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q-SERIES RECIPROCITIES AND FURTHER π-FORMULAE

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Abstract

By examining reciprocal relations of basic well-poised, quadratic and cubic series, we establish q-analogues of three infinite series for $1/\pi^2$ due to Guillera (2003) and λ-parameter extensions of three infinite series for $1/\pi$ due to Ramanujan (1914). Several further infinite series identities of Ramanujan-type are also derived as consequences.

1. Introduction and motivation

One century ago, Ramanujan [19] discovered 17 remarkable infinite series for $1/\pi$. Recently, there has been growing interest in finding new identities of Ramanujan-type. By using the WZ-method, Guillera [15–18] detected numerous identities of Ramanujan-type. Borweins [4, 5], Chan *et al* [7, 8] and Zudilin [22] found further formulae via modular equations. More comprehensive investigation has been made by Chu and Zhang [9, 10, 12] through the hypergeometric series approach.

Now that most hypergeometric series results have their q-series counterparts (see Gasper-Rahman [14] for example), it is natural to ask whether these infinite series expressions concerning π admit q-analogues. We find that some transformations of q-series due to Chu-Zhang [11] can be employed to answer affirmatively this question in some cases. Instead of presenting a full coverage of this topic, we are limited here to exemplify a few representative series.

The objective of this paper is twofold. First in Section 2, we shall derive q-analogues of three infinite series for $1/\pi^2$ of Guillera [15, 16] and λ-parameter extensions of three Ramanujan's series by examining carefully one reciprocal relation of well-poised series and two reciprocal relations of quadratic series.

Then in Section 3, we shall explore the reciprocal relation of iteration pattern **[30111]** that is not covered by the compendium [12]. Even though this transformation of cubic series is quite complicated, its reduction form (as $q \to 1$) will be shown to be very useful, via bisection and trisection series, to deduce several

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new identities of Ramanujan-type. Four elegant formulae are anticipated below, where the first one confirms the infinite series expression for $\zeta(3)$ conjectured experimentally by Sun [20, Conjecture 8]:

- Example 14:

$$\frac{21}{2}\zeta(3)=\sum_{k=0}^{\infty}\begin{bmatrix}1, & 1, & 1, & 1, & 1\\ \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}, & \frac{4}{3}, & \frac{5}{3}\end{bmatrix}_k\frac{13+38k+28k^2}{(-27)^k}.$$

- Example 22:

$$\frac{9\sqrt{3}}{2^{4/3}\pi}=\sum_{k=0}^{\infty}\begin{bmatrix}\frac{1}{3}, & \frac{2}{3}, & \frac{1}{6}\\ 1, & 1, & 1\end{bmatrix}_k\frac{2+21k}{(-27)^k}.$$

- Example 23:

$$\frac{27\sqrt{3}}{2^{5/3}\pi}=\sum_{k=0}^{\infty}\begin{bmatrix}\frac{1}{3}, & \frac{2}{3}, & \frac{5}{6}\\ 1, & 1, & 1\end{bmatrix}_k\frac{5+42k}{(-27)^k}.$$

- Example 26:

$$\frac{3}{\pi}=\sum_{k=0}^{\infty}\begin{bmatrix}\frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}\\ 1, & 1, & 1, & \frac{5}{6}, & \frac{7}{6}\end{bmatrix}_k\frac{1+10k+28k^2}{(-27)^k}.$$

Throughout the paper, the following notations will be utilized. Let $\mathbf{N}$ be the set of natural numbers. The shifted factorial in base q is defined by

$$(x;q)_0=1 \quad\text{and}\quad (x;q)_n=(1-x)(1-qx)\cdots(1-q^{n-1}x) \quad \text{for } n\in\mathbf{N}.$$

When $|q|<1$, we have two well-defined infinite products

$$(x;q)_\infty=\prod_{k=0}^{\infty}(1-q^kx) \quad\text{and}\quad (x;q)_n=(x;q)_\infty/(q^nx;q)_\infty.$$

The product and quotient of shifted factorials are abbreviated respectively to

$$[\alpha,\beta,\ldots,\gamma;q]_n=(\alpha;q)_n(\beta;q)_n\cdots(\gamma;q)_n,$$

$$\left[\begin{matrix}\alpha,\beta,\ldots,\gamma\\ A,B,\ldots,C\end{matrix}\,\middle|\,q\right]_n=\frac{(\alpha;q)_n(\beta;q)_n\cdots(\gamma;q)_n}{(A;q)_n(B;q)_n\cdots(C;q)_n}.$$

When $q\to1$, the ordinary corresponding shifted factorial reads as

$$(x)_0=1=1 \quad\text{and}\quad (x)_n=x(x+1)\cdots(x+n-1) \quad \text{for } n\in\mathbf{N}$$

with its multiparameter forms being given by

$$[\alpha,\beta,\ldots,\gamma]_n=(\alpha)_n(\beta)_n\cdots(\gamma)_n,$$

$$\begin{bmatrix}\alpha,\beta,\ldots,\gamma\\ A,B,\ldots,C\end{bmatrix}_n=\frac{(\alpha)_n(\beta)_n\cdots(\gamma)_n}{(A)_n(B)_n\cdots(C)_n}.$$

The shifted factorial can also be expressed as the following quotient

$$(x)_n = \Gamma(x+n)/\Gamma(x)$$

where the Γ-function is defined by the Euler integral

$$\Gamma(x) = \int_0^{\infty} u^{x-1} e^{-u}\,\mathrm{d}u \quad \text{with } \Re(x) > 0$$

and admits the well-known reciprocal relations

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \quad \text{and} \quad \Gamma\left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}-x\right) = \frac{\pi}{\cos \pi x}.$$

2. Three reciprocal relations and consequences

For the partial sum of Bailey's bilateral well-poised series defined by

$$\Omega(a;b,c,d,e) := \sum_{k\geq 0} \frac{1-q^{2k}a}{1-a}\left[\begin{matrix} b, & c, & d, & e \\ qa/b, & qa/c, & qa/d, & qa/e \end{matrix}\,\middle|\, q\right]_k \left(\frac{qa^2}{bcde}\right)^k$$

the well known q-Dougall formula (cf. [14, II-20]) can be restated as

$$\Omega(a;b,c,d,a) = \left[\begin{matrix} qa, qa/bc, qa/bd, qa/cd \\ qa/b, qa/c, qa/d, qa/bcd \end{matrix}\,\middle|\, q\right]_\infty \quad \text{where } |qa/bcd| < 1. \tag{1}$$

By means of the modified Abel lemma on summation by parts, Chu and Zhang [11] established several transformation formulae from $\Omega(a;b,c,d,e)$ into fast convergent series. Each transformation is derived by iterating a recurrence relation, characterized by an "iteration pattern" $[n_a, n_b, n_c, n_d, n_e]$, that expresses $\Omega(a;b,c,d,e)$ in terms of another shifted one $\Omega(aq^{n_a}; bq^{n_b}, cq^{n_c}, dq^{n_d}, eq^{n_e})$, with $\{n_a, n_b, n_c, n_d, n_e\}$ being five integers. This section will carefully examine three of them that leads to q-analogues of three Guillera's series for $1/\pi^2$ and λ-parameter extensions of three Ramanujan's series.

For the sake of brevity, denote the quotient of shifted factorials in base q by

$$W(x,y) := \left[\begin{matrix} q^2x, qx/y^2, qx/y^2, qx/y^2 \\ qx/y, qx/y, qx/y, q^2x/y^3 \end{matrix}\,\middle|\, q\right]_\infty = \frac{1-qx/y^3}{1-qx}\Omega(x;y,y,y,x) \tag{2}$$

provided that $|qx/y^3| < 1$ for the convergence of Ω-series.

In view of the q-Gamma function [14, §1.10]

$$\Gamma_q(\lambda) = (1-q)^{1-\lambda}\frac{(q;q)_\infty}{(q^\lambda;q)_\infty} \quad \text{and} \quad \lim_{q\to 1^-}\Gamma_q(\lambda) = \Gamma(\lambda) \tag{3}$$

we can compute the following limits for $W(x,y)$

$$\lim_{q\to 1^-} W(q,q^{1/2}) = \lim_{q\to 1^-} \frac{\Gamma_q(\frac{3}{2})\Gamma_q(\frac{3}{2})\Gamma_q(\frac{3}{2})\Gamma_q(\frac{3}{2})}{\Gamma_q(1)\Gamma_q(1)\Gamma_q(1)\Gamma_q(3)} = \frac{\pi^2}{32}, \tag{4}$$

$$\lim_{q\to 1^-} W(q^{1/2},q^{1/2}) = \lim_{q\to 1^-} \frac{\Gamma_q(1)\Gamma_q(1)\Gamma_q(1)\Gamma_q(1)}{\Gamma_q(\frac{1}{2})\Gamma_q(\frac{1}{2})\Gamma_q(\frac{1}{2})\Gamma_q(\frac{5}{2})} = \frac{4}{3\pi^2}; \tag{5}$$

that will be used to deduce limiting relations of q-series without explanation.

§2.1. Well-poised transformation from iteration pattern [20111]. We start from the following expression of $\Omega(a;b,c,d,e)$ in terms of well-poised series.

LEMMA 1 (Chu-Zhang [11, Theorem 15]: $|qa^2/bcde| < 1$).

$$\begin{aligned}\Omega(a;b,c,d,e) = {} & \left[\begin{matrix} a/e, qa^2/bcd \\ a, qa^2/bcde \end{matrix}\middle| q\right]_1 \sum_{k=0}^{\infty} \frac{1-q^{2k+1}a^2/bcd}{1-qa^2/bcd} q^{\binom{k}{2}} \\ & \times \frac{[c,d,e,qa/bc,qa/bd,qa/be;q]_k}{[qa/c,qa/d,a/e,q^2a^2/bcde;q]_k} \frac{(-qa^2/cde)^k}{(qa/b;q)_{2k}} \\ & + \frac{a}{e}\left[\begin{matrix} e, qa/bc, qa/bd \\ a, qa/b, qa^2/bcde \end{matrix}\middle| q\right]_1 \sum_{k=0}^{\infty} \frac{(-q^2a^2/cde)^k}{(q^2a/b;q)_{2k}} \\ & \times \frac{[c,d,qe,q^2a/bc,q^2a/bd,qa/be;q]_k}{[qa/c,qa/d,qa/e,q^2a^2/bcde;q]_k} q^{\binom{k}{2}}.\end{aligned}$$

Letting $a = b = x$ and $c = d = e = y$ in Lemma 1, we can reformulate the resulting equation as follows.

THEOREM 2 (Reciprocal relation of well-poised series).

$$\begin{aligned}W(x,y) = {} & \sum_{k=0}^{\infty} \frac{(1-q^kx/y)(1-q^{2k+1}x/y^2)}{(1-x)(1-qx)} \frac{q^{\binom{k}{2}}}{(q;q)_{2k}} \\ & \times \frac{[y,y,y,q/y,q/y,q/y;q]_k}{[qx/y,qx/y,qx/y,q^2x/y^3;q]_k} \left(\frac{qx^2}{-y^3}\right)^k \\ & + \frac{x}{y} \sum_{k=0}^{\infty} \frac{(-q^2x^2/y^3)^k}{(1-x)(1-qx)} \frac{q^{\binom{k}{2}}}{(q;q)_{2k+1}} \\ & \times \frac{[y,y,q/y;q]_k[y,q/y,q/y;q]_{k+1}}{[qx/y,qx/y,qx/y,q^2x/y^3;q]_k}.\end{aligned}$$

Remark. According to the definition (2) of $W(x,y)$, the last equality holds for $|qx/y^3| < 1$. However, the series on the right hand side converges uniformly for both x and y except for the following poles

$$\{x=1,q^{-1}\}\cup\{y=0\}\cup\{x/y=q^{-m}\mid m\in\mathbf{N}\}\cup\{qx/y^3=q^{-n}\mid n\in\mathbf{N}\}.$$

Therefore, the formula displayed in Theorem 2 is valid, by analytic continuation, for all the x and y excluding the above poles. The same explanation applies also to the formulae appearing in Theorems 6 and 10.

The series in Theorem 2 can be further reduced so that q-analogues are derived for two well-known infinite series for $1/\pi$ and $1/\pi^2$.

Corollary 3 ($x=y=q^{1/2}$ in Theorem 2).

$$W(q^{1/2},q^{1/2})=\sum_{k=0}^{\infty}\frac{(-1)^k q^{k^2/2}}{(-q^{1/2};q^{1/2})_{2k}}\frac{(q^{1/2};q)_k^5}{(q;q)_k^5}\times\frac{1+q^{k+1/2}+q^{3k+1/2}+q^{4k+1}-4q^{2k+1/2}}{(1-q^{1/2})(1-q^{3/2})(1+q^{k+1/2})}.$$

This is the q-analogue of the following beautiful formula discovered first by Guillera (cf. Chu-Zhang [12, Example 5]).

Example 1 (Guillera [15, 17]).

$$\frac{8}{\pi^2}=\sum_{k=0}^{\infty}\left(\frac{-1}{4}\right)^k\begin{bmatrix}\frac12, & \frac12, & \frac12, & \frac12, & \frac12\\ 1, & 1, & 1, & 1, & 1\end{bmatrix}_k\{1+8k+20k^2\}.$$

Corollary 4 ($x=y^2=q$ in Theorem 2).

$$W(q,q^{1/2})=\sum_{k=0}^{\infty}\frac{(-q)^k q^{k^2/2}}{(-q^{1/2};q^{1/2})_{2k}}\frac{(q^{1/2};q)_k^3}{(q;q)_k(q^{3/2};q)_k^2}\frac{(1-q^{1/2})(1+q^{2k+1}+3q^{k+1/2})}{(1-q^2)(1+q^{1/2})(1+q^{k+1/2})}.$$

This is the q-analogue of another interesting formula.

Example 2 (cf. Chu-Zhang [12, Example 3]).

$$\frac{\pi^2}{10}=\sum_{k=0}^{\infty}\left(\frac{-1}{4}\right)^k\begin{bmatrix}\frac12, & \frac12, & \frac12\\ 1, & \frac32, & \frac32\end{bmatrix}_k=\sum_{k=0}^{\infty}\left(\frac{-1}{16}\right)^k\frac{\binom{2k}{k}}{(1+2k)^2}.$$

§2.2. Quadratic transformation from iteration pattern [20101]. Recall the following expression of $\Omega(a;b,c,d,e)$ in terms of quadratic series.

LEMMA 5 (Chu-Zhang [11, Theorem 9]: $|qa^2/bcde| < 1$).

$$\Omega(a;b,c,d,e) = \left[\begin{matrix} a/e, qa^2/bcd \\ a, qa^2/bcde \end{matrix}\middle| q\right]_1 \sum_{k=0}^{\infty} \frac{1-q^{3k+1}a^2/bcd}{1-qa^2/bcd}\left(\frac{q^k a^2}{ce}\right)^k$$
$$\times \frac{[c,e,qa/bc,qa/be,qa/cd,qa/de;q]_k}{[qa/b,qa/d,q^2a^2/bcde;q]_{2k}} \frac{(qa/bd;q)_{2k}}{[qa/c,a/e;q]_k}$$
$$+\frac{a}{e}\left[\begin{matrix} e, qa/bc, qa/bd, qa/cd, q^2a^2/bde \\ a, qa/b, qa/d, qa^2/bcde, q^2a^2/bcde \end{matrix}\middle| q\right]_1$$
$$\times \sum_{k=0}^{\infty} \frac{1-q^{3k+2}a^2/bde}{1-q^2a^2/bde} \frac{(q^2a/bd;q)_{2k}}{[qa/c,qa/e;q]_k}\left(\frac{q^{1+k}a^2}{ce}\right)^k$$
$$\times \frac{[c,qe,q^2a/bc,qa/be,q^2a/cd,qa/de;q]_k}{[q^2a/b,q^2a/d,q^3a^2/bcde;q]_{2k}}.$$

Specifying $a = c = x$ and $b = d = e = y$ in Lemma 5 and then making some simplifications, we may state the result in the following theorem.

THEOREM 6 (Reciprocal relation of quadratic series).

$$W(x,y) = \sum_{k=0}^{\infty} \frac{(1-x/y)(1-q^{3k+1}x/y^2)}{(1-x)(1-qx)}\left(\frac{q^k x}{y}\right)^k$$
$$\times \frac{[x,y,q/y,q/y,qx/y^2,qx/y^2;q]_k}{[qx/y,qx/y,q^2x/y^3;q]_{2k}} \frac{(qx/y^2;q)_{2k}}{[q,x/y;q]_k}$$
$$+\frac{x}{y}\sum_{k=0}^{\infty} \frac{1-q^{3k+2}x^2/y^3}{(1-x)(1-qx)} \frac{(qx/y^2;q)_{2k+1}}{[q,qx/y;q]_k}\left(\frac{q^{k+1}x}{y}\right)^k$$
$$\times \frac{[x,qx/y^2,qx/y^2;q]_k[y,q/y,q/y;q]_{k+1}}{[qx/y,qx/y,q^2x/y^3;q]_{2k+1}}.$$

This theorem can be analogously utilized to derive two elegant q-series formulae. The first one is given in the following corollary.

COROLLARY 7 ($x = y = q^{1/2}$ in Theorem 6).

$$W(q^{1/2},q^{1/2}) = \sum_{k=0}^{\infty} q^{k^2+k} \frac{(q^{1/2};q)_k^3(q^{1/2};q)_{2k}}{(q;q)_k^5(-q^{1/2};q^{1/2})_{2k+1}^3} \frac{(1-q^{2k+1/2})(1-q^{3k+3/2})}{(1-q^{1/2})(1-q^{3/2})}$$
$$\times\left\{1+q^{-k}\frac{(1-q^k)(1-q^{3k+1/2})(1+q^{k+1/2})^3}{(1-q^{2k+1/2})(1-q^{3k+3/2})}\right\}.$$

This is the q-analogue of the following infinite series identity originally found by Guillera (cf. Chu-Zhang [12, Example 16]).

Example 3 (Guillera [15, 17]).

$$\frac{32}{\pi^2}=\sum_{k=0}^{\infty}\left(\frac{1}{16}\right)^k\begin{bmatrix}\frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{4}, & \frac{3}{4}\\ 1, & 1, & 1, & 1, & 1\end{bmatrix}_k\{3+34k+120k^2\}.$$

The second one is deduced by putting $x=y^2=q$ in Theorem 6. The corresponding euqation can be expressed as

$$W(q,q^{1/2})=\sum_{k=0}^{\infty}\frac{(q^{1/2};q^{1/2})_{2k}^3}{(q^{3/4};q^{1/2})_{2k}^3}\frac{(-q^{1/2};q^{1/2})_{2k}}{(-q^{3/4};q^{1/2})_{2k}^3}\frac{q^{k^2+k/2}(1-q^{3k+1})}{(1-q^2)(1+q^{1/2})}$$
$$\times\left\{1+q^{k+1/2}\frac{(1+q^{k+1/2})(1-q^{k+1/2})^3(1-q^{3k+5/2})}{(1-q^{2k+3/2})^3(1-q^{3k+1})}\right\}.$$

For the sequence $\{\Lambda(k)\}$ defined by the quotient of shifted factorials

$$\Lambda(k):=\frac{(q^{1/2};q^{1/2})_k^3}{(q^{3/4};q^{1/2})_k^3}\frac{(-q^{1/2};q^{1/2})_k}{(-q^{3/4};q^{1/2})_k^3}\frac{q^{(1/2)\binom{k+1}{2}}(1-q^{3k/2+1})}{(1+q^{1/2})(1-q^2)}$$

it is not hard to verify that the last series can be expressed as

$$\sum_{k\geq 0}\Lambda(k)=\sum_{k\geq 0}\{\Lambda(2k)+\Lambda(2k+1)\}=\sum_{k\geq 0}\Lambda(2k)\left\{1+\frac{\Lambda(2k+1)}{\Lambda(2k)}\right\}$$

in view of the bisection series. This leads to the following elegant formula.

Corollary 8 ($x=y^2=q$ in Theorem 6: Bisection series).

$$W(q,q^{1/2})=\sum_{k=0}^{\infty}\frac{(q^{1/2};q^{1/2})_k^3}{(q^{3/4};q^{1/2})_k^3}\frac{(-q^{1/2};q^{1/2})_k}{(-q^{3/4};q^{1/2})_k^3}\frac{q^{(1/2)\binom{k+1}{2}}(1-q^{3k/2+1})}{(1+q^{1/2})(1-q^2)}.$$

This is the q-analogue of another formula (cf. Chu-Zhang [12, Example 73]).

Example 4 (Guillera [18, §2.1]).

$$\frac{\pi^2}{4}=\sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^k\begin{bmatrix}1, & 1, & 1\\ \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}\end{bmatrix}_k\{2+3k\}.$$

§2.3. Quadratic transformation from iteration pattern [31111]. There is another expression of $\Omega(a;b,c,d,e)$ in terms of quadratic series that we record it below.

LEMMA 9 (Chu-Zhang [11, Theorem 17]: $|qa^2/bcde| < 1$).

$$\Omega(a;b,c,d,e)$$

$$= \frac{[a/e, qa^2/bcd;q]_1}{[a, qa^2/bcde;q]_1} \sum_{k=0}^{\infty} \frac{1-q^{3k+1}a^2/bcd}{1-qa^2/bcd} \left(-\frac{q^3a^3}{bcde}\right)^k$$

$$\times \frac{[b,c,d,e,qa/bc,qa/bd,qa/be,qa/cd,qa/ce,qa/de;q]_k}{[qa/b,qa/c,qa/d,a/e,q^2a^2/bcde;q]_{2k}} q^{5\binom{k}{2}}$$

$$+ \frac{a}{e} \frac{[e,qa/bc,qa/bd,qa/cd,q^2a^2/bde;q]_1}{[a,qa/b,qa/d;q]_1 (qa^2/bcde;q)_2} \sum_{k=0}^{\infty} \frac{1-q^{3k+2}a^2/bde}{1-q^2a^2/bde} q^{5\binom{k}{2}}$$

$$\times \frac{[b,c,d,qe,q^2a/bc,q^2a/bd,qa/be,q^2a/cd,qa/ce,qa/de;q]_k}{[q^2a/b,qa/c,q^2a/d,qa/e,q^3a^2/bcde;q]_{2k}} \left(-\frac{q^5a^3}{bcde}\right)^k$$

$$+ \frac{qa^2}{ce} \frac{[c,e,qa/bc,qa/bd,qa/be,qa/cd,qa/de;q]_1}{[a,qa/b,qa/c,qa/d,qa/e;q]_1 (qa^2/bcde;q)_2} \sum_{k=0}^{\infty} \left(-\frac{q^7a^3}{bcde}\right)^k q^{5\binom{k}{2}}$$

$$\times \frac{[b,qc,d,qe,q^2a/bc,q^2a/bd,q^2a/be,q^2a/cd,qa/ce,q^2a/de;q]_k}{[q^2a/b,q^2a/c,q^2a/d,q^2a/e,q^3a^2/bcde;q]_{2k}}.$$

For $a = d = x$ and $b = c = e = y$ in Lemma 9, the last transformation can be simplified into the following one.

THEOREM 10 (Reciprocal relation of quadratic series).

$$W(x,y) = \sum_{k=0}^{\infty} \frac{(1-q^{2k}x/y)(1-q^{3k+1}x/y^2)}{(1-x)(1-qx)} \left(\frac{q^3x^2}{-y^3}\right)^k q^{5\binom{k}{2}}$$

$$\times \frac{[x,y,y,y,q/y,q/y,q/y,qx/y^2,qx/y^2,qx/y^2;q]_k}{[q,qx/y,qx/y,qx/y,q^2x/y^3;q]_{2k}}$$

$$+ \frac{x}{y} \sum_{k=0}^{\infty} \frac{(1-q^{3k+2}x/y^2)}{(1-x)(1-qx)} \frac{[y,q/y,q/y,qx/y^2;q]_{k+1}}{[q,qx/y,q^2x/y^3;q]_{2k+1}}$$

$$\times \frac{[x,y,y,q/y,qx/y^2,qx/y^2;q]_k}{[qx/y,qx/y;q]_{2k}} \left(\frac{q^5x^2}{-y^3}\right)^k q^{5\binom{k}{2}}$$

$$+ \frac{qx^2}{y^2} \sum_{k=0}^{\infty} \frac{[x,y,qx/y^2;q]_k}{(1-x)(1-qx)} \left(\frac{q^7x^2}{-y^3}\right)^k q^{5\binom{k}{2}}$$

$$\times \frac{[y,y,q/y,q/y,q/y,qx/y^2,qx/y^2;q]_{k+1}}{[q,qx/y,qx/y,qx/y,q^2x/y^3;q]_{2k+1}}.$$

Similarly, this theorem can be specialized to two interesting q-series formulae.

Corollary 11 ($x = y = q^{1/2}$ in Theorem 10).

$$W(q^{1/2}, q^{1/2}) = \sum_{k=0}^{\infty}(-1)^k \frac{(q^{1/2};q)_k^{10}}{(q;q)_{2k}^5} \frac{q^{(5k^2+4k)/2}(1-q^{k+1/2})(1-q^{3k+3/2})}{(1+q^{k+1/2})^3(1-q^{1/2})(1-q^{3/2})}$$
$$\times \left\{1 - \frac{(1+q^{k+1/2})^3(1-q^{-2k})(1-q^{3k+1/2})}{(1-q^{k+1/2})(1-q^{3k+3/2})} + \frac{q^{2k+1}(1-q^{k+1/2})}{(1+q^{k+1/2})^2(1-q^{3k+3/2})}\right\}.$$

This is the q-analogue of the following formula also due to Guillera (cf. Chu-Zhang [12, Example 64]).

Example 5 (Guillera [15, 17]).

$$\frac{128}{\pi^2} = \sum_{k=0}^{\infty}\left(\frac{-1}{1024}\right)^k \begin{bmatrix} \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ 1, & 1, & 1, & 1, & 1 \end{bmatrix}_k \{13 + 180k + 820k^2\}.$$

Corollary 12 ($x = y^2 = q$ in Theorem 10).

$$W(q, q^{1/2}) = \sum_{k=0}^{\infty}(-q)^k q^{5k^2/2} \frac{(q;q)_k^4 (q^{1/2};q)_k^6}{(q;q)_{2k}(q^{3/2};q)_{2k}^4} \frac{(1-q^{2k+1/2})(1-q^{3k+1})}{(1-q)(1-q^2)}$$
$$\times \left\{1 + \frac{q^{3k+1/2}(1-q^{k+1})(1-q^{k+1/2})^3(1-q^{3k+2})}{(1-q^{2k+1})(1-q^{2k+1/2})(1-q^{2k+3/2})^2(1-q^{3k+1})} + \frac{q^{5k+2}(1-q^{k+1})^2(1-q^{k+1/2})^5}{(1-q^{2k+1})(1-q^{2k+1/2})(1-q^{2k+3/2})^4(1-q^{3k+1})}\right\}.$$

When $q \to 1$, this corollary gives rise to a new infinite series identity below.

Example 6 (Ramanujan-like series for π^2).

$$\frac{81\pi^2}{16} = \sum_{k=0}^{\infty}\left(\frac{-1}{1024}\right)^k \begin{bmatrix} 1, & 1, & 1, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ \frac{5}{4}, & \frac{5}{4}, & \frac{5}{4}, & \frac{5}{4}, & \frac{7}{4}, & \frac{7}{4}, & \frac{7}{4}, & \frac{7}{4} \end{bmatrix}_k$$
$$\times \{50 + 587k + 2762k^2 + 6664k^3 + 8738k^4 + 5936k^5 + 1640k^6\}.$$

§2.4. Limiting expressions. Letting $e \to \infty$ in Lemmas 1, 5 and 9, we get the following three reduced reciprocal relations.

PROPOSITION 13 ($a = c = x,\ b = d = y$ and $e \to \infty$ in Lemma 1).

$$\left[\begin{matrix} qx, qx/y^2 \\ qx/y, qx/y \end{matrix}\,\middle|\, q\right]_\infty = \sum_{k=0}^{\infty} \frac{1-q^{2k+1}x/y^2}{1-x}\left(\frac{q^k x}{y}\right)^k \frac{[x, y, q/y, qx/y^2; q]_k}{[q, qx/y; q]_k (qx/y; q)_{2k}}$$
$$- x\sum_{k=0}^{\infty} \frac{[x, y; q]_k [q/y, qx/y^2; q]_{k+1}}{[q, qx/y; q]_k (qx/y; q)_{2k+1}} \frac{(q^{2+k}x/y)^k}{1-x}.$$

PROPOSITION 14 ($a = b = x,\ c = d = y$ and $e \to \infty$ in Lemma 5).

$$\left[\begin{matrix} qx, qx/y^2 \\ qx/y, qx/y \end{matrix}\,\middle|\, q\right]_\infty = \sum_{k=0}^{\infty} \frac{1-q^{3k+1}x/y^2}{1-x}\left(\frac{qx^2}{-y}\right)^k \frac{[y, q/y, qx/y^2; q]_k (q/y; q)_{2k}}{(qx/y; q)_k [q, qx/y; q]_{2k}} q^{3\binom{k}{2}}$$
$$- x\sum_{k=0}^{\infty} \frac{(-q^3x^2/y)^k}{1-x} \frac{(y; q)_k [q/y, qx/y^2; q]_{k+1} (q/y; q)_{2k+1}}{(qx/y; q)_k [q, qx/y; q]_{2k+1}} q^{3\binom{k}{2}}.$$

PROPOSITION 15 ($a = d = x,\ b = c = y$ and $e \to \infty$ in Lemma 9).

$$\left[\begin{matrix} qx, qx/y^2 \\ qx/y, qx/y \end{matrix}\,\middle|\, q\right]_\infty = \sum_{k=0}^{\infty} \frac{1-q^{3k+1}x/y^2}{1-x}\left(\frac{q^{3k}x^2}{y^2}\right)^k \frac{[x, y, y, q/y, q/y, qx/y^2; q]_k}{[q, qx/y, qx/y; q]_{2k}}$$
$$- x\sum_{k=0}^{\infty} \frac{(q^{3+3k}x^2/y^2)^k}{1-x} \frac{[x, y, y; q]_k [q/y, q/y, qx/y^2; q]_{k+1}}{(qx/y; q)_{2k} [q, qx/y; q]_{2k+1}}$$
$$- \frac{qx^2}{y}\sum_{k=0}^{\infty} \frac{(q^{5+3k}x^2/y^2)^k}{1-x} \frac{[x, y; q]_k [y, q/y, q/y, qx/y^2; q]_{k+1}}{[q, qx/y, qx/y; q]_{2k+1}}.$$

In view of the limiting relation

$$\lim_{q\to 1^-}\left[\begin{matrix} q, & q^2 \\ q^{3/2}, & q^{3/2} \end{matrix}\,\middle|\, q\right]_\infty = \lim_{q\to 1^-} \frac{\Gamma_q\left(\frac{3}{2}\right)\Gamma_q\left(\frac{3}{2}\right)}{\Gamma_q(1)\Gamma_q(2)} = \frac{\pi}{4}$$

we recover the following three infinite series expression for π.

Example 7 ($x = y^2 = q$ and $q \to 1$ in Proposition 13: Chu [10, Example 3]).

$$\frac{3\pi}{4} = \sum_{k=0}^{\infty} \left[\begin{matrix} 1, & \frac{1}{2} \\ \frac{5}{4}, & \frac{7}{4} \end{matrix}\right]_k \frac{2+3k}{4^k}.$$

Example 8 ($x = y^2 = q$ and $q \to 1$ in Proposition 14: Chu-Zhang [10, Example 76]).

$$\frac{3\pi}{2} = \sum_{k=0}^{\infty} \left[\begin{matrix} \frac{1}{2}, & \frac{1}{4}, & \frac{3}{4} \\ \frac{3}{2}, & \frac{5}{4}, & \frac{7}{4} \end{matrix}\right]_k \frac{5+21k+20k^2}{(-4)^k}.$$

Example 9 ($x = y^2 = q$ and $q \to 1$ in Proposition 15: Chu [10, Example 50]).

$$\frac{9\pi}{4} = \sum_{k=0}^{\infty} \begin{bmatrix} 1, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ \frac{5}{4}, & \frac{5}{4}, & \frac{7}{4}, & \frac{7}{4} \end{bmatrix}_k \frac{7 + 42k + 75k^2 + 42k^3}{64^k}.$$

We remark that the series in Example 8 is equivalent to the formula of BBP-type (see also Examples 29 and 30) due to Adamchik and Wagon [1]:

$$\frac{3\pi}{2} = \sum_{k=0}^{\infty} \left(\frac{-1}{4}\right)^k \left\{\frac{1}{1+4k} + \frac{2}{2+4k} + \frac{1}{3+4k}\right\}.$$

Further formulae of BBP-type can be found in the papers [1, 2, 6, 21].

Instead, letting $x = y = q^\lambda$ in Propositions 13, 14 and 15, and then taking into account of the limiting relation

$$\lim_{q \to 1^-} \left[\begin{matrix} q^{1+\lambda}, & q^{1-\lambda} \\ q, & q \end{matrix} \middle| q\right]_\infty = \lim_{q \to 1^-} \frac{\Gamma_q(1)\Gamma_q(1)}{\Gamma_q(1+\lambda)\Gamma_q(1-\lambda)} = \frac{\sin \pi\lambda}{\pi\lambda}$$

we find the following three infinite series identities containing a free parameter λ.

Corollary 16 ($x = y = q^\lambda$ and $q \to 1$ in Proposition 13).

$$\frac{\sin \pi\lambda}{\pi} = \sum_{k=0}^{\infty} \frac{\lambda - \lambda^2 + k(2+3k)}{4^k} \begin{bmatrix} \lambda, & \lambda, & 1-\lambda, & 1-\lambda \\ 1, & 1, & 1, & \frac{3}{2} \end{bmatrix}_k.$$

Corollary 17 ($x = y = q^\lambda$ and $q \to 1$ in Proposition 14).

$$\frac{\sin \pi\lambda}{\pi} = \sum_{k=0}^{\infty} \begin{bmatrix} \lambda, & 1-\lambda, & 1-\lambda, & \frac{1-\lambda}{2}, & \frac{2-\lambda}{2} \\ 1, & 1, & 1, & \frac{3}{2}, & \frac{3}{2} \end{bmatrix}_k$$
$$\times \frac{(1+3k-\lambda)(1+2k)^2 - (1+2k-\lambda)(1+k-\lambda)^2}{(-4)^k}.$$

Corollary 18 ($x = y = q^\lambda$ and $q \to 1$ in Proposition 15).

$$\frac{\sin \pi\lambda}{\pi} = \sum_{k=0}^{\infty} \begin{bmatrix} \lambda, & \lambda, & \lambda, & 1-\lambda, & 1-\lambda, & 1-\lambda \\ 1, & 1, & 1, & \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2} \end{bmatrix}_k$$
$$\times \frac{(1+3k-\lambda)(1+2k)^3 - (1+3k+\lambda)(1+k-\lambda)^3}{64^k}.$$

These formulae can be considered as λ-parameter extensions of three Ramanujan's identities for $1/\pi$. In fact, they reduce, when $\lambda = 1/2$, to the three typical known formulae due to Ramanujan [19, 1914].

Example 10 (Ramanujan [19, Equation 28]: cf. Chu-Zhang [12, Example 9]).

$$\frac{4}{\pi}=\sum_{k=0}^{\infty}\begin{bmatrix}\frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}\\ 1, & 1, & 1\end{bmatrix}_k\frac{1+6k}{4^k}.$$

Example 11 (Ramanujan [19, Equation 35]: cf. Chu-Zhang [12, Example 10]).

$$\frac{8}{\pi}=\sum_{k=0}^{\infty}\begin{bmatrix}\frac{1}{2}, & \frac{1}{4}, & \frac{3}{4}\\ 1, & 1, & 1\end{bmatrix}_k\frac{3+20k}{(-4)^k}.$$

Example 12 (Ramanujan [19, Equation 29]: cf. Chu-Zhang [12, Example 116]).

$$\frac{16}{\pi}=\sum_{k=0}^{\infty}\begin{bmatrix}\frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}\\ 1, & 1, & 1\end{bmatrix}_k\frac{5+42k}{64^k}.$$

3. Cubic transformation and further formulae

In the same paper [11], Chu and Zhang established, under the iteration patterns **[30111]** and **[41111]**, two further expressions of $\Omega(a;b,c,d,e)$ in terms of cubic and quartic series in Theorem 11 and Theorem 13. The latter does not produce interesting results due to its complexity. However from the former one, we can derive, through multisection series, several remarkable infinite series identities, that are not covered in the compendium by Chu-Zhang [12].

First we record the following transformation of cubic series.

LEMMA 19 (Chu-Zhang [11, Theorem 11]: $|qa^2/bcde|<1$).

$$\begin{aligned}\Omega(a;b,c,d,e)=&\left[\begin{matrix}a/e, qa^2/bcd\\ a, qa^2/bcde\end{matrix}\,\middle|\, q\right]_1\sum_{k=0}^{\infty}\frac{[c,d,e,qa/cd,qa/ce,qa/de;q]_k}{[qa/b,q^2a^2/bcde;q]_{3k}}\\&\times\left[\begin{matrix}qa/bc, & qa/bd, & qa/be\\ qa/c, & qa/d, & a/e\end{matrix}\,\middle|\, q\right]_{2k}\frac{1-q^{4k+1}a^2/bcd}{1-qa^2/bcd}\left(\frac{q^{3k}a^3}{cde}\right)^k\\&+\frac{a}{e}\left[\begin{matrix}e,qa/bc,qa/bd,qa/cd,q^2a^2/bde\\ a,qa/b,qa/d,qa^2/bcde,q^2a^2/bcde\end{matrix}\,\middle|\, q\right]_1\\&\times\sum_{k=0}^{\infty}\frac{1-q^{4k+2}a^2/bde}{1-q^2a^2/bde}\left[\begin{matrix}q^2a/bc, & q^2a/bd, & qa/be\\ qa/c, & q^2a/d, & qa/e\end{matrix}\,\middle|\, q\right]_{2k}\\&\times\frac{[c,d,qe,q^2a/cd,qa/ce,qa/de;q]_k}{[q^2a/b,q^3a^2/bdce;q]_{3k}}\left(\frac{q^{3k+2}a^3}{cde}\right)^k\end{aligned}$$

$$+\frac{qa^2}{ce}\frac{[c,e,qa/bc,qa/be,qa/cd,qa/de,q^3a^2/bce;q]_1(qa/bd;q)_2}{[a,qa/c,qa/d,qa/e;q]_1(qa/b;q)_2(qa^2/bcde;q)_3}$$
$$\times\sum_{k=0}^{\infty}\frac{1-q^{4k+3}a^2/bce}{1-q^3a^2/bce}\left[\begin{matrix}q^2a/bc, & q^3a/bd, & q^2a/be\\ q^2a/c, & q^2a/d, & q^2a/e\end{matrix}\middle| q\right]_{2k}$$
$$\times\frac{[qc,d,qe,q^2a/cd,qa/ce,q^2a/de;q]_k}{[q^3a/b,q^4a^2/bcde;q]_{3k}}\left(\frac{q^{3k+4}a^3}{cde}\right)^k.$$

For this lemma, we shall not examine its particular cases of q-series in order to avoid lengthy expressions. However its limiting case $q\to 1$ can be written, after some routine simplifications, as the following transformation formula, that will be utilized to derive several remarkable infinite series identities.

THEOREM 20. *For five complex parameters $\{a,b,c,d,e\}$ satisfying the condition $\Re(1+2a-b-c-d-e)>0$, there holds the transformation formula*

$$\sum_{k=0}^{\infty}(a+2k)\left[\begin{matrix}b, & c, & d, & e\\ 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e\end{matrix}\right]_k$$
$$=\sum_{k=0}^{\infty}\mathscr{P}(k)\frac{[1+a-b-c,1+a-b-d,1+a-b-e]_{2k}}{[1+a-c,1+a-d,1+a-e]_{2k+1}}$$
$$\times\frac{[c,d,e,1+a-c-d,1+a-c-e,1+a-d-e]_k}{(1+a-b)_{3k+2}(1+2a-b-c-d-e)_{3k+3}}$$

where $\mathscr{P}(k)$ is a polynomial of degree 9 in k, given explicitly by

$$\begin{aligned}\mathscr{P}(k):=&(1+2a-b-c-d+4k)(1+a-c+2k)\\ &\times(1+a-d+2k)(a-e+2k)(1+a-e+2k)\\ &\times(1+a-b+3k)(2+a-b+3k)(2+2a-b-c-d-e+3k)\\ &\times(3+2a-b-c-d-e+3k)\\ &+(2+2a-b-d-e+4k)(e+k)(1+a-c-d+k)\\ &\times(1+a-c+2k)(1+a-e+2k)\\ &\times(1+a-b-c+2k)(1+a-b-d+2k)\\ &\times(2+a-b+3k)(3+2a-b-c-d-e+3k)\\ &+(3+2a-b-c-e+4k)(c+k)(e+k)\\ &\times(1+a-c-d+k)(1+a-d-e+k)\\ &\times(1+a-b-c+2k)(1+a-b-d+2k)\\ &\times(2+a-b-d+2k)(1+a-b-e+2k).\end{aligned}$$

Remark. When one of parameters $\{b,c,d,e\}$ is equal to a (for example $e=a$) in the last theorem, the sum on the left can be evaluated by Dougall's formula (cf. [3, §4.4]) as the Γ-function quotient

$$\Gamma\left[\begin{matrix} 1+a-b, 1+a-c, 1+a-d, 1+a-b-c-d \\ a, 1+a-b-c, 1+a-b-d, 1+a-c-d \end{matrix}\right].$$

This is analytic in the whole complex space of dimension 4 except for the hyperplanes determined by

$$\{b-a\in\mathbf{N}\}\cup\{c-a\in\mathbf{N}\}\cup\{d-a\in\mathbf{N}\}\cup\{b+c+d-a\in\mathbf{N}\}$$

which is covered also by the convergence domain of the infinite series displayed on the right hand side of Theorem 20. Therefore in this case, the parameter restriction $\Re(1+2a-b-c-d-e)>0$ can be removed by analytic continuation, which justify the validity of Examples 18 and 22 where the parameter settings don't obey the condition $\Re(1+2a-b-c-d-e)>0$.

By specifying the parameters in Theorem 20 and then considering eventually bisection and trisection series, we find numerous new formulae of Ramanujan-type concerning π, $\zeta(3)$ and the Catalan constant G. The selected examples are displayed below with parameter settings and references being highlighted in the headers.

Example 13 ($a=2,\ b=c=d=e=1$: Chu-Zhang [12, Example 118]).

$$144\zeta(3)=\sum_{k=0}^{\infty}\left[\begin{matrix} 1, & 1, & 1, & 1 \\ \frac{4}{3}, & \frac{4}{3}, & \frac{5}{3}, & \frac{5}{3} \end{matrix}\right]_k \frac{173+501k+364k^2}{(1+2k)(729)^k}.$$

Example 14 ($a=1,\ b=c=d=e=\frac{1}{2}$: Bisection series).

$$\frac{21}{2}\zeta(3)=\sum_{k=0}^{\infty}\left[\begin{matrix} 1, & 1, & 1, & 1, & 1 \\ \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}, & \frac{4}{3}, & \frac{5}{3} \end{matrix}\right]_k \frac{13+38k+28k^2}{(-27)^k}.$$

This series has been conjectured experimentally by Sun [20, Conjecture 8].

Example 15 ($a=c=1,\ d=\frac{1}{3},\ e=\frac{2}{3},\ b\to-\infty$: Trisection series [13, p. 89]).

$$\frac{2\pi}{3\sqrt{3}}=\sum_{k=0}^{\infty}\frac{(1)_k}{(3/2)_k}\left(\frac{1}{4}\right)^k \quad \text{Apéry series:}\quad \frac{\pi}{3\sqrt{3}}=\sum_{n=1}^{\infty}\frac{1}{n\binom{2n}{n}}.$$

Example 16 ($a=c=d=e=\frac{1}{4},\ b\to-\infty$: Bisection series [12, Example 87]).

$$\frac{2\sqrt{2}}{\pi}=\sum_{k=0}^{\infty}\left[\begin{matrix} \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ 1, & 1, & 1 \end{matrix}\right]_k \frac{1+6k}{(-8)^k}.$$

Example 17 ($a=c=d=e=\frac{1}{2},\ b\to-\infty$: Bisection series [12, Example 88]).

$$2G=\sum_{k=0}^{\infty}\begin{bmatrix}1, & 1, & 1\\ \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}\end{bmatrix}_k\frac{2+3k}{(-8)^k}.$$

Example 18 ($a=b=c=\frac{1}{2},\ d=\frac{1}{6},\ e=\frac{5}{6}$: Trisection series [19, Equation 40]).

$$\frac{2\sqrt{3}}{\pi}=\sum_{k=0}^{\infty}\begin{bmatrix}\frac{1}{2}, & \frac{1}{4}, & \frac{3}{4}\\ 1, & 1, & 1\end{bmatrix}_k\frac{1+8k}{9^k}.$$

Example 19 ($a=c=1,\ b=\frac{1}{2},\ d=\frac{1}{3},\ e=\frac{2}{3}$: Trisection series).

$$\pi\sqrt{3}=\sum_{k=0}^{\infty}\begin{bmatrix}1, & \frac{3}{4}, & \frac{5}{4}\\ \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}\end{bmatrix}_k\frac{5+8k}{9^k}.$$

Example 20 ($a=2,\ b=c=d=1,\ e\to-\infty$: Chu-Zhang [12, Example 24]).

$$\frac{\pi^2}{2}=\sum_{k=0}^{\infty}\begin{bmatrix}1, & 1, & \frac{1}{2}\\ \frac{3}{2}, & \frac{4}{3}, & \frac{5}{3}\end{bmatrix}_k\frac{5+7k}{(-27)^k}.$$

Example 21 ($a=b=1,\ c=d=\frac{1}{2},\ e\to-\infty$: [10, Example 31] & [12, Example 27]).

$$\frac{15\pi}{8}=\sum_{k=0}^{\infty}\begin{bmatrix}1, & \frac{1}{2}\\ \frac{7}{6}, & \frac{11}{6}\end{bmatrix}_k\frac{6+7k}{(-27)^k}.$$

Example 22 ($a=b=c=d=\frac{5}{6},\ e=\frac{2}{3}$: Bisection series).

$$\frac{9\sqrt{3}}{2^{4/3}\pi}=\sum_{k=0}^{\infty}\begin{bmatrix}\frac{1}{3}, & \frac{2}{3}, & \frac{1}{6}\\ 1, & 1, & 1\end{bmatrix}_k\frac{2+21k}{(-27)^k}.$$

Example 23 ($a=b=c=d=\frac{1}{6},\ e=\frac{1}{3}$: Bisection series).

$$\frac{27\sqrt{3}}{2^{5/3}\pi}=\sum_{k=0}^{\infty}\begin{bmatrix}\frac{1}{3}, & \frac{2}{3}, & \frac{5}{6}\\ 1, & 1, & 1\end{bmatrix}_k\frac{5+42k}{(-27)^k}.$$

Example 24 ($a=c=\frac{1}{2},\ b=\frac{3}{4},\ d=\frac{1}{4},\ e\to-\infty$).

$$\frac{21}{\sqrt{2}}=\sum_{k=0}^{\infty}\begin{bmatrix}\frac{1}{4}, & \frac{3}{4}, & \frac{3}{4}\\ 1, & \frac{11}{12}, & \frac{19}{12}\end{bmatrix}_k\frac{15+28k}{(-27)^k}.$$

Example 25 ($a = c = \frac{1}{2},\ b = \frac{1}{4},\ d = \frac{3}{4},\ e \to -\infty$).

$$\frac{15}{\sqrt{2}} = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{4}, & \frac{1}{4}, & \frac{3}{4} \\ 1, & \frac{13}{12}, & \frac{17}{12} \end{bmatrix}_k \frac{11 + 124k + 224k^2}{(-27)^k}.$$

Example 26 ($a = c = d = e = \frac{1}{4},\ b = \frac{1}{2}$: Bisection series).

$$\frac{3}{\pi} = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ 1, & 1, & 1, & \frac{5}{6}, & \frac{7}{6} \end{bmatrix}_k \frac{1 + 10k + 28k^2}{(-27)^k}.$$

Example 27 ($a = c = 1,\ b = \frac{5}{6},\ d = \frac{1}{3},\ e = \frac{1}{6}$: Bisection series).

$$\frac{1036 \cdot 2^{2/3}\pi}{999\sqrt{3}} = \sum_{k=0}^{\infty} \begin{bmatrix} 2, & \frac{1}{3}, & \frac{1}{6} \\ \frac{10}{9}, & \frac{13}{9}, & \frac{16}{9} \end{bmatrix}_k \frac{3 + 7k}{(-27)^k}.$$

Example 28 ($a = c = 1,\ b = \frac{1}{6},\ d = \frac{2}{3},\ e = \frac{5}{6}$: Bisection series).

$$\frac{80 \cdot 2^{1/3}\pi}{9\sqrt{3}} = \sum_{k=0}^{\infty} \begin{bmatrix} 1, & \frac{2}{3}, & \frac{5}{6} \\ \frac{11}{9}, & \frac{14}{9}, & \frac{17}{9} \end{bmatrix}_k \frac{21 + 61k + 42k^2}{(-27)^k}.$$

Example 29 ($a = b = 1,\ c = \frac{1}{2},\ d = \frac{1}{3},\ e = \frac{2}{3}$: Bisection series).

$$2\pi\sqrt{3} = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{2}, & \frac{1}{3}, & \frac{2}{3} \\ \frac{3}{2}, & \frac{4}{3}, & \frac{5}{3} \end{bmatrix}_k \frac{11 + 44k + 42k^2}{(-27)^k}.$$

This is equivalent to the following BBP-type formula:

$$\pi\sqrt{3} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \left\{ \frac{2}{1 + 2k} + \frac{3}{1 + 3k} + \frac{1}{2 + 3k} \right\}.$$

Example 30 ($a = b = 1,\ c = \frac{1}{2},\ d = \frac{1}{6},\ e = \frac{5}{6}$: Bisection series).

$$\frac{45\pi}{2\sqrt{3}} = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{2}, & \frac{1}{6}, & \frac{5}{6} \\ \frac{3}{2}, & \frac{7}{6}, & \frac{11}{6} \end{bmatrix}_k \frac{41 + 116k + 84k^2}{(-27)^k}.$$

This is equivalent to another BBP-type formula:

$$\frac{9\pi}{2\sqrt{3}} = \sum_{k=0}^{\infty} \left(\frac{-1}{27}\right)^k \left\{ \frac{9}{1 + 6k} - \frac{3}{3 + 6k} + \frac{1}{5 + 6k} \right\}.$$

Example 31 ($a = c = \frac{7}{12},\ b = \frac{1}{3},\ d = e = \frac{1}{4}$: Bisection series).

$$\frac{15\pi}{4\sqrt{3}} = \sum_{k=0}^{\infty} \begin{bmatrix} 1, & \frac{1}{2}, & \frac{2}{3} \\ \frac{4}{3}, & \frac{4}{3}, & \frac{11}{6} \end{bmatrix}_k \frac{7 + 26k + 21k^2}{(-27)^k}.$$

Example 32 $(a=c=\frac{11}{12},\ b=\frac{2}{3},\ d=e=\frac{1}{4}$: Bisection series).

$$\frac{6\pi}{\sqrt{3}}=\sum_{k=0}^{\infty}\begin{bmatrix}1, & \frac{1}{2}, & \frac{1}{3}\\ \frac{5}{3}, & \frac{5}{3}, & \frac{7}{6}\end{bmatrix}_k\frac{11+31k+21k^2}{(-27)^k}.$$

Example 33 $(a=\frac{3}{2},\ b=c=d=1,\ e\to-\infty$: Chu-Zhang [12, Example 29]).

$$30G=\sum_{k=0}^{\infty}\begin{bmatrix}1, & 1\\ \frac{7}{6}, & \frac{11}{6}\end{bmatrix}_k\frac{83+192k+112k^2}{(1+4k)(3+4k)(-27)^k}.$$

Example 34 $(a=b=1,\ c=\frac{1}{2},\ d=\frac{1}{4},\ e=\frac{3}{4}$: Bisection series).

$$2\pi=\sum_{k=0}^{\infty}\begin{bmatrix}\frac{1}{2}, & \frac{1}{2}\\ \frac{4}{3}, & \frac{5}{3}\end{bmatrix}_k\frac{19+107k+192k^2+112k^3}{(1+4k)(3+4k)(-27)^k}.$$

Example 35 $(a=c=1,\ b=\frac{1}{3},\ d=\frac{2}{3},\ e\to-\infty)$.

$$\frac{160\pi}{9\sqrt{3}}=\sum_{k=0}^{\infty}\begin{bmatrix}1, & \frac{1}{3}, & \frac{1}{3}, & \frac{5}{6}\\ \frac{7}{6}, & \frac{11}{9}, & \frac{14}{9}, & \frac{17}{9}\end{bmatrix}_k\frac{33+227k+438k^2+252k^3}{(-27)^k}.$$

Example 36 $(a=c=1,\ b=\frac{2}{3},\ d=\frac{1}{3},\ e\to-\infty)$.

$$\frac{280\pi}{9\sqrt{3}}=\sum_{k=0}^{\infty}\begin{bmatrix}1, & \frac{2}{3}, & \frac{2}{3}, & \frac{1}{6}\\ \frac{11}{6}, & \frac{10}{9}, & \frac{13}{9}, & \frac{16}{9}\end{bmatrix}_k\frac{57+308k+498k^2+252k^3}{(-27)^k}.$$

Example 37 $(a=2,\ c=d=e=1,\ b\to-\infty$: Chu-Zhang [12, Example 114]).

$$\frac{4\pi^2}{3}=\sum_{k=0}^{\infty}\begin{bmatrix}1, & 1, & 1\\ \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}\end{bmatrix}_k\frac{13+21k}{64^k}.$$

Example 38 $(a=d=1,\ c=\frac{1}{3},\ e=\frac{2}{3},\ b\to-\infty$: Chu-Zhang [12, Example 115]).

$$\frac{40\pi}{9\sqrt{3}}=\sum_{k=0}^{\infty}\begin{bmatrix}1, & \frac{1}{3}, & \frac{2}{3}\\ \frac{3}{2}, & \frac{7}{6}, & \frac{11}{6}\end{bmatrix}_k\frac{8+27k+21k^2}{64^k}.$$

Example 39 $(a=d=1,\ c=e=\frac{1}{2},\ b\to-\infty$: Chu [10, Example 50]).

$$\frac{9\pi}{4}=\sum_{k=0}^{\infty}\begin{bmatrix}1, & \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2}\\ \frac{5}{4}, & \frac{5}{4}, & \frac{7}{4}, & \frac{7}{4}\end{bmatrix}_k\frac{7+42k+75k^2+42k^3}{64^k}.$$

Example 40 ($a = c = \frac{1}{2},\ d = \frac{1}{4},\ e = \frac{3}{4},\ b \to -\infty$).

$$15\sqrt{2} = \sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{2}, & \frac{1}{4}, & \frac{1}{4}, & \frac{3}{4}, & \frac{3}{4} \\ 1, & \frac{7}{8}, & \frac{9}{8}, & \frac{11}{8}, & \frac{13}{8} \end{bmatrix}_k \frac{21 + 242k + 752k^2 + 672k^3}{64^k}.$$

Concluding comments. We have examined three instances of q-analogues for infinite series expressions of $1/\pi$ and $1/\pi^2$, respectively, discovered by Ramanujan and Guillera. For other identities, it is possible to work out their q-analogues by following the approaches devised by Chu-Zhang [11, 12]. The interested reader is encouraged to try further.

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