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ZETA FUNCTIONS FOR KÄHLER GRAPHS

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Abstract

To create a discrete analogue of magnetic fields on Riemannian manifolds is a challenging problem. The notion of Kähler graphs introduced by the second author is one of trials of this discretization. In this article we study the asymptotic behavior of the weighted number of prime cycles with respect to their lengths by use of a zeta function.

1. Introduction

A graph is a 1-dimensional CW-complex which consists of a set of vertices and a set of edges. In the field of geometry, graphs are considered as discrete models of Riemannian manifolds. Paths on this graph, which are chains of edges, correspond to geodesics. When we study Riemannian manifolds we frequently consider some geometric structures on them, complex structures, contact structures and so on. The second author is hence interested in giving discrete models of Riemannian manifolds which inherit geometric structures.

Some geometric structures induce closed 2-forms. On a Riemannian manifold, a closed 2-form is said to be a magnetic field because it can be regarded as a generalization of static magnetic fields on a Euclidean 3-space (see [12], for example). Since geodesics are motions with constant velocities, we are interested in motions of constant accelerations. This is a way of classical treatment of magnetic fields. When a magnetic field is uniform, that is, its strength does not depend on points and directions, a charged particle gets a uniform Lorentz force and its motion is of constant acceleration. Typical examples of uniform magnetic fields are constant multiples of the Kähler form on a Kähler manifold. Such magnetic fields are said to be Kähler magnetic fields (see [1, 3]). The second author intend to give a discrete model corresponding to Kähler manifolds admitting Kähler magnetic fields.

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In his paper [2] he introduced the notion of Kähler graphs. A Kähler graph $G = (V, E^{(p)} \cup E^{(a)})$ is a compound graph which consists of a set V of vertices, a set $E^{(p)}$ of principal edges and a set $E^{(a)}$ of auxiliary edges. Since graphs are 1-dimensional objects, we give magnetic fields on Kähler graphs by showing trajectories for magnetic fields. We consider paths on the principal graph $(V, E^{(p)})$ of a Kähler graph as geodesics on this graph. In order to show a uniform magnetic field of strength q/p with relatively prime positive integers p, q, we give trajectories for this magnetic field. We take a p-step path on the principal graph. This corresponds to a geodesic segment on a Riemannian manifold. We choose a q-step path on the auxiliary graph whose origin is the terminus of the above p-step path, and make a $(p+q)$-step "bicolored" path on a Kähler graph. We consider chains of such paths as trajectories for the magnetic field. This means that a p-step path is bended by Lorentz force and its terminus reaches to the terminus of a $(p+q)$-bicolored path whose first p-step coincides with the original one. Of course there are many q-step paths for each p-step path. Since Kähler graphs do not have 2-dimensional objects, we can not show the direction of Lorentz force, we therefore consider all such q-step paths and treat them probabilistically. Thus, every $(p+q)$-step bicolored path, hence every trajectory, has its probabilistic weight so that the sum of probabilities of $(p+q)$-bicolored paths with a given first p-step path is equal to one.

In this paper we count probabilistic weights of prime cycles on a Kähler graph. We define a zeta function for bicolored closed paths which has information of lengths and weights of closed paths. Along the ordinary way (cf. [10, 15]) we show the asymptotic behavior of probabilistic weights of prime cycles with respect to their lengths. Our discretization of magnetic fields on Riemannian manifolds is done from the viewpoint of classical treatment. As our discretization is still only a trial, the reader should confer [13] for another discretization.

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2. Derived graphs of a Kähler graph

Let $G = (V, E)$ be a graph whose edges are not directed. We say an edge $e \in E$ to be a loop if its both ends coincide. For two vertices if there exist two and more edges joining them, we say these edges to be multiple edges. When a graph does not have loops and multiple edges, it is called simple (for more on graphs see [6], for example). We call a simple undirected graph $G = (V, E)$ *Kähler* if the set E of edges is divided into two disjoint subsets $E^{(p)}$, $E^{(a)}$ and satisfies the following condition: At each vertex $v \in V$, there are at least two edges in $E^{(p)}$ and two edges in $E^{(a)}$ both of which are emanating from v. We call $(V, E^{(p)})$ and $(V, E^{(a)})$ the *principal graph* and the *auxiliary graph* of a Kähler graph, respectively. Given two vertices $v, w \in V$ we denote by $v \sim_p w$ if they are adjacent to each other in the principal graph $(V, E^{(p)})$, and denote by $v \sim_a w$ if they are adjacent to each other in the auxiliary graph $(V, E^{(a)})$. Here, two vertices in a graph are said to be adjacent to each other if there exists an edge joining them. When a Kähler graph $G = (V, E^{(p)} \cup E^{(a)})$ is locally finite, for each vertex $v \in V$, we denote by $d_G^{(p)}(v)$ the cardinality of the set $\{w \in V \mid w \sim_p v\}$ and by $d_G^{(a)}(v)$ the cardinality of the set $\{w \in V \mid w \sim_a v\}$. We call them the principal degree and the auxiliary degree at v, respectively. The degree $d_G(v)$ of G at v is hence $d_G^{(p)}(v) + d_G^{(a)}(v)$. Typical examples of Kähler graphs are given by taking complements of graphs. For an ordinary undirected simple graph $G = (V, E)$, its complement graph $G^c = (V, E^c)$ is defined by the following manner: Two distinct vertices are adjacent to each other in G^c if and only if they are not adjacent to each other in G. If we set $G^K = (V, E \cup E^c)$, then it is a Kähler graph when $d_G(v) \geq 2$ and $d_{G^c}(v) \geq 2$ at each vertex $v \in V$. We call this a *complement-filled* Kähler graph. An ordinary graph is said to be regular if its degree function is constant. We call a Kähler graph *regular* if both of its principal and auxiliary graphs are regular, and call it *complete* if it is regular and if its arbitrary two distinct vertices are joined by an edge. Since we have many examples of regular ordinary graphs, we can construct many examples of complete Kähler graphs by taking their complement-filled Kähler graphs. For construction of regular Kähler graphs, see [16, 18].

For a pair (p, q) of relatively prime positive integers, we say that a $(p+q)$-step path $\gamma = (v_0, v_1, \dots, v_{p+q})$ is a *(p, q)-primitive bicolored path* if it satisfies the following conditions:

i) $v_{i+1} \neq v_{i-1}$ for $1 \leq i \leq p+q-1$,

ii) $v_{i-1} \sim_p v_i$ for $1 \leq i \leq p$,

iii) $v_{i-1} \sim_a v_i$ for $p+1 \leq i \leq p+q$.

The first condition shows that this path does not have back-tracking. We put $o(\gamma) = v_0$ and $t(\gamma) = v_{p+q}$ and call them the origin and the terminus of γ, respectively. For this (p, q)-primitive bicolored path γ we set its *probabilistic weight* $\omega(\gamma)$ by

$$\omega(\gamma) = \left\{ d_G^{(a)}(v_p) \prod_{j=p+1}^{p+q-1} \{d_G^{(a)}(v_j) - 1\} \right\}^{-1}.$$

As $d_G^{(a)}(v) \geq 2$ for every vertex v of G, we see $\omega(\gamma) \leq 1/2$ for every primitive bicolored path γ. We note that for each p-step path in the principal graph the sum of probabilistic weights of (p, q)-primitive bicolored paths whose first p-step coincide with the given one is equal to 1. Also, we should note that the probabilistic weight $\omega(\gamma)$ of a (p, q)-primitive bicolored path γ does not coincide in general with the reciprocal of the number of (p, q)-primitive bicolored paths emanating from $o(\gamma)$. We say an $m(p+q)$-step path $\gamma = (v_0, \dots, v_{m(p+q)})$ to be a (p, q)-bicolored path if all its subpaths $\gamma_j = (v_{(j-1)(p+q)}, \dots, v_{j(p+q)})$, $j = 1, \dots, m$ are (p, q)-primitive bicolored paths. For this bicolored path γ, we define its probabilistic weight $\omega(\gamma)$ by $\omega(\gamma) = \prod_{j=1}^{m} \omega(\gamma_j)$. It satisfies $\omega(\gamma) \leq 1/2^m$.

We here make mention of geometric meaning of bicolored paths and their probabilistic weights. If we consider principal graphs as discrete models of Riemannian manifolds, we can regard paths as geodesics, which are trajectories of motions of particles without influence of outer force. Under the influence of magnetic fields, motions of charged particles are bended. When the strength of a magnetic field is q/p, every p-step path without back-tracking in a principal graph is bended and reaches to the terminus of a $(p+q)$-primitive bicolored path whose first p-step coincides with the original path in a principal graph. Since we can not show the direction of the action of this magnetic field on the Kähler graph, because graphs are 1-dimensional objects, we consider a bended path reaches to one of terminuses of primitive bicolored paths and treat bended direction probabilistically. Formally, we can define (p,q)-primitive bicolored paths even if p, q are not relatively prime. But when $p = ap'$ and $q = aq'$ with some positive integers a, p', q' with $a \geq 2$, we note that the set of terminuses of (p,q)-primitive bicolored paths is different from the set of $a(p'+q')$-step (p',q')-primitive bicolored paths. As we need to define "curved paths" on a graph, the authors consider that we should show curved property in minimum steps. We hence restrict ourselves to pairs of relatively prime positive integers. Geometrical point of view, it is more natural to suppose that the cardinality of the set of vertices is sufficiently large compared with $p+q$ and degrees of vertices. In this sense, the case $(p,q) = (1,1)$ is most important. Still, we do not make mention of this point any more in this paper.

For a Kähler graph $G = (V, E^{(p)} \cup E^{(a)})$ we have an directed graph $G_{p,q} = (V, E_{p,q})$ with the set $E_{p,q}$ of all (p,q)-primitive bicolored paths on G. We call this directed graph the (p,q)-*derived graph* of G. This graph may have loops and multiple edges, and each edge has its weight. We note that this graph is not a circuit by the condition of Kähler graphs. Here, we say a graph to be a circuit if it is homeomorphic to a circle S^1 as a CW-complex. When G is a finite Kähler graph, we define (p,q)-adjacency operator $\mathscr{A}_{p,q}$ acting on the set $C(V)$ of all functions on V by

$$\mathscr{A}_{p,q} f(v) = \sum_{\gamma} \omega(\gamma) f(t(\gamma)),$$

where γ runs over the set of all (p,q)-primitive bicolored paths emanating from v. Being different from adjacency operators of ordinary graphs, this (p,q)-adjacency operator is not symmetric in general. More precisely, this operator is a composition of an operator on the principal graph and an operator on the auxiliary graph. We define operators $\mathscr{A}_p$ and $\mathscr{Q}_q$ acting on $C(V)$ by

$$\mathscr{A}_p f(v) = \sum_{\rho} f(t(\rho)) \quad \text{and} \quad \mathscr{Q}_q f(v) = \sum_{\tau} \omega(\tau) f(t(\tau)),$$

where ρ runs over the set of all p-step paths on the principal graph which are emanating from v and do not have backtracking, τ runs over the set of all q-step paths on the auxiliary graph which are emanating from v and do not have backtracking, and $\omega(\tau)$ is the probabilistic weight of τ by regarding it as a $(0,q)$-primitive bicolored path. Then we have $\mathscr{A}_{p,q} = \mathscr{A}_p \circ \mathscr{Q}_q$ (see [16, 17] for more on this operator).

We call a directed graph (V, E) irreducible (or strongly connected) if for an arbitrary pair (v, w) of distinct vertices there exists a directed path whose origin is v and whose terminus is w. When $G_{p,q}$ is irreducible, by Perron-Frobenius Theorem, we see that there is a positive eigenvalue $\lambda_{p,q}(G)$ of $\mathscr{A}_{p,q}$ satisfying the following conditions:

i) Eigenvalues of $\mathscr{A}_{p,q}$ whose absolute values are $\lambda_{p,q}(G)$ are of the form $e^{2\sqrt{-1}\pi j/k_{p,q}}\lambda_{p,q}(G)$ $(j = 0, 1, \dots, k_{p,q} - 1)$ with some positive integer $k_{p,q}$;

ii) These eigenvalues $e^{2\sqrt{-1}\pi j/k_{p,q}}\lambda_{p,q}(G)$ are simple;

iii) The absolute values of other eigenvalues are less than $\lambda_{p,q}(G)$.

Since $\mathscr{A}_{p,q}$ is decomposed as $\mathscr{A}_p \circ \mathscr{Q}_q$, the eigenvalue $\lambda_{p,q}(G)$ is estimated as

$$\min_{v\in V} d_G^{(p)}(v)\left\{\min_{v\in V} d_G^{(p)}(v) - 1\right\}^{p-1} \le \lambda_{p,q}(G) \le \max_{v\in V} d_G^{(p)}(v)\left\{\max_{v\in V} d_G^{(p)}(v) - 1\right\}^{p-1}.$$

When the principal graph of a Kähler graph G is regular of degree $d_G^{(p)}$, the eigenvalue $\lambda_{p,q}(G)$ is hence given by $\lambda_{p,q}(G) = d_G^{(p)}(d_G^{(p)} - 1)^{p-1}$. But when the principal graph of a Kähler graph G is not regular, we have $\lambda_{p,q}(G) \neq \lambda_{p,q'}(G)$ for $q \neq q'$, in general.

Example 1. Let G_1 be a regular Kähler graph of $d_{G_1}^{(p)} = d_{G_1}^{(a)} = 2$ given in Fig. 1. Then its $(1,1)$-derived graph is irreducible, but its $(2,1)$-derived graph has 4 connected components. The eigenvalues of $(1,1)$ and $(2,1)$ adjacency operators are

$$\mathrm{Ev}(\mathscr{A}_{1,1}) = \{2, 0, 0, 0, 0, 0, 0, -2\}, \quad \mathrm{Ev}(\mathscr{A}_{2,1}) = \{2, 2, 2, 2, 0, 0, 0, 0\}.$$

Example 2. Let G_2 be a Kähler graph given in Fig. 2 whose principal graph is regular of $d_{G_2}^{(p)} = 3$ but whose auxiliary graph is not regular. The eigenvalues of $(1,1)$, $(2,1)$ and $(1,2)$ adjacency operators are

$$\mathrm{Ev}(\mathscr{A}_{1,1}) = \left\{3, 0, 0, 0, -\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}, -1\right\}, \quad \mathrm{Ev}(\mathscr{A}_{2,1}) = \left\{6, 2, \frac{4}{3}, \frac{2}{3}, \frac{2}{3}, 0, 0, -\frac{4}{3}\right\},$$

$$\mathrm{Ev}(\mathscr{A}_{1,2}) = \left\{3, \frac{2+\sqrt{7}}{3}, \frac{3+\sqrt{21}}{6}, 1, \frac{1}{6}, -\frac{1}{6}, -\frac{2-\sqrt{7}}{3}, -\frac{3-\sqrt{21}}{6}\right\}.$$

Hence $\lambda_{1,1}(G_2) = \lambda_{1,2}(G_2) = 3$, $\lambda_{2,1}(G_2) = 6$.

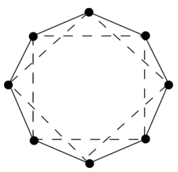

FIGURE 1. G_1

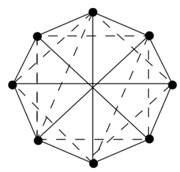

FIGURE 2. G_2

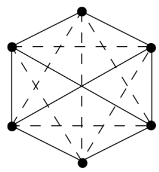

FIGURE 3. G_3

Example 3. Let G_3 be a complement-filled Kähler graph given in Fig. 3. Its principal graph is not regular. The $(1,1)$, $(1,2)$ and $(2,1)$ adjacency operators are expressed by matrices

$$A_{1,1}=\begin{pmatrix} 0 & \frac{1}{2} & 0 & 1 & 0 & \frac{1}{2} \\ 1 & 0 & \frac{5}{6} & \frac{1}{3} & \frac{5}{6} & 0 \\ \frac{1}{3} & \frac{5}{6} & 0 & 1 & 0 & \frac{5}{6} \\ 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{5}{6} & 0 & 1 & 0 & \frac{5}{6} \\ 1 & 0 & \frac{5}{6} & \frac{1}{3} & \frac{5}{6} & 0 \end{pmatrix}, \quad A_{1,2}=\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & 1 & 0 & \frac{1}{4} \\ 1 & \frac{1}{6} & \frac{7}{12} & \frac{1}{2} & \frac{7}{12} & \frac{1}{6} \\ \frac{1}{2} & \frac{7}{12} & \frac{1}{6} & 1 & \frac{1}{6} & \frac{7}{12} \\ 1 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{7}{12} & \frac{1}{6} & 1 & \frac{1}{6} & \frac{7}{12} \\ 1 & \frac{1}{6} & \frac{7}{12} & \frac{1}{2} & \frac{7}{12} & \frac{1}{6} \end{pmatrix},$$

$$A_{2,1}=\begin{pmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ \frac{2}{3} & \frac{13}{6} & 0 & \frac{3}{2} & 0 & \frac{3}{2} \\ 0 & \frac{3}{2} & \frac{2}{3} & \frac{13}{6} & \frac{2}{3} & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ \frac{3}{2} & 0 & \frac{2}{3} & \frac{2}{3} & \frac{13}{6} & 0 \\ \frac{2}{3} & \frac{2}{3} & 0 & \frac{3}{2} & 0 & \frac{13}{6} \end{pmatrix},$$

hence their eigenvalues are

$$\mathrm{Ev}(\mathscr{A}_{1,1})=\left\{\frac{4+\sqrt{13}}{3},\frac{4-\sqrt{13}}{3},0,0,\frac{-4+\sqrt{7}}{3},\frac{-4-\sqrt{7}}{3}\right\},$$

$$\mathrm{Ev}(\mathscr{A}_{1,2})=\left\{\frac{3+\sqrt{3}}{2},\frac{3-\sqrt{3}}{2},0,0,\frac{-4+\sqrt{10}}{6},\frac{-4-\sqrt{10}}{6}\right\},$$

$$\mathrm{Ev}(\mathscr{A}_{2,1})=\left\{\frac{29+\sqrt{649}}{12},3,\frac{3}{2},\frac{11+\sqrt{-23}}{12},\frac{11-\sqrt{-23}}{12},\frac{29-\sqrt{649}}{12}\right\}.$$

Thus we have

$$\lambda_{1,1}(G_3)=(4+\sqrt{13})/3,\quad \lambda_{1,2}(G_3)=(3+\sqrt{3})/2,\quad \lambda_{2,1}(G_3)=(29+\sqrt{649})/12.$$

3. Zeta functions

Let $G=(V,E^{(p)}\cup E^{(a)})$ be a Kähler graph. For an $m(p+q)$ step (p,q)-bicolored path $\gamma=\gamma_1\cdot\gamma_2\cdots\gamma_m$ with (p,q)-primitive bicolored paths $\gamma_1,\dots,\gamma_m$, we put $o(\gamma)=o(\gamma_1)$ and $t(\gamma)=t(\gamma_m)$, and call them the origin and the terminus of γ, respectively. For this bicolored path γ, we set $\ell(\gamma)=m(p+q)$. We call γ *closed* if $o(\gamma)=t(\gamma)$, and call it *prime* if in addition there are no divisor k of m satisfying $\gamma_{i+k}=\gamma_i$ for all i by considering the indices by modulo m. We denote by $\mathfrak{C}_m^{(p,q)}(G)$ the set of all $m(p+q)$-step (p,q)-bicolored closed paths. We say that two bicolored paths $\gamma^{(1)}$, $\gamma^{(2)}$ are congruent to each other if both of them belong to $\mathfrak{C}_m^{(p,q)}(G)$ with some m and if we denote them as $\gamma^{(1)}=\gamma_1^{(1)}\cdot\gamma_2^{(1)}\cdots\gamma_m^{(1)}$,

$\gamma^{(2)} = \gamma_1^{(2)} \cdot \gamma_2^{(2)} \cdots \gamma_m^{(2)} \in \mathfrak{C}_m^{(p,q)}(G)$ there is i_0 satisfying $\gamma_{i+i_0}^{(1)} = \gamma_i^{(2)}$ by considering the indices by modulo m. We shall call a congruence class of closed paths a cycle. It is clear that $\ell(\gamma^{(1)}) = \ell(\gamma^{(2)})$ if two (p,q)-bicolored closed paths $\gamma^{(1)}$ and $\gamma^{(2)}$ are congruent to each other. Moreover, as (p,q)-bicolored paths start with principal edges, we find $\omega(\gamma^{(1)}) = \omega(\gamma^{(2)})$ in this case. We denote by $\mathfrak{P}_m^{(p,q)}(G)$ the set of all congruence classes of $m(p+q)$-step (p,q)-bicolored prime closed paths, and put $\mathfrak{P}^{(p,q)}(G) = \bigcup_{m=1}^{\infty} \mathfrak{P}_m^{(p,q)}(G)$.

We define the (p,q)-zeta function $\zeta_G(u;p,q)$ of a finite Kähler graph $G = (V, E^{(p)} \cup E^{(a)})$ by

$$\zeta_G(s;p,q) = \prod_{\mathfrak{p} \in \mathfrak{P}^{(p,q)}(G)} \{1 - \omega(\mathfrak{p}) e^{-s\ell(\mathfrak{p})}\}^{-1}.$$

We may say that this is a L-function of a directed graph $G_{p,q}$. But here, we do not consider ω as a character (cf. [8]).

Lemma. *Suppose that the (p,q)-derived graph $G_{p,q}$ of a finite Kähler graph G is irreducible. If* $\mathrm{Re}(s) > \log \lambda_{p,q}(G)/(p+q)$, *we have*

$$\zeta_G(s;p,q) = \det(I - e^{-s(p+q)} \mathscr{A}_{p,q})^{-1}.$$

Proof. By direct computation we have

$$\begin{aligned}
\log \zeta_G(s;p,q) &= -\sum_{\mathfrak{p} \in \mathfrak{P}^{(p,q)}(G)} \log\{1 - \omega(\mathfrak{p}) e^{-s\ell(\mathfrak{p})}\} = \sum_{\mathfrak{p}} \sum_{n=1}^{\infty} \frac{\omega(\mathfrak{p})^n}{n} e^{-ns\ell(\mathfrak{p})} \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{\infty} \sum_{\mathfrak{p}:\ell(\mathfrak{p})=k(p+q)} \omega(\mathfrak{p})^n e^{-nks(p+q)} \\
&= \sum_{m=1}^{\infty} \frac{1}{m} \sum_{k:k|m} \left\{ k \sum_{\mathfrak{p}:\ell(\mathfrak{p})=k(p+q)} \omega(\mathfrak{p})^{m/k} \right\} e^{-ms(p+q)}.
\end{aligned}$$

On the other hand, if $\gamma \in \mathfrak{C}_m^{(p,q)}(G)$ is not prime, there is a divisor k of m and a prime closed path $\sigma \in \mathfrak{C}_k^{(p,q)}(G)$ satisfying that γ is a m/k-multiple of σ. In this case, as (p,q)-bicolored paths start with principal edges, we have $\omega(\gamma) = \omega(\sigma)^{m/k}$ by definition. Since there are k distinct closed (p,q)-bicolored paths which are congruent to σ, we have

$$\mathrm{trace}(\mathscr{A}_{p,q}^m) = \sum_{\gamma \in \mathfrak{C}_m^{(p,q)}(G)} \omega(\gamma) = \sum_{k:k|m} \left\{ k \sum_{\mathfrak{p}:\ell(\mathfrak{p})=k(p+q)} \omega(\mathfrak{p})^{m/k} \right\},$$

hence we obtain

$$\log \zeta_G(s;p,q) = \sum_{m=1}^{\infty} \frac{1}{m} \,\mathrm{trace}(\mathscr{A}_{p,q}^m) e^{-ms(p+q)}.$$

This leads us to the conclusion. □

We here study zeta functions for some Kähler graphs.

Example 4. By the above Lemma, if we put $u = u_{p,q} := e^{-s(p+q)}$, we find that the zeta functions for Kähler graphs given in §2 are expressed as follows:

$$\zeta_{G_1}(u;1,1) = \frac{1}{(1-2u)(1+2u)}, \quad \zeta_{G_1}(u;1,2) = \frac{1}{(1-2u)(1+2u)(1-2u^2)^2},$$

$$\zeta_{G_2}(u;1,1) = \frac{9}{(3-8u+u^2)(3+8u+u^2)},$$

$$\zeta_{G_2}(u;2,1) = \frac{12}{(6+8u+u^2)(2-6u+3u^2)},$$

$$\zeta_{G_2}(u;1,2) = \frac{72}{(2-3u)(1-3u)(6-11u+6u^2)(6-29u+8u^2)}.$$

Proposition 1. *Let $G = (V, E)$ be a connected regular finite ordinary graph whose degree d_G satisfies $2 \le d_G \le n_G - 2$, where n_G denotes the cardinality of the set V of vertices. If we denote the eigenvalues of the adjacency operator $\mathscr{A}_G$ of G by $\lambda_1 = d_G, \lambda_2, \dots, \lambda_{n_G}$, then the $(1,1)$-zeta function of its complement filled Kähler graph G^K is given as*

$$\zeta_G(u;1,1) = \frac{1}{(1-d_G u)\prod_{i=2}^{n_G}\left(1 + \dfrac{\lambda_i(\lambda_i+1)u}{n_G - d_G - 1}\right)} \quad (u = e^{-2s}).$$

Proof. The adjacency operator $\mathscr{A}_{G^c}$ of the complement graph G^c of G is given as $\mathscr{A}_{G^c} = \mathscr{M} - I - \mathscr{A}_G$, where the operator $\mathscr{M}$ is defined by $\mathscr{M}g(v) = \sum_{w \in V} g(w)$ for $g \in C(V)$. We take an eigenfunction f_i corresponding to λ_i. Since G is connected, we see f_1 is a constant function and find that $\mathscr{M}f_1 = n_G f_1$ and $\mathscr{M}f_i = 0$ for $i \ge 2$. Thus, we obtain

$$\mathscr{A}_{1,1}f_i = \frac{1}{n_G - d_G - 1}\mathscr{A}_G\mathscr{A}_{G^c}f_i = \begin{cases} d_G f_i, & \text{when } i = 1, \\ \dfrac{-\lambda_i(\lambda_i+1)}{n_G - d_G - 1} f_i, & \text{when } i \ge 2, \end{cases}$$

and get the conclusion. □

Since we can express higher steps adjacency and probabilistic transition operators $\mathscr{A}_p$, $\mathscr{Q}_q$ of regular Kähler graphs by using the adjacency operators $\mathscr{A}^{(p)}$, $\mathscr{A}^{(a)}$ of its principal and auxiliary graphs ([17]), we can express the (p,q)-zeta functions of complement filled Kähler graphs of regular ordinary graphs.

We here give a property of zeta functions of general Kähler graphs. In view of results for smooth Anosov flows on compact manifolds ([5]) and Lemma, we define (p,q)-entropy of a finite Kähler graph G by

$$h_{p,q}(G) = (\log \lambda_{p,q}(G))/(p+q).$$

When the principal graph of G is regular, we see

$$h_{p,q}(G) = \{\log d_G^{(p)} + (p-1)\log(d_G^{(p)} - 1)\}/(p+q).$$

As a consequence of Lemma we have the following result.

PROPOSITION 2. *Let $G = (V, E^{(p)} \cup E^{(a)})$ be a finite Kähler graph. If the (p,q)-derived graph $G_{p,q}$ of G is irreducible, then the followings hold*:

(1) *$\zeta_G(s; p, q)$ converges absolutely and is holomorphic on $\mathrm{Re}(s) > h_{p,q}(G)$*;

(2) *It is extended meromorphically to the whole plane*;

(3) *It has a simple pole at $s = h_{p,q}(G) + \sqrt{-1}\beta$, where $\beta \equiv 2\pi j/k_{p,q}$ $(j = 0, 1, \dots, k_{p,q} - 1)$ modulo $2\pi/(p+q)$, and except such poles it is holomorphic in the neighborhood of $\mathrm{Re}(s) \geq h_{p,q}(G)$.*

4. Counting prime cycles

We are now in the position to study the asymptotic behavior of the number of prime cycles in a Kähler graph $G = (V, E^{(p)} \cup E^{(a)})$ by an ordinary way (cf. [5, 11]). For a positive number x we set

$$\pi_G^{(p,q)}(x) = \sum_{\substack{\mathfrak{p} \in \mathfrak{P}^{(p,q)}(G) \\ \ell(\mathfrak{p}) \leq x}} \omega(\mathfrak{p}),$$

which shows the "weighted" counting of the number of (p,q)-bicolored prime cycles. Since we consider behaviors of trajectories probabilistically, this counting corresponds to the counting of prime cycles on an ordinary graph. For functions $f, g : [0, \infty) \to \mathbf{R}$, we denote by $f \sim g$ $(x \to \infty)$ if they satisfy $\lim_{x\to\infty} f(x)/g(x) = 1$.

THEOREM. *If the (p,q)-derived graph $G_{p,q}$ of a finite Kähler graph $G = (V, E^{(p)} \cup E^{(a)})$ is irreducible, then we have*

$$\pi_G^{(p,q)}(x) \sim e^{h_{p,q}(G)x}/(h_{p,q}(G)x) \quad (x \to \infty).$$

Proof. We denote $h_{p,q}(G)$ by $h_{p,q}$ for the sake of simplicity. For a positive y, we set

$$\varphi(y) = h_{p,q} \sum_{\substack{\mathfrak{p} \in \mathfrak{P}^{(p,q)}(G), n \geq 1 \\ n\ell(\mathfrak{p}) \leq y/h_{p,q}}} \omega(\mathfrak{p})^n \ell(\mathfrak{p}).$$

We then have $\varphi(0) = 0$, and as we see

$$\log \zeta_G(s; p, q) = \sum_{\mathfrak{p} \in \mathfrak{P}^{(p,q)}(G)} \sum_{n=1}^{\infty} \frac{\omega(\mathfrak{p})^n}{n} e^{-ns\ell(\mathfrak{p})}$$

in the proof of Lemma, we have

$$-\frac{d}{du}\log\zeta_G(h_{p,q}u;p,q) = h_{p,q}\sum_{\mathfrak{p}}\sum_{n=1}^{\infty}\omega(\mathfrak{p})^n\ell(\mathfrak{p})e^{-nh_{p,q}u\ell(\mathfrak{p})} = \int_0^\infty e^{-uy}\,d\varphi(y).$$

Hence the function $u \mapsto \int_0^\infty e^{-uy}\,d\varphi(y)$ converges absolutely on $\mathrm{Re}(u) > 1$ by Proposition 2. We see that $\varphi(y) \sim e^y$ $(y \to \infty)$ by Proposition 2 and by Wiener-Ikehara's Tauberian theorem (see Appendix A in [11] or §15.2 in [7]).

We set $\hat{\pi}_G(y) = \pi_G^{(p,q)}(y/h_{p,q})$. Since lengths of (p,q)-primitive bicolored paths are $p+q$, we trivially have $\ell(\mathfrak{p}) \geq p+q$ for every $\mathfrak{p} \in \mathfrak{P}^{(p,q)}(G)$. Thus we have

$$\begin{aligned}\hat{\pi}_G(y) &= \sum_{\mathfrak{p}:\ell(\mathfrak{p})\leq y/h_{p,q}}\omega(\mathfrak{p}) \leq \sum_{\mathfrak{p}:\ell(\mathfrak{p})\leq y/h_{p,q}}\omega(\mathfrak{p})\frac{\ell(\mathfrak{p})}{p+q}\\ &\leq \frac{1}{p+q}\sum_{\substack{\mathfrak{p},n\geq 1\\ n\ell(\mathfrak{p})\leq y/h_{p,q}}}\omega(\mathfrak{p})^n\ell(\mathfrak{p}) = \frac{\varphi(y)}{h_{p,q}(p+q)},\end{aligned}$$

and find that there is a positive C satisfying $\hat{\pi}_G(y) \leq Ce^y$ because $\varphi(y) \sim e^y$ $(y \to \infty)$.

We now estimate $\hat{\pi}_G(y)$. We have

$$\begin{aligned}y\hat{\pi}_G(y) &= y\sum_{\mathfrak{p}:\ell(\mathfrak{p})\leq y/h_{p,q}}\omega(\mathfrak{p}) \geq h_{p,q}\sum_{\mathfrak{p}:\ell(\mathfrak{p})\leq y/h_{p,q}}\omega(\mathfrak{p})\ell(\mathfrak{p})\\ &= \varphi(y) - h_{p,q}\sum_{\substack{n\geq 2,\mathfrak{p}\\ n\ell(\mathfrak{p})\leq y/h_{p,q}}}\omega(\mathfrak{p})^n\ell(\mathfrak{p}).\end{aligned}$$

Here, as we have $\omega(\mathfrak{p}) \leq 1/2$, we obtain

$$\begin{aligned}\sum_{\substack{n\geq 2,\mathfrak{p}\\ n\ell(\mathfrak{p})\leq y/h_{p,q}}}\omega(\mathfrak{p})^n\ell(\mathfrak{p}) &\leq \sum_{\substack{n\geq 2,\mathfrak{p}\\ n\ell(\mathfrak{p})\leq y/h_{p,q}}}\omega(\mathfrak{p})\ell(\mathfrak{p})\\ &= \sum_{\mathfrak{p}:\ell(\mathfrak{p})\leq y/(2h_{p,q})}\omega(\mathfrak{p})\sum_{n:2\leq n\leq y/(h_{p,q}\ell(\mathfrak{p}))}\ell(\mathfrak{p}) \leq \frac{y}{h_{p,q}}\hat{\pi}_G\left(\frac{y}{2}\right).\end{aligned}$$

We hence find

$$\frac{y\hat{\pi}_G(y)}{e^y} \geq \frac{\varphi(y)}{e^y} - \frac{y}{e^y}\hat{\pi}_G\left(\frac{y}{2}\right) \geq \frac{\varphi(y)}{e^y} - \frac{Cy}{e^{y/2}}. \tag{4.1}$$

On the other hand, we have

$$\begin{aligned}\hat{\pi}_G(y) &= \hat{\pi}_G(y - 2\log y) + \sum_{\mathfrak{p}:(y-2\log y)/h_{p,q} \le \ell(\mathfrak{p}) \le y/h_{p,q}} \omega(\mathfrak{p}) \\ &\le \hat{\pi}_G(y - 2\log y) + \frac{h_{p,q}}{y - 2\log y} \sum_{\mathfrak{p}:(y-2\log y)/h_{p,q} \le \ell(\mathfrak{p}) \le y/h_{p,q}} \omega(\mathfrak{p})\ell(\mathfrak{p}) \\ &\le \hat{\pi}_G(y - 2\log y) + \frac{1}{y - 2\log y}\varphi(y),\end{aligned}$$

hence obtain

$$\begin{aligned}\frac{y\hat{\pi}_G(y)}{e^y} &\le \frac{y}{y-2\log y}\frac{\varphi(y)}{e^y} + \frac{y}{e^y}\hat{\pi}_G(y - 2\log y) \\ &\le \frac{y}{y-2\log y}\frac{\varphi(y)}{e^y} + \frac{C}{y}.\end{aligned} \tag{4.2}$$

By (4.1) and (4.2) we see $\hat{\pi}_G(y) \sim e^y/y$, and get the conclusion. $\square$

Corollary 1. *When the principal graph of a Kähler graph G is regular, the asymptotic behavior of $\pi_G^{(p,q)}$ does not depend on q provided that $G_{p,q}$ is irreducible.*

5. Concluding remarks

In this section we shall compare our result with the result on Kähler magnetic flows for a Kähler manifold of negative curvature. Let M be a Kähler manifold of negative curvature with complex structure J. A constant multiple $\mathbf{B}_\kappa = \kappa\mathbf{B}_J$ of its Kähler form $\mathbf{B}_J$ is called a Kähgler magnetic field. We say a smooth curve γ parameterized by its arclength to be a trajectory for $\mathbf{B}_\kappa$ if it satisfies $\nabla_{\dot\gamma}\dot\gamma = \kappa J\dot\gamma$. Since trajectories for the trivial magnetic field $\mathbf{B}_0$ are geodesics, we may say that trajectories are generalizations of geodesics. Just like geodesics induce the geodesic flow φ_t on the unit tangent bundle UM, trajectories for a Kähler magnetic field $\mathbf{B}_\kappa$ induce a magnetic flow $\mathbf{B}_\kappa\varphi_t$ on UM. When M is compact and $\mathbf{B}_\kappa\varphi_t$ is hyperbolic, we have a zeta function as a dynamical system which is defined by $\zeta_{\mathbf{B}_\kappa\varphi_t}(s) = \prod_{\mathfrak{p}}\{1 - e^{\ell(\mathfrak{p})}\}^{-1}$, where $\mathfrak{p}$ runs over the set of all congruence classes of prime periodic orbits and $\ell(\mathfrak{p})$ denotes the period of a prime periodic orbit contained in $\mathfrak{p}$. If we denote by $h_\kappa(M)$ the topological entropy of $\mathbf{B}_\kappa\varphi_t$, this zeta function satisfies the following (see [9]):

(1) It converges absolutely and is holomorphic on $\mathrm{Re}(s) > h_\kappa(M)$;
(2) It is extended meromorphically to a open neighborhood containing $\mathrm{Re}(s) \ge h_\kappa(M)$;
(3) It has a simple pole at $s = h_\kappa(M)$.

For a positive number x, we denote by $\pi_{\mathbf{B}_\kappa\varphi_t}(x)$ the number of congruence classes of prime closed orbits whose periods are not longer than x. As a direct consequence of these properties we find that the asymptotic behavior of the function

$\pi_{\mathbf{B}_\kappa\varphi_t}$ is given as

$$\pi_{\mathbf{B}_\kappa\varphi_t}(x) \sim e^{h_\kappa(M)x}/(h_\kappa(M)x) \quad (x \to \infty).$$

We here compare closed paths on a Kähler graph $G = (V, E^{(p)} \cup E^{(a)})$ whose principal graph is regular and periodic orbits of Kähler magnetic flows for a compact quotient $M = \Gamma\backslash\mathbf{C}H^n(c)$ of a complex hyperbolic space $\mathbf{C}H^n(c)$ of constant holomorphic sectional curvature c. When $|\kappa| < \sqrt{|c|}$, we see that the Kähler magnetic flow $\mathbf{B}_\kappa\varphi_t$ for M is hyperbolic and that its topological entropy $h_\kappa(M)$ is given as $h_\kappa(M) = n\sqrt{|c| - \kappa^2}$ (see [1]). Thus we have

$$h_\kappa(M) = \sqrt{1 - (\kappa^2/|c|)}h_0(M)$$

with the topological entropy $h_0(M)$ of the geodesic flow φ_t for M. On the other hand, if the regular principal graph $(V, E^{(p)})$ of a Kähler graph G is connected and is not a bipartite, then the asymptotic behavior of the number $\pi_G^{(1,0)}(x)$ of prime cycles whose lengths are not longer than x is $e^{h_{1,0}(G)x}/(h_{1,0}(G)x)$ with $h_{1,0}(G) = \log(d_G^{(p)} - 1)$. Thus we find

$$h_{p,q}(G) = \frac{1}{1 + (q/p)} h_{1,0}(G) + \frac{1}{p+q} \log(d_G^{(p)}/(d_G^{(p)} - 1)).$$

We next consider the relationship between the distance $d(P, Q)$ of two points $P, Q \in M$ and the length length(γ) of the trajectory-segment γ joining them. According to [3] we have

$$\frac{2}{\sqrt{|c|}} \sinh\left(\frac{1}{2}\sqrt{|c|}d(P, Q)\right) = \frac{2}{\sqrt{|c| - \kappa^2}} \sinh\left(\frac{1}{2}\sqrt{|c| - \kappa^2}\,\mathrm{length}(\gamma)\right).$$

Quite roughly, we can consider that $d(P, Q) \fallingdotseq \sqrt{1 - (\kappa^2/|c|)}\,\mathrm{length}(\gamma)$ by taking expansions of both sides of the above equality. On the other hand, a (p, q)-primitive bicolored path is of step $(p + q)$. It is natural to consider the distance between its origin and terminus is p, because it moves on the principal graph p-steps. Thus the relation between the length $\ell(\sigma)$ of (p, q)-bicolored path σ and the moving-distance $d(\sigma)$ is given by $d(\sigma) = \dfrac{1}{1 + (q/p)}\ell(\sigma)$. The authors consider that q/p corresponds to $\kappa^2/|c|$ by these relations on entropies and on distances and lengths.

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