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KODAI MATH. J.
41 (2018), 284–300

A NON-INTEGRATED HYPERSURFACE DEFECT RELATION FOR MEROMORPHIC MAPS OVER COMPLETE KÄHLER MANIFOLDS INTO PROJECTIVE ALGEBRAIC VARIETIES

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Abstract

In this paper, a non-integrated defect relation for meromorphic maps from complete Kähler manifolds M into smooth projective algebraic varieties V intersecting hypersurfaces located in k-subgeneral position (see (1.5) below) is proved. The novelty of this result lies in that both the upper bound and the truncation level of our defect relation depend only on k, $\dim_{\mathbf{C}}(V)$ and the degrees of the hypersurfaces considered; besides, this defect relation recovers Hirotaka Fujimoto [6, Theorem 1.1] when subjected to the same conditions.

1. Introduction

Fujimoto [4, 5, 6, 8] introduced the innovative notion of non-integrated, or modified, defect for meromorphic maps over a complex Kähler manifold into the complex projective space. Recent extensions and generalizations may be found in Ru and Sogome [17] as well as Tan and Truong [19]. Below we will replicate the essential elements in this aspect from those references.

Denote M an m-dimensional Kähler manifold with Kähler form $\boldsymbol{\omega} = \frac{\sqrt{-1}}{2}\sum_{i,j} h_{i\bar{j}}\, \boldsymbol{d}z_i \wedge \boldsymbol{d}\bar{z}_j$. Write $\mathrm{Ric}\,\boldsymbol{\omega} = \boldsymbol{dd^c}\log(\det(h_{i\bar{j}}))$ with $\boldsymbol{d} = \partial + \bar{\partial}$ and $\boldsymbol{d^c} = \frac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial)$. Let $f : M \to \mathbf{P}^n(\mathbf{C})$ be a meromorphic map, and let D be a hypersurface in $\mathbf{P}^n(\mathbf{C})$ of degree d with $f(M) \not\subseteq D$. Take ν_D^f to be the intersection divisor generated through f and D, and take $\mu_0 > 0$ to be an integer. Denote $\mathcal{A}(D, \mu_0)$ the family of constants $\eta \geq 0$ such that there exists a bounded, nonnegative, continuous function h on M, with zeros of order no less than $\min\{\nu_D^f, \mu_0\}$, satisfying

$$d\eta\Omega_f + \boldsymbol{dd^c}\log \mathsf{h}^2 \geq [\min\{\nu_D^f, \mu_0\}]. \tag{1.1}$$

2010 *Mathematics Subject Classification*. 32H30, 32H04, 32H25, 32A22.

Key words and phrases. Kähler manifold, projective algebraic variety, meromorphic map, Nevanlinna theory, hypersurface, k-subgeneral position, non-integrated defect relation.

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Received March 1, 2017.

Here, Ω_f denotes the pull-back of the normalized Fubini-Study metric form on $\mathbf{P}^n(\mathbf{C})$, and $[\nu]$ denotes the $(1,1)$-current associated with the divisor $\nu \geq 0$.

Note condition (1.1) says that for each nonzero holomorphic function ψ on an open set U of M with $\nu_\psi^0 = \min\{\nu_D^f, \mu_0\}$ outside an analytic subset of codimension at least 2, the function $\mathsf{v} := \log\left(\frac{\mathsf{h}^2\|\mathfrak{f}\|^{2d\eta}}{|\psi|^2}\right)$ is continuous and plurisubharmonic, where $\|\mathfrak{f}\|^2 = \sum_{\imath=0}^n |\mathsf{f}_\imath|^2$ for a (local) reduced representation $\mathfrak{f} = (\mathsf{f}_0, \mathsf{f}_1, \ldots, \mathsf{f}_n) : M \to \mathbf{C}^{n+1}$ of $f = [\mathsf{f}_0 : \mathsf{f}_1 : \cdots : \mathsf{f}_n]$.

The *non-integrated defect* of f regarding D, truncated at level μ_0, is defined as

$$\delta_{\mu_0}^f(D) = 1 - \inf\{\eta \geq 0 : \eta \in \mathcal{A}(D, \mu_0)\}. \tag{1.2}$$

Then, like Nevanlinna's or Stoll's classical defects, $0 \leq \delta_{\mu_0+1}^f(D) \leq \delta_{\mu_0}^f(D) \leq 1$, $\delta_{\mu_0}^f(D) = 1$ if $f(M) \cap D = \emptyset$, and $\delta_{\mu_0}^f(D) \geq 1 - \frac{\mu_0}{\mu}$ for any integer $\mu \geq \mu_0$ if $[\nu_D^f - \mu] \geq 0$ on $f^{-1}(D)$.

Further, we say $f : M \to \mathbf{P}^n(\mathbf{C})$ satisfies the "**condition** $\mathbf{C}(\rho)$" provided for some constant $\rho \geq 0$, there is a bounded, nonnegative, continuous function h on M such that

$$\rho\Omega_f + dd^c \log \mathsf{h}^2 \geq \mathrm{Ric}\, \omega. \tag{1.3}$$

Now, the original result of Fujimoto [6, Theorem 1.1] can be stated as follows.

THEOREM 1.1. *Assume M is an m-dimensional complete Kähler manifold such that the universal covering of M is biholomorphically isomorphic to a ball in $\mathbf{C}^m$. Let $f : M \to \mathbf{P}^n(\mathbf{C})$ be a linearly non-degenerate meromorphic map such that the* **condition** $\mathbf{C}(\rho)$ *is satisfied, and let $H_1, H_2, \ldots, H_q$ be $q(\geq n+1)$ hyperplanes in $\mathbf{P}^n(\mathbf{C})$ that are located in general position. Then, one has the following defect relation*

$$\sum_{j=1}^q \delta_n^f(H_j) \leq n + 1 + \rho n(n+1). \tag{1.4}$$

Ru and Sogome [17] (see also Yan [20]), and Tan and Truong [19] generalized independently the preceding Theorem 1.1 in the way that $\mathbf{P}^n(\mathbf{C})$ is replaced by a projective algebraic variety $V \subseteq \mathbf{P}^N(\mathbf{C})$ and hyperplanes in $\mathbf{P}^n(\mathbf{C})$ located in general position are extended to hypersurfaces in $\mathbf{P}^N(\mathbf{C})$ located in different types of k-subgeneral positions. One recalls that the k-subgeneral position condition used in [19] comes from Dethloff, Tan and Thai [3, Definition 1.1].

It is noteworthy that both the upper bounds and the truncation levels of the defect relations obtained in [17, Theorem 1.1], [19, Definition 1.1 and Theorem 1.2] and [20, Definition 1.2 and Theorem 1.1] depend on a given constant $\varepsilon > 0$, and both blow up to $+\infty$ as $\varepsilon \to 0$. Also, it's not clear to us if those results can recover Theorem 1.1 under the same assumptions.

In the sequel, assume that $V \subseteq \mathbf{P}^N(\mathbf{C})$ is a smooth projective algebraic variety of dimension $n(\leq N)$. $q(>k)$ hypersurfaces $D_1, D_2, \ldots, D_q$ in $\mathbf{P}^N(\mathbf{C})$ are said to be located in *k-subgeneral position* $(k \geq n)$ with respect to V provided for every $1 \leq j_0 < j_1 < \cdots < j_k \leq q$,

$$\left(\bigcap_{s=0}^{k} \operatorname{supp}(D_{j_s})\right) \cap V = \emptyset^1. \tag{1.5}$$

Here, supp(D) is the support of the divisor D. One says $D_1, D_2, \ldots, D_q$ are in general position with respect to V, if they are located in n-subgeneral position with respect to V.

The purpose of this paper is by combining the techniques used in [19] and [20] to describe a hypersurface defect relation, with definite truncation level and explicit upper bound, that will be exactly Fujimoto's original Theorem 1.1 when $d = 1$, $k = n = N$ and $V = \mathbf{P}^n(\mathbf{C})$.

Fix an integer $d \geq 1$. Write $\mathcal{H}_d$ the vector space of homogeneous polynomials of degree d in $\mathbf{C}[w_0, w_1, \ldots, w_N]$ and $\mathcal{I}_V$ the prime ideal in $\mathbf{C}[w_0, w_1, \ldots, w_N]$ defining V. Denote

$$H_V(d) := \dim_{\mathbf{C}}\left(\frac{\mathcal{H}_d}{\mathcal{H}_d \cap \mathcal{I}_V}\right)$$

to be the Hilbert function of V. Recall $H_V(d) = n + 1$ when $d = 1$, $n = N$ and $V = \mathbf{P}^n(\mathbf{C})$.

Finally, we can formulate our main theorem of this paper as the following result.

THEOREM 1.2. *Assume M is an m-dimensional complete Kähler manifold such that the universal covering of M is biholomorphically isomorphic to a ball in $\mathbf{C}^m$, and assume $V \subseteq \mathbf{P}^N(\mathbf{C})$ is an irreducible projective algebraic variety of dimension $n(\leq N)$. Let $f : M \to V$ be an algebraically non-degenerate meromorphic map such that the* **condition** $\mathbf{C}(\rho)$ *is satisfied, and let $D_1, D_2, \ldots, D_q$ be $q(\geq k+1)$ hypersurfaces in $\mathbf{P}^N(\mathbf{C})$ that are located in k-subgeneral position $(k \geq n)$ regarding V and have degrees $d_1, d_2, \ldots, d_q$ respectively. Denote by d the least common multiple of $d_1, d_2, \ldots, d_q$. Then, one has the following defect relation*

$$\sum_{j=1}^{q} \delta^f_{H_V(d)-1}(D_j) \leq \frac{2k-n+1}{n+1}\left\{H_V(d) + \frac{\rho}{d} H_V(d)(H_V(d)-1)\right\}. \tag{1.6}$$

It is worthwhile to mention when $d = 1$, $k = n = N$ and $V = \mathbf{P}^n(\mathbf{C})$, Theorem 1.2 recovers exactly Fujimoto's initial work. As $H_V(d) \leq \binom{d+N}{N}$, the

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[1] As far as we can check, this condition (1.5) appeared first in Chen, Ru and Yan [2].

truncation level in Theorem 1.2 is smaller than that in [19, Theorem 1.2] and also better than those in [17, 20], yet the upper bound in (1.6) might be larger than those in [17, 19, 20] (depending on their ε).

2. Preliminaries

In this auxiliary section, we describe some basic notations and necessary results that are used afterwards throughout this paper.

Denote $\|z\|^2 = \sum_{j=1}^m |z_j|^2$ for $z = (z_1, z_2, \dots, z_m) \in \mathbf{C}^m$. Write $B(r) = \{z \in \mathbf{C}^m : \|z\| < r\}$ and $S(r) = \{z \in \mathbf{C}^m : \|z\| = r\}$ for $r \in (0, \infty)$, and $B(\infty) = \mathbf{C}^m$. Define

$$\upsilon_j = (\boldsymbol{dd^c}\|z\|^2)^j \quad \text{for } j = 1, 2, \dots, m \text{ on } \mathbf{C}^m, \text{ and}$$

$$\sigma_m = \boldsymbol{d^c} \log\|z\|^2 \wedge (\boldsymbol{dd^c} \log\|z\|^2)^{m-1} \quad \text{on } \mathbf{C}^m \backslash \{0\}.$$

Suppose $f : B(R_0) \to \mathbf{P}^n(\mathbf{C})$ is a meromorphic map with $0 < R_0 \leq \infty$. Choose holomorphic functions $\mathfrak{f}_0, \mathfrak{f}_1, \dots, \mathfrak{f}_n$ with $\tilde{\mathfrak{f}} = (\mathfrak{f}_0, \mathfrak{f}_1, \dots, \mathfrak{f}_n) : B(R_0) \backslash \mathfrak{I}_f \to \mathbf{C}^{n+1}$ a reduced representation of f. Notice the singularity set $\mathfrak{I}_f := \{z \in B(R_0) : \mathfrak{f}_0(z) = \mathfrak{f}_1(z) = \cdots = \mathfrak{f}_n(z) = 0\}$ of f is of dimension at most $m - 2$. Fix this reduced representation $\tilde{\mathfrak{f}}$ of f. Then, $\Omega_f = \boldsymbol{dd^c} \log\|\tilde{\mathfrak{f}}\|^2$ will be the pull-back of the normalized Fubini-Study metric form on $\mathbf{P}^n(\mathbf{C})$ through f.

Given $r_0 \in (0, R_0)$, the *characteristic function* of f for $r \in (r_0, R_0)$ is defined as

$$T_f(r, r_0) = \int_{r_0}^r \frac{\boldsymbol{dt}}{t^{2m-1}} \int_{B(t)} \Omega_f \wedge \upsilon_{m-1}, \tag{2.1}$$

which can also be written as

$$T_f(r, r_0) = \int_{S(r)} \log\|\tilde{\mathfrak{f}}\|\sigma_m - \int_{S(r_0)} \log\|\tilde{\mathfrak{f}}\|\sigma_m. \tag{2.2}$$

For a holomorphic function ψ on an open subset U of $\mathbf{C}^m$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m) \in \mathbf{Z}_{\geq 0}^m$, an m-tuple of nonnegative integers, set $|\alpha| := \sum_{j=1}^m \alpha_j$ and $D^\alpha\psi := D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_m^{\alpha_m}\psi$ where $D_j\psi = \dfrac{\partial\psi}{\partial z_j}$ for $j = 1, 2, \dots, m$. Define $\nu_\psi^0 : U \to \mathbf{Z}_{\geq 0}$ by $\nu_\psi^0(z) := \max\{\kappa : D^\alpha\psi(z) = 0\}$ for all possible $\alpha \in \mathbf{Z}_{\geq 0}^m$ with $|\alpha| < \kappa$, and write $\mathrm{supp}(\nu_\psi^0) := \overline{\{z \in U : \nu_\psi^0(z) > 0\}}$.

For a meromorphic function φ on U, there exist two coprime holomorphic functions ψ_1, ψ_2 on U with $\varphi = \dfrac{\psi_1}{\psi_2}$ such that $\nu_\varphi^\infty := \nu_{\psi_2}^0$ and $\mathrm{supp}(\nu_\varphi^\infty) := \mathrm{supp}(\nu_{\psi_2}^0)$.

Take $\mu_0 > 0$ an integer or ∞. For a meromorphic map $f : B(R_0) \to \mathbf{P}^n(\mathbf{C})$ with a reduced representation $\tilde{\mathfrak{f}}$ and a hypersurface D in $\mathbf{P}^n(\mathbf{C})$ of degree d with Q its defining homogeneous polynomial, let $\nu_D^f := \nu_{Q(\tilde{\mathfrak{f}})}^0$ be the intersection divisor associated with f and D on $B(R_0) \backslash \mathfrak{I}_f$. The *valence function* of f regarding D, with truncation level μ_0, is defined to be

$$N_f^{\mu_0}(r, r_0; D) = \int_{r_0}^{r} \frac{n_f^{\mu_0}(t; D)}{t} \, \boldsymbol{d}t, \tag{2.3}$$

where

$$n_f^{\mu_0}(t; D) := \begin{cases} \dfrac{1}{t^{2m-2}} \displaystyle\int_{\mathrm{supp}(\nu_D^f) \cap B(t)} \min\{\nu_D^f, \mu_0\} \upsilon_{m-1} & \text{when } m \geq 2, \\ \sum_{\|z\|<t} \min\{\nu_D^f(z), \mu_0\} & \text{when } m = 1. \end{cases}$$

The first main theorem says $N_f^{\mu_0}(r, r_0; D) \leq dT_f(r, r_0) + O(1)$ (see [9, 10]). Let $\hat{\delta}_{\mu_0}^f(D)$ be Nevanlinna's defect or its high dimensional extension by Stoll that is defined as

$$\hat{\delta}_{\mu_0}^f(D) = 1 - \limsup_{r \to R_0} \frac{N_f^{\mu_0}(r, r_0; D)}{dT_f(r, r_0)}.$$

When $\lim_{r \to R_0} T_f(r, r_0) = \infty$, then [6, Proposition 5.6] or [17, Proposition 2.1] yield

$$0 \leq \delta_{\mu_0}^f(D) \leq \hat{\delta}_{\mu_0}^f(D) \leq 1. \tag{2.4}$$

Below, we recall two results of An, Quang and Thai [1, 16]. The first one is an extension to hypersurfaces of the celebrated Nochka weights [13, 14] concerning hyperplanes.

Proposition 2.1 ([1, Lemma 3.3] or [16, Lemma 3]). *Assume that* $V \subseteq \mathbf{P}^N(\mathbf{C})$ *is an irreducible projective algebraic variety of dimension* n $(n \leq N)$. *Let* $D_1, D_2, \ldots, D_q$ *be* $q > 2k - n + 1$ $(k \geq n)$ *hypersurfaces in* $\mathbf{P}^N(\mathbf{C})$ *of common degree* d *that are located in* k*-subgeneral position with respect to* V. *Then, there exist* q *rational numbers* $0 < \omega_1, \omega_2, \ldots, \omega_q \leq 1$ *such that*

(a.) *for* $\varpi := \max_{j \in \{1,2,\ldots,q\}}\{\omega_j\}$, *one has*

$$\omega_j \leq \varpi = \frac{\sum_{j=1}^q \omega_j - n - 1}{q - 2k + n - 1} \quad \textit{and} \quad \frac{n+1}{2k - n + 1} \leq \varpi \leq \frac{n}{k}\text{[2]}; \tag{2.5}$$

(b.) *for each subset* $\mathcal{R}$ *of* $\{1, 2, \ldots, q\}$ *with* $\#\mathcal{R} = k + 1$, *one has* $\sum_{j \in \mathcal{R}} \omega_j \leq n + 1$;

(c.) *for* q *arbitrarily given constants* $E_1, E_2, \ldots, E_q \geq 1$ *and each set* $\mathcal{R}$ *as in* (b.), *there exists a subset* $\mathcal{T}$ *of* $\mathcal{R}$ *with* $\#\mathcal{T} = \mathrm{rank}\{Q_j\}_{j \in \mathcal{T}} = n + 1$ *satisfying*

$$\prod_{j \in \mathcal{R}} E_j^{\omega_j} \leq \prod_{j \in \mathcal{T}} E_j, \tag{2.6}$$

[2] Note this upper bound in the second estimate of (2.5) was discovered by Toda [15].

where Q_j *is the defining homogeneous polynomial of* D_j *in* $\mathbf{P}^N(\mathbf{C})$ *for* $j = 1, 2, \dots, q$.

Lemma 2.2 ([1, Lemma 4.2] or [16, Lemma 5]). *Under the same assumptions of Proposition* 2.1, *for each subset* $\mathcal{T} \subseteq \{1, 2, \dots, q\}$ *with* $\#\mathcal{T} = \operatorname{rank}\{Q_j\}_{j \in \mathcal{T}} = n+1$, *there are* $H_V(d) - n - 1$ *hypersurfaces* $D_1^*, D_2^*, \dots, D_{H_V(d)-n-1}^*$ *in* $\mathbf{P}^N(\mathbf{C})$ *such that*

$$\operatorname{rank}\{\{Q_j\}_{j \in \mathcal{T}} \cup \{Q_i^*\}_{i=1}^{H_V(d)-n-1}\} = H_V(d).$$

Here, Q_j *and* Q_i^* *are the homogeneous polynomials defining* D_j *and* D_i^* *respectively.*

3. Proof of Theorem 1.2

First, it's interesting to notice the following consequence of our Theorem 1.2.

Theorem 3.1. *Suppose* M *is an* m*-dimensional complete Kähler manifold such that the universal covering of* M *is biholomorphically isomorphic to a ball in* $\mathbf{C}^m$. *Let* $f : M \to \mathbf{P}^N(\mathbf{C})$ *be a meromorphic map that satisfies the* **condition** $\mathbf{C}(\rho)$ *whose image spans a linear subspace of dimension* n, *not contained in any of* $H_1, H_2, \dots, H_q$, *where* $H_1, H_2, \dots, H_q$ *are* q *hyperplanes in* $\mathbf{P}^N(\mathbf{C})$ *located in general position. Then, one has the following defect relation*

$$\sum_{j=1}^{q} \delta_n^f(H_j) \le 2N - n + 1 + \rho n(2N - n + 1). \tag{3.1}$$

This is a Cartan-Nochka type result. For the classical defect relation, the associated second main theorem was originally suggested by Cartan and proved by Nochka; for that with truncation, the associated second main theorem was initially shown by Fujimoto [8, Theorem 3.2.12] and refined by Noguchi [15, Theorem 3.1] with a better estimate about error terms.

For each $j = 1, 2, \dots, q$, set Q_j to be the homogeneous polynomial of degree d_j defining D_j in $\mathbf{P}^N(\mathbf{C})$; replacing Q_j by Q_j^{d/d_j} when necessary, we may assume $Q_1, Q_2, \dots, Q_q \in \mathcal{H}_d$, where from now on we use d to represent the least common multiple of $d_1, d_2, \dots, d_q$.

Now, we will proceed to prove Theorem 1.2 by considering two situations

$$\limsup_{r \to R_0} \frac{T_f(r, r_0)}{\log \dfrac{1}{R_0 - r}} < \infty \quad \text{and} \quad \limsup_{r \to R_0} \frac{T_f(r, r_0)}{\log \dfrac{1}{R_0 - r}} = \infty \tag{3.2}$$

when the universal covering of M is biholomorphic to a finite ball $B(R_0)$ in $\mathbf{C}^m$.

Let $\pi : \tilde{M} \to M$ be the universal covering of M. Then, $f \circ \pi : \tilde{M} \to V$ is again algebraically non-degenerate since $f : M \to V$ is algebraically non-degenerate; also, one has $\delta^{f}_{H_V(d)-1}(D_j) \le \delta^{f\circ\pi}_{H_V(d)-1}(D_j)$. Hence, by lifting f to the covering, we fix $M = B(1)$ subsequently.

Consider first the former case in (3.2) that is more important.

Assume $\mathfrak{f} = (\mathfrak{f}_0, \mathfrak{f}_1, \dots, \mathfrak{f}_N)$ and $Q_j = \sum_{\beta\in\mathscr{I}_d} a_{j\beta} w^\beta$, where $\mathscr{I}_d$ is the set of $(N+1)$-tuples $\beta = (\beta_0, \beta_1, \dots, \beta_N) \in \mathbf{Z}^{N+1}_{\ge 0}$ with $|\beta| := \sum_{\imath=0}^N \beta_\imath = d$ and $w^\beta := w_0^{\beta_0} w_1^{\beta_1} \cdots w_N^{\beta_N}$. For every $j = 1, 2, \dots, q$, notice $|Q_j(\mathfrak{f})| = |\sum_{\beta\in\mathscr{I}_d} a_{j\beta}\mathfrak{f}^\beta| \le (\sum_{\beta\in\mathscr{I}_d} \varrho_\beta |a_{j\beta}|)\|\mathfrak{f}\|^d$ so that

$$|Q_j(\mathfrak{f})| \le \varrho\|\mathfrak{f}\|^d \quad \text{with } \varrho := \sum_{j=1}^q \sum_{\beta\in\mathscr{I}_d} \varrho_\beta |a_{j\beta}| > 0. \tag{3.3}$$

Fix a basis $\{\phi_1, \phi_2, \dots, \phi_{H_V(d)}\} \subseteq \mathcal{H}_d$ of $\dfrac{\mathcal{H}_d}{\mathcal{H}_d \cap \mathcal{I}_V}$. Because f is algebraically non-degenerate, $F := [\phi_1(\mathfrak{f}) : \phi_2(\mathfrak{f}) : \cdots : \phi_{H_V(d)}(\mathfrak{f})] : M \to \mathbf{P}^{H_V(d)-1}(\mathbf{C})$ is linearly non-degenerate. In view of [6, Proposition 4.5], there exist $H_V(d)$ m-tuples $\alpha^l = (\alpha^l_1, \alpha^l_2, \dots, \alpha^l_m) \in \mathbf{Z}^m_{\ge 0}$ with

$$|\alpha^l| = \sum_{\jmath=1}^m \alpha^l_\jmath < l \quad \text{and} \quad \sum_{l=1}^{H_V(d)} |\alpha^l| \le \frac{H_V(d)(H_V(d)-1)}{2}, \tag{3.4}$$

such that the Wronskian $W_{\alpha^1\dots\alpha^{H_V(d)}}(F)$ of F is not identically zero on M, where

$$W_{\alpha^1\dots\alpha^{H_V(d)}}(F) := \det(D^{\alpha^l}\phi_\ell(\mathfrak{f}))_{1\le l,\ell\le H_V(d)}. \tag{3.5}$$

For any subset $\mathcal{T} \subseteq \{1, 2, \dots, q\}$ with $\#\mathcal{T} = \operatorname{rank}\{Q_j\}_{j\in\mathcal{T}} = n+1$, use the hypersurfaces in Lemma 2.2 to define $F_\mathcal{T} := [\{Q_j(\mathfrak{f})\}_{j\in\mathcal{T}} : Q^*_1(\mathfrak{f}) : \cdots : Q^*_{H_V(d)-n-1}(\mathfrak{f})]$ (by abuse of notations). Then, there is a constant $C_\mathcal{T} \ne 0$ such that $W_{\alpha^1\dots\alpha^{H_V(d)}}(F_\mathcal{T}) = C_\mathcal{T} W_{\alpha^1\dots\alpha^{H_V(d)}}(F)$.

Fix $\mathfrak{w} \in V \cap f(M)$. Abusing the notation, $\mathfrak{w} = cw$ for some $w \in \mathbf{C}^{N+1}\backslash\{0\}$ and all complex numbers $c \ne 0$. Pick a subset $\mathcal{R}$ of $\{1, 2, \dots, q\}$ with $\#\mathcal{R} = k+1$ such that $|Q_j(w)| \le |Q_s(w)|$ when $j \in \mathcal{R}$ and $s \in \{1, 2, \dots, q\}\backslash\mathcal{R}$; using the k-subgeneral position hypothesis (1.5) and the continuity of $\dfrac{|Q_s(w)|^2}{(|w_0|^2 + |w_1|^2 + \cdots + |w_N|^2)^d}$, there exists a constant $\gamma_Y > 0$ such that

$$\gamma_Y\|\mathfrak{f}(z)\|^d \le \min_{s\in\{1,2,\dots,q\}\backslash\mathcal{R}} |Q_s(\mathfrak{f})(z)| \tag{3.6}$$

for all $z \in f^{-1}(Y)\backslash I_f$, where Y is an appropriate open neighborhood of $\mathfrak{w}$ in V.

Take such a z and set $E_j := \dfrac{\varrho\|\mathfrak{f}(z)\|^d}{|Q_j(\mathfrak{f})(z)|} \ge 1$ for $j \in \mathcal{R}$; then, Proposition 2.1—Parts (b.)&(c.) yields a subset $\mathcal{T}$ of $\mathcal{R}$ with $\#\mathcal{T} = n+1$ such that the estimate (2.6) holds. Noting (3.3), (3.6) and the estimate concerning these E_j for $\mathcal{R}$, $\mathcal{T}$, one observes that

$$\frac{\|\mathfrak{f}(z)\|^{d\sum_{j=1}^{q}\omega_j}|W_{\alpha^1\dots\alpha^{H_V(d)}}(F)(z)|}{|Q_1(\mathfrak{f})(z)|^{\omega_1}\cdots|Q_q(\mathfrak{f})(z)|^{\omega_q}} \le \prod_{j\in\mathcal{R}}\left(\frac{\varrho\|\mathfrak{f}(z)\|^d}{|Q_j(\mathfrak{f})(z)|}\right)^{\omega_j}\frac{|W_{\alpha^1\dots\alpha^{H_V(d)}}(F)(z)|}{\varrho^{\sum_{j\in\mathcal{R}}\omega_j}\gamma_Y^{\sum_{s\in\{1,2,\dots,q\}\setminus\mathcal{R}}\omega_s}}$$

$$\le K\frac{\|\mathfrak{f}(z)\|^{d(n+1)}|W_{\alpha^1\dots\alpha^{H_V(d)}}(F)(z)|}{\prod_{j\in\mathcal{T}}|Q_j(\mathfrak{f})(z)|}$$

$$\le K\frac{\|\mathfrak{f}(z)\|^{dH_V(d)}|W_{\alpha^1\dots\alpha^{H_V(d)}}(F_{\mathcal{T}})(z)|}{\prod_{j\in\mathcal{T}}|Q_j(\mathfrak{f})(z)|\prod_{i=1}^{H_V(d)-n-1}|Q_i^*(\mathfrak{f})(z)|}.$$

Here, and hereafter, $K > 0$ represents an absolute constant whose value may change from line to line but (in general) can be interpreted appropriately within the context.

For simplicity, put

$$\varphi := \frac{W_{\alpha^1\dots\alpha^{H_V(d)}}(F)}{Q_1^{\omega_1}(\mathfrak{f})\cdots Q_q^{\omega_q}(\mathfrak{f})} \quad \text{and} \quad \aleph(F_{\mathcal{T}}) := \frac{W_{\alpha^1\dots\alpha^{H_V(d)}}(F_{\mathcal{T}})}{\prod_{j\in\mathcal{T}}Q_j(\mathfrak{f})\prod_{i=1}^{H_V(d)-n-1}Q_i^*(\mathfrak{f})}.$$

Considering the compactness of V, we have

$$\|\mathfrak{f}(z)\|^{d\{\sum_{j=1}^{q}\omega_j - H_V(d)\}}|\varphi(z)| \le K\sum_{\mathcal{R},\mathcal{T}}|\aleph(F_{\mathcal{T}})(z)| \quad \forall z \in M\backslash I_f. \tag{3.7}$$

Here, the summation is taken over all the subsets $\mathcal{T} \subseteq \mathcal{R} \subseteq \{1, 2, \dots, q\}$ with $\#\mathcal{R} = k+1$ and $\#\mathcal{T} = n+1$. Since q, k, n are all finite, there can only be finitely many possibilities.

On the other hand, one may observe that

$$\nu_{\varphi}^{\infty} \le \sum_{j=1}^{q}\omega_j \min\{\nu_{D_j}^{f}, H_V(d)-1\} \tag{3.8}$$

outside an analytic subset of codimension at least 2. As a matter of fact, when $\zeta \in M\backslash I_f$ is a zero of some $Q_j(\mathfrak{f})$, it can be a zero of no more than $k+1$ functions $Q_j(\mathfrak{f})$ by (1.5). Assume $Q_j(\mathfrak{f})$ vanishes at ζ for $j \in \tilde{\mathcal{R}} \subseteq \{1, 2, \dots, q\}$ with $\#\tilde{\mathcal{R}} = k+1$ yet $Q_s(\mathfrak{f})(\zeta) \ne 0$ for $s \in \{1, 2, \dots, q\}\backslash\tilde{\mathcal{R}}$. By virtue of Proposition 2.1—Part (c.), putting $\tilde{E}_j := \exp(\max\{\nu_{D_j}^{f}(\zeta) - H_V(d) + 1, 0\}) \ge 1$ for $j \in \tilde{\mathcal{R}}$, there exists a subset $\tilde{\mathcal{T}}$ of $\tilde{\mathcal{R}}$ with $\#\tilde{\mathcal{T}} = \operatorname{rank}\{Q_j\}_{j\in\tilde{\mathcal{T}}} = n+1$ such that

$$\sum_{j\in\tilde{\mathcal{R}}}\omega_j \max\{\nu_{D_j}^{f}(\zeta) - H_V(d) + 1, 0\} \le \sum_{j\in\tilde{\mathcal{T}}}\max\{\nu_{D_j}^{f}(\zeta) - H_V(d) + 1, 0\},$$

from which it follows that, in view of $\nu^0_{W_{\alpha^1\dots\alpha^{H_V(d)}}(F)} = \nu^0_{W_{\alpha^1\dots\alpha^{H_V(d)}}(F_{\tilde{\mathcal{T}}})}$,

$$\sum_{j\in\tilde{\mathcal{R}}}\omega_j \max\{\nu_{D_j}^{f}(\zeta) - H_V(d) + 1, 0\} \le \nu^0_{W_{\alpha^1\dots\alpha^{H_V(d)}}(F)}(\zeta).$$

This estimate clearly leads to (3.8) upon verifying the following computations at ζ

$$\begin{aligned}v_{\varphi}^{\infty} &\leq \sum_{j=1}^{q}\omega_j v_{D_j}^{f} - v_{W_{\alpha^1\dots\alpha^{H_V(d)}}(F)}^{0} = \sum_{j\in\tilde{\mathcal{R}}}\omega_j v_{D_j}^{f} - v_{W_{\alpha^1\dots\alpha^{H_V(d)}}(F)}^{0}\\ &= \sum_{j\in\tilde{\mathcal{R}}}\omega_j(\min\{v_{D_j}^{f}, H_V(d)-1\} + \max\{v_{D_j}^{f} - H_V(d)+1, 0\}) - v_{W_{\alpha^1\dots\alpha^{H_V(d)}}(F)}^{0}\\ &\leq \sum_{j\in\tilde{\mathcal{R}}}\omega_j \min\{v_{D_j}^{f}, H_V(d)-1\} \leq \sum_{j=1}^{q}\omega_j \min\{v_{D_j}^{f}, H_V(d)-1\}.\end{aligned}$$

Next, we suppose that

$$\sum_{j=1}^{q}\omega_j\delta_{H_V(d)-1}^{f}(D_j) \leq H_V(d) + \frac{\rho}{d}H_V(d)(H_V(d)-1). \tag{3.9}$$

When (3.9) is true, then by (1.2) and the first relation in (2.5), it yields that

$$\sum_{j=1}^{q}\eta_j\omega_j \geq \varpi(q-2k+n-1) + n + 1 - H_V(d) - \frac{\rho}{d}H_V(d)(H_V(d)-1)$$

for all nonnegative constants $\eta_j \in \mathcal{A}(D_j, H_V(d)-1)$; that is,

$$\sum_{j=1}^{q}\eta_j \geq q-2k+n-1+\frac{1}{\varpi}\left\{n+1-H_V(d)-\frac{\rho}{d}H_V(d)(H_V(d)-1)\right\}.$$

This further implies that

$$\sum_{j=1}^{q}(1-\eta_j) \leq 2k-n+1+\frac{1}{\varpi}\left\{H_V(d)-n-1+\frac{\rho}{d}H_V(d)(H_V(d)-1)\right\},$$

which, along with the lower bound in the second estimate of (2.5), leads to (1.6).

In the sequel, we show by contradiction the validity of (3.9).

Suppose it doesn't hold. Then, by definition of non-integrated defect, there are nonnegative constants $\tilde{\eta}_j \in \mathcal{A}(D_j, H_V(d)-1)$ and continuous, pluri-subharmonic functions $\tilde{\mathsf{u}}_j \not\equiv -\infty$, for every $j = 1, 2, \dots, q$, such that $e^{\tilde{\mathsf{u}}_j}|\psi_j| \leq \|\mathfrak{f}\|^{d\tilde{\eta}_j}$ and

$$\sum_{j=1}^{q}(1-\tilde{\eta}_j)\omega_j > H_V(d) + \frac{\rho}{d}H_V(d)(H_V(d)-1). \tag{3.10}$$

Here, ψ_j is a nonzero holomorphic function that satisfies $v_{\psi_j}^{0} = \min\{v_{D_j}^{f}, H_V(d)-1\}$. Define $\mathsf{u}_j := \tilde{\mathsf{u}}_j + \log|\psi_j| \not\equiv -\infty$ that is continuous and pluri-subharmonic, and satisfies $e^{\mathsf{u}_j} \leq \|\mathfrak{f}\|^{d\tilde{\eta}_j}$. So, for $\vartheta_1(z) := \log|z^{\boldsymbol{\alpha}}\varphi(z)| + \sum_{j=1}^{q}\omega_j\mathsf{u}_j(z)$ with $\boldsymbol{\alpha} :=$

$\sum_{l=1}^{H_V(d)} \alpha^l \in \mathbf{Z}_{\geq 0}^m$, seeing the preceding analyses and (3.8), one clearly deduces that ϑ_1 is pluri-subharmonic on M.

Note we assumed the **condition** $\mathbf{C}(\rho)$ satisfied; that is, (1.3) holds. By [6, p252, Remark], there exists a continuous, pluri-subharmonic function $\vartheta_2 \not\equiv -\infty$ such that $e^{\vartheta_2}\, \boldsymbol{dV} \leq \|\mathfrak{f}\|^{2\rho} \upsilon_m$. Here, and henceforth, we use $\boldsymbol{dV}$ to denote the canonical volume form on M.

Set $t_0 := \dfrac{2\rho}{d\{\sum_{j=1}^q (1-\tilde{\eta}_j)\omega_j - H_V(d)\}} > 0$ and write $\theta := \vartheta_2 + t_0\vartheta_1$. Then, θ is pluri-subharmonic and thus a subharmonic function on $M = B(1)$. In addition, one has

$$e^{\theta}\, \boldsymbol{dV} = e^{\vartheta_2 + t_0\vartheta_1}\, \boldsymbol{dV} \leq e^{t_0\vartheta_1} \|\mathfrak{f}\|^{2\rho} \upsilon_m = |z^{\boldsymbol{\alpha}}\varphi|^{t_0} e^{t_0 \sum_{j=1}^q \omega_j \mathfrak{u}_j} \|\mathfrak{f}\|^{2\rho} \upsilon_m$$
$$\leq |z^{\boldsymbol{\alpha}}\varphi|^{t_0} \|\mathfrak{f}\|^{t_0 d \sum_{j=1}^q \omega_j \tilde{\eta}_j + 2\rho} \upsilon_m = |z^{\boldsymbol{\alpha}}\varphi|^{t_0} \|\mathfrak{f}\|^{t_0 d\{\sum_{j=1}^q \omega_j - H_V(d)\}} \upsilon_m.$$

By (3.4) and (3.10), we easily get $t_0(\sum_{l=1}^{H_V(d)} |\alpha^l|) < \varsigma < 1$ for some constant $\varsigma > 0$. Therefore, recalling $\upsilon_m = 2m\|z\|^{2m-1}\sigma_m \wedge \boldsymbol{d}\|z\|$ and (3.7), we have

$$\text{(3.11)} \quad \int_M e^{\theta}\, \boldsymbol{dV} \leq \int_M |z^{\boldsymbol{\alpha}}|^{t_0} |\varphi(z)| \|\mathfrak{f}(z)\|^{d\{\sum_{j=1}^q \omega_j - H_V(d)\}}|^{t_0} \upsilon_m$$
$$\leq K \sum_{\mathcal{R},\mathcal{T}} \int_0^1 r^{2m-1} \left(\int_{S(r)} |z^{\boldsymbol{\alpha}} \aleph(F_{\mathcal{T}})(z)|^{t_0} \sigma_m \right) \boldsymbol{dr}$$
$$\leq K \int_0^1 r^{2m-1} \left(\frac{R^{2m-1}}{R-r} T_F(R, r_0) \right)^{\varsigma} \boldsymbol{dr} \leq K \int_0^1 \left(\frac{1}{R-r} T_F(R, r_0) \right)^{\varsigma} \boldsymbol{dr}$$

when $r_0 < r < R < 1$, where we applied [6, Proposition 6.1] (see also [17, Proposition 3.3]) for the derivation of the third, or the second last, estimate in (3.11).

Finally, seeing Hayman [11, Lemma 2.4 (ii)] and letting $R = r + \dfrac{1-r}{eT_F(r, r_0)}$, one has

$$T_F(R, r_0) \leq 2T_F(r, r_0) \leq 2dT_f(r, r_0)$$

outside a set with finite logarithmic measure. Recall we assumed the case $\limsup_{r\to 1} \dfrac{T_f(r, r_0)}{\log \frac{1}{1-r}} < \infty$ in (3.2). The preceding analyses combined with [5, Proposition 5.5] yields that

$$\text{(3.12)} \quad \int_M e^{\theta}\, \boldsymbol{dV} \leq K \int_0^1 \left(\frac{2}{\frac{1-r}{eT_F(r, r_0)}} T_F(r, r_0) \right)^{\varsigma} \boldsymbol{dr} \leq K \int_0^1 \left(\frac{2d^2 e}{1-r} T_f^2(r, r_0) \right)^{\varsigma} \boldsymbol{dr}$$
$$\leq K \int_0^1 \frac{1}{(1-r)^{\varsigma}} \left(\log \frac{1}{1-r} \right)^{2\varsigma} \boldsymbol{dr} = \frac{K}{(1-\varsigma)^{2\varsigma+1}} \Gamma(2\varsigma + 1) < \infty.$$

This result however would contradict Yau [21] and Karp [12, Theorem B], as $M = B(1)$ has infinite volume with respect to the given complete Kähler metric; see [17, p1147].

From now on, we shall consider the latter case in (3.2) and the situation when the universal covering of M is biholomorphic to $\mathbf{C}^m$ simultaneously, since both may be treated essentially in the same way through traditional defect relation and (2.4). As can be seen from the following discussions, we don't need f to satisfy the growth **condition** $\mathbf{C}(\rho)$ in these settings.

Noting the description below (3.2), we without loss of generality assume $M = B(R_0)$ for some $0 < R_0 \le \infty$ afterwards. Moreover, when $R_0 = \infty$, we can use the flat metric to see $\operatorname{Ric}\boldsymbol{\omega} \equiv 0$; that is, all meromorphic maps $f : \mathbf{C}^m \to V$ satisfy the **condition** $\mathbf{C}(0)$ automatically.

PROPOSITION 3.2. *Under the same hypotheses of Theorem* 1.2 *concerning the algebraic variety V in $\mathbf{P}^N(\mathbf{C})$ and the hypersurfaces $D_1, D_2, \dots, D_q$ in $\mathbf{P}^N(\mathbf{C})$, let $f : B(R_0)(\subseteq \mathbf{C}^m) \to V$ be an algebraically non-degenerate meromorphic map with $0 < R_0 \le \infty$. Then, one has*

$$\left\{q - \frac{2k-n+1}{n+1} H_V(d)\right\} T_f(r, r_0) \le \sum_{j=1}^{q} \frac{1}{d_j} N_f^{H_V(d)-1}(r, r_0; D_j) + S_f(r, r_0), \tag{3.13}$$

where $S_f(r, r_0) \ge 0$ satisfies $S_f(r, r_0) \le K\{\log^+ T_f(r, r_0) + \log^+ r\}$ for all $r \in (r_0, \infty)$ outside a set of finite linear measure when $R_0 = \infty$ and

$$S_f(r, r_0) \le K \log^+ T_f(r, r_0) + \frac{2k-n+1}{2d(n+1)} H_V(d)(H_V(d)-1) \log^+ \frac{1}{R_0 - r} \tag{3.14}$$

for all $r \in (r_0, R_0)$ outside a set of finite logarithmic measure when $R_0 < \infty$.

Proof. Like the first case in (3.2), by (3.4), (3.7) and the argument in (3.11), one has

$$\int_{S(r)} |z^{\boldsymbol{\alpha}} \varphi(z)| \|\mathfrak{f}(z)\|^{d\{\sum_{j=1}^{q} \omega_j - H_V(d)\}} |^{\check{t}_0} \sigma_m \le K \left(\frac{R^{2m-1}}{R-r} T_F(R, r_0)\right)^{\check{\varsigma}}$$

for $r_0 < r < R < R_0$, which further implies that, applying the concavity of logarithm,

$$\int_{S(r)} \log|z^{\boldsymbol{\alpha}} \times \varphi(z)| \sigma_m + \int_{S(r)} \log\|\mathfrak{f}\|^{d\{\sum_{j=1}^{q} \omega_j - H_V(d)\}} \sigma_m \le \frac{\check{\varsigma}}{\check{t}_0} \log^+ \frac{1}{R-r} + K\{\log^+ T_F(R, r_0) + \log^+ R\}. \tag{3.15}$$

Here, $\check{t}_0, \check{\varsigma} > 0$ are arbitrarily given constants satisfying $\check{t}_0(\sum_{l=1}^{H_V(d)} |\alpha^l|) < \check{\varsigma} < 1$. Besides, use Jensen's formula and (3.8) to derive that

$$\int_{S(r)} \log|z^{\boldsymbol{\alpha}} \times \varphi(z)| \sigma_m \ge -\sum_{j=1}^{q} \omega_j N_f^{H_V(d)-1}(r, r_0; D_j) + O(1),$$

which combined with the first relation in (2.5) and (3.15) altogether leads to

$$\begin{aligned}&\{\varpi(q-2k+n-1)+n+1-H_V(d)\}T_f(r,r_0)\\&\quad\le\sum_{j=1}^{q}\frac{\omega_j}{d}N_f^{H_V(d)-1}(r,r_0;D_j)+\frac{H_V(d)(H_V(d)-1)}{2d}\log^+\frac{1}{R-r}\\&\qquad+K\{\log^+T_F(R,r_0)+\log^+R\}\end{aligned}$$

when $\frac{\check{\varsigma}}{\check{t}_0}$ approaches $\frac{H_V(d)(H_V(d)-1)}{2}$ from the above. Since $d_j\le d$, $\omega_j\le\varpi$ and $\frac{1}{\varpi}\le\frac{2k-n+1}{n+1}$, (3.13) follows immediately from the above inequality with

$$S_f(r,r_0):=\frac{1}{2d\varpi}H_V(d)(H_V(d)-1)\log^+\frac{1}{R-r}+K\{\log^+T_F(R,r_0)+\log^+R\}.$$

The remaining estimates about $S_f(r,r_0)$ appear to be exactly the same as those, for instance, in [6, Proposition 6.2] or [17, Theorem 4.5] by virtue of [11, Lemma 2.4]. □

A natural consequence of Proposition 3.2 is the standard defect relation

$$\sum_{j=1}^{q}\hat{\delta}^f_{H_V(d)-1}(D_j)\le\frac{2k-n+1}{n+1}H_V(d), \tag{3.16}$$

provided either $R_0=\infty$ and f is transcendental[3], or $R_0<\infty$ and $\limsup_{r\to R_0}\frac{T_f(r,r_0)}{\log\frac{1}{R_0-r}}=\infty$. Thus, (1.6) follows from (2.4) and (3.16) so that our proof is finished completely.

4. Some related uniqueness results

In 1986, Fujimoto [7] generalized the well-known five-value theorem of Nevanlinna to the situation of meromorphic maps over a complete, connected Kähler manifold M (whose universal covering is biholomorphic to a finite ball in $\mathbf{C}^m$) into $\mathbf{P}^n(\mathbf{C})$ that satisfy the growth **condition** $\mathbf{C}(\rho)$ and share hyperplanes; other closely related results can be found in [18, 20].

In this last section, under the same setting as this result of Fujimoto, we use the techniques in the proof of Theorem 1.2 to describe two uniqueness results regarding hypersurfaces located in k-subgeneral position, following essentially the approach applied in [7, 18, 20].

[3] Notice when f is rational, then one can choose $S_f(r,r_0)=O(1)$ to have (3.16).

Considering the comments made in [7, Section 5], we will without loss of generality suppose that either $M = B(1) \subseteq \mathbf{C}^m$ (finite ball covering of M) or $M = \mathbf{C}^m$ subsequently.

In fact, when $f, g : M \to V$ are the given meromorphic maps, then $f \circ \pi$, $g \circ \pi : \tilde{M} \to V$ will satisfy all the hypotheses as meromorphic maps over the lifted, complete universal covering $\tilde{M}$ of M. Since $f \circ \pi \equiv g \circ \pi$ on $\tilde{M}$ implies $f \equiv g$ on M, we simply assume $M = \tilde{M}$.

THEOREM 4.1. *Assume $V \subseteq \mathbf{P}^N(\mathbf{C})$ is an irreducible projective algebraic variety of dimension $n(\leq N)$. Let $f, g : B(1)(\subseteq \mathbf{C}^m) \to V$ be two algebraically non-degenerate meromorphic maps, both satisfying the* **condition** $\mathbf{C}(\rho)$. *Let $D_1, D_2, \ldots, D_q$ be q hypersurfaces in $\mathbf{P}^N(\mathbf{C})$ of degrees $d_1, d_2, \ldots, d_q$, located in k-subgeneral position $(k \geq n)$ with respect to V. Suppose further that $\limsup_{r\to 1} \dfrac{T_f(r, r_0) + T_g(r, r_0)}{\log \frac{1}{1-r}} < \infty$ and f, g satisfy the following conditions*

(1.) *$f^{-1}(D_j) = g^{-1}(D_j)$ for $j = 1, 2, \ldots, q$,*
(2.) *$f = g$ on $\bigcup_{j=1}^q f^{-1}(D_j)$,*
(3.) *$f^{-1}(D_j \cap D_{j'})$ has dimension at most $m - 2$ for $1 \leq j \neq j' \leq q$.*

Then, one has $f \equiv g$ provided, for the least common multiple d of $d_1, d_2, \ldots, d_q$,

$$q > \frac{2k - n + 1}{n + 1}\left\{H_V(d) + \frac{\rho}{d} H_V(d)(H_V(d) - 1)\right\} + \frac{2}{d}(H_V(d) - 1). \tag{4.1}$$

Proof. Assume $f = [\mathfrak{f}_0 : \mathfrak{f}_1 : \cdots : \mathfrak{f}_N]$ and $g = [\mathfrak{g}_0 : \mathfrak{g}_1 : \cdots : \mathfrak{g}_N]$, with reduced representations $\mathfrak{f} = (\mathfrak{f}_0, \mathfrak{f}_1, \ldots, \mathfrak{f}_N)$ and $\mathfrak{g} = (\mathfrak{g}_0, \mathfrak{g}_1, \ldots, \mathfrak{g}_N)$. Suppose in the following $f \not\equiv g$. Then, there exist at least two distinct indices $0 \leq \imath \neq \breve{\imath} \leq N$ such that the holomorphic function $\chi := \mathfrak{f}_\imath \mathfrak{g}_{\breve{\imath}} - \mathfrak{f}_{\breve{\imath}} \mathfrak{g}_\imath$ is not identically zero and satisfies $|\chi| \leq 2\|\mathfrak{f}\|\,\|\mathfrak{g}\|$ on $M = B(1)$.

Employ the previous notations to have F as before and $G := [\phi_1(\mathfrak{g}) : \phi_2(\mathfrak{g}) : \cdots : \phi_{H_V(d)}(\mathfrak{g})]$, both being linearly non-degenerate maps to $\mathbf{P}^{H_V(d)-1}(\mathbf{C})$. Thus, one finds two sets of $H_V(d)$ m-tuples $\alpha^l, \tilde{\alpha}^l \in \mathbf{Z}^m_{\geq 0}$ with (3.4) satisfied for each one, and $W_{\alpha^1 \ldots \alpha^{H_V(d)}}(F) \times W_{\tilde{\alpha}^1 \ldots \tilde{\alpha}^{H_V(d)}}(G) \not\equiv 0$ with $W_{\tilde{\alpha}^1 \ldots \tilde{\alpha}^{H_V(d)}}(G) := \det(D^{\tilde{\alpha}^l} \phi_\ell(\mathfrak{g}))_{1 \leq l, \ell \leq H_V(d)}$. Besides, for every subset $\mathcal{T} \subseteq \{1, 2, \ldots, q\}$ with $\#\mathcal{T} = \operatorname{rank}\{Q_j\}_{j \in \mathcal{T}} = n + 1$, use the hypersurfaces in Lemma 2.2 to define $G_\mathcal{T}$ similarly, and there is a constant $\tilde{C}_\mathcal{T} \neq 0$ such that $W_{\tilde{\alpha}^1 \ldots \tilde{\alpha}^{H_V(d)}}(G_\mathcal{T}) = \tilde{C}_\mathcal{T} W_{\tilde{\alpha}^1 \ldots \tilde{\alpha}^{H_V(d)}}(G)$. Recall $\varphi = \dfrac{W_{\alpha^1 \ldots \alpha^{H_V(d)}}(F)}{Q_1^{\omega_1}(\mathfrak{f}) \cdots Q_q^{\omega_q}(\mathfrak{f})}$ and $\aleph(F_\mathcal{T}) = \dfrac{W_{\alpha^1 \ldots \alpha^{H_V(d)}}(F_\mathcal{T})}{\prod_{j \in \mathcal{T}} Q_j(\mathfrak{f}) \prod_{i=1}^{H_V(d)-n-1} Q_i^*(\mathfrak{f})}$. Analogously, set $\tilde{\varphi} := \dfrac{W_{\tilde{\alpha}^1 \ldots \tilde{\alpha}^{H_V(d)}}(G)}{Q_1^{\omega_1}(\mathfrak{g}) \cdots Q_q^{\omega_q}(\mathfrak{g})}$ and $\tilde{\aleph}(G_\mathcal{T}) := \dfrac{W_{\tilde{\alpha}^1 \ldots \tilde{\alpha}^{H_V(d)}}(G_\mathcal{T})}{\prod_{j \in \mathcal{T}} Q_j(\mathfrak{g}) \prod_{i=1}^{H_V(d)-n-1} Q_i^*(\mathfrak{g})}$. Then, one has (3.7) and

$$\|\mathfrak{g}(z)\|^{d\{\sum_{j=1}^q \omega_j - H_V(d)\}} |\tilde{\varphi}(z)| \leq K \sum_{\mathcal{R}, \mathcal{T}} |\tilde{\aleph}(G_\mathcal{T})(z)| \quad \forall z \in M \backslash \mathsf{I}_g. \tag{4.2}$$

Now, it is routine to see our condition (4.1) and (2.5) imply that

$$\sum_{j=1}^{q} \omega_j > H_V(d) + \frac{2\varpi}{d}(H_V(d)-1) + \frac{\rho}{d} H_V(d)(H_V(d)-1). \tag{4.3}$$

From our assumptions, we know $\chi(z) = 0$ for all $z \in \bigcup_{j=1}^{q} f^{-1}(D_j)$. As $\omega_j \le \varpi$, we can infer that $v_{\varphi}^{\infty}, v_{\tilde{\varphi}}^{\infty} \le \varpi(H_V(d)-1)v_{\chi}^{0}$ and thus $\varphi\chi^{\varpi(H_V(d)-1)}$, $\tilde{\varphi}\chi^{\varpi(H_V(d)-1)}$ are both holomorphic functions on $B(1)$. Recall the Kähler form $\boldsymbol{\omega} = \frac{\sqrt{-1}}{2}\sum_{i,j} h_{i\bar{j}}\, \boldsymbol{d}z_i \wedge \boldsymbol{d}\bar{z}_j$ on $B(1)$. By hypothesis, there exist two continuous, pluri-subharmonic functions $\tau_1, \tau_2 \not\equiv -\infty$ such that

$$e^{\tau_1}\sqrt{\det(h_{i\bar{j}})} \le \|\mathfrak{f}\|^{\rho} \quad \text{and} \quad e^{\tau_2}\sqrt{\det(h_{i\bar{j}})} \le \|\mathfrak{g}\|^{\rho}.$$

Take $\tau := \log|z^{\boldsymbol{\alpha}+\tilde{\boldsymbol{\alpha}}}\varphi\tilde{\varphi}\chi^{2\varpi(H_V(d)-1)}|^{\hat{t}_0}$ for $\hat{t}_0 := \dfrac{\rho}{d\{\sum_{j=1}^{q}\omega_j - H_V(d)\} - 2\varpi(H_V(d)-1)} > 0$ with $\boldsymbol{\alpha} = \sum_{l=1}^{H_V(d)} \alpha^l$, $\tilde{\boldsymbol{\alpha}} := \sum_{l=1}^{H_V(d)} \tilde{\alpha}^l \in \mathbf{Z}_{\ge 0}^m$. Then, τ is pluri-subharmonic and one has

$$\begin{aligned}
\det(h_{i\bar{j}})e^{\tau+\tau_1+\tau_2} &\le |z^{\boldsymbol{\alpha}}\varphi|^{\hat{t}_0}|z^{\tilde{\boldsymbol{\alpha}}}\tilde{\varphi}|^{\hat{t}_0}|\chi|^{2\hat{t}_0\varpi(H_V(d)-1)}\|\mathfrak{f}\|^{\rho}\|\mathfrak{g}\|^{\rho} \\
&\le K|z^{\boldsymbol{\alpha}}\varphi|^{\hat{t}_0}\|\mathfrak{f}\|^{\rho+2\hat{t}_0\varpi(H_V(d)-1)}|z^{\tilde{\boldsymbol{\alpha}}}\tilde{\varphi}|^{\hat{t}_0}\|\mathfrak{g}\|^{\rho+2\hat{t}_0\varpi(H_V(d)-1)} \\
&= K|z^{\boldsymbol{\alpha}}\varphi|^{\hat{t}_0}\|\mathfrak{f}\|^{d\hat{t}_0\{\sum_{j=1}^{q}\omega_j - H_V(d)\}}|z^{\tilde{\boldsymbol{\alpha}}}\tilde{\varphi}|^{\hat{t}_0}\|\mathfrak{g}\|^{d\hat{t}_0\{\sum_{j=1}^{q}\omega_j - H_V(d)\}}.
\end{aligned}$$

Via (4.3), we get $\hat{t}_0 H_V(d)(H_V(d)-1) < \hat{\varsigma} < 1$ for some constant $\hat{\varsigma} > 0$. So, seeing $\boldsymbol{dV} = c_m \det(h_{i\bar{j}})\upsilon_m$ for an absolute constant $c_m > 0$, (3.4), (3.7) and (4.2), we have

$$\begin{aligned}
\int_M e^{\tau+\tau_1+\tau_2}\,\boldsymbol{dV} &\le K\left(\int_M |z^{\boldsymbol{\alpha}}|^{2\hat{t}_0}|\varphi(z)\|\mathfrak{f}(z)\|^{d\{\sum_{j=1}^{q}\omega_j - H_V(d)\}}|^{2\hat{t}_0}\upsilon_m\right)^{1/2} \\
&\quad \times \left(\int_M |z^{\tilde{\boldsymbol{\alpha}}}|^{2\hat{t}_0}|\tilde{\varphi}(z)\|\mathfrak{g}(z)\|^{d\{\sum_{j=1}^{q}\omega_j - H_V(d)\}}|^{2\hat{t}_0}\upsilon_m\right)^{1/2} \\
&\le K\left\{\sum_{\mathcal{R},\mathcal{T}}\int_0^1 r^{2m-1}\left(\int_{S(r)} |z^{\boldsymbol{\alpha}}\aleph(F_{\mathcal{T}})(z)|^{2\hat{t}_0}\sigma_m\right)\boldsymbol{d}r\right\}^{1/2} \\
&\quad \times \left\{\sum_{\mathcal{R},\mathcal{T}}\int_0^1 r^{2m-1}\left(\int_{S(r)} |z^{\tilde{\boldsymbol{\alpha}}}\tilde{\aleph}(G_{\mathcal{T}})(z)|^{2\hat{t}_0}\sigma_m\right)\boldsymbol{d}r\right\}^{1/2} \\
&\le K\int_0^1 \frac{1}{(1-r)^{\hat{\varsigma}}}\left(\log\frac{1}{1-r}\right)^{2\hat{\varsigma}}\boldsymbol{d}r = \frac{K}{(1-\hat{\varsigma})^{2\hat{\varsigma}+1}}\Gamma(2\hat{\varsigma}+1) < \infty
\end{aligned} \tag{4.4}$$

by Hölder's inequality, where a parallel argument concerning (3.11) and (3.12) is used to derive (4.4). This contradicts the results of Yau [21] and Karp [12], and thus $f \equiv g$. □

Finally, we describe a uniqueness result when the growth **condition** $\mathbf{C}(\rho)$ is dropped. Since it follows directly from the discussions in [7, Section 4] (see also [18, Section 3] or [20, Theorem 4.2]) and our Proposition 3.2 (in particular (3.14)), we only outline its proof.

PROPOSITION 4.2. *Under the same hypotheses of Theorem* 4.1 *concerning the algebraic variety* V *in* $\mathbf{P}^N(\mathbf{C})$ *and the hypersurfaces* $D_1, D_2, \ldots, D_q$ *in* $\mathbf{P}^N(\mathbf{C})$, *suppose* $f, g : B(R_0)(\subseteq \mathbf{C}^m) \to V$ *are algebraically non-degenerate meromorphic maps satisfying the conditions* (1.)–(3.). *Fix* d *the least common multiple of* $d_1, d_2, \ldots, d_q$. *Then, one has* $f \equiv g$ *provided either*

$$q > \frac{2k-n+1}{n+1} H_V(d) + \frac{2}{d}(H_V(d) - 1) \tag{4.5}$$

when $R_0 = \infty$ *or*

$$q > \frac{2k-n+1}{n+1}\left\{H_V(d) + \frac{\lambda}{d} H_V(d)(H_V(d) - 1)\right\} + \frac{2}{d}(H_V(d) - 1) \tag{4.6}$$

when $R_0 = 1$ *with* $\lambda := \liminf_{r \to 1} \dfrac{\log \dfrac{1}{1-r}}{T_f(r, r_0) + T_g(r, r_0)}$ *outside a set of finite logarithmic measure.*

Proof. From the derivation of (3.13) and the facts that $\chi = 0$ on $\bigcup_{j=1}^q f^{-1}(D_j)$ and $T_\chi(r, r_0) \le T_f(r, r_0) + T_g(r, r_0)$, one has for the valence function $N\left(r, r_0; \dfrac{1}{\chi}\right)$ of zeros of χ

$$\begin{aligned}&\left\{q - \frac{2k-n+1}{n+1} H_V(d)\right\}\{T_f(r, r_0) + T_g(r, r_0)\} \\ &\quad \le \frac{2}{d}(H_V(d) - 1) N\left(r, r_0; \frac{1}{\chi}\right) + S_f(r, r_0) + S_g(r, r_0)\end{aligned}$$

when we suppose $f \not\equiv g$; that is, considering the first main theorem,

$$q \le \frac{2k-n+1}{n+1} H_V(d) + \frac{2}{d}(H_V(d) - 1) + \liminf_{r \to R_0} \frac{S_f(r, r_0) + S_g(r, r_0)}{T_f(r, r_0) + T_g(r, r_0)}.$$

If $R_0 = \infty$, a contradiction against (4.5) follows[4]; on the other hand, if $R_0 = 1$,

[4] Recall when f, g are rational, then $S_f(r, r_0) = S_g(r, r_0) = O(1)$.

(3.14) yields a contradiction against (4.6) as $\liminf_{r\to 1} \frac{S_f(r,r_0)+S_g(r,r_0)}{T_f(r,r_0)+T_g(r,r_0)} \leq \frac{\lambda(2k-n+1)}{d(n+1)} H_V(d)(H_V(d)-1)$. □

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