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FOXBY EQUIVALENCES ASSOCIATED TO STRONGLY GORENSTEIN MODULES

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Abstract

In order to establish the Foxby equivalences associated to strongly Gorenstein modules, we introduce the notions of strongly $\mathcal{W}_P$-Gorenstein, $\mathcal{W}_I$-Gorenstein and $\mathcal{W}_F$-Gorenstein modules and discuss some basic properties of these modules. We show that the subcategory of strongly Gorenstein projective left R-modules in the left Auslander class and the subcategory of strongly $\mathcal{W}_P$-Gorenstein left S-modules are equivalent under Foxby equivalence. The injective and flat case are also studied.

1. Introduction

Throughout this paper, all rings are associative with identity and all modules are unitary. Let $\mathscr{A}$ be an abelian category and $\mathcal{C}$ an additive full subcategory of $\mathscr{A}$. Sather-Wagstaff, Sharif and White [12, Definition 4.1] introduced the Gorenstein subcategory $\mathcal{G}(\mathcal{C})$ of $\mathscr{A}$, and investigated its stability. As a special case of the Gorenstein subcategory $\mathcal{G}(\mathcal{C})$, Geng and Ding [8] introduced $\mathcal{W}$-Gorenstein modules for a self-orthogonal class $\mathcal{W}$ of left R-modules. These modules generalize some known modules such as Gorenstein projective (injective) modules [3, 4, 9], V-Gorenstein projective (injective) modules [6, 14], Ω-Gorenstein projective (injective) modules [5, 15] and so on. As the special examples of $\mathcal{W}$-Gorenstein modules, $\mathcal{W}_P$-Gorenstein modules and $\mathcal{W}_I$-Gorenstein modules are investigated and some new Foxby equivalences of categories are established in [8]. In parallel with $\mathcal{W}_P$-Gorenstein modules and $\mathcal{W}_I$-Gorenstein modules, Di et al. [2] introduced $\mathcal{W}_F$-Gorenstein modules over a commutative ring, established some Foxby equivalences of categories and discussed the stability of the category of $\mathcal{W}_F$-Gorenstein modules. For more details about $\mathcal{W}_P$-Gorenstein, $\mathcal{W}_I$-Gorenstein and $\mathcal{W}_F$-Gorenstein modules, we refer the reader to [2, 8, 12].

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Foxby equivalences between some special classes of modules in the Auslander class and that in the Bass class have been studied by many authors, see [2, 7, 8, 10, 11, 16]. Let ${}_SC_R$ be a faithfully semidualizing bimodule. Denote by $\mathcal{GP}(R)$, $\mathcal{GI}(S)$, $\mathcal{G}(\mathcal{W}_P)$ and $\mathcal{G}(\mathcal{W}_I)$ the classes of Gorenstein projective left R-modules, Gorenstein injective left S-modules, $\mathcal{W}_P$-Gorenstein left S-modules and $\mathcal{W}_I$-Gorenstein left R-modules, respectively. According to [8, Theorem 3.11], we have the following Foxby equivalences diagram:

$$\mathcal{A}_C^l(R)\cap\mathcal{GP}(R) \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R-}{\rightleftarrows}} \mathcal{G}(\mathcal{W}_P),$$

$$\mathcal{G}(\mathcal{W}_I) \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R-}{\rightleftarrows}} \mathcal{B}_C^l(S)\cap\mathcal{GI}(S),$$

where $\mathcal{A}_C^l(R)$ and $\mathcal{B}_C^l(S)$ denote the left Auslander class and left Bass class with respect to a semidualizing bimodule ${}_SC_R$, respectively. Let R be a commutative ring. Di et al. [2, Theorem 3.5] also obtained the following Foxby equivalences of categories:

$$\mathcal{A}_C^l(R)\cap\mathcal{GF}(R) \underset{\mathrm{Hom}_R(C,-)}{\overset{C\otimes_R-}{\rightleftarrows}} \mathcal{G}(\mathcal{W}_F).$$

where $\mathcal{GF}(R)$ denotes the class of Gorenstein flat R-modules and $\mathcal{G}(\mathcal{W}_F)$ the class of $\mathcal{W}_F$-Gorenstein R-modules.

Bennis and Mahdou [1] studied a particular case of Gorenstein projective, injective and flat modules, which they called strongly Gorenstein projective, injective and flat modules. They proved that every Gorenstein projective (resp., injective, flat) module is a direct summand of a strongly Gorenstein projective (resp., injective, flat) module. Denote by $\mathcal{SGP}(R)$, $\mathcal{SGI}(S)$ and $\mathcal{SGF}(R)$ the classes of strongly Gorenstein projective left R-modules, strongly Gorenstein injective left S-modules and strongly Gorenstein flat left R-modules, respectively. A natural question arises: what are the counterparts to the strongly Gorenstein projective, strongly Gorenstein injective and strongly Gorenstein flat modules in the following Foxby equivalences diagram:

$$\mathcal{A}_C^l(R)\cap\mathcal{SGP}(R) \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R-}{\rightleftarrows}} *,$$

$$\bullet \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R-}{\rightleftarrows}} \mathcal{B}_C^l(S)\cap\mathcal{SGI}(S),$$

$$\mathcal{A}_C^l(R)\cap\mathcal{SGF}(R) \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R-}{\rightleftarrows}} \#.$$

The aim of this paper is to give the characterizations of $*$, $\bullet$ and $\#$ in above diagram. To this end, we introduce the concepts of strongly $\mathcal{W}_P$-Gorenstein,

$\mathcal{W}_I$-Gorenstein and $\mathcal{W}_F$-Gorenstein modules, and show that these classes of modules exactly play the roles of $*$, $\bullet$ and $\#$ in above diagram.

This paper is organized as follows:

In section 2, we recall some basic notions which we need in the later sections.

In section 3, we introduce the notions of strongly $\mathcal{W}_P$-Gorenstein, $\mathcal{W}_I$-Gorenstein and $\mathcal{W}_F$-Gorenstein modules and discuss the basic properties of these modules. Some results related to strongly Gorenstein projective, injective and flat modules are extended to strongly $\mathcal{W}_P$-Gorenstein, $\mathcal{W}_I$-Gorenstein and $\mathcal{W}_F$-Gorenstein modules.

In section 4, we investigate the Foxby equivalences between the subclasses of the Auslander class and that of the Bass class. Let ${}_SC_R$ be a semidualizing bimodule. We show that there are equivalences of categories:

$$\mathcal{A}_C^l(R) \cap \mathcal{SGP}(R) \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{SG}(\mathcal{W}_P),$$

$$\mathcal{SG}(\mathcal{W}_I) \underset{\mathrm{Hom}_R(C,-)}{\overset{-\otimes_S C}{\rightleftarrows}} \mathcal{B}_C^r(R) \cap \mathcal{SGI}(R),$$

$$\mathcal{A}_C^l(R) \cap \mathcal{SGF}(R) \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{SG}(\mathcal{W}_F),$$

where $\mathcal{SG}(\mathcal{W}_P)$, $\mathcal{SG}(\mathcal{W}_I)$ and $\mathcal{SG}(\mathcal{W}_F)$ denote the classes of strongly $\mathcal{W}_P$-Gorenstein left S-modules, strongly $\mathcal{W}_I$-Gorenstein right S-modules and strongly $\mathcal{W}_F$-Gorenstein left S-modules, respectively.

2. Preliminaries

In this section, we will recall some notions and terminologies which we need in the later sections.

Semidualizing bimodules. An (S,R)-bimodule $C = {}_SC_R$ is semidualizing if

(1) ${}_SC$ admits a degreewise finite S-projective resolution.
(2) C_R admits a degreewise finite R-projective resolution.
(3) The homothety map ${}_SS_S \xrightarrow{{}_S\gamma} \mathrm{Hom}_R(C,C)$ is an isomorphism.
(4) The homothety map ${}_RR_R \xrightarrow{\gamma_R} \mathrm{Hom}_S(C,C)$ is an isomorphism.
(5) $\mathrm{Ext}_S^{\geqslant 1}(C,C) = 0$.
(6) $\mathrm{Ext}_R^{\geqslant 1}(C,C) = 0$.

A semidualizing bimodule ${}_SC_R$ is faithfully semidualizing if it satisfies the following conditions for all modules ${}_SN$ and M_R

(a) If $\mathrm{Hom}_S(C,N) = 0$, then $N = 0$.
(b) If $\mathrm{Hom}_R(C,M) = 0$, then $M = 0$.

In what follows, C stands for a semidualizing bimodule ${}_SC_R$.

Auslander class and Bass class with respect to C. For a semidualizing bimodule ${}_SC_R$, the left, right Auslander class $\mathcal{A}_C^l(R)$, $\mathcal{A}_C^r(S)$ and the left, right

Bass class $\mathcal{B}_C^l(S)$, $\mathcal{B}_C^r(R)$ with respect to C are defined as follows:

$$\mathcal{A}_C^l(R) = \{M \in R\text{-Mod} \mid \mathrm{Tor}_{\geqslant 1}^R(C, M) = \mathrm{Ext}_S^{\geqslant 1}(C, C \otimes_R M) = 0, \\ M \cong \mathrm{Hom}_S(C, C \otimes_R M)\},$$

$$\mathcal{A}_C^r(S) = \{M \in \text{Mod-}S \mid \mathrm{Tor}_{\geqslant 1}^S(M, C) = \mathrm{Ext}_R^{\geqslant 1}(C, M \otimes_S C) = 0, \\ M \cong \mathrm{Hom}_R(C, M \otimes_S C)\},$$

$$\mathcal{B}_C^l(S) = \{N \in S\text{-Mod} \mid \mathrm{Ext}_S^{\geqslant 1}(C, N) = \mathrm{Tor}_{\geqslant 1}^R(C, \mathrm{Hom}_S(C, N)) = 0, \\ C \otimes_R \mathrm{Hom}_S(C, N) \cong N\},$$

$$\mathcal{B}_C^r(R) = \{N \in \text{Mod-}R \mid \mathrm{Ext}_R^{\geqslant 1}(C, N) = \mathrm{Tor}_{\geqslant 1}^S(\mathrm{Hom}_R(C, N), C) = 0, \\ \mathrm{Hom}_R(C, N) \otimes_S C \cong N\},$$

where R-Mod (resp., S-Mod) denotes the category of left R-modules (resp., left S-modules), and Mod-R (resp., Mod-S) denotes the category of right R-modules (resp., right S-modules).

According to [10, Lemma 4.1], the class $\mathcal{A}_C^l(R)$ contains all flat left R-modules, and the class $\mathcal{B}_C^l(S)$ contains all injective left S-modules.

C-projectives, C-injectives, C-flats. A left S-module is C-flat (resp., C-projective) if it has the form $C \otimes_R F$ for some flat (resp., projective) left R-module F. A right S-module is C-injective if it has the form $\mathrm{Hom}_R(C, E)$ for some injective right R-module E. Set

$$\mathcal{F}_C(S) = \{C \otimes_R F \mid F \text{ is a flat left } R\text{-module}\},$$

$$\mathcal{P}_C(S) = \{C \otimes_R P \mid P \text{ is a projective left } R\text{-module}\},$$

$$\mathcal{I}_C(S) = \{\mathrm{Hom}_R(C, E) \mid E \text{ is an injective right } R\text{-module}\}.$$

Then $\mathcal{P}_C(S) \subseteq \mathcal{F}_C(S) \subseteq \mathcal{B}_C^l(S)$ and $\mathcal{I}_C(S) \subseteq \mathcal{A}_C^r(S)$.

$\mathcal{W}_P$-Gorenstein, $\mathcal{W}_I$-Gorenstein and $\mathcal{W}_F$-Gorenstein modules. We recall from [8] and [2] that a left S-module M is called a $\mathcal{W}_P$-Gorenstein module if there exists an exact sequence

$$\cdots \to W_1 \to W_0 \to W^0 \to W^1 \to \cdots$$

of modules in $\mathcal{P}_C(S)$ with $M \cong \mathrm{Ker}(W^0 \to W^1)$ such that $\mathrm{Hom}_S(W, -)$ and $\mathrm{Hom}_S(-, W)$ leave the sequence exact whenever $W \in \mathcal{P}_C(S)$. A right S-module M is called a $\mathcal{W}_I$-Gorenstein module if there exists an exact sequence

$$\cdots \to U_1 \to U_0 \to U^0 \to U^1 \to \cdots$$

of modules in $\mathcal{I}_C(S)$ with $M \cong \mathrm{Ker}(U^0 \to U^1)$ such that $\mathrm{Hom}_S(U, -)$ and $\mathrm{Hom}_S(-, U)$ leave the sequence exact whenever $U \in \mathcal{I}_C(S)$. A left S-module

M is called a $\mathcal{W}_F$-Gorenstein module if there exists an exact sequence

$$\cdots \to X_1 \to X_0 \to X^0 \to X^1 \to \cdots$$

of modules in $\mathcal{F}_C(S)$ with $M \cong \mathrm{Ker}(X^0 \to X^1)$ such that $\mathrm{Hom}_S(W, -)$ and $U \otimes_S -$ leave the sequence exact whenever $U \in \mathcal{I}_C(S)$ and $W \in \mathcal{P}_C(S)$.

3. Basic Properties

Definition 3.1. A left S-module M is called a strongly $\mathcal{W}_P$-Gorenstein module if there exists an exact sequence

$$\cdots \to W \xrightarrow{f} W \xrightarrow{f} W \xrightarrow{f} W \to \cdots$$

of module in $\mathcal{P}_C(S)$ with $M \cong \mathrm{Ker}\, f$ such that $\mathrm{Hom}_S(W', -)$ and $\mathrm{Hom}_S(-, W')$ leave the sequence exact whenever $W' \in \mathcal{P}_C(S)$. We denote the class of strongly $\mathcal{W}_P$-Gorenstein left S-modules by $\mathcal{SG}(\mathcal{W}_P)$.

Remark 3.2. (1) If $M \in \mathcal{P}_C(S)$, then M is a strongly $\mathcal{W}_P$-Gorenstein module.

(2) Note that if $C = {}_RR_R$, then strongly $\mathcal{W}_P$-Gorenstein modules coincide with strongly Gorenstein projective R-modules.

Lemma 3.3 ([8, Proposition 3.5]). *Let* $\cdots \to W_1 \to W_0 \to W^0 \to W^1 \to \cdots$ *be an exact sequence of modules in* $\mathcal{P}_C(S)$ *with* $M \cong \mathrm{Ker}(W^0 \to W^1)$. *Then* $M \in \mathcal{B}_C^l(S)$ *if and only if* $\mathrm{Hom}_S(W, -)$ *leaves the sequence exact for any* $W \in \mathcal{P}_C(S)$.

Proposition 3.4. *Let M be a left S-module. Then the following are equivalent*:

(1) *M is a strongly $\mathcal{W}_P$-Gorenstein module.*

(2) $M \in \mathcal{B}_C^l(S)$ *and there exists an exact sequence* $0 \to M \to C \otimes_R P \to M \to 0$ *with P projective and* $\mathrm{Ext}_S^i(M, W) = 0$ *for all* $i \geq 1$ *and any* $W \in \mathcal{P}_C(S)$.

(3) $M \in \mathcal{B}_C^l(S)$ *and there exists an exact sequence* $0 \to M \to C \otimes_R P \to M \to 0$ *with P projective and* $\mathrm{Ext}_S^1(M, W) = 0$ *for any* $W \in \mathcal{P}_C(S)$.

Proof. (1) $\Rightarrow$ (2) It is obvious by Lemma 3.3 and dimension shifting. (2) $\Rightarrow$ (3) $\Rightarrow$ (1) are obvious. □

Proposition 3.5. *The class $\mathcal{SG}(\mathcal{W}_P)$ is closed under direct sums.*

Proof. Let M_i $(i \in I)$ be a family of strongly $\mathcal{W}_P$-Gorenstein modules, then $M_i \in \mathcal{B}_C^l(S)$ for any $i \in I$ and there exist exact sequences $0 \to M_i \to W_i \to M_i \to 0$ with $W_i \in \mathcal{P}_C(S)$ such that $\mathrm{Ext}_S^k(M_i, W) = 0$ for all $k \geq 1$ and any $W \in \mathcal{P}_C(S)$. Then we obtain an exact sequence $0 \to \bigoplus_{i \in I} M_i \to \bigoplus_{i \in I} W_i \to \bigoplus_{i \in I} M_i \to 0$

with $\bigoplus_{i\in I} W_i \in \mathcal{P}_C(S)$. According to [10, Proposition 4.2], we have $\bigoplus_{i\in I} M_i \in \mathcal{B}_C^l(S)$. Moreover, $\mathrm{Ext}_S^k(\bigoplus_{i\in I} M_i, W) \cong \prod_{i\in I} \mathrm{Ext}_S^k(M_i, W) = 0$ for all $k \geq 1$ and any $W \in \mathcal{P}_C(S)$. Thus $\bigoplus_{i\in I} M_i$ is a strongly $\mathcal{W}_P$-Gorenstein module by Proposition 3.4. □

According to [1, Theorem 2.7], M is a Gorenstein projective module if and only if M is a direct summand of a strongly Gorenstein projective module. Now we extend this result as follows.

Theorem 3.6. *Let M be a left S-module. The following are equivalent*:
(1) *M is a $\mathcal{W}_P$-Gorenstein module.*
(2) *M is a direct summand of a strongly $\mathcal{W}_P$-Gorenstein module.*

Proof. We employ the method in the proof of [1, Theorem 2.7].
(1) $\Rightarrow$ (2) If M is a $\mathcal{W}_P$-Gorenstein module, then there exists an exact sequence

$$\mathbf{P}: \cdots \longrightarrow W_{-2} \xrightarrow{d_{-2}} W_{-1} \xrightarrow{d_{-1}} W_0 \xrightarrow{d_0} W_1 \xrightarrow{d_1} W_2 \xrightarrow{d_2} W_3 \longrightarrow \cdots$$

of modules in $\mathcal{P}_C(S)$ with $M \cong \mathrm{Ker}\, d_0$ such that $\mathrm{Hom}_S(W, -)$ and $\mathrm{Hom}_S(-, W)$ leave the sequence exact for any $W \in \mathcal{P}_C(S)$. For any $n \in \mathbf{Z}$, denote by $\Sigma^n\mathbf{P}$ the exact sequence obtained from $\mathbf{P}$ by increasing all indexes by n:

$$(\Sigma^n\mathbf{P})_i = W_{i-n}, \quad d_i^{\Sigma^n\mathbf{P}} = d_{i-n}$$

for any $i \in \mathbf{Z}$. Consider the following exact sequence

$$\mathbf{F} = \oplus(\Sigma^n\mathbf{P}): \cdots \longrightarrow \oplus W_i \xrightarrow{\oplus d_i} \oplus W_i \xrightarrow{\oplus d_i} \oplus W_i \longrightarrow \cdots.$$

Note that $\mathrm{Ker}(\oplus d_i) \cong \oplus \mathrm{Ker}\, d_i$, so M is isomorphic to a direct summand of $\mathrm{Ker}(\oplus d_i)$. Moreover, we have $\mathrm{Hom}_S(\mathbf{F}, W) \cong \prod_{n\in\mathbf{Z}} \mathrm{Hom}_S(\Sigma^n\mathbf{P}, W)$ is exact for any $W \in \mathcal{P}_C(S)$. Since $\mathrm{Ker}\, d_i$ is a $\mathcal{W}_P$-Gorenstein module by [8, Proposition 2.7], so is $\mathrm{Ker}(\oplus d_i)$. Then $\mathrm{Ker}(\oplus d_i) \in \mathcal{B}_C^l(S)$ by Lemma 3.3. Thus $\mathrm{Hom}_S(C, \mathbf{F})$ is exact, and then $\mathrm{Hom}_S(W, \mathbf{F})$ is exact for any $W \in \mathcal{P}_C(S)$ by [10, Lemma 6.1]. Thus M is a direct summand of strongly $\mathcal{W}_P$-Gorenstein module $\mathrm{Ker}(\oplus d_i)$.
(2) $\Rightarrow$ (1) Since every strongly $\mathcal{W}_P$-Gorenstein module is a $\mathcal{W}_P$-Gorenstein module, and the class of $\mathcal{W}_P$-Gorenstein modules is closed under direct summands. Then the result follows. □

Theorem 3.7. *Let $0 \to N \to M \to Q \to 0$ be an exact sequence with $Q \in \mathcal{P}_C(S)$. If C is a faithfully semidualizing bimodule, then N is a strongly $\mathcal{W}_P$-Gorenstein module if and only if M is a strongly $\mathcal{W}_P$-Gorenstein module.*

Proof. First, we show that if $N \in \mathcal{B}_C^l(S)$, then the exact sequence $0 \to N \to M \to Q \to 0$ is split. Suppose $Q = C \otimes_R P$ with P projective, then $Q \in \mathcal{B}_C^l(S)$.

If $N \in \mathcal{B}_C^l(S)$, then

$$\begin{aligned}\operatorname{Ext}_S^i(Q, N) &= \operatorname{Ext}_S^i(C \otimes_R P, N) \cong \operatorname{Ext}_R^i(\operatorname{Hom}_S(C, C \otimes_R P), \operatorname{Hom}_S(C, N)) \\ &\cong \operatorname{Ext}_R^i(P, \operatorname{Hom}_S(C, N)) = 0\end{aligned}$$

for any $i \geq 1$ by [10, Theorem 6.4]. Then the result follows.

If N is a strongly $\mathcal{W}_P$-Gorenstein module, then $N \in \mathcal{B}_C^l(S)$ by Proposition 3.4, and so $M \cong N \oplus Q$. Then M is a strongly $\mathcal{W}_P$-Gorenstein module by Proposition 3.5.

Conversely, if M is a strongly $\mathcal{W}_P$-Gorenstein module, then $M \in \mathcal{B}_C^l(S)$ and there exists an exact sequence $0 \to M \to G \to M \to 0$ with $G \in \mathcal{P}_C(S)$. Note that $Q \in \mathcal{B}_C^l(S)$, then $N \in \mathcal{B}_C^l(S)$ by [10, Corollary 6.3]. Thus $M \cong N \oplus Q$, and there exists an exact sequence $0 \to Q \to M \to N \to 0$. Consider the following pushout diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & Q & \to & M & \to & N & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & Q & \to & G & \to & D & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & M & = & M & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Clearly, N is a $\mathcal{W}_P$-Gorenstein module by Theorem 3.6, so is D. Thus $\operatorname{Ext}_S^i(D, Q) = 0$ for any $i \geq 1$. Then the exact sequence $0 \to Q \to G \to D \to 0$ is split. Thus $D \in \mathcal{P}_C(S)$ by [10, Proposition 5.1]. Consider the following pullback diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & N & = & N & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & B & \to & D & \to & Q & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & N & \to & M & \to & Q & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

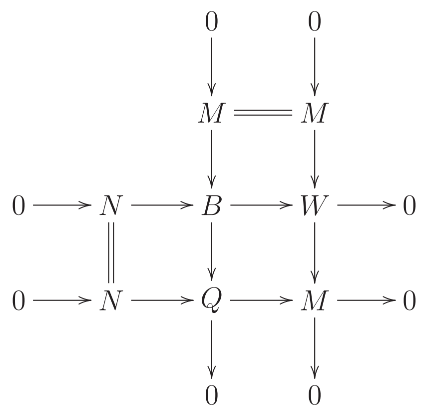
Since $D, Q \in \mathcal{P}_C(S)$, then $B \in \mathcal{P}_C(S)$ by [10, Corollary 6.4]. Consider the exact sequence $0 \to N \to B \to N \to 0$. Note that N is a $\mathcal{W}_P$-Gorenstein module, then $\mathrm{Ext}_S^i(N, W) = 0$ for any $i \geq 1$ and $W \in \mathcal{P}_C(S)$. This means that N is a strongly $\mathcal{W}_P$-Gorenstein module by Proposition 3.4. □

Corollary 3.8. *Let M be a left S-module and $Q \in \mathcal{P}_C(S)$. If C is a faithfully semidualizing bimodule, then M is a strongly $\mathcal{W}_P$-Gorenstein module if and only if $M \oplus Q$ is a strongly $\mathcal{W}_P$-Gorenstein module.*

Proof. Consider the exact sequence $0 \to M \to M \oplus Q \to Q \to 0$ with $Q \in \mathcal{P}_C(S)$. Then the result follows from Theorem 3.7. □

Corollary 3.9. *Let $0 \to N \to Q \to M \to 0$ be an exact sequence with $Q \in \mathcal{P}_C(S)$. If C is a faithfully semidualizing bimodule and M is a strongly $\mathcal{W}_P$-Gorenstein module, then N is a strongly $\mathcal{W}_P$-Gorenstein module.*

Proof. If M is a strongly $\mathcal{W}_P$-Gorenstein module, then there exists an exact sequence $0 \to M \to W \to M \to 0$ with $W \in \mathcal{P}_C(S)$. Consider the following pull-back diagram



By the middle vertical sequence and Theorem 3.7, we have B is a strongly $\mathcal{W}_P$-Gorenstein module. Thus by the middle horizontal sequence and Theorem 3.7, N is a strongly $\mathcal{W}_P$-Gorenstein module follows. □

Corollary 3.10. *Let C be a faithfully semidualizing bimodule. Then M is a strongly $\mathcal{W}_P$-Gorenstein module if and only if there exists an exact sequence $0 \to M \to C \otimes_R P \to N \to 0$ with P is projective and N is a strongly $\mathcal{W}_P$-Gorenstein module.*

Proof. The result follows from Corollary 3.9 and definition. □

Clearly, every strongly $\mathcal{W}_P$-Gorenstein module is $\mathcal{W}_P$-Gorenstein. However, [1, Example 2.13] showed that the converse is not true in general. But we have the following result.

THEOREM 3.11. *Let C be a faithfully semidualizing bimodule. The following conditions are equivalent*:

(1) *Every $\mathcal{W}_P$-Gorenstein left S-module is a strongly $\mathcal{W}_P$-Gorenstein module.*

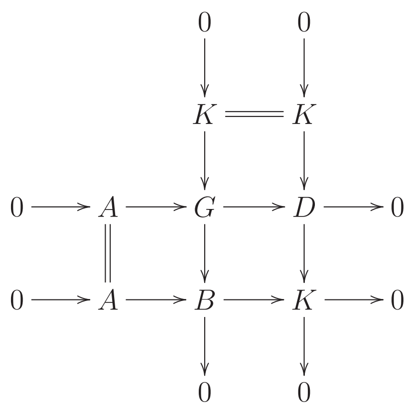
(2) *For any exact sequence $0 \to N \to Q \to M \to 0$ with $N, Q \in \mathcal{SG}(\mathcal{W}_P)$. If $\mathrm{Ext}_S^1(M, W) = 0$ for any $W \in \mathcal{P}_C(S)$, then $M \in \mathcal{SG}(\mathcal{W}_P)$.*

(3) *For any exact sequence $0 \to A \to B \to K \to 0$ with $A, K \in \mathcal{SG}(\mathcal{W}_P)$, then $B \in \mathcal{SG}(\mathcal{W}_P)$.*

(4) *For any exact sequence $0 \to A \to B \to K \to 0$ with $B, K \in \mathcal{SG}(\mathcal{W}_P)$, then $A \in \mathcal{SG}(\mathcal{W}_P)$.*

Proof. (1) $\Rightarrow$ (2) Since N, Q are strongly $\mathcal{W}_P$-Gorenstein modules, so N, Q are $\mathcal{W}_P$-Gorenstein modules. Note that $\mathrm{Ext}_S^1(M, W) = 0$ for any $W \in \mathcal{P}_C(S)$, then M is a $\mathcal{W}_P$-Gorenstein module by [8, Corollary 2.6]. Thus $M \in \mathcal{SG}(\mathcal{W}_P)$ by (1).

(2) $\Rightarrow$ (3) Let $0 \to A \to B \to K \to 0$ be an exact sequence with $A, K \in \mathcal{SG}(\mathcal{W}_P)$. From $\mathrm{Ext}_S^1(K, W) = \mathrm{Ext}_S^1(A, W) = 0$, we deduce that $\mathrm{Ext}_S^1(B, W) = 0$ for any $W \in \mathcal{P}_C(S)$. On the other hand, since $K \in \mathcal{SG}(\mathcal{W}_P)$, then there exists an exact sequence $0 \to K \to D \to K \to 0$ with $D \in \mathcal{P}_C(S)$. Consider the following pullback diagram



Since $A \in \mathcal{SG}(\mathcal{W}_P)$ and $D \in \mathcal{P}_C(S)$, it follows that $G \in \mathcal{SG}(\mathcal{W}_P)$ by Theorem 3.7. Since $\mathrm{Ext}_S^1(B, W) = 0$ for any $W \in \mathcal{P}_C(S)$, we have $B \in \mathcal{SG}(\mathcal{W}_P)$ by (2).

(3) $\Rightarrow$ (4) Let $0 \to A \to B \to K \to 0$ be an exact sequence with $B, K \in \mathcal{SG}(\mathcal{W}_P)$. Then there exists an exact sequence $0 \to K \to D \to K \to 0$ with $D \in \mathcal{P}_C(S)$. Consider the following pullback diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & G & \to & D & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & B & \to & K & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

From the middle vertical sequence, we have $G \in \mathcal{SG}(\mathcal{W}_P)$ by (3). Note that $D \in \mathcal{P}_C(S)$, from the middle horizontal sequence and Theorem 3.7, it follows that $A \in \mathcal{SG}(\mathcal{W}_P)$.

(4) $\Rightarrow$ (1) Let M be a $\mathcal{W}_P$-Gorenstein left S-module. Then there exists a $\mathcal{W}_P$-Gorenstein module N such that $M \oplus N$ is a strongly $\mathcal{W}_P$-Gorenstein module by Theorem 3.6. Set $L = (M \oplus N) \oplus (M \oplus N) \oplus \cdots$. Then L is a strongly $\mathcal{W}_P$-Gorenstein module by Proposition 3.5. Consider the exact sequence $0 \to M \to M \oplus N \oplus L \to N \oplus L \to 0$. Since $M \oplus N \oplus L \cong L$ and $N \oplus L \cong L$, so we have an exact sequence $0 \to M \to L \to L \to 0$. Thus $M \in \mathcal{SG}(\mathcal{W}_P)$ by (4). □

Definition 3.12. A right S-module M is called a strongly $\mathcal{W}_I$-Gorenstein module if there exists an exact sequence

$$\cdots \to U \xrightarrow{f} U \xrightarrow{f} U \xrightarrow{f} U \to \cdots$$

of module in $\mathcal{I}_C(S)$ with $M \cong \operatorname{Ker} f$ such that $\operatorname{Hom}_S(U', -)$ and $\operatorname{Hom}_S(-, U')$ leave the sequence exact whenever $U' \in \mathcal{I}_C(S)$. We denote the class of strongly $\mathcal{W}_I$-Gorenstein right S-modules by $\mathcal{SG}(\mathcal{W}_I)$.

Definition 3.13. A left S-module M is called a strongly $\mathcal{W}_F$-Gorenstein module if there exists an exact sequence

$$\cdots \to W \xrightarrow{f} W \xrightarrow{f} W \xrightarrow{f} W \to \cdots$$

of module in $\mathcal{F}_C(S)$ with $M \cong \operatorname{Ker} f$ such that $\operatorname{Hom}_S(W', -)$ and $U \otimes_S -$ leave the sequence exact whenever $W' \in \mathcal{P}_C(S)$ and $U \in \mathcal{I}_C(S)$. We denote the class of strongly $\mathcal{W}_F$-Gorenstein left S-modules by $\mathcal{SG}(\mathcal{W}_F)$.

Remark 3.14. (1) By a dual argument of the proofs of all the foregoing results on strongly $\mathcal{W}_P$-Gorenstein modules, we have the dual versions on strongly $\mathcal{W}_I$-Gorenstein modules.

(2) Analogue to the proofs of Proposition 3.4 and Proposition 3.5, we can show the similar results for strongly $\mathcal{W}_F$-Gorenstein modules. If R is a right coherent ring and C is a faithfully semidualizing bimodule, by a similar argument of the proofs of Theorem 3.6, Theorem 3.7, Corollary 3.8, Corollary 3.9, Corollary 3.10 and Theorem 3.11, we obtain the similar results for strongly $\mathcal{W}_F$-Gorenstein modules.

4. Foxby equivalence

Let ${}_SC_R$ be a semidualizing bimodule (not necessarily faithfully). In this section, we will establish the Foxby equivalences associated to strongly $\mathcal{W}_P$-Gorenstein, $\mathcal{W}_I$-Gorenstein and $\mathcal{W}_F$-Gorenstein modules, respectively. At first, we need to prove the following results.

Proposition 4.1. *Let M be a left S-module. Then the following are equivalent*:

(1) *M is a strongly $\mathcal{W}_P$-Gorenstein module.*

(2) *$M \in \mathcal{B}_C^l(S)$ and $\mathrm{Hom}_S(C,M)$ is a strongly Gorenstein projective module.*

Proof. (1) $\Rightarrow$ (2) If M is a strongly $\mathcal{W}_P$-Gorenstein module, then $M \in \mathcal{B}_C^l(S)$ by Proposition 3.4 and there exists an exact sequence $0 \to M \to C \otimes_R P \to M \to 0$ with P projective such that $\mathrm{Ext}_S^i(M,W) = 0$ for all $i \geq 1$ and any $W \in \mathcal{P}_C(S)$. Since $M \in \mathcal{B}_C^l(S)$, then $\mathrm{Ext}_S^{\geqslant 1}(C,M) = 0$. Applying the functor $\mathrm{Hom}_S(C,-)$ to the exact sequence, we obtain an exact sequence $0 \to \mathrm{Hom}_S(C,M) \to P \to \mathrm{Hom}_S(C,M) \to 0$. We only need to show $\mathrm{Ext}_R^i(\mathrm{Hom}_S(C,M),Q) = 0$ for all $i \geq 1$ and any projective R-module Q. Note that $M \in \mathcal{B}_C^l(S)$, then $\mathrm{Hom}_S(C,M) \in \mathcal{A}_C^l(R)$ and $M \cong C \otimes_R \mathrm{Hom}_S(C,M)$. Given any projective R-module Q, by [10, Theorem 6.4], we have

$$\begin{aligned} 0 = \mathrm{Ext}_S^i(M, C \otimes_R Q) &\cong \mathrm{Ext}_S^i(C \otimes_R \mathrm{Hom}_S(C,M), C \otimes_R Q) \\ &\cong \mathrm{Ext}_R^i(\mathrm{Hom}_S(C,M),Q) \end{aligned}$$

for all $i \geq 1$. Thus $\mathrm{Hom}_S(C,M)$ is a strongly Gorenstein projective module by [1, Proposition 2.9].

(2) $\Rightarrow$ (1) Since $\mathrm{Hom}_S(C,M)$ is a strongly Gorenstein projective module, then there exists an exact sequence $0 \to \mathrm{Hom}_S(C,M) \to P \to \mathrm{Hom}_S(C,M) \to 0$ with P projective. Note that $M \in \mathcal{B}_C^l(S)$, so $\mathrm{Hom}_S(C,M) \in \mathcal{A}_C^l(R)$. Then $\mathrm{Tor}_{\geqslant 1}^R(C, \mathrm{Hom}_S(C,M)) = 0$. Applying the functor $C \otimes_R -$ to the above exact sequence, we obtain an exact sequence $0 \to M \to C \otimes_R P \to M \to 0$. In the following, we will show $\mathrm{Ext}_S^i(M, C \otimes_R Q) = 0$ for all $i \geq 1$ and any projective R-module Q. Since $\mathrm{Hom}_S(C,M)$ is a strongly Gorenstein projective module, so $\mathrm{Ext}_R^i(\mathrm{Hom}_S(C,M),Q) = 0$ for all $i \geq 1$ and any projective R-module Q. Then

$$\begin{aligned} 0 = \mathrm{Ext}_R^i(\mathrm{Hom}_S(C,M),Q) &\cong \mathrm{Ext}_S^i(C \otimes_R \mathrm{Hom}_S(C,M), C \otimes_R Q) \\ &\cong \mathrm{Ext}_S^i(M, C \otimes_R Q) \end{aligned}$$

by [10, Theorem 6.4]. This means that M is a strongly $\mathcal{W}_P$-Gorenstein module by Proposition 3.4. □

Corollary 4.2. *Let $M \in \mathcal{A}_C^l(R)$ be a left R-module. Then the following are equivalent*:

(1) *M is a strongly Gorenstein projective module.*

(2) *$C \otimes_R M$ is a strongly $\mathcal{W}_P$-Gorenstein module.*

Proof. (1) ⇒ (2) Since $M \in \mathcal{A}_C^l(R)$, then $C \otimes_R M \in \mathcal{B}_C^l(S)$ and $\mathrm{Hom}_S(C, C \otimes_R M) \cong M$ is a strongly Gorenstein projective module. Thus $C \otimes_R M$ is a strongly $\mathcal{W}_P$-Gorenstein module by Proposition 4.1.

(2) ⇒ (1) Since $M \in \mathcal{A}_C^l(R)$, then $M \cong \mathrm{Hom}_S(C, C \otimes_R M)$. Note that $C \otimes_R M$ is a strongly $\mathcal{W}_P$-Gorenstein module, it follows that $\mathrm{Hom}_S(C, C \otimes_R M) \cong M$ is a strongly Gorenstein projective module by Proposition 4.1. □

The proof of the Proposition 4.3 and Corollary 4.4 are dual to the proof of Proposition 4.1 and Corollary 4.2, so we omit all the proofs.

Proposition 4.3. *Let M be a right S-module. Then the following are equivalent*:

(1) *M is a strongly $\mathcal{W}_I$-Gorenstein module.*

(2) *$M \in \mathcal{A}_C^r(S)$ and $M \otimes_S C$ is a strongly Gorenstein injective module.*

Corollary 4.4. *Let $M \in \mathcal{B}_C^r(R)$ be a right R-module. Then the following are equivalent*:

(1) *M is a strongly Gorenstein injective module.*

(2) $\mathrm{Hom}_R(C, M)$ *is a strongly $\mathcal{W}_I$-Gorenstein module.*

Proposition 4.5. *Let M be a left S-module. Then the following are equivalent*:

(1) *M is a strongly $\mathcal{W}_F$-Gorenstein module.*

(2) *$M \in \mathcal{B}_C^l(S)$ and* $\mathrm{Hom}_S(C, M)$ *is a strongly Gorenstein flat module.*

Proof. (1) ⇒ (2) If M is a strongly $\mathcal{W}_F$-Gorenstein module, then $M \in \mathcal{B}_C^l(S)$ by Remark 3.14 and there exists an exact sequence $0 \to M \to C \otimes_R F \to M \to 0$ with F flat. Since $M \in \mathcal{B}_C^l(S)$, then $\mathrm{Ext}_S^{\geqslant 1}(C, M) = 0$. If we apply the functor $\mathrm{Hom}_S(C, -)$ to the exact sequence, we obtain an exact sequence $0 \to \mathrm{Hom}_S(C, M) \to F \to \mathrm{Hom}_S(C, M) \to 0$. We only need to prove $\mathrm{Tor}_i^R(E, \mathrm{Hom}_S(C, M)) = 0$ for any injective R-module E and any $i \geq 1$. Note that $E \in \mathcal{B}_C^r(R)$, then $E \cong \mathrm{Hom}_R(C, E) \otimes_S C$. Since M is a strongly $\mathcal{W}_F$-Gorenstein module, then $\mathrm{Tor}_i^S(\mathrm{Hom}_R(C, E), M) = 0$ for all $i \geq 1$ and any injective R-module E. Thus

$$\begin{aligned}\mathrm{Tor}_i^R(E, \mathrm{Hom}_S(C, M)) &\cong \mathrm{Tor}_i^R(\mathrm{Hom}_R(C, E) \otimes_S C, \mathrm{Hom}_S(C, M)) \\ &\cong \mathrm{Tor}_i^S(\mathrm{Hom}_R(C, E), M) = 0\end{aligned}$$

by [10, Theorem 6.4]. This means that $\mathrm{Hom}_S(C, M)$ is a strongly Gorenstein flat module.

(2) $\Rightarrow$ (1) Since $\mathrm{Hom}_S(C, M)$ is a strongly Gorenstein flat module, then there exists an exact sequence $0 \to \mathrm{Hom}_S(C, M) \to F \to \mathrm{Hom}_S(C, M) \to 0$ with F flat. Note that $M \in \mathcal{B}_C^l(S)$, then $\mathrm{Hom}_S(C, M) \in \mathcal{A}_C^l(R)$, and so $\mathrm{Tor}_{\geqslant 1}^R(C, \mathrm{Hom}_S(C, M)) = 0$. If we apply the functor $C \otimes_R -$ to this sequence, we obtain an exact sequence $0 \to M \to C \otimes_R F \to M \to 0$. In the following, we will show $\mathrm{Tor}_i^S(\mathrm{Hom}_R(C, E), M) = 0$ for any injective R-module E. Since $\mathrm{Hom}_S(C, M)$ is a strongly Gorenstein flat module, then $\mathrm{Tor}_i^R(E, \mathrm{Hom}_S(C, M)) = 0$ for any injective R-module E. Thus

$$\begin{aligned} 0 = \mathrm{Tor}_i^R(E, \mathrm{Hom}_S(C, M)) &\cong \mathrm{Tor}_i^R(\mathrm{Hom}_R(C, E) \otimes_S C, \mathrm{Hom}_S(C, M)) \\ &\cong \mathrm{Tor}_i^S(\mathrm{Hom}_R(C, E), M) \end{aligned}$$

by [10, Theorem 6.4]. This implies that M is a strongly $\mathcal{W}_F$-Gorenstein module. □

Corollary 4.6. *Let $M \in \mathcal{A}_C^l(R)$ be a left R-module. Then the following are equivalent*:

(1) *M is a strongly Gorenstein flat module.*

(2) *$C \otimes_R M$ is a strongly $\mathcal{W}_F$-Gorenstein module.*

Proof. It is similar to the proof of Corollary 4.2. □

We use $\mathcal{SGP}(R)$, $\mathcal{SGI}(R)$ and $\mathcal{SGF}(R)$ to denote the classes of strongly Gorenstein projective left R-modules, strongly Gorenstein injective right R-modules and strongly Gorenstein flat left R-modules respectively.

Theorem 4.7. *There are equivalences of categories*:

$$\mathcal{A}_C^l(R) \cap \mathcal{SGP}(R) \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{SG}(\mathcal{W}_P),$$

$$\mathcal{SG}(\mathcal{W}_I) \underset{\mathrm{Hom}_R(C,-)}{\overset{-\otimes_S C}{\rightleftarrows}} \mathcal{B}_C^r(R) \cap \mathcal{SGI}(R),$$

$$\mathcal{A}_C^l(R) \cap \mathcal{SGF}(R) \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} \mathcal{SG}(\mathcal{W}_F).$$

Proof. By Proposition 4.1 and Corollary 4.2, it is obvious that the functor $C \otimes_R -$ maps $\mathcal{A}_C^l(R) \cap \mathcal{SGP}(R)$ to $\mathcal{SG}(\mathcal{W}_P)$, also the functor $\mathrm{Hom}_S(C, -)$ maps $\mathcal{SG}(\mathcal{W}_P)$ to $\mathcal{A}_C^l(R) \cap \mathcal{SGP}(R)$. Moreover, note that if $M \in \mathcal{A}_C^l(R) \cap \mathcal{SGP}(R)$ and $N \in \mathcal{SG}(\mathcal{W}_P)$, then there exist natural isomorphisms $M \cong \mathrm{Hom}_S(C, C \otimes_R M)$

and $N \cong C \otimes_R \mathrm{Hom}_S(C, N)$. Then the desired first equivalence of categories follows.

Similarly, we can show the other two equivalences of categories by using Proposition 4.3, Corollary 4.4, Proposition 4.5 and Corollary 4.6. □

Remark 4.8. We can define strongly $\mathcal{W}_P$-Gorenstein right R-modules, strongly $\mathcal{W}_F$-Gorenstein right R-modules and strongly $\mathcal{W}_I$-Gorenstein left R-modules. Using right Auslander class $\mathcal{A}_C^r(S)$ and left Bass class $\mathcal{B}_C^l(S)$, by a similar argument, we can obtain the similar equivalences of categories.

We use $\mathcal{P}(R)$, $\mathcal{I}(R)$ and $\mathcal{F}(R)$ to denote the classes of projective left R-modules, injective right R-modules and flat left R-modules respectively. Let n be a nonnegative integer. Denote by $\mathcal{G}(\mathcal{W}_P)_{\leq n}$, $\mathcal{G}(\mathcal{W}_I)_{\leq n}$ and $\mathcal{G}(\mathcal{W}_F)_{\leq n}$ the classes of left S-modules with $\mathcal{G}(\mathcal{W}_P)$-projective dimension at most n, right S-modules with $\mathcal{G}(\mathcal{W}_I)$-injective dimension at most n and left S-modules with $\mathcal{G}(\mathcal{W}_F)$-projective dimension at most n, respectively. $\mathcal{GP}(R)_{\leq n}$, $\mathcal{GI}(R)_{\leq n}$ and $\mathcal{GF}(R)_{\leq n}$ denote the classes of left R-modules with Gorenstein projective dimension at most n, right R-modules with Gorenstein injective dimension at most n, and left R-modules with Gorenstein flat dimension at most n, respectively.

Let C be a semidualizing module over a commutative ring R. Di et al. in [2, Theorem 3.9] established some Foxby equivalences associated to $\mathcal{W}_F$-Gorenstein modules. In fact, if ${}_SC_R$ is a faithfully semidualizing bimodule over arbitrary associative rings, by an argument similar to Proposition 4.5 and Corollary 4.6, we can obtain the same results in [2, Theorem 3.9]. By assembling the information above and [2, Theorem 3.9], [8, Remark 3.13], we have the following extension of the Foxby equivalences.

THEOREM 4.9. *If ${}_SC_R$ is a faithfully semidualizing bimodule, then there are equivalences of categories*:

$$
\begin{array}{ccc}
\mathcal{F}(R) & \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} & \mathcal{F}_C(S) \\
\downarrow & & \downarrow \\
\mathcal{A}_C^l(R) \cap \mathcal{SGF}(R) & \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} & \mathcal{SG}(\mathcal{W}_F) \\
\downarrow & & \downarrow \\
\mathcal{A}_C^l(R) \cap \mathcal{GF}(R) & \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} & \mathcal{G}(\mathcal{W}_F) \\
\downarrow & & \downarrow \\
\mathcal{A}_C^l(R) \cap \mathcal{GF}(R)_{\leq n} & \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} & \mathcal{G}(\mathcal{W}_F)_{\leq n} \\
\downarrow & & \downarrow \\
\mathcal{A}_C^l(R) & \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} & \mathcal{B}_C^l(S),
\end{array}
$$

$$
\begin{array}{ccc}
\mathcal{P}(R) & \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} & \mathcal{P}_C(S) \\
\downarrow & & \downarrow \\
\mathcal{A}^l_C(R)\cap\mathcal{SGP}(R) & \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} & \mathcal{SG}(\mathcal{W}_P) \\
\downarrow & & \downarrow \\
\mathcal{A}^l_C(R)\cap\mathcal{GP}(R) & \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} & \mathcal{G}(\mathcal{W}_P) \\
\downarrow & & \downarrow \\
\mathcal{A}^l_C(R)\cap\mathcal{GP}(R)_{\leq n} & \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} & \mathcal{G}(\mathcal{W}_P)_{\leq n} \\
\downarrow & & \downarrow \\
\mathcal{A}^l_C(R) & \underset{\mathrm{Hom}_S(C,-)}{\overset{C\otimes_R -}{\rightleftarrows}} & \mathcal{B}^l_C(S),
\end{array}
$$

$$
\begin{array}{ccc}
\mathcal{A}^r_C(S) & \underset{\mathrm{Hom}_R(C,-)}{\overset{-\otimes_S C}{\rightleftarrows}} & \mathcal{B}^r_C(R) \\
\uparrow & & \uparrow \\
\mathcal{G}(\mathcal{W}_I)_{\leq n} & \underset{\mathrm{Hom}_R(C,-)}{\overset{-\otimes_S C}{\rightleftarrows}} & \mathcal{B}^r_C(R)\cap\mathcal{GI}(R)_{\leq n} \\
\uparrow & & \uparrow \\
\mathcal{G}(\mathcal{W}_I) & \underset{\mathrm{Hom}_R(C,-)}{\overset{-\otimes_S C}{\rightleftarrows}} & \mathcal{B}^r_C(R)\cap\mathcal{GI}(R) \\
\uparrow & & \uparrow \\
\mathcal{SG}(\mathcal{W}_I) & \underset{\mathrm{Hom}_R(C,-)}{\overset{-\otimes_S C}{\rightleftarrows}} & \mathcal{B}^r_C(R)\cap\mathcal{SGI}(R) \\
\uparrow & & \uparrow \\
\mathcal{I}_C(S) & \underset{\mathrm{Hom}_R(C,-)}{\overset{-\otimes_S C}{\rightleftarrows}} & \mathcal{I}(R).
\end{array}
$$

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