

Define $\ell(I) := \max\{\ell_{ij} \mid i \in I,\ j \in J_i\}$. For our present purpose, we take the following definition of Łojasiewicz non-degeneracy. Consider a strictly positive weight vector $Q \in N^{*I}$. Consider $F(j) := z_j \left(\frac{\partial f}{\partial z_j}\right)^I + \bar{z}_j \left(\frac{\partial f}{\partial \bar{z}_j}\right)^I$. Note that $\frac{\partial F(j)_Q}{\partial z_j}$ and $\frac{\partial F(j)_Q}{\partial \bar{z}_j}$ are polynomials in $\mathbf{C}[z_1, \bar{z}_1, \dots, z_m, \bar{z}_m]$. Here $F(j)_Q := (F(j))_Q$ with the weight of z_j being 0 by definition. We use the notations $f_j := \frac{\partial f}{\partial z_j}$ and $f_{\bar{j}} := \frac{\partial f}{\partial \bar{z}_j}$ for simplicity. Note that

$$\frac{\partial F(j)_Q}{\partial z_j} = \begin{cases} (f_j)_Q, & \operatorname{rdeg}_Q f_j \le \operatorname{rdeg}_Q f_{\bar{j}} \\ 0, & \operatorname{rdeg}_Q f_j > \operatorname{rdeg}_Q f_{\bar{j}} \end{cases}$$

$$\frac{\partial F(j)_Q}{\partial \bar{z}_j} = \begin{cases} (f_{\bar{j}})_Q, & \mathrm{rdeg}_Q\, f_j \geq \mathrm{rdeg}_Q\, f_{\bar{j}} \\ 0, & \mathrm{rdeg}_Q\, f_j < \mathrm{rdeg}_Q\, f_{\bar{j}} \end{cases}.$$

Put $I' := \{i \mid q_i = m(Q)\}$ and $J(Q) := \bigcup_{i \in I'} J_i$. $f(\mathbf{z}, \bar{\mathbf{z}})$ is called *Łojasiewicz non-degenerate* if under any such situation and for any $\mathbf{a} \in \mathbf{C}^{*I}$, there exists $i_0 \in I'$, $j_0 \in J(Q)$ such that

$$\left|\frac{\partial F(j_0)_Q}{\partial z_{j_0}}(\mathbf{a})\right| \neq \left|\frac{\partial F(j_0)_Q}{\partial \bar{z}_{j_0}}(\mathbf{a})\right|. \tag{36}$$

Writing $F(j) := g(j) + ih(j)$, this is equivalent to

Proposition 30. *Under the above notations,*

$$\left\{\frac{\partial g(j_0)_Q}{\partial \bar{z}_{j_0}}(\mathbf{a}), \frac{\partial h(j_0)_Q}{\partial \bar{z}_{j_0}}(\mathbf{a})\right\} \tag{37}$$

are linearly independent over $\mathbf{R}$.

The assertion follows from Proposition 1 [20]. Assume that such a Q is associated with an analytic family $\mathbf{z}(t)$ and $d = \mathrm{rdeg}_Q\, F(j)$. Then $d \leq \ell_{i_0, j_0} q_{i_0}$ and

$$\frac{\partial g}{\partial \bar{z}_j}(\mathbf{z}(t)) = \frac{\partial g(j)_Q}{\partial \bar{z}_j}(\mathbf{a})t^d + (\text{higher terms}), \quad \mathrm{ord}\, \frac{\partial g}{\partial \bar{z}_j}(\mathbf{z}(t)) = d$$

$$\frac{\partial h}{\partial \bar{z}_j}(\mathbf{z}(t)) = \frac{\partial h(j)_Q}{\partial \bar{z}_j}(\mathbf{a})t^d + (\text{higher terms}), \quad \mathrm{ord}\, \frac{\partial h}{\partial \bar{z}_j}(\mathbf{z}(t)) = d.$$

Proposition 31. *Using Proposition* 30, *there exists a good modified gradient pair* $\{\bar{\partial} g(\mathbf{z}(t)), \bar{\partial} h(\mathbf{z}(t))_{c(t)}\}$ *or* $\{\bar{\partial} g(\mathbf{z}(t))_{c(t)}, \bar{\partial} h(\mathbf{z}(t))\}$. *Their order in t has a upper bound* $d = \mathrm{rdeg}_Q\, F(j_0) \leq \ell_{i_0, j_0} q_{i_0}$.

4.4. Jacobian dual Newton diagram. We consider the derivatives $f_i(\mathbf{z}) := \dfrac{\partial f}{\partial z_i}(\mathbf{z})$ and $f_{\bar{i}}(\mathbf{z}) := \dfrac{\partial f}{\partial \bar{z}_i}(\mathbf{z})$, $i = 1, \dots, n$. Put $F_i(\mathbf{z}, \bar{\mathbf{z}}) = f_i(\mathbf{z}, \bar{\mathbf{z}}) f_{\bar{i}}(\mathbf{z}, \bar{\mathbf{z}})$. If one of the derivatives vanishes identically, we consider only non-zero derivatives. For example, if $f_{\bar{i}} \equiv 0$, we put $F_i = f_i$. We consider their Newton boundary $\Gamma(F_i)$, $i = 1, \dots, n$. Two weight vectors P, Q are *Jacobian equivalent* if $\Delta(P, F_i) = \Delta(Q, F_i)$ for any $i = 1, \dots, n$ and $\Delta(P, f) = \Delta(Q, f)$. We denote it by $P \underset{J}{\sim} Q$. This gives a polyhedral cone subdivision of N_+ and we denote this as $\Gamma_J^*(f)$ and we call it *the Jacobian dual Newton diagram of* f. $\Gamma_J^*(f)$ is a polyhedral cone subdivision of N_+ which is finer than $\Gamma^*(f)$.

Alternatively we can consider the function $F(\mathbf{z}) = f(\mathbf{z}) F_1(\mathbf{z}) \cdots F_n(\mathbf{z})$. Then $\Gamma_J^*(f)$ is nothing but the dual Newton diagram $\Gamma^*(F)$ of F. For any weight vector P, we have $\Delta(P, F) = \Delta(P, f) + \Delta(P, F_1) + \cdots + \Delta(P, F_n)$ where the sum is Minkowski sum. For a weight vector P, the set of equivalent weight vectors in $\Gamma^*(f)$ and $\Gamma^*_J(f)$ is denoted as $[P]$ and $[P]_J$ respectively. We consider the vertices of these dual Newton diagrams. We denote the set of strictly positive vertices of $\Gamma^*(f)$ and $\Gamma^*_J(f)$ by $\mathscr{V}^+$, $\mathscr{V}_J^+$ as before. Now we can generalize Theorem 14. Let $\mathscr{V}_J^{++} \subset \mathscr{V}_J^+$ be the set of the vertices of $\Gamma^*_J(f)$ which are in a vanishing boundary region of $\Gamma^*(f)$ as in the holomorphic case. The numbers of $\mathscr{V}^+$, $\mathscr{V}_J^+$, $\mathscr{V}_J^{++}$ are finite. We define basic invariants, as before

$$\begin{aligned}
\eta_{J,max}(f) &:= \max\{\eta(P) \mid P \in \mathscr{V}^+ \cup \mathscr{V}_J^{++}\},\\
\eta_{max} &:= \max\{\eta(P) \mid P \in \mathscr{V}^+\}\\
\eta'_{J,max} &:= \max\{\eta'_{k,i}(R) \mid R \in \mathscr{V}_J^{++},\ k, i = 1, \dots, n\}\\
\eta''_{J,max}(f) &:= \max\{\eta_{J,max}(f), \eta'_{J,max}(f)\}, \quad \text{where}\\
\eta'_{k,i}(R) &:= \frac{\min\{d(R, f_i), d(R, f_{\bar{i}})\}}{m(R)}.
\end{aligned}$$

THEOREM 32. *Let $f(\mathbf{z})$ be a non-degenerate, Łojasiewicz non-degenerate mixed function with an isolated singularity at the origin. Then Łojasiewicz exponent $\ell_0(f)$ satisfies the estimation $\ell_0(f) \le \eta''_{J,max}(f)$.*

Proof. The proof is completely parallel to that of Theorem 12. We consider an analytic curve $C(t)$ parametrized as $\mathbf{z}(t) = (z_1(t), \dots, z_n(t))$ and put $I := \{i \mid z_i(t) \not\equiv 0\}$. Consider the Taylor expansion of $z_i(t)$ as before:

$$\begin{cases} \mathbf{z}(t) = (z_1(t), \dots, z_n(t)), \quad \mathbf{z}(0) = 0,\ \mathbf{z}(t) \in \mathbf{C}^{*I} \\ z_i(t) = a_i t^{p_i} + (\text{higher terms}), \quad i \in I \end{cases} \tag{38}$$

We put $P = (p_i) \in N_+^{*I}$ as before. Assume first $I = \{1, \dots, n\}$. We divide the situation into three cases as before.

- C-1 $[P]$ is an inner region. That is, $\overline{[P]}$ has only strictly positive weight vectors in the boundary.
- C-2 $[P]$ is a regular boundary region.
- C-3 $[P]$ is a vanishing boundary region. In this case, we need to consider the subdivision by $[P]_J$. There are three subcases.
  - C-3-1. $[P]_J$ is an inner region.
  - C-3-2. $[P]_J$ is a regular boundary region.
  - C-3-3. $[P]_J$ is also a vanishing boundary region.

Then the proof goes exactly as that of Theorem 12, using Lemma 27. For the cases C-1, C-3-1, C-2, C-3-2, we start from a given good modified gradient pair and the estimation of these gradient by Lemma 27. Then the argument is completely the same. We have the estimation $\ell_0(C(t)) \le \eta''_{J,max}(f)$ in these cases.

For the case C-3-3, consider the situation that P is not a simplicially positive and R, Q as in Lemma 15 so that P is on the line segment $\overline{RQ}$, R is simplicially positive and Q is not strictly positive. If Q is non-vanishing, it reduced to Case 3-2. Thus we assume that Q is a vanishing weight vector and $d(Q,f) > 0$. Assume that $Q = (q_1,\dots,q_n)$ and $I = \{i \mid q_i = 0\}$ and assume $I = \{1,\dots,m\}$ for simplicity. Note that $\mathbf{C}^I$ is a vanishing coordinate subspace. For each $i \in I$, there exists some $j \notin I$ and a monomial $z_i^{n_{i,j}} z_j$ with a non-zero coefficient, as f has an isolated singularity at the origin. Put J_i be the set of such j for a fixed $i \in I$ and put $J(I) = \bigcup_{i\in I} J_i$. Here $n_{i,j}$ is assumed to be the smallest when j is fixed. Put $\xi_I := \max\{n_{i,j} \mid i \in I, j \in J_i\}$ and $\xi(f)$ be the maximum of ξ_I where I moves in the coordinate subspaces corresponding to vanishing coordinate subspaces. Put $\eta'_{J,max}(f) := \max\{\eta'_{k,i}(R) \mid R \in \mathscr{V}^{++}\}$ where $\eta'_{j,i}(R) = d(R,f_i)/r_j$. Under the above situation, we will prove, as in the holomorphic case, that

$$(\star) \quad \ell_0(C(t)) \le \max\{\xi(f), \eta_{J,max}(f), \eta'_{J,max}\}.$$

Consider the normalized weight vector $\hat{R}_s := (1-s)\hat{R} + s\hat{Q}$, $0 \le s \le 1$. Note that $\hat{R}_0 = \hat{R}$, $\hat{R}_1 = \hat{Q}$ and putting $\hat{R}_s = (\hat{r}_{s,1},\dots,\hat{r}_{s,n})$,

$$\hat{r}_{s,i} = \begin{cases} (1-s)\hat{r}_i, & 1 \le i \le m \\ (1-s)\hat{r}_i + s\hat{q}_i, & m < i \le n. \end{cases}$$

Put $I' = \{i \in I \mid \hat{r}_i = m(\hat{R}_I)\}$ and $J' = \bigcup_{i\in I'} J_i$. Thus for $i \le m$, the normalized weight $\hat{r}_{sj}$ goes to 0, when s approaches to 1. On the other hand, for $j > m$, $\hat{r}_{sj} \ge \delta$, $0 \le \forall s \le 1$ for some $\delta > 0$. Thus there exists an ε, $1 > \varepsilon > 0$ so that for $1 - \varepsilon \le s \le 1$, $m(\hat{R}_s)$ is taken by $i \in I'$. Note that for $j \in J'$, there exists a small enough ε_2, $\varepsilon_2 \le \varepsilon_1$ so that $(f_j)_{\hat{R}_s} = ((f_j)^I)_{\hat{R}_s}$ as $\hat{r}_{sj} \ge \delta$ for $j > m$ for $1 - \varepsilon_2 \le s \le 1$. Here $(f_j)^I$ is the restriction of f_j to $\mathbf{C}^I$ and $(\hat{R}_s)_I$ is the I projection of $\hat{R}_s$ to N_+^I. That is, $(f_j)_{\hat{R}_s}$ contains only variable $z_1,\dots,z_m$. By the Łojasiewicz non-degeneracy, there exists $i_0 \in I'$ and $j_0 \in J_{i_0}$ such that

$$\left\{ \frac{\partial g(j_0)_Q}{\partial \bar{z}_{j_0}}(\mathbf{a}), \frac{\partial h(j_0)_Q}{\partial \bar{z}_{j_0}}(\mathbf{a}) \right\} \tag{39}$$

are linearly independent over $\mathbf{R}$. Here we use the same notation as in (39). By the definition of Jacobian dual Newton diagram and Proposition 39, there is a good modified gradient pair, say $\bar{\partial} g(\mathbf{z}(t))$, $\bar{\partial} h(\mathbf{z}(t))_{c(t)}$ (we assume $\operatorname{ord} \bar{\partial} g(\mathbf{z}(t)) \le \operatorname{ord} \bar{\partial} h(\mathbf{z}(t))$ for simplicity) so that *their orders are estimated from above by* $\ell_{i_0,j_0} q_{i_0}$.

The rest of the argument is simply the evaluation of the number $\ell_{i_0,j_0} q_{i_0} / m(\hat{R}_{s_0})$ and the proof is completed by the exact same argument as in the proof of Theorem 12, Case 3-3-3 using the following. The case $C(t) \in \mathbf{C}^{*I}$ with $I^c \ne \emptyset$ is treated also by the exactly same argument. □

LEMMA 33 (Restatement of Lemma 17). *We have the inequality*: $\ell(I) \le \eta_{max}(f)$.

Theorem 19 for non-degenerate weighted homogeneous polynomial and Łojasiewicz Join Theorem 21 also hold in the exactly same way for mixed functions. For example, we get the following as a corollary of Theorem 19.

THEOREM 34 ([4]). *Let $f(\mathbf{z})$ be a strongly non-degenerate mixed weighted homogeneous polynomial with isolated singularity at the origin and* $\dim\Gamma(f) = n-1$. *Let R be the weight vector of f. Then we have the estimation $\ell_0(f) \le \eta(R)$.*

4.5. Making f convenient. We also generalize Theorem 21. Take an integer $N > \eta''_{J,max}(f)+1$. Consider mixed polynomial $R(\mathbf{z},\bar{\mathbf{z}}) := \sum_{i=1}^{n} c_i z_i^{m_i}\bar{z}_i^{n_i}$ where n_i, m_i are any fixed non-negative integers with $m_i + n_i = N_i \ge N$ and $n_i \ne m_i$. We choose such $\{(m_i,n_i)\,|\,i=1,\dots,n\}$ and fix them. The coefficients $c_1,\dots,c_n$ are generic so that $f_1 := f(\mathbf{z},\bar{\mathbf{z}}) + R(\mathbf{z},\bar{\mathbf{z}})$ is strongly non-degenerate. Consider the family $f_s(\mathbf{z},\bar{\mathbf{z}}) = f(\mathbf{z},\bar{\mathbf{z}}) + sR(\mathbf{z},\bar{\mathbf{z}})$. Then we have the following.

THEOREM 35. *There exists a $r_0 > 0$ such that for any $r \le r_0$, the sphere S_r and the family of hypersurface $V_s := f_s^{-1}(0)$ intersect transversely for any $0 \le s \le 1$. In particular, the links of f and f_1 are isotopic and their Milnor fibrations are isomorphic.*

The proof is similar and we leave it to the reader.

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