L. MA
KODAI MATH. J.
41 (2018), 348–358

CONVEXITY AND THE DIRICHLET PROBLEM OF TRANSLATING MEAN CURVATURE FLOWS

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Abstract

In this work, we propose a new evolving geometric flow (called translating mean curvature flow) for the translating solitons of hypersurfaces in R^{n+1}. We study the basic properties, such as positivity preserving property, of the translating mean curvature flow. The Dirichlet problem for the graphical translating mean curvature flow is studied and the global existence of the flow and the convergence property are also considered.

1. Introduction

In this note, we propose a new evolving flow (called translating mean curvature flow) for the translating solitons of hypersurfaces in R^{n+1}. This flow is a modification of mean curvature flow with a translation by a fixed vector. We study the basic properties of the translating mean curvature flow. The Dirichlet problem for the graphical translating mean curvature flow is studied and the global existence of the flow and the convergence property are also presented. This work can be considered as a continuation of our paper [11]. For interesting result about self similar solutions for the mean curvature flow in Riemannian cone manifolds, we can see the paper of Futaki, Hattori, and Yamamoto [4].

We propose the translating mean curvature flow in the following way. Given a fixed nonzero vector $V \in R^{n+1}$. The translating mean curvature flow for translating soliton is defined as a one parameter family of properly immersed hypersurface $M_t = X(\Sigma, t)$, where $0 < t < T$ and $X : \Sigma \times [0, T) \to R^{n+1}$ evolved by the evolution equation

$$X_t = \overline{H}(X) + V^N, \quad t > 0 \tag{1}$$

2000 *Mathematics Subject Classification*. 35J60, 53C21, 58J05.

Key words and phrases. mean curvature flow, translating solitons, convexity, the maximum principle.

The research is partially supported by the National Natural Science Foundation of China (No. 11271111 and No. 11771124).

Received May 22, 2017; revised June 26, 2017.

where $\overline{H}(X)$ is the mean curvature vector of the hypersurface M_t at the position vector X and V^N is the normal component of the vector V. We denote $V^T = V - V^N$ the tangential part of the vector V. Recall that for the outer unit normal $v := v(\cdot, t)$ on M_t, the mean curvature is defined by $H = div(v)$ and the mean curvature vector is $\overline{H} = -Hv$. Let (e_j) be a local orthonormal frame on M_t. Let $a_{ij} = \langle D_{ei}e_j, v\rangle$ and let $A = (a_{ij})$ be the second fundamental form on M_t. Then $H = -\sum a_{jj}$. Define $\overline{X} = X - tV$. Then we have the mean curvature flow

$$\overline{X}_t = \overline{H}(\overline{X}), \quad t > 0$$

with the same initial hypersurface X_0 up to diffeomorphisms on X_0. Therefore, many geometric properties such as convexity, mean convexity, are preserved by the flows. However, the global behaviors of two flows $X(t)$ and $\overline{X}(t)$ are different. Hence the flows (1) need to be considered independently.

Applying the maximum principle (and Hamilton's tensor maximum principle) to derived evolution equations from (1) we obtain the following result.

Theorem 1. *Given a translating mean curvature flow M_t with bounded second fundamental form A, $t \in [0, T)$ with $T > 0$.*

(1). (i). *If $\langle V, v\rangle \geq 0$ on the initial hypersurface M_0, then $\langle V, v\rangle \geq 0$ on the hypersurface M_t for any $t > 0$. Similarly, assume that $H \geq 0$ on the initial hypersurface M_0. Then $H \geq 0$ on the hypersurface M_t for any $t > 0$.* (ii). *Assume that $A \geq 0$ on the initial hypersurface M_0. Then $A \geq 0$ on the hypersurface M_t for any $t > 0$.*

(2). *Assume that for some constant β, $H - \beta\langle V, v\rangle \leq 0$ on the initial hypersurface M_0 and $H - \beta\langle V, v\rangle < 0$ at at least point in M_0. Then $H - \beta\langle V, v\rangle < 0$ on M_t for $t > 0$.*

(3). *If we assume $A + \beta\frac{\langle V, v\rangle}{n}g \geq 0$ at the initial hypersurface M_0 and $A + \beta\frac{\langle V, v\rangle}{n}g > 0$ at least one point $p \in M_0$, we have $A + \beta\frac{\langle V, v\rangle}{n}g > 0$ on M_t for $t > 0$.*

To derive this result, we shall do computations as in [3]. As we have pointed out above, the property (1) can be derived from the mean curvature flow. For completeness, we give a full proof. Related Harnack inequalities for translating mean curvature flow similar to results in [6] may be the same.

One example for hypersurfaces with $H - \langle V, v\rangle < 0$ is the graph of the parabolic function $u(x) = \frac{\lambda}{2}|x|^2$, where $x \in R^n$, $n \geq 2$ with $\lambda = 1$ and $V = -e_{n+1} = (0, \dots 0, -1)$. In this case, $Du(x) = x$, $v = \sqrt{1 + |x|^2}$, $v = (-x, 1)/v$,

$$\langle v, V\rangle = -1/v,$$

and

$$-H = div\left(\frac{x}{\sqrt{1+|x|^2}}\right) = \frac{1}{v}\left(\frac{n+(n-1)|x|^2}{1+|x|^2}\right) > \frac{1}{v}.$$

One can compute that for $\lambda > 0$ small we have $H - \langle V, v\rangle > 0$.

The Dirichlet problem for translating solitons on convex domain has been studied by X. J. Wang (see Theorem 5.2 in [13] from the viewpoint of Monge-Ampere equations). B. White [14] has given a geometric measure theory argument for the existence of minimizers of the weighted area

$$\int e^{-\lambda x_{n+1}}\, dA(x)$$

amongst integral currents over the mean convex domain. Namely, letting W be a bounded domain in R^n with piecewise smooth mean convex boundary and letting Γ be a smooth closed $(n-1)$ manifold in $\partial W \times R$ that is a graph-like. Then he has used the globally defined radially symmetric solitons $y = \varphi(x)$ as barriers for the minimizing process of integral currents which lie in the region $\mathbf{R}$ defined by

$$\mathbf{R} = \{(x, y) \in \overline{W} \times R;\ b \le y \le \varphi(x)\}$$

where $b = \inf\{y; (x, y) \in \Gamma\}$. We remark that his region $\mathbf{R}$ (in the proof of Theorem 10 in [14]) may be replaced by the region

$$\check{R} = \{(x, y) \in \overline{W} \times R;\ \varphi(x) - C \le y \le \varphi(x)\}$$

for suitable constant $C > 0$. The choice of the lower barrier $\varphi(x) - C$ is nice in the sense that it is a sub-solution to the mean curvature soliton equation. One may get the minimizers by using BV functions. Our approach for the existence of translating solitons with the Dirichlet boundary condition on convex domains is the heat flow method. That is, we propose the translating mean curvature flow to get the solitons as the limits. The uniqueness and convexity of the translating solitons with convex boundary data ϕ remain as open questions.

The Dirichlet problem for the graphical mean curvature flow on mean convex domains has been studied by G. Huisken [7] and Lieberman [10]. Their results show that the Dirichlet problem of the graphical mean curvature flow on mean convex domains has a global flow and it converges to a minimal surface at time infinity. Their result can not been directly applied to the following graphical translating mean curvature flow.

$$(2) \qquad \partial_t u = \sqrt{1+|Du|^2}\, div\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) - 1, \quad in\ \Omega \times [0, \infty)$$

with the Dirichlet boundary condition

$$u = \phi, \quad on\ \partial\Omega,\ t \ge 0$$

and the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega.$$

Here we assume $\Omega \subset R^n$ is a bounded domain with C^2 boundary, $\phi \in C^{2,\alpha}(\overline{\Omega})$, $u_0 \in C^{2,\alpha}(\overline{\Omega})$, and $u_0 = \phi$ on $\partial\Omega$. The flow (2) corresponds to the negative gradient flow of the weighted area functional

$$F(u) = \int_\Omega \sqrt{1+|Du|^2}e^{u(x)}\,dx.$$

If we let $f = u + t$, then f satisfies

$$\partial_t u = \sqrt{1+|Du|^2}\, div\left(\frac{Du}{\sqrt{1+|Du|^2}}\right), \quad in\ \Omega \times [0,\infty)$$

with the Dirichlet boundary condition

$$f = \phi + t, \quad on\ \partial\Omega,\ t \geq 0$$

and the initial condition

$$f(x,0) = u_0(x), \quad x \in \Omega.$$

Observe that the boundary condition now depends on time variable and the known result [7] can not be applied directly to it.

We have the following result.

THEOREM 2. *Assume $\Omega \subset R^n$ be a bounded convex domain with C^2 boundary. Assume that $\phi \in C^{2,\alpha}(\overline{\Omega})$, $u_0 \in C^{2,\alpha}(\overline{\Omega})$, and $u_0 = \phi$ on $\partial\Omega$. Then the Dirichlet problem of* (2) *has a smooth solution and $u(\cdot,t)$ converges to the translating soliton with boundary data ϕ as $t \to \infty$.*

The plan of this note is below. In section 2 we discuss the positivity preserving properties of the general translating mean curvature flow. In section 3 we consider the global existence of the Dirichlet problem of graphic mean curvature flows on bounded convex domains in R^n.

2. Positivity preserving property of the translating mean curvature flow

We shall use Hamilton's tensor maximum principle as below (see [2] for full statement and the proof).

PROPOSITION 3. *Let $(M, g(t))$ be a one parameter family of complete noncompact Riemannian manifolds with bounded curvature. Suppose $S = S_{ij}(x,t)\,dx^i dx^j$ is a smooth time-dependent symmetric 2-tensor field such that*

$$(\partial_t - \Delta_{g(t)})S \geq \nabla_X S + B(S,t)$$

where $B(S,t)$ is locally Lipschitz in (S,t) and $X = X(t)$ is a smooth time dependent vector field on M. Assume that B satisfies the null-eigenvector assumption in the sense that for some time-parallel vector field v and at some point $x \in M$ such that if $S \geq 0$ and $S(v,\cdot) = 0$, then $B(S,t)(v,v) \geq 0$. Assume that $S \geq 0$ at the initial time $t = 0$. Then $S \geq 0$ for all $t > 0$.

Proof of Theorem 1. Recall the following formulae for the flow $X_t = f\nu$ with local coordinates (x_j) on M_t, we have for the evolving metric $g_{ij} = \langle \partial_{x_i} X, \partial_{x_j} X\rangle$, outer unit normal ν, and the second fundamental form (a_{ij}), we have

$$\partial_t g_{ij} = -2f a_{ij},$$
$$\partial_t \nu = -\nabla f,$$

and

$$\partial_t a_{ij} = f_{ij} - f a_{ik} a_{kj}.$$

We shall let $f = \langle V, \nu\rangle - H$, which is our translating mean curvature flow case. Let $(g^{ij}) = (g_{ij})^{-1}$.

We now use moving frames to compute formulae for the flow. As in [3] and [8] we take (e_i) to be the evolving frame on M_t such that

$$\partial_t e_i = \frac{1}{2} g^{jk} \partial_t g_{ij} e_k = -f g^{jk} a_{ij} e_k.$$

Then we have

$$\partial_t g_{ij} = 0.$$

At a fixed point $p \in M_t$ we may assume that $\langle e_i, e_j\rangle = \delta_{ij}$ and $\nabla_{e_i} e_j = 0$. Then

$$\partial_t A(e_i, e_j) = f_{ij} + f a_{ik} a_{kj}.$$

Note that

$$\nabla_i \langle V, \nu\rangle = \langle V, D_{e_i} \nu\rangle = -\langle V, e_k\rangle a_{ik},$$

and at p,

$$\nabla_j \nabla_i \langle V, \nu\rangle = -\langle V, e_k\rangle a_{ik,j} - \langle V, D_{e_j} e_k\rangle a_{ik} = -a_{ij,V^T} - \langle V, \nu\rangle a_{ik} a_{kj}.$$

Then we have

$$\Delta \langle V, \nu\rangle = \nabla_{V^T} H - \langle V, \nu\rangle |A|^2.$$

Since $\partial_t \langle V, \nu\rangle = -\langle V, \nabla f\rangle = -\nabla_{V^T} f$ and $f + H = \langle V, \nu\rangle$, we get

$$\begin{aligned}(\partial_t - \Delta)\langle V, \nu\rangle &= -\nabla_{V^T} f - \nabla_{V^T} H + \langle V, \nu\rangle |A|^2 \\ &= -\nabla_{V^T} \langle V, \nu\rangle + \langle V, \nu\rangle |A|^2.\end{aligned}$$

That is,

$$(\partial_t - \Delta)\langle V, \nu\rangle = -\nabla_{V^T} \langle V, \nu\rangle + \langle V, \nu\rangle |A|^2. \tag{3}$$

Recall the well-known formulae that

$$(\Delta A)_{ij} = -|A|^2 a_{ij} - H a_{ik} a_{kj} - H_{ij}.$$

Then we have

$$(\partial_t A - \Delta A)(e_i, e_j) = (f + H)_{ij} + (f + H) a_{ik} a_{kj} + |A|^2 a_{ij},$$

which implies that for the normalized mean curvature flow,

$$(\partial_t A - \Delta A)(e_i, e_j) = -\nabla_{V^T} a_{ij} + |A|^2 a_{ij},$$

that is,

$$(\partial_t A - \Delta A) = -\nabla_{V^T} A + |A|^2 A. \tag{4}$$

Applying Hamilton's tensor maximum principle above (see also Proposition 12.31 in [2]) we know that $A \geq 0$ is preserved along the translating mean curvature flow. Note that by taking the trace of (4), we have

$$(\partial_t - \Delta) H = -\nabla_{V^T} H + |A|^2 H.$$

We can apply the scalar maximum principle to this equation and to (3) too. This gives the property (1) in Theorem 1.

By these formulae for A, H, and $\langle V, \nu \rangle$ we obtain that

$$(\partial_t - \Delta)\left(A + \beta \frac{\langle V, \nu \rangle}{n} g\right) = -\nabla_{V^T}\left(A + \beta \frac{\langle V, \nu \rangle}{n} g\right) + |A|^2 \left(A + \beta \frac{\langle V, \nu \rangle}{n} g\right),$$

and

$$(\partial_t - \Delta)(H - \beta\langle V, \nu \rangle) = -\nabla_{V^T}(H - \beta\langle V, \nu \rangle) + |A|^2 (H - \beta\langle V, \nu \rangle).$$

Define the operator

$$L = \Delta - \nabla_{V^T} + |A|^2 = \mathbf{L} + |A|^2.$$

Then the above equations can be rewritten as

$$(\partial_t - L)\left(A + \beta \frac{\langle V, \nu \rangle}{n} g\right) = 0$$

and

$$(\partial_t - L)(\beta\langle V, \nu \rangle - H) = 0.$$

Applying the maximum principle (and Hamilton's tensor maximum principle as above) to above two equations, we complete the proof of Theorem 1. □

One immediate consequence is the following pinching estimate.

Corollary 4. *Given a translating mean curvature flow M_t with bounded second fundamental form A, $t \in [0, T)$ with $T > 0$. Assume that for some uniform constants β_1 and β_2, $\beta_1 \langle V, \nu \rangle \leq H \leq \beta_2 \langle V, \nu \rangle$ on the initial hypersurface M_0. Then $\beta_1 \langle V, \nu \rangle \leq H \leq \beta_2 \langle V, \nu \rangle$ on M_t for $t > 0$.*

The proof is the same as (2) in Theorem 1.

As in [3] we have for any symmetric 2-tensor f and positive function h on the manifold M,

$$(\partial_t - \mathbf{L})|f|^2 \le 2\langle f, (\partial_t - \mathbf{L})f\rangle,$$

$$(\partial_t - \mathbf{L})\left|\frac{f}{h}\right|^2 \le 2\left\langle \frac{f}{h}, (\partial_t - \mathbf{L})\frac{f}{h}\right\rangle,$$

and

$$(\partial_t - \mathbf{L})\frac{f}{h} = \frac{(\partial_t - \mathbf{L})f}{h} - \frac{f(\partial_t - \mathbf{L})h}{h^2} + \frac{2}{h}\left\langle \nabla h, \nabla\frac{f}{h}\right\rangle.$$

Then we have

$$(\partial_t - \mathbf{L})\left|\frac{f}{h}\right|^2 \le 2\left\langle \nabla\left|\frac{f}{h}\right|^2, \nabla \log h\right\rangle$$

when we put

$$f = A + \lambda\frac{\langle V, v\rangle}{n}g$$

for any real number λ and

$$h = \beta\langle V, v\rangle - H.$$

Let

$$B = \frac{A + \lambda\frac{\langle V, v\rangle}{n}g}{\beta\langle V, v\rangle - H}.$$

By the maximum principle, we have the following.

LEMMA 5. *Let $M_t \subset R^{n+1}$ be a one parameter family of hypersurfaces evolved by the translating mean curvature flow* (1). *Assume that $\beta\langle V, v\rangle - H > 0$ on the initial hypersurface for some constant β, and $|A|^2$ are bounded on each M_t. Then*

$$(\partial_t - \mathbf{L})|B|^2 \le 2\langle \nabla|B|^2, \nabla \log(\beta\langle V, v\rangle - H)\rangle, \quad \text{on } M_t.$$

We now point out the geometric meaning of the operator $\Delta - \nabla_{V^T} + |A|^2$ on the hypersurface M. Define the operator

$$L = \Delta - \nabla_{V^T} + |A|^2,$$

which is the Jacobian operator for the weighted volume

$$F(M) = \int_M e^{-\langle V, X\rangle}\, dX.$$

Then, $F' = -\overline{H} - V^N = Hv - V^N$. In fact, for $X_t = X' = fv$ and $H' = \partial_t H$, we have

$$H' = -\Delta f - |A|^2 f$$

and

$$v' = -\nabla f.$$

Then

$$(H - \langle V, v \rangle)' = -\Delta f - |A|^2 f + \langle V, \nabla f \rangle = -Lf.$$

At the critical point of F where

$$H = \langle V, v \rangle,$$

we have

$$F'' = -\int_M \langle f, Lf \rangle \, dm$$

where $dm = e^{-\langle V, X \rangle} \, dX$.

3. The Dirichlet problem for the translating graphical mean curvature flow

Recall that $\Omega \subset R^n$ is a bounded convex domain with C^2 boundary.

Note that the flow (2) corresponds to the negative gradient flow of the weighted area functional

$$F(u) = \int_\Omega \sqrt{1 + |Du|^2} e^{u(x)} \, dx.$$

In fact,

$$\delta F(u) \delta u = -\int_\Omega \left[div \left(\frac{Du}{\sqrt{1 + |Du|^2}} \right) - \frac{1}{v} \right] \delta u e^{u(x)} \, dx,$$

where $v = \sqrt{1 + |Du|^2}$. The functional $F(u)$ corresponds to the functional $F(M)$ with $V = -e_{n+1}$ in the previous section.

We point out a similarity between the translating mean curvature flow (2) and the graphical mean curvature flow. Fix any $t_0 > 0$. Define $U = u - t_0 + t$. Then U satisfies the following

$$\partial_t U = \sqrt{1 + |DU|^2} \, div \left(\frac{DU}{\sqrt{1 + |DU|^2}} \right), \quad in \ \Omega \times [0, \infty) \tag{5}$$

with the Dirichlet boundary condition

$$U = \phi - t_0 + t, \quad on \ \partial\Omega, \ t \geq 0$$

and the initial condition

$$U(x,0) = u_0(x) - t_0, \quad x \in \Omega.$$

Define $Q = \sqrt{1+|DU|^2}$. Then Q satisfies

$$(6) \qquad \partial_t Q = D_i(a^{ij}D_jQ) + Ha^l D_l Q - a^{ij}a^{kl}D_iD_kuD_jD_lu \cdot Q, \quad in \ \Omega \times (0,T)$$

where $a^i = Q^{-1}D_iU$ and $a^{ij} = \partial a^i/\partial x_j$. We shall use (6) to get the uniform gradient bound of u. We need to control $\sup_{\partial\Omega}|Du|$ first. Because of the equation (6) (being the same as the case of mean curvature flow) we believe the result of Theorem 2 should also be true for mean convex domains. However, we shall not discuss this in this note.

We now begin the *proof of Theorem* 2.

The existence of short time solution to (2) can be obtained by the standard method. Let $T > 0$ be the maximal existence time of the solution $u(x,t)$. We claim that $T = +\infty$. To obtain this, we need to find a priori estimates for $\sup_\Omega|u|$ and $\sup_\Omega|Du|$.

Define

$$Au = -div\left(\frac{Du}{\sqrt{1+|Du|^2}}\right).$$

Let w be the bowl soliton constructed by Altschuler-Wu [1]. Then we have

$$-\sqrt{1+|Dw|^2}Aw = 1, \quad in \ R^n.$$

Note that w satisfies (2). By adding to w some uniform constant C we may assume $w - C \le -\sup_\Omega|u_0|$ and $w + C \ge \sup_\Omega|u_0|$. Using $w \pm C$ as the barriers, we obtain that

$$w - C \le u \le w + C, \quad in \ \Omega \times [0,T).$$

This gives us the uniform bound of $\sup_\Omega|u|$.

We now use the fact that the domain Ω is convex. Recall that by the result of J. Serrin [12] or applying Cor. 14.3 in [5] to the operator

$$Q(u) = -(\sqrt{1+|Du|^2})^3 Au - (1+|Du|^2),$$

we can construct barriers δ_+ and δ_- such that $\delta_\pm|_{\partial\Omega} = \phi$,

$$-(\sqrt{1+|D\delta_+|^2})^3 A\delta_+ \le 1 + |D\delta_+|^2, \quad \delta_+ \ge \phi$$

and

$$-(\sqrt{1+|D\delta_-|^2})^3 A\delta_- \ge 1 + |D\delta_-|^2, \quad \delta_- \le \phi$$

in Ω. We may also assume $\delta_- \le u_0 \le \delta_+$ (see [7]).

Applying the maximum principle to the evolution equation (2) we know that $\delta_- \le u \le \delta_+$ on $\Omega \times [0,T)$. Thus, we know that there is a uniform constant C_0

depending only on $\partial\Omega$, u_0, and ϕ such that

$$|Du| \le C_0, \quad on\ \partial\Omega \times [0, T).$$

Applying the maximum principle to the equation (6) for Q, we obtain the uniform bound for $\sup_\Omega |Du|$. Once these are done, we then get the existence of the unique solution to the Dirichlet problem of (2) for all times $0 < t < \infty$ with the uniform gradient bound on $\sup_\Omega |Du|$. The standard parabolic equation theory [9] guarantees uniform bounds of all higher derivatives of u. Since $\partial_t u = 0$ on $\partial\Omega$, by the equation we have $H + \frac{1}{v} = 0$ on $\partial\Omega$ and for $dm := e^u\, dx$,

$$\frac{d}{dt}\int_\Omega v\, dm = -\int_\Omega \left(H + \frac{1}{v}\right)^2 dm.$$

Then

$$\int_0^\infty \int_\Omega \left(H + \frac{1}{v}\right)^2 dm \le \int_\Omega v\, dm(0).$$

Using the uniform bound about v, we can conclude that $\sup_\Omega |\partial_t u|$ and $\sup_\Omega \left|H + \frac{1}{v}\right|$ converges to zero uniformly as $t \to \infty$. This completes the proof of Theorem 2. □

Acknowledgments. The author is grateful to the unknown referees for helpful suggestions.

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