N. PALI
KODAI MATH. J.
41 (2018), 201–226

ON COMPLEX DEFORMATIONS OF KÄHLER-RICCI SOLITONS

NEFTON PALI

Abstract

We obtain a formal obstruction, i.e. a necessary condition for the existence of polarized complex deformations of Kähler-Ricci solitons. This obstruction is expressed in terms of the harmonic part of the variation of the complex structure.

1. The obstruction result

Despite the remarkable work of Podesta-Spiro, [9], not much is known on the existence of complex deformations of Kähler-Ricci solitons. In this paper, we provide an effective result on this topic. Namely, given any polarized family of complex deformations over a Kähler-Ricci soliton (polarized by the symplectic form of the initial Kähler-Ricci soliton), we can effectively establish a necessary condition for this family to exist.

Let (X, J, g, ω) be a Fano manifold with $\omega = \mathrm{Ric}_J(\Omega)$, where $\Omega > 0$ is the unique volume form such that $\int_X \Omega = 1$. (We denote by $\mathrm{Ric}_J(\Omega)$ the Chern-Ricci form associated to the volume form Ω). We introduce the Ω-divergence operator acting on vector fields ξ as

$$\mathrm{div}^\Omega\, \xi := \frac{d(\xi \neg \Omega)}{\Omega},$$

where $\neg$ denotes the natural contraction operation. (We invite the readers to see the identity (9) below for a link with the usual notion div_g^f, where $f := \log \dfrac{dV_g}{\Omega}$. See also the remark 2 at the end of the paper for a mathematical explanation why the index Ω should be used instead of f in the related operators).

It is well known (see [3]), that the Lie algebra of J-holomorphic vector fields $H^0(X, T^{1,0}_{X,J})$ identifies with the space of complex valued functions

$$\Lambda^\Omega_{g,J} := \overline{-\mathrm{div}^\Omega\, H^0(X, T^{1,0}_{X,J})} \subset C^\infty_\Omega(X, \mathbf{C})_0,$$

2010 *Mathematics Subject Classification*. 53C25, 53C55, 32J15.

Key words and phrases. Polarized complex deformations, Chern-Ricci form, Kähler-Ricci solitons.

Received October 11, 2016; revised May 11, 2017.

where $C_\Omega^\infty(X, \mathbf{C})_0$ is the space of smooth complex valued functions with vanishing integral with respect to Ω. We denote by $\mathscr{H}^{0,1}_{g,\Omega}(T_{X,J})$ the space of T_X-valued $(0,1)$-forms which are harmonic with respect to the Hodge-Witten Laplacian determined by the volume form Ω.

Assume now (X, J, g, ω) is a compact Kähler-Ricci soliton and we consider the functions $f := \log \dfrac{dV_g}{\Omega}$, $F := f - \int_X f\Omega$. The solution of the variational stability problem in [6], theorem 1 in section 1, shows that the vanishing harmonic cone

$$\mathscr{H}^{0,1}_{g,\Omega}(T_{X,J})_0 := \left\{ A \in \mathscr{H}^{0,1}_{g,\Omega}(T_{X,J}) \,\middle|\, \int_X |A|_g^2 F\Omega = 0 \right\},$$

is relevant for the deformation theory of compact Kähler-Ricci solitons. In the Dancer-Wang Kähler-Ricci soliton case $\mathscr{H}^{0,1}_{g,\Omega}(T_{X,J})_0 \neq \{0\}$, thanks to a result in [4].

For any $A \in \mathscr{H}^{0,1}_{g,\Omega}(T_{X,J})$ we define the $\mathbf{R}$-linear functional

$$\Phi_A : \Lambda^\Omega_{g,J} \to \mathbf{R},$$

$$\Phi_A(u) := \int_X [2 \operatorname{Re} u \langle \nabla_g^2 f, A^2 \rangle_g - \langle J\nabla_g f \neg \nabla_g A, i\bar{u} \times_J A \rangle_g]\Omega,$$

where $(a + ib) \times_J A := aA + bJA$ for any $a, b \in \mathbf{R}$. With these notations we can state our obstruction result.

Theorem 1. *Let (X, J, g, ω) be a compact Kähler-Ricci soliton, let $(J_t, \omega)_{t \in (-\varepsilon, \varepsilon)}$ be a smooth family of Kähler-Ricci solitons with $J_0 = J$ and let $A \in \mathscr{H}^{0,1}_{g,\Omega}(T_{X,J})$ be the harmonic part of the variation $\dot{J}_0$. Then $A \in \mathscr{H}^{0,1}_{g,\Omega}(T_{X,J})_0$ and $\Phi_A = 0$.*

The fact that $A \in \mathscr{H}^{0,1}_{g,\Omega}(T_{X,J})_0$ is a statement in [6], Theorem 1, section 1. We will show also that for any $A \in \mathscr{H}^{0,1}_{g,\Omega}(T_{X,J})$ hold the identity

$$\int_X |A|_g^2 F\Omega = -\int_X [2\langle \nabla_g^2 f, A^2 \rangle_g - \langle J\nabla_g f \neg \nabla_g A, JA \rangle_g]\Omega,$$

whose right-hand side shows strong similarity with the integral $\Phi_A(u)$.

2. Properties of the first variation of Perelman's map H

We need to remind a few basic facts proved in [6]. We first remind some of the notations in [6]. We start with some algebraic operations on tensors over a smooth Riemannian manifold (X, g).

In this paper, as in [6], we identify bilinear forms sections on T_X with morphisms $T_X \to T_X^*$, via the natural contraction operation. We notice in particular that the metric g determines an invertible element $g : T_X \to T_X^*$ given by $g\xi := g(\xi, \bullet)$, for any $\xi \in T_X$. Such element can be composed with an endomorphism section A of T_X. This yields an element $gA : T_X \to T_X^*$, which in its turn can be identified with a bilinear form section of T_X. In the same way the element $g^{-1} : T_X^* \to T_X$ can be composed with T_X^*-valued tensors. Indeed, given any $\theta \in (T_X^*)^{\otimes p}$ over X, we define the element $\theta_g^* \in (T_X^*)^{\otimes p-1} \otimes T_X$ as

$$\theta_g^*(\xi_1, \dots, \xi_{p-1}) := g^{-1}[\theta(\xi_1, \dots, \xi_{p-1}, \bullet)],$$

for all $\xi_1, \dots, \xi_{p-1} \in T_X$. We will use sometimes the notation $\theta_g^* := g^{-1}\theta$, to denote this operation.

On the other hand we notice that g^{-1} identifies in a natural way an element $g^{-1} \in C^\infty(X, S^2 T_X)$. In this paper the symbol $\neg$ denotes also the natural contraction operation on the first two entires of a tensor $\theta \in (T_X^*)^{\otimes p}$, $p \geq 2$. Then the usual trace operation $\mathrm{Tr}_g\, \theta$ with respect to g satisfies the identity $\mathrm{Tr}_g\, \theta = g^{-1} \neg \theta$. This is useful when we will derive the trace operator.

For any endomorphism A of the tangent bundle and for any bilinear form B over it we define the contraction operation $A \neg B := \mathrm{Alt}(BA)$, where Alt is the alternating operator.

We define now some fundamental linear and non-linear differential operators. Let $\Omega > 0$ be a smooth volume form over an oriented compact and connected Riemannian manifold (X, g). We equip the set of smooth Riemannian metrics $\mathcal{M}$ over X with the scalar product

$$(u, v) \mapsto \int_X \langle u, v \rangle_g \Omega, \tag{1}$$

for all $u, v \in L^2(X, S^2_{\mathbf{R}} T_X^*)$. Let P_g^* be the formal adjoint of some operator P with respect to the metric g. We observe that the operator $P_g^{*\Omega} := e^f P_g^*(e^{-f} \bullet)$, with $f := \log \dfrac{dV_g}{\Omega}$, is the formal adjoint of P with respect to the scalar product (1). We define the real weighted Laplacian operator $\Delta_g^\Omega := \nabla_g^{*\Omega} \nabla_g$. We notice in particular the identity $\mathrm{div}^\Omega \nabla_g u = -\Delta_g^\Omega u$, for all functions u.

Over a Fano manifold (X, J, g, ω), with $\omega = \mathrm{Ric}_J(\Omega)$, $\int_X \Omega = 1$. we define the linear operator $B_{g,J}^\Omega$ acting on smooth complex valued functions u as $B_{g,J}^\Omega u := \mathrm{div}^\Omega(J\nabla_g u)$. This is a first order differential operator. Indeed

$$\begin{aligned} B_{g,J}^\Omega u &= \mathrm{Tr}_{\mathbf{R}}(J \nabla_g^2 u) - df \cdot J \nabla_g u \\ &= g(\nabla_g u, J \nabla_g f), \end{aligned}$$

since J is g-anti-symmetric. We define the weighted complex Laplacian operator $\Delta_{g,J}^\Omega := \Delta_g^\Omega - i B_{J,g}^\Omega$, acting on smooth complex valued functions. We remind the identity $\Lambda_{g,J}^\Omega = \mathrm{Ker}(\Delta_{g,J}^\Omega - 2\mathbf{I})$, (see [3]). We denote by

$$\Lambda_{g,J}^{\Omega,\perp} := [\mathrm{Ker}(\Delta_{g,J}^\Omega - 2\mathbf{I})]^{\perp_\Omega} \subset C_\Omega^\infty(X, \mathbf{C})_0,$$

the L_Ω^2-orthogonal space to $\Lambda_{g,J}^\Omega$ inside $C_\Omega^\infty(X, \mathbf{C})_0$.

We remind now that the Ω-Bakry-Emery-Ricci tensor of the metric g is defined by the formula

$$\operatorname{Ric}_g(\Omega) := \operatorname{Ric}(g) + \nabla_g d \log \frac{dV_g}{\Omega}.$$

A Riemannian metric g is called a Ω-shrinking Ricci soliton if $g = \operatorname{Ric}_g(\Omega)$. We define the following fundamental objects

$$h \equiv h_{g,\Omega} := \operatorname{Ric}_g(\Omega) - g,$$

$$2H \equiv 2H_{g,\Omega} := -\Delta_g^{\Omega} f + \operatorname{Tr}_g h + 2f.$$

(We remind $f := \log \dfrac{dV_g}{\Omega}$). We define also the integral normalized function $\underline{H} := H - \int_X H\Omega$. We denote by $\mathscr{V}_1$ the space of smooth positive volume forms with unit integral over X. For any $V \in T_{\mathscr{V}_1}$, we define

$$V_\Omega^* := V/\Omega.$$

We notice now that over a polarized Fano manifold (X, ω), $\omega \in 2\pi c_1(X)$, the space of ω-compatible complex structures $\mathscr{J}_\omega$ embeds naturally inside $\mathscr{M} \times \mathscr{V}_1$ via the Chern-Ricci form. The image of this embedding is

$$\mathscr{S}_\omega := \{(g, \Omega) \in \mathscr{M}_\omega \times \mathscr{V}_1 \mid \omega = \operatorname{Ric}_J(\Omega), J = -\omega^{-1} g\},$$

with $\mathscr{M}_\omega := -\omega \cdot \mathscr{J}_\omega \subset \mathscr{M}$. The fact that the space $\mathscr{J}_\omega$ may be singular in general implies that also the space $\mathscr{S}_\omega$ may be singular. We denote by $\mathrm{TC}_{\mathscr{S}_\omega, (g,\Omega)}$ the tangent cone of $\mathscr{S}_\omega$ at an arbitrary point $(g, \Omega) \in \mathscr{S}_\omega$. This is by definition the union of all tangent vectors of $\mathscr{S}_\omega$ at the point (g, Ω). We notice that, (see for example [5], lemma 7 section 5), the tangent cone $\mathrm{TC}_{\mathscr{M}_\omega, g}$ of $\mathscr{M}_\omega$ at an arbitrary point $g \in \mathscr{M}_\omega$ satisfies the inclusion

$$\mathrm{TC}_{\mathscr{M}_\omega, g} \subseteq \mathbf{D}^J_{g,[0]}, \tag{2}$$

with

$$\mathbf{D}^J_{g,[0]} := \{v \in C^\infty(X, S^2_{\mathbf{R}} T^*_X) \mid v = -J^* v J, \bar{\partial}_{T_{X,J}} v^*_g = 0\},$$

It has been shown in [6], lemma 17 section 16, that for any $(g, \Omega) \in \mathscr{S}_\omega$ the inclusion holds

$$\mathrm{TC}_{\mathscr{S}_\omega, (g,\Omega)} \subseteq \mathbf{T}^J_{g,\Omega}, \tag{3}$$

with

$$\mathbf{T}^J_{g,\Omega} := \{(v, V) \in \mathbf{D}^J_{g,[0]} \times T_{\mathscr{V}_1} \mid 2dd^c_J V^*_\Omega = -d(\nabla_g^{*_\Omega} v^*_g \neg \omega)\}.$$

(We will use the definition $2d^c_J := i(\bar{\partial}_J - \partial_J)$ in this paper). We remind (see [6], identity 1.2 in section 1) that a point $(g, \Omega) \in \mathscr{S}_\omega$ is a Kähler-Ricci soliton if and only if $\underline{H}_{g,\Omega} = 0$. Furthermore,

$$2\underline{H}_{g,\Omega} = -(\Delta^\Omega_{g,J} - 2\mathbf{I}) F \in \Lambda^{\Omega,\perp}_{g,J} \cap C^\infty_\Omega(X, \mathbf{R})_0,$$

for all $(g,\Omega) \in \mathscr{S}_\omega$. The infinitesimal properties of the map $(g,\Omega) \in \mathscr{S}_\omega \mapsto \underline{H}_{g,\Omega}$ are explained in the next sub-section.

2.1. Triple splitting of the space $\mathbf{T}^J_{g,\Omega}$

In [6], section 1, we introduce a pseudo-Riemannian metric G over $\mathscr{M} \times \mathscr{V}_1$ which is positive defined over $\mathbf{T}^J_{g,\Omega}$ for any $(g,\Omega) \in \mathscr{S}_\omega$, with $J := -\omega^{-1}g$. By abuse of notations we will denote by $G_{g,\Omega}$ the scalar product over $\Lambda^{\Omega,\perp}_{g,J}$, induced by the isomorphism

$$\eta : \Lambda^{\Omega,\perp}_{g,J} \oplus \mathscr{H}^{0,1}_{g,\Omega}(T_{X,J}) \to \mathbf{T}^J_{g,\Omega}$$
$$(\psi, A) \mapsto \left(g(\bar{\partial}_{T_{X,J}} \nabla_{g,J}\bar{\psi} + A), -\frac{1}{2}\,\mathrm{Re}[(\Delta^{\Omega}_{g,J} - 2\mathbf{I})\psi]\Omega\right).$$

Explicitly (see [6] sub-section 18.2),

$$G_{g,\Omega}(\varphi,\psi) = \frac{1}{2}\int_X [(\Delta^{\Omega}_{g,J} - 2\mathbf{I})\varphi \cdot \bar{\psi} + (\Delta^{\Omega}_{g,J} - 2\mathbf{I})\psi \cdot \bar{\varphi}]\Omega$$
$$+ \frac{1}{2}\int_X \mathrm{Im}[(\Delta^{\Omega}_{g,J} - 2\mathbf{I})\varphi]\,\mathrm{Im}[(\Delta^{\Omega}_{g,J} - 2\mathbf{I})\psi]\Omega.$$

For any $(g,\Omega) \in \mathscr{S}_\omega$, we introduce in [6], sub-section 18.2, the vector spaces

$$\mathbf{E}^J_{g,\Omega} := \{u \in \Lambda^{\Omega,\perp}_{g,J} \mid (\Delta^{\Omega}_{g,J} - 2\mathbf{I})u = \overline{(\Delta^{\Omega}_{g,J} - 2\mathbf{I})u}\},$$
$$\mathbf{O}^J_{g,\Omega} := (\mathbf{E}^J_{g,\Omega})^{\perp_G} \cap \Lambda^{\Omega,\perp}_{g,J},$$

and we denote by $[g,\Omega]_\omega := \mathrm{Symp}^0(X,\omega) \cdot (g,\Omega) \subset \mathscr{S}_\omega$ the orbit of the point (g,Ω) under the action of the identity component of the group of smooth symplectomorphisms $\mathrm{Symp}^0(X,\omega)$ of X. The map η restricts to a G-isometry

$$\eta : \mathbf{O}^J_{g,\Omega} \to T_{[g,\Omega]_\omega,(g,\Omega)}.$$

The positivity of the metric $G_{g,\Omega}$ over $\Lambda^{\Omega,\perp}_{g,J}$, combined with an elliptic argument (see [6], sub-section 18.2) implies the decomposition

$$\Lambda^{\Omega,\perp}_{g,J} = \mathbf{O}^J_{g,\Omega} \oplus_G \mathbf{E}^J_{g,\Omega},$$

Over a compact Kähler-Ricci soliton (X,J,g,ω), we introduce the operator

$$P^{\Omega}_{g,J} := (\Delta^{\Omega}_{g,J} - 2\mathbf{I})\overline{(\Delta^{\Omega}_{g,J} - 2\mathbf{I})}.$$

This is a non-negative self-adjoint real elliptic operator with respect to the L^2_Ω-hermitian product. The restriction of the differential of the map $(g,\Omega) \in \mathscr{S}_\omega \mapsto \underline{H}_{g,\Omega}$ over the space $\Lambda^{\Omega,\perp}_{g,J}$, identifies, via the isomorphism η, with the map

$$D_{g,\Omega}\underline{H} : \Lambda^{\Omega,\perp}_{g,J} \to \Lambda^{\Omega,\perp}_{g,J} \cap C^\infty_\Omega(X,\mathbf{R})_0$$
$$\psi \mapsto \frac{1}{4} P^{\Omega}_{g,J}\,\mathrm{Re}\,\psi.$$

This map restricts to an isomorphism

$$D_{g,\Omega}\underline{H} : \mathbf{E}^J_{g,\Omega} \to \Lambda^{\Omega,\perp}_{g,J} \cap C^\infty_\Omega(X,\mathbf{R})_0,$$

(see [6], lemma 24, section 19, for the technical details), and

$$\mathbf{O}^J_{g,\Omega} = \operatorname{Ker} D_{g,\Omega}\underline{H} \cap \Lambda^{\Omega,\perp}_{g,J}.$$

Moreover, $\operatorname{Ker} P^\Omega_{g,J} \cap C^\infty_\Omega(X,\mathbf{R})_0 = \{\operatorname{Re} u \mid u \in \Lambda^\Omega_{g,J}\} =: \operatorname{Re} \Lambda^\Omega_{g,J}$ and

$$P^\Omega_{g,J} C^\infty_\Omega(X,\mathbf{R})_0 = \Lambda^{\Omega,\perp}_{g,J} \cap C^\infty_\Omega(X,\mathbf{R})_0.$$

In general for any $(g,\Omega) \in \mathscr{S}_\omega$ Kähler-Ricci soliton the identity holds

$$\operatorname{Ker} D_{g,\Omega}\underline{H} \cap \mathbf{T}^J_{g,\Omega} = T_{[g,\Omega]_\omega,(g,\Omega)} \oplus_G \mathscr{H}^{0,1}_{g,\Omega}(T_{X,J}),$$

with $J := -\omega^{-1}g$. We finally notice that applying the finiteness theorem (see for example [2], proposition 6.6, page 26), to the real elliptic operator $P^\Omega_{g,J} : C^\infty_\Omega(X,\mathbf{R})_0 \to C^\infty_\Omega(X,\mathbf{R})_0$, we deduce the L^2_Ω-orthogonal decomposition

$$C^\infty_\Omega(X,\mathbf{R})_0 = [\Lambda^{\Omega,\perp}_{g,J} \cap C^\infty_\Omega(X,\mathbf{R})_0] \oplus_\Omega \operatorname{Re} \Lambda^\Omega_{g,J}. \tag{4}$$

Remark 1. We denote by $\Lambda^\Omega_{g,\mathbf{R}} := \operatorname{Ker}_{\mathbf{R}}(\Delta^\Omega_g - 2\mathbf{I}) \subset C^\infty_\Omega(X,\mathbf{R})_0$, and by

$$\Lambda^{\Omega,\perp}_{g,\mathbf{R}} := [\operatorname{Ker}_{\mathbf{R}}(\Delta^\Omega_g - 2\mathbf{I})]^{\perp_\Omega} \subset C^\infty_\Omega(X,\mathbf{R})_0,$$

its L^2_Ω-orthogonal inside $C^\infty_\Omega(X,R)_0$. It is easy to see that the map

$$\chi : \Lambda^{\Omega,\perp}_{g,\mathbf{R}} \cap C^\infty_\Omega(X,\mathbf{R})_0 \to T_{[g,\Omega]_\omega,(g,\Omega)},$$
$$u \mapsto (2\omega\bar\partial_{T_{X,J}}\nabla_g u, (B^\Omega_{g,J}u)\Omega),$$

is an isomorphism. Thus, there exists an isomorphism map

$$\tau : \mathbf{O}^J_{g,\Omega} \to i\Lambda^{\Omega,\perp}_{g,\mathbf{R}}$$
$$\theta \mapsto iu : \theta - iu \in \Lambda^\Omega_{g,J}.$$

3. Variation formulas for the Ω-divergence operators

For any endomorphism section A of the tangent bundle we denote by A^T_g the transposed endomorphism section with respect to the metric g. For any $u, v \in C^\infty(X, S^2T^*_X)$ we define in [6], section 10, the real valued 1-form

$$M_g(u,v)(\xi) := 2\nabla_g v(e_k, u^*_g e_k, \xi) + \nabla_g u(\xi, v^*_g e_k, e_k),$$

for all $\xi \in T_X$. (Here $(e_k)_k$ is a g-orthonormal local frame of the tangent bundle). We show now the following important lemma.

Lemma 1. *The first variation of the operator valued map*

$$(g,\Omega) \mapsto \nabla^{*_\Omega}_g : C^\infty(X, S^2T_X) \to C^\infty(X, T^*_X),$$

in arbitrary directions (v, V) *is given by the formula*

$$2[(D_{g,\Omega}\nabla^{*\bullet}_{\bullet})(v, V)]u = M_g(v, u) - 2u \cdot (\nabla^{*\Omega}_g v^*_g + \nabla_g V^*_{\Omega}). \tag{5}$$

Proof. We first differentiate the identity defining the covariant derivative of a symmetric 2-tensor u in the direction v. We infer

$$\dot{\nabla}_g u(\xi, \eta, \mu) = -u(\dot{\nabla}_g(\xi, \eta), \mu) - u(\eta, \dot{\nabla}_g(\xi, \mu)),$$

where $\dot{\nabla}_g := (D_g \nabla_\bullet)(v)$. Using the variation formula for the Levi-Civita connection in [1], we obtain

$$\begin{aligned} 2\dot{\nabla}_g u(\xi, \eta, \mu) = & -u(\nabla_{g,\xi} v^*_g \eta + \nabla_{g,\eta} v^*_g \xi - (\nabla_g v^*_g \eta)^T_g \xi, \mu) \\ & - u(\eta, \nabla_{g,\xi} v^*_g \mu + \nabla_{g,\mu} v^*_g \xi - (\nabla_g v^*_g \mu)^T_g \xi). \end{aligned}$$

We transform the term

$$\begin{aligned} u((\nabla_g v^*_g \eta)^T_g \xi, \mu) &= g(u^*_g (\nabla_g v^*_g \eta)^T_g \xi, \mu) \\ &= g((\nabla_g v^*_g \eta)^T_g \xi, u^*_g \mu) \\ &= g(\xi, \nabla_g v^*_g (u^*_g \mu, \eta)) \\ &= \nabla_g v(u^*_g \mu, \eta, \xi). \end{aligned}$$

We deduce the variation formula

$$\begin{aligned} 2\dot{\nabla}_g u(\xi, \eta, \mu) = & -u(\nabla_{g,\xi} v^*_g \eta + \nabla_{g,\eta} v^*_g \xi, \mu) + \nabla_g v(u^*_g \mu, \xi, \eta) \\ & - u(\eta, \nabla_{g,\xi} v^*_g \mu + \nabla_{g,\mu} v^*_g \xi) + \nabla_g v(u^*_g \eta, \xi, \mu). \end{aligned}$$

Thus, using the fact that u is symmetric we infer

$$\begin{aligned} 2(g^{-1} \neg \dot{\nabla}_g u)\mu = & \, 2u(\nabla^*_g v^*_g, \mu) + \nabla_g v(u^*_g \mu, e_k, e_k) \\ & - u(\nabla_{g,e_k} v^*_g \mu + \nabla_{g,\mu} v^*_g e_k, e_k) + \nabla_g v(e_k, u^*_g e_k, \mu), \end{aligned}$$

where $(e_k)_k$ is a g-orthonormal basis of $T_{X,p}$ which diagonalizes u at the point p. We observe however that the right hand-side of the previous equality is independent of the choice of the g-orthonormal basis $(e_k)_k$ thanks to the intrinsic definition of trace. Simplifying, we deduce

$$2(g^{-1} \neg \dot{\nabla}_g u)\mu = 2u(\nabla^*_g v^*_g, \mu) + \nabla_g v(u^*_g \mu, e_k, e_k) - \nabla_g v(\mu, u^*_g e_k, e_k). \tag{6}$$

We can compute now the first variation of the expression

$$\nabla^{*\Omega}_g u = -g^{-1} \neg \nabla_g u + \nabla_g f \neg u,$$

with $f \equiv f_{g,\Omega} := \log \dfrac{dV_g}{\Omega}$. We observe the identity

$$[(D_{g,\Omega}\nabla^{*\bullet}_{\bullet})(v,V)]u = (v^*_g g^{-1})\neg\nabla_g u - g^{-1}\neg\dot{\nabla}_g u + [(D_{g,\Omega}\nabla_{\bullet} f_{\bullet,\bullet})(v,V)]\neg u.$$

Let $(e_k)_k$ be a g-orthonormal local frame of T_X such that $\nabla_g e_k(p) = 0$, for some arbitrary point p. Using (6) and the variation formulas

$$\frac{d}{dt}(\nabla_{g_t} f_t) = \nabla_{g_t}\dot{f}_t - \dot{g}^*_t \nabla_{g_t} f_t, \tag{7}$$

$$\dot{f}_t = \frac{1}{2}\,\mathrm{Tr}_{g_t}\,\dot{g}_t - \dot{\Omega}^*_t, \tag{8}$$

we obtain the equalities at the point p,

$$\begin{aligned}
2[(D_{g,\Omega}\nabla^{*\bullet}_{\bullet})(v,V)]u(\mu) &= 2\nabla_g u(e_k, v^*_g e_k, \mu) - 2u(\nabla^*_g v^*_g, \mu)\\
&\quad - \nabla_g v(u^*_g\mu, e_k, e_k) + \nabla_g v(\mu, u^*_g e_k, e_k)\\
&\quad + u(\nabla_g(\mathrm{Tr}_g\, v - 2V^*_\Omega) - 2v^*_g\nabla_g f, \mu)\\
&= 2\nabla_g u(e_k, v^*_g e_k, \mu) - 2u(\nabla^{*\Omega}_g v^*_g, \mu)\\
&\quad - \nabla_g v(u^*_g\mu, e_k, e_k) + \nabla_g v(\mu, u^*_g e_k, e_k)\\
&\quad + u(\nabla_g(\mathrm{Tr}_g\, v - 2V^*_\Omega), \mu)\\
&= M_g(v,u)(\mu) - 2u(\nabla^{*\Omega}_g v^*_g + \nabla_g V^*_\Omega, \mu),
\end{aligned}$$

thanks to the identity at the point p,

$$\nabla_g v(u^*_g\mu, e_k, e_k) = u(\nabla_g\,\mathrm{Tr}_g\, v, \mu).$$

In order to see this last fact, we observe the equalities

$$\begin{aligned}
u(\nabla_g\,\mathrm{Tr}_g\, v, \mu) &= g(u^*_g\nabla_g\,\mathrm{Tr}_g\, v, \mu)\\
&= g(\nabla_g\,\mathrm{Tr}_g\, v, u^*_g\mu)\\
&= (d\,\mathrm{Tr}_g\, v)(u^*_g\mu)\\
&= (u^*_g\mu).v(e_k, e_k)\\
&= \nabla_g v(u^*_g\mu, e_k, e_k),
\end{aligned}$$

at the point p. We obtain the required variation formula. □

In a similar way we compute the first variation formula for the operator div^Ω_g acting on 1-forms α

$$\mathrm{div}^\Omega_g\,\alpha := g^{-1}\neg\nabla_g\alpha - \alpha\cdot\nabla_g f. \tag{9}$$

We notice the elementary identity $\mathrm{div}^\Omega_g\,\alpha = \mathrm{div}^\Omega\,\alpha^*_g$. With these notations hold the following lemma.

LEMMA 2. *The first variation of the operator valued map*

$$(g, \Omega) \mapsto \operatorname{div}_g^{\Omega} : C^{\infty}(X, T_X^*) \to C^{\infty}(X, \mathbf{R}),$$

in arbitrary directions (v, V) *is given by the formula*

$$[(D_{g,\Omega} \operatorname{div}_{\bullet}^{\bullet})(v, V)]\alpha = -\langle \nabla_g \alpha_g^*, v_g^* \rangle_g + 2\alpha \cdot (\nabla_g^{*\Omega} v_g^* + \nabla_g V_{\Omega}^*).$$

We include the proof for readers convenience.

Proof. Let α be a 1-form and let ξ, η be two smooth vector fields. Differentiating the identity

$$\xi.(\alpha \cdot \eta) = \nabla_{g,\xi}\alpha \cdot \eta + \alpha \cdot \nabla_{g,\xi}\eta,$$

with respect to the variable g we obtain

$$\begin{aligned} 2\dot{\nabla}_g \alpha(\xi, \eta) &= -\alpha \cdot 2\dot{\nabla}_g(\xi, \eta) \\ &= -\alpha \cdot (\nabla_{g,\xi} v_g^* \cdot \eta + \nabla_{g,\eta} v_g^* \cdot \xi) + \nabla_g v(\alpha_g^*, \xi, \eta). \end{aligned}$$

We notice indeed the equalities

$$\begin{aligned} \alpha \cdot [(\nabla_{g,\bullet} v_g^* \cdot \eta)_g^T \cdot \xi] &= g(\alpha_g^*, (\nabla_{g,\bullet} v_g^* \cdot \eta)_g^T \cdot \xi) \\ &= g(\nabla_{g,\alpha_g^*} v_g^* \cdot \eta, \xi) \\ &= \nabla_g v(\alpha_g^*, \xi, \eta). \end{aligned}$$

We deduce

$$\begin{aligned} 2(g^{-1} \neg \dot{\nabla}_g \alpha) &= 2\alpha \cdot \nabla_g^* v_g^* + \alpha_g^*. \operatorname{Tr}_g v \\ &= \alpha \cdot (2\nabla_g^* v_g^* + \nabla_g \operatorname{Tr}_g v). \end{aligned}$$

We can compute now the first variation of the definition (9). We observe the identities

$$\begin{aligned} 2[(D_{g,\Omega} \operatorname{div}_{\bullet}^{\bullet})(v, V)]\alpha &= -2(v_g^* g^{-1}) \neg \nabla_g \alpha + 2g^{-1} \neg \dot{\nabla}_g \alpha \\ &\quad - 2\alpha \cdot [(D_{g,\Omega} \nabla_{\bullet} f_{\bullet,\bullet})(v, V)] \\ &= -2\nabla_g \alpha(e_k, v_g^* e_k) + \alpha \cdot (2\nabla_g^* v_g^* + \nabla_g \operatorname{Tr}_g v) \\ &\quad - \alpha \cdot (\nabla_g(\operatorname{Tr}_g v - 2V_{\Omega}^*) - 2v_g^* \cdot \nabla_g f). \end{aligned}$$

We infer the required variation formula. □

We can compute now a first variation formula for the double divergence operator $\operatorname{div}_g^{\Omega} \nabla_g^{*\Omega}$. We observe first the trivial identity

$$\begin{aligned} [D_{g,\Omega}(\operatorname{div}_{\bullet}^{\bullet} \nabla_{\bullet}^{*\bullet})(v, V)]v &= [(D_{g,\Omega} \operatorname{div}_{\bullet}^{\bullet})(v, V)]\nabla_g^{*\Omega} v \\ &\quad + \operatorname{div}_g^{\Omega}\{[(D_{g,\Omega} \nabla_{\bullet}^{*\bullet})(v, V)]v\}, \end{aligned}$$

and we explicit the last term;

$$
\begin{aligned}
& 2\operatorname{div}_g^{\Omega}\{[(D_{g,\Omega}\nabla^*_{\bullet}\bullet)(v,V)]v\}\\
&\quad = e_l.[2\nabla_g v(e_k, v_g^* e_k, e_l) + \nabla_g v(e_l, v_g^* e_k, e_k) - 2v(\nabla_g^{*\Omega} v_g^* + V_\Omega^*, e_l)]\\
&\qquad - 2\nabla_g v(e_k, v_g^* e_k, \nabla_g f) - \nabla_g v(\nabla_g f, v_g^* e_k, e_k) + 2v(\nabla_g^{*\Omega} v_g^* + V_\Omega^*, \nabla_g f).
\end{aligned}
$$

Developing further we obtain

$$
\begin{aligned}
& 2\operatorname{div}_g^{\Omega}\{[(D_{g,\Omega}\nabla^*_{\bullet}\bullet)(v,V)]v\}\\
&\quad = 2g(\nabla_{g,e_l}\nabla_{g,e_k} v_g^* \cdot v_g^* e_k, e_l) + 2g(\nabla_{g,e_k} v_g^* \cdot \nabla_{g,e_l} v_g^* e_k, e_l)\\
&\qquad + \nabla^2_{g,e_l,e_l} v(v_g^* e_k, e_k) + g(\nabla_{g,e_l} v_g^* \cdot \nabla_{g,e_l} v_g^* e_k, e_k)\\
&\qquad - 2\nabla_{g,e_l} v(e_l, \nabla_g^{*\Omega} v_g^* + V_\Omega^*) - 2v(\nabla_{g,e_l}(\nabla_g^{*\Omega} v_g^* + V_\Omega^*), e_l)\\
&\qquad - 2g(\nabla_{g,e_k} v_g^* \cdot v_g^* e_k, \nabla_g f) - \nabla_g v(\nabla_g f, v_g^* e_k, e_k) + 2v(\nabla_g^{*\Omega} v_g^* + V_\Omega^*, \nabla_g f)\\
&\quad = 2g(v_g^* e_k, \nabla_{g,e_l}\nabla_{g,e_k} v_g^* e_l) + 2g(\nabla_{g,e_l} v_g^* e_k, \nabla_{g,e_k} v_g^* e_l)\\
&\qquad - \Delta_g^{\Omega} v(v_g^* e_k, e_k) + g(\nabla_{g,e_l} v_g^* e_k, \nabla_{g,e_l} v_g^* e_k)\\
&\qquad + 2\nabla_g^{*\Omega} v \cdot (\nabla_g^{*\Omega} v_g^* + V_\Omega^*) - 2g(\nabla_{g,e_l}(\nabla_g^{*\Omega} v_g^* + V_\Omega^*), v_g^* e_l)\\
&\qquad - 2g(v_g^* e_k, \nabla_{g,e_k} v_g^* \cdot \nabla_g f).
\end{aligned}
$$

If we set

$$
\widehat{\nabla_g v_g^*}(\xi,\eta) := \nabla_g v_g^*(\eta,\xi),
$$

then the last expression writes as

$$
\begin{aligned}
& 2\operatorname{div}_g^{\Omega}\{[(D_{g,\Omega}\nabla^*_{\bullet}\bullet)(v,V)]v\}\\
&\quad = 2g(v_g^* e_k, \nabla_{g,e_l}\widehat{\nabla_g v_g^*}(e_l,e_k)) + 2g(\nabla_g v_g^*(e_l,e_k), \widehat{\nabla_g v_g^*}(e_l,e_k))\\
&\qquad - g(\Delta_g^{\Omega} v_g^* \cdot v_g^* e_k, e_k) + |\nabla_g v_g^*|_g^2\\
&\qquad + 2\nabla_g^{*\Omega} v \cdot (\nabla_g^{*\Omega} v_g^* + V_\Omega^*) - 2\langle \nabla_g(\nabla_g^{*\Omega} v_g^* + V_\Omega^*), v_g^* \rangle_g\\
&\qquad - 2g(v_g^* e_k, \widehat{\nabla_g v_g^*}(\nabla_g f, e_k)).
\end{aligned}
$$

We infer the formula

$$
\begin{aligned}
\operatorname{div}_g^{\Omega}\{[(D_{g,\Omega}\nabla^*_{\bullet}\bullet)(v,V)]v\} = {} & -\frac{1}{4}\Delta_g^{\Omega}|v|_g^2\\
& - \langle \nabla_g^{*\Omega}\widehat{\nabla_g v_g^*}, v_g^* \rangle_g + \langle \widehat{\nabla_g v_g^*}, \nabla_g v_g^* \rangle_g\\
& + \nabla_g^{*\Omega} v \cdot (\nabla_g^{*\Omega} v_g^* + \nabla_g V_\Omega^*)\\
& - \langle \nabla_g(\nabla_g^{*\Omega} v_g^* + \nabla_g V_\Omega^*), v_g^* \rangle_g.
\end{aligned}
$$

We obtain in conclusion the variation identity

$$
\begin{aligned}
(10)\qquad [D_{g,\Omega}(\operatorname{div}^{\Omega}_{\bullet} \nabla^{*}_{\bullet})(v, V)]v = &-\frac{1}{4}\Delta^{\Omega}_{g}|v|^2_g \\
&- \langle \nabla^{*\Omega}_{g}\widehat{\nabla_g v^*_g}, v^*_g\rangle_g + \langle \widehat{\nabla_g v^*_g}, \nabla_g v^*_g\rangle_g \\
&+ 2\nabla^{*\Omega}_{g} v \cdot (\nabla^{*\Omega}_{g} v^*_g + \nabla_g V^*_{\Omega}) \\
&- \langle \nabla_g(2\nabla^{*\Omega}_{g} v^*_g + \nabla_g V^*_{\Omega}), v^*_g\rangle_g.
\end{aligned}
$$

4. The second variation of Perelman's map H

Lemma 3. *The Hessian form $\nabla_G DH(g,\Omega)$ of Perelman's map $(g,\Omega) \in \mathcal{M} \times \mathcal{V}_1 \mapsto H_{g,\Omega}$, with respect to the pseudo-Riemannian structure G at the point $(g,\Omega) \in \mathcal{M} \times \mathcal{V}_1$ in arbitrary directions (v, V) is given by the expression*

$$
\begin{aligned}
2\nabla_G DH(g,\Omega)(v, V; v, V) = &-\frac{1}{2}\langle \mathcal{L}^{\Omega}_{g} v, v\rangle_g - \Delta^{\Omega}_{g}\left[\frac{1}{4}|v|^2_g + (V^*_{\Omega})^2\right] \\
&+\frac{1}{2}|v|^2_g + (V^*_{\Omega})^2 - \frac{1}{2}G_{g,\Omega}(v, V; v, V) \\
&- 2|\nabla^{*\Omega}_{g} v^*_g + \nabla_g V^*_{\Omega}|^2_g \\
&+ \langle \nabla_g(2\nabla^{*\Omega}_{g} v^*_g + 3\nabla_g V^*_{\Omega}), v^*_g\rangle_g \\
&+ \langle \nabla^{*\Omega}_{g} v^*_g, \nabla^{*\Omega}_{g} v^*_g + 2\nabla_g V^*_{\Omega}\rangle_g \\
&+ V^*_{\Omega}(\operatorname{div}^{\Omega} \nabla^{*\Omega}_{g} v^*_g + \langle v, h_{g,\Omega}\rangle_g).
\end{aligned}
$$

Proof. We consider a smooth curve $(g_t, \Omega_t)_{t\in\mathbf{R}} \subset \mathcal{M} \times \mathcal{V}_1$ with $(g_0, \Omega_0) = (g, \Omega)$ and with arbitrary speed $(\dot{g}_0, \dot{\Omega}_0) = (v, V)$. We show in [6], section 6 that the G-covariant derivative ∇_G of its speed, in the speed direction, is given by the expressions

$$
\begin{aligned}
(\theta_t, \Theta_t) &\equiv \nabla_G(\dot{g}_t, \dot{\Omega}_t)(\dot{g}_t, \dot{\Omega}_t),\\
\theta_t &:= \ddot{g}_t + \dot{g}_t(\dot{\Omega}^*_t - \dot{g}^*_t),\\
\Theta_t &:= \ddot{\Omega}_t + \frac{1}{4}[|\dot{g}_t|^2_t - 2(\dot{\Omega}^*_t)^2 - G_{g_t,\Omega_t}(\dot{g}_t, \dot{\Omega}_t; \dot{g}_t, \dot{\Omega}_t)]\Omega_t.
\end{aligned}
$$

Then

$$
\nabla_G DH(g_t, \Omega_t)(\dot{g}_t, \dot{\Omega}_t; \dot{g}_t, \dot{\Omega}_t) = \frac{d^2}{dt^2}H(g_t, \Omega_t) - D_{g_t,\Omega_t}H(\theta_t, \Theta_t).
$$

Using the first variation formula (1.5), section 1 in [6] for the map H we obtain the equalities

$$2\nabla_G DH(g_t,\Omega_t)(\dot g_t,\dot\Omega_t;\dot g_t,\dot\Omega_t)$$
$$=\frac{d}{dt}[(\Delta^{\Omega_t}_{g_t}-2\mathbf{I})\dot\Omega^*_t-\operatorname{div}^{\Omega_t}_{g_t}(\nabla^{*\Omega_t}_{g_t}\dot g_t+d\dot\Omega^*_t)-\langle\dot g_t,h_t\rangle_{g_t}]$$
$$-(\Delta^{\Omega_t}_{g_t}-2\mathbf{I})\Theta^*_t+\operatorname{div}^{\Omega_t}_{g_t}(\nabla^{*\Omega_t}_{g_t}\theta_t+d\Theta^*_t)+\langle\theta_t,h_t\rangle_{g_t}.$$

Using the identity (10) we obtain

$$
\begin{aligned}
&2\nabla_G DH(g_t,\Omega_t)(\dot g_t,\dot\Omega_t;\dot g_t,\dot\Omega_t)\\
&=2(\Delta^{\Omega_t}_{g_t}-\mathbf{I})\left(\frac{d}{dt}\dot\Omega^*_t-\Theta^*_t\right)\\
&+2\langle\nabla_g d\dot\Omega^*_t,\dot g_t\rangle_{g_t}-2\langle\nabla^{*\Omega_t}_{g_t}\dot g^*_t+\nabla_{g_t}\dot\Omega^*_t,\nabla_{g_t}\dot\Omega^*_t\rangle_{g_t}\\
&+\operatorname{div}^{\Omega_t}_{g_t}\ \nabla^{*\Omega_t}_{g_t}(\theta_t-\ddot g_t)\\
&+\frac14\Delta^{\Omega_t}_{g_t}|\dot g_t|^2_{g_t}+\langle\nabla^{*\Omega_t}_{g_t}\widehat{\nabla_{g_t}\dot g^*_t},\dot g^*_t\rangle_{g_t}-\langle\widehat{\nabla_{g_t}\dot g^*_t},\nabla_{g_t}\dot g^*_t\rangle_{g_t}\\
&-2\nabla^{*\Omega_t}_{g_t}\dot g_t\cdot(\nabla^{*\Omega_t}_{g_t}\dot g^*_t+\nabla_{g_t}\dot\Omega^*_t)+\langle\nabla_{g_t}(2\nabla^{*\Omega_t}_{g_t}\dot g^*_t+\nabla_{g_t}\dot\Omega^*_t),\dot g^*_t\rangle_{g_t}\\
&+\operatorname{Tr}_{\mathbf{R}}\left[\left(\theta^*_t-\frac{d}{dt}\dot g^*_t\right)h^*_t-\dot g^*_t(\dot h^*_t-\dot g^*_t h^*_t)\right].
\end{aligned}
$$

Rearranging the previous expression, we obtain

$$
\begin{aligned}
&2\nabla_G DH(g_t,\Omega_t)(\dot g_t,\dot\Omega_t;\dot g_t,\dot\Omega_t)\\
&=2(\Delta^{\Omega_t}_{g_t}-\mathbf{I})\left(\frac{d}{dt}\dot\Omega^*_t-\Theta^*_t\right)+\frac14\Delta^{\Omega_t}_{g_t}|\dot g_t|^2_{g_t}\\
&+\langle\nabla_{g_t}d\dot\Omega^*_t,\dot g_t\rangle_{g_t}-2|\nabla^{*\Omega_t}_{g_t}\dot g^*_t+\nabla_{g_t}\dot\Omega^*_t|^2_{g_t}\\
&+\operatorname{div}^{\Omega_t}_{g_t}\ \nabla^{*\Omega_t}_{g_t}(\theta_t-\ddot g_t)\\
&+\langle\nabla^{*\Omega_t}_{g_t}\widehat{\nabla_{g_t}\dot g^*_t},\dot g^*_t\rangle_{g_t}-\langle\widehat{\nabla_{g_t}\dot g^*_t},\nabla_{g_t}\dot g^*_t\rangle_{g_t}\\
&+2\langle\nabla_{g_t}(\nabla^{*\Omega_t}_{g_t}\dot g^*_t+\nabla_{g_t}\dot\Omega^*_t),\dot g^*_t\rangle_{g_t}\\
&+\operatorname{Tr}_{\mathbf{R}}\left[\left(\theta^*_t-\frac{d}{dt}\dot g^*_t\right)h^*_t-\dot g^*_t(\dot h^*_t-\dot g^*_t h^*_t)\right].
\end{aligned}
$$

Using the expression of θ_t, we develop the term

$$\operatorname{div}^{\Omega_t}_{g_t}\ \nabla^{*\Omega_t}_{g_t}(\theta_t-\ddot g_t)=\operatorname{div}^{\Omega_t}\ \nabla^{*\Omega_t}_{g_t}[\dot\Omega^*_t\dot g^*_t-(\dot g^*_t)^2].$$

For this purpose we remind a few elementary divergence type identities. For any smooth, function u, vector field ξ and endomorphism section A of T_X, the following identities hold

$$\nabla_g^{*\Omega}(uA) = -A \cdot \nabla_g u + u \nabla_g^{*\Omega} A,$$
$$\operatorname{div}^{\Omega}(u\xi) = \langle \nabla_g u, \xi \rangle_g + u \operatorname{div}^{\Omega} \xi,$$
$$\nabla_g^{*\Omega} A^2 = -\operatorname{Tr}_g(\nabla_g A \cdot A) + A \nabla_g^{*\Omega} A.$$

Furthermore if A is g-symmetric then also the formulas hold

$$\operatorname{div}^{\Omega}(A \cdot \xi) = -\langle \nabla_g^{*\Omega} A, \xi \rangle_g + \langle A, \nabla_g \xi \rangle_g, \tag{11}$$

$$\operatorname{div}^{\Omega} \operatorname{Tr}_g(\nabla_g A \cdot A) = -\langle \nabla_g^{*\Omega} \widehat{\nabla_g A}, A \rangle_g + \langle \widehat{\nabla_g A}, \nabla_g A \rangle_g. \tag{12}$$

For readers convenience we show (11) and (12) in the appendix. Using the previous formulas we obtain the equalities

$$\begin{aligned}
\operatorname{div}^{\Omega_t} \nabla_{g_t}^{*\Omega_t}[\dot{\Omega}_t^* \dot{g}_t^* - (\dot{g}_t^*)^2] &= \operatorname{div}^{\Omega_t}[-\dot{g}_t^* \nabla_{g_t} \dot{\Omega}_t^* + \dot{\Omega}_t^* \nabla_{g_t}^{*\Omega_t} \dot{g}_t^*] \\
&\quad + \operatorname{div}^{\Omega_t}[\operatorname{Tr}_{g_t}(\nabla_{g_t} \dot{g}_t^* \cdot \dot{g}_t^*) - \dot{g}_t^* \nabla_{g_t}^{*\Omega_t} \dot{g}_t^*] \\
&= -\langle \nabla_{g_t} d\dot{\Omega}_t^*, \dot{g}_t \rangle_{g_t} + 2\langle \nabla_{g_t}^{*\Omega_t} \dot{g}_t^*, \nabla_{g_t} \dot{\Omega}_t^* \rangle_{g_t} \\
&\quad + \dot{\Omega}_t^* \operatorname{div}^{\Omega_t} \nabla_{g_t}^{*\Omega_t} \dot{g}_t^* \\
&\quad - \langle \nabla_{g_t}^{*\Omega_t} \widehat{\nabla_{g_t} \dot{g}_t^*}, \dot{g}_t^* \rangle_{g_t} + \langle \widehat{\nabla_{g_t} \dot{g}_t^*}, \nabla_{g_t} \dot{g}_t^* \rangle_{g_t} \\
&\quad + |\nabla_{g_t}^{*\Omega_t} \dot{g}_t^*|_{g_t}^2 - \langle \dot{g}_t^*, \nabla_{g_t} \nabla_{g_t}^{*\Omega_t} \dot{g}_t^* \rangle_{g_t} \\
&= -\langle \nabla_{g_t}(\nabla_{g_t}^{*\Omega_t} \dot{g}_t^* + \nabla_{g_t} \dot{\Omega}_t^*), \dot{g}_t^* \rangle_{g_t} \\
&\quad + \langle \nabla_{g_t}^{*\Omega_t} \dot{g}_t^*, \nabla_{g_t}^{*\Omega_t} \dot{g}_t^* + 2\nabla_{g_t} \dot{\Omega}_t^* \rangle_{g_t} \\
&\quad + \dot{\Omega}_t^* \operatorname{div}^{\Omega_t} \nabla_{g_t}^{*\Omega_t} \dot{g}_t^* \\
&\quad - \langle \nabla_{g_t}^{*\Omega_t} \widehat{\nabla_{g_t} \dot{g}_t^*}, \dot{g}_t^* \rangle_{g_t} + \langle \widehat{\nabla_{g_t} \dot{g}_t^*}, \nabla_{g_t} \dot{g}_t^* \rangle_{g_t}.
\end{aligned}$$

Plunging this identity in the last expression of the Hessian of the map H we obtain

$$\begin{aligned}
&2\nabla_G DH(g_t, \Omega_t)(\dot{g}_t, \dot{\Omega}_t; \dot{g}_t, \dot{\Omega}_t) \\
&= 2(\Delta_{g_t}^{\Omega_t} - \mathbf{I})\left(\frac{d}{dt}\dot{\Omega}_t^* - \Theta_t^*\right) + \frac{1}{4}\Delta_{g_t}^{\Omega_t}|\dot{g}_t|_{g_t}^2 \\
&\quad + \langle \nabla_{g_t} d\dot{\Omega}_t^*, \dot{g}_t \rangle_{g_t} - 2|\nabla_{g_t}^{*\Omega_t} \dot{g}_t^* + \nabla_{g_t} \dot{\Omega}_t^*|_{g_t}^2 \\
&\quad + \langle \nabla_{g_t}(\nabla_{g_t}^{*\Omega_t} \dot{g}_t^* + \nabla_{g_t} \dot{\Omega}_t^*), \dot{g}_t^* \rangle_{g_t} \\
&\quad + \langle \nabla_{g_t}^{*\Omega_t} \dot{g}_t^*, \nabla_{g_t}^{*\Omega_t} \dot{g}_t^* + 2\nabla_{g_t} \dot{\Omega}_t^* \rangle_{g_t} + \dot{\Omega}_t^* \operatorname{div}^{\Omega_t} \nabla_{g_t}^{*\Omega_t} \dot{g}_t^* \\
&\quad + \dot{\Omega}_t^* \langle \dot{g}_t, h_t \rangle_{g_t} - \frac{1}{2}\langle \mathscr{L}_{g_t}^{\Omega_t} \dot{g}_t - L_{\nabla_{g_t}^{*\Omega_t} \dot{g}_t^* + \nabla_{g_t} \dot{\Omega}_t^*} g_t, \dot{g}_t \rangle_{g_t},
\end{aligned}$$

thanks to the variation formula (1.4), section 1 in [6] for the map h. Rearranging the previous expression, we infer

$$
\begin{aligned}
&2\nabla_G DH(g_t,\Omega_t)(\dot{g}_t,\dot{\Omega}_t;\dot{g}_t,\dot{\Omega}_t)\\
&\quad= 2(\Delta^{\Omega_t}_{g_t}-\mathbf{I})\left(\frac{d}{dt}\dot{\Omega}^*_t-\Theta^*_t\right)+\frac{1}{4}\Delta^{\Omega_t}_{g_t}|\dot{g}_t|^2_{g_t}\\
&\qquad-2|\nabla^{*\Omega_t}_{g_t}\dot{g}^*_t+\nabla_{g_t}\dot{\Omega}^*_t|^2_{g_t}\\
&\qquad+\langle\nabla_{g_t}(2\nabla^{*\Omega_t}_{g_t}\dot{g}^*_t+3\nabla_{g_t}\dot{\Omega}^*_t),\dot{g}^*_t\rangle_{g_t}\\
&\qquad+\langle\nabla^{*\Omega_t}_{g_t}\dot{g}^*_t,\nabla^{*\Omega_t}_{g_t}\dot{g}^*_t+2\nabla_{g_t}\dot{\Omega}^*_t\rangle_{g_t}\\
&\qquad+\dot{\Omega}^*_t(\operatorname{div}^{\Omega_t}\nabla^{*\Omega_t}_{g_t}\dot{g}^*_t+\langle\dot{g}_t,h_t\rangle_{g_t})-\frac{1}{2}\langle\mathscr{L}^{\Omega_t}_{g_t}\dot{g}_t,\dot{g}_t\rangle_{g_t}.
\end{aligned}
$$

Then the conclusion follows from the expression of Θ_t. □

In [6], section 7, we show that the space G-orthogonal to the tangent to the orbit of a point $(g,\Omega)\in\mathscr{M}\times\mathscr{V}_1$, under the action of the identity component of the diffeomorphism group, is

$$\mathbf{F}_{g,\Omega}:=\{(v,V)\in T_{\mathscr{M}\times\mathscr{V}_1}\mid\nabla^{*\Omega}_g v^*_g+\nabla_g V^*_\Omega=0\}.$$

Corollary 1. *The Hessian form $\nabla_G DH(g,\Omega)$ of Perelman's map $(g,\Omega)\in\mathscr{M}\times\mathscr{V}_1\mapsto H_{g,\Omega}$, with respect to the pseudo-Riemannian structure G at the point $(g,\Omega)\in\mathscr{M}\times\mathscr{V}_1$ in arbitrary directions $(v,V)\in\mathbf{F}_{g,\Omega}$, is given by the expression*

$$
\begin{aligned}
&2\nabla_G DH(g,\Omega)(v,V;v,V)\\
&\quad=-\frac{1}{2}\langle(\mathscr{L}^{\Omega}_g+2\nabla_g\nabla^{*\Omega}_g)v,v\rangle_g\\
&\qquad-\frac{1}{2}(\Delta^{\Omega}_g-2\mathbf{I})\left[\frac{1}{2}|v|^2_g+(V^*_\Omega)^2-\frac{1}{2}G_{g,\Omega}(v,V;v,V)\right]\\
&\qquad+V^*_\Omega\langle v,h_{g,\Omega}\rangle_g.
\end{aligned}
$$

5. Application of the weighted Bochner identity

We observe that the formal adjoint of the $\bar{\partial}_{T_{X,J}}$ operator with respect to the hermitian product

$$\langle\cdot,\cdot\rangle_{\omega,\Omega}:=\int_X\langle\cdot,\cdot\rangle_\omega\Omega, \tag{13}$$

is the operator

$$\bar{\partial}^{*_{g,\Omega}}_{T_{X,J}}:=e^f\,\bar{\partial}^{*_g}_{T_{X,J}}(e^{-f}\bullet).$$

With this notation, we define the anti-holomorphic Ω-Hodge-Witten Laplacian operator acting on T_X-valued q-forms as

$$\Delta^{\Omega,-J}_{T_{X,g}}:=\frac{1}{q}\bar{\partial}_{T_{X,J}}\bar{\partial}^{*_{g,\Omega}}_{T_{X,J}}+\frac{1}{q+1}\bar{\partial}^{*_{g,\Omega}}_{T_{X,J}}\bar{\partial}_{T_{X,J}},$$

with the usual convention $\infty \cdot 0 = 0$, and the functorial convention on the scalar product in the subsection 7.1 of the appendix in [5]. We will omit the symbol Ω in the Hodge-Witten Laplacian operator, when $\Omega = \mathrm{Cst}\, dV_g$. We define the vector space

$$\mathscr{H}^{0,1}_{g,\Omega}(T_{X,J}) := \mathrm{Ker}\, \Delta^{\Omega,-J}_{T_{X,g}} \cap C^\infty(X, T^*_{X,-J} \otimes T_{X,J}).$$

It has been shown in [6], lemma 14, section 14, that for any smooth J-anti-linear endomorphism section A of the tangent bundle hold the fundamental Bochner type formula

$$\mathscr{L}^\Omega_g A = 2\Delta^{-J}_{T_{X,g}} A + [\mathrm{Ric}^*(g), A] + \nabla_g f \neg \nabla_g A. \tag{14}$$

We observe that for bi-degree reasons we have the equalities

$$\begin{aligned}\bar\partial^{*_{g,\Omega}}_{T_{X,J}} A &= \nabla^{*_\Omega}_g A\\ &= \nabla^*_g A + A\nabla_g f\\ &= \bar\partial^{*_g}_{T_{X,J}} A + A\nabla_g f.\end{aligned}$$

Using the last equality we obtain the expression

$$\bar\partial_{T_{X,J}} \bar\partial^{*_{g,\Omega}}_{T_{X,J}} A = \bar\partial_{T_{X,J}} \bar\partial^{*_g}_{T_{X,J}} A + \nabla^{0,1}_{g,J} A\nabla_g f + A\partial^g_{T_{X,J}} \nabla_g f.$$

We observe indeed

$$\begin{aligned}2\bar\partial_{T_{X,J}}(A\nabla_g f) &= \nabla_g(A\nabla_g f) + J\nabla_{g,J\cdot}(A\nabla_g f)\\ &= \nabla_g A\nabla_g f + A\nabla^2_g f + J\nabla_{g,J\cdot} A\nabla_g f + JA\nabla_{g,J\cdot}\nabla_g f\\ &= 2\nabla^{0,1}_{g,J} A\nabla_g f + A(\nabla^2_g f - J\nabla_{g,J\cdot}\nabla_g f)\\ &= 2\nabla^{0,1}_{g,J} A\nabla_g f + 2A\partial^g_{T_{X,J}}\nabla_g f.\end{aligned}$$

Still for bi-degree reasons, the identities hold

$$\begin{aligned}\frac{1}{2}\bar\partial^{*_{g,\Omega}}_{T_{X,J}} \bar\partial_{T_{X,J}} A &= \frac{1}{2}\nabla^{*_\Omega}_{T_X,g} \bar\partial_{T_{X,J}} A\\ &= \nabla^{*_\Omega}_g \bar\partial_{T_{X,J}} A\\ &= \nabla^*_g \bar\partial_{T_{X,J}} A + \nabla_g f \neg \bar\partial_{T_{X,J}} A.\end{aligned}$$

Thus

$$\begin{aligned}\frac{1}{2}\bar\partial^{*_{g,\Omega}}_{T_{X,J}} \bar\partial_{T_{X,J}} A &= \frac{1}{2}\nabla^*_{T_X,g} \bar\partial_{T_{X,J}} A + \nabla_g f \neg \bar\partial_{T_{X,J}} A\\ &= \frac{1}{2}\bar\partial^{*_g}_{T_{X,J}} \bar\partial_{T_{X,J}} A + \nabla_g f \neg \nabla^{0,1}_{g,J} A - \nabla^{0,1}_{g,J} A\nabla_g f.\end{aligned}$$

Combining the identities obtained so far we deduce the expression

$$\Delta^{\Omega,-J}_{T_{X,g}} A = \Delta^{-J}_{T_{X,g}} A + \nabla_g f \neg \nabla^{0,1}_{g,J} A + A\partial^g_{T_{X,J}} \nabla_g f. \tag{15}$$

Plugging this in the fundamental identity (14) we obtain the equalities

$$\begin{aligned}\mathscr{L}_g^{\Omega} A &= 2\Delta_{T_{X,g}}^{\Omega,-J} A + [\mathrm{Ric}^*(g), A] - 2A\partial_{T_{X,J}}^{g} \nabla_g f \\ &\quad - \nabla_g f \neg (\nabla_{g,J}^{0,1} - \nabla_{g,J}^{1,0}) A \\ &= 2\Delta_{T_{X,g}}^{\Omega,-J} A + [\mathrm{Ric}^*(g), A] - 2A\partial_{T_{X,J}}^{g} \nabla_g f \\ &\quad - (J\nabla_g f) \neg J \nabla_g A.\end{aligned}$$

Thus, if $A \in \mathscr{H}_{g,\Omega}^{0,1}(T_{X,J})$, then the stability identity holds

$$\langle \mathscr{L}_g^{\Omega} A, A \rangle_g = -2\langle \nabla_g^2 f, A^2 \rangle_g + \langle J\nabla_g f \neg \nabla_g A, JA \rangle_g. \tag{16}$$

6. Variations of ω-compatible complex structures

Let (X, J, g, ω) be a Fano manifold such that $\omega = \mathrm{Ric}_J(\Omega)$, with $\Omega \in \mathscr{V}_1$ and let $(J_t)_t \subset \mathscr{J}_\omega$ be a smooth curve such that $J_0 = J$. We differentiate the definition $g_t := -\omega J_t$. We obtain $\dot{g}_t^* := g_t^{-1}\dot{g}_t = -J_t\dot{J}_t$ and $\ddot{g}_t^* := g_t^{-1}\ddot{g}_t = -J_t\ddot{J}_t$. On the other hand, deriving twice the condition $J_t^2 = -\mathbf{I}$, we obtain $-(J_t\ddot{J}_t)_{J_t}^{1,0} = \dot{J}_t^2$ and thus $(\ddot{g}_t^*)_{J_t}^{1,0} = (\dot{g}_t^*)^2$. The latter gives

$$(\ddot{g}_t^*)_{J_t}^{0,1} = \ddot{g}_t^* - (\dot{g}_t^*)_t^2 = \frac{d}{dt}\dot{g}_t^*.$$

Let N_J be the Nijenhuis tensor of an arbitrary almost complex structure J. Then the general formula

$$2\frac{d}{dt} N_{J_t} = \bar{\partial}_{T_{X,J_t}}(J_t\dot{J}_t) + J_t\dot{J}_t N_{J_t} - (J_t\dot{J}_t)\neg N_{J_t},$$

(see the proof of lemma 7, section 5 in [5]), implies $\bar{\partial}_{T_{X,J_t}}\dot{g}_t^* \equiv 0$, in our case. Thus time deriving the identity

$$\frac{d}{dt}\bar{\partial}_{T_{X,J_t}}\dot{g}_t^* \equiv 0,$$

we obtain the property

$$\bar{\partial}_{T_{X,J_t}}\frac{d}{dt}\dot{g}_t^* = \dot{g}_t^* \neg \nabla_{g_t,J_t}^{1,0}\dot{g}_t^*. \tag{17}$$

Indeed we prove the variation formula

$$\left(\frac{d}{dt}\bar{\partial}_{T_{X,J_t}}\right)\dot{g}_t^* = -\dot{g}_t^* \neg \nabla_{g_t,J_t}^{1,0}\dot{g}_t^*.$$

For this purpose we expand the derivative of $\bar{\partial}_{T_{X,J_t}}$ acting on a smooth endomorphism section A of T_X. We obtain

$$
\begin{aligned}
2\left[\left(\frac{d}{dt}\bar{\partial}_{T_{X,J_t}}\right)A\right](\xi,\eta) &= 2\frac{d}{dt}\ \mathrm{Alt}[\nabla^{0,1}_{g_t,J_t}A(\xi,\eta)] \\
&= \mathrm{Alt}\frac{d}{dt}[\nabla_{g_t}A(\xi,\eta) + J_t\nabla_{g_t}A(J_t\xi,\eta)] \\
&= \mathrm{Alt}[\dot{\nabla}_{g_t}A(\xi,\eta) + \dot{J}_t\nabla_{g_t}A(J_t\xi,\eta)] \\
&\quad + \mathrm{Alt}[J_t\dot{\nabla}_{g_t}A(J_t\xi,\eta) + J_t\nabla_{g_t}A(\dot{J}_t\xi,\eta)].
\end{aligned}
$$

Using the variation formula

$$\dot{\nabla}_{g_t}A(\xi,\eta) = \dot{\nabla}_{g_t}(\xi,A\eta) - A\dot{\nabla}_{g_t}(\xi,\eta),$$

and the fact that the bilinear form $\dot{\nabla}_{g_t}$ is symmetric we deduce the formula

$$
\begin{aligned}
&2\left[\left(\frac{d}{dt}\bar{\partial}_{T_{X,J_t}}\right)A\right](\xi,\eta) \\
&\quad = \mathrm{Alt}[\dot{\nabla}_{g_t}(\xi,A\eta) + \dot{J}_t\nabla_{g_t}A(J_t\xi,\eta)] \\
&\qquad + \mathrm{Alt}[J_t\dot{\nabla}_{g_t}(J_t\xi,A\eta) - J_tA\dot{\nabla}_{g_t}(J_t\xi,\eta) + J_t\nabla_{g_t}A(\dot{J}_t\xi,\eta)].
\end{aligned}
$$

We remind now (see the proof of lemma 1 in [7]), that time deriving the Kähler condition $\nabla_{g_t}J_t \equiv 0$, we obtain the identity

$$\dot{\nabla}_{g_t}(\eta,\xi) + J_t\dot{\nabla}_{g_t}(J_t\eta,\xi) + J_t\nabla_{g_t}\dot{J}_t(\xi,\eta) = 0,$$

Using this in the previous formula with $A = \dot{g}_t^* = -J_t\dot{J}_t$ we obtain

$$
\begin{aligned}
&2\left[\left(\frac{d}{dt}\bar{\partial}_{T_{X,J_t}}\right)\dot{g}_t^*\right](\xi,\eta) \\
&\quad = \mathrm{Alt}[-J_t\nabla_{g_t}\dot{J}_t(\dot{g}_t^*\eta,\xi) - \dot{g}_t^*J_t\nabla_{g_t}\dot{J}_t(\eta,\xi)] \\
&\qquad + \mathrm{Alt}[\dot{J}_t\nabla_{g_t}\dot{g}_t^*(J_t\xi,\eta) + J_t\nabla_{g_t}\dot{g}_t^*(\dot{J}_t\xi,\eta)] \\
&\quad = \mathrm{Alt}[\nabla_{g_t}\dot{g}_t^*(\dot{g}_t^*\eta,\xi) + \dot{J}_t\nabla_{g_t}\dot{g}_t^*(J_t\xi,\eta) + J_t\nabla_{g_t}\dot{g}_t^*(\dot{J}_t\xi,\eta)] \\
&\qquad - \dot{g}_t^*\partial^{g_t}_{T_{X,J_t}}\dot{g}_t^*(\xi,\eta) \\
&\quad = \mathrm{Alt}[\nabla^{1,0}_{g_t,J_t}\dot{g}_t^*(\dot{g}_t^*\eta,\xi) - \nabla^{1,0}_{g_t}\dot{g}_t^*(\dot{g}_t^*\xi,\eta)] \\
&\quad = -2[\dot{g}_t^*\neg\nabla^{1,0}_{g_t,J_t}\dot{g}_t^*](\xi,\eta),
\end{aligned}
$$

and thus the required formula. The latter can also be obtained deriving the Maurer-Cartan equation, which writes in the Kähler case (see the appendix) as

$$\bar{\partial}_{T_{X,J}}\mu_t + \mu_t\neg\nabla^{1,0}_{g,J}\mu_t = 0,$$

with μ_t the Caley transform of J_t with respect to J.

We remind now (see [6], identity (14.7), section 14), that for any smooth family $(g_t,\Omega_t)_t \subset \mathscr{S}_\omega$, hold the identity

$$\Delta^{\Omega_t,-J_t}_{T_{X,g_t}}\dot{g}_t^* = (\Delta^{\Omega_t,-J_t}_{T_{X,g_t}}\dot{g}_t^*)^T_{g_t},$$

with $J_t := -\omega^{-1} g_t$. The latter rewrites as

$$\bar{\partial}_{T_{X,J_t}} \nabla^{*\Omega_t}_{g_t} \dot{g}^*_t = (\bar{\partial}_{T_{X,J_t}} \nabla^{*\Omega_t}_{g_t} \dot{g}^*_t)^T_{g_t}. \tag{18}$$

LEMMA 4. *For any smooth family $(g_t, \Omega_t)_t \subset \mathscr{S}_\omega$, with $(g, \Omega) = (g_0, \Omega_0)$ and $(\dot{g}_0, \dot{\Omega}_0) \in \mathbf{F}^J_{g,\Omega}$, we have the symmetry property*

$$\bar{\partial}_{T_{X,J_t}} \nabla^{*\Omega_t}_{g_t} \frac{d}{dt_{|_{t=0}}} \dot{g}^*_t = \left(\bar{\partial}_{T_{X,J_t}} \nabla^{*\Omega_t}_{g_t} \frac{d}{dt_{|_{t=0}}} \dot{g}^*_t \right)^T_g + [\partial^g_{T_{X,J}} \nabla^{*\Omega}_g \dot{g}^*_0, \dot{g}^*_0]. \tag{19}$$

Proof. Let A be a smooth g-symmetric endomorphism section of T_X. Differentiating in the variables (g, Ω) the trivial identity $\nabla^{*\Omega}_g A = g^{-1} \nabla^{*\Omega}_g (gA)$, we obtain

$$\begin{aligned}[(D_{g,\Omega} \nabla^{*\bullet}_\bullet)(v, V)]A = & -v^*_g \nabla^{*\Omega}_g A + g^{-1}[(D_{g,\Omega} \nabla^{*\bullet}_\bullet)(v, V)](gA) \\ & + \nabla^{*\Omega}_g (v^*_g A).\end{aligned}$$

We observe now the identities

$$\begin{aligned} M_g(v, v) &= 2g \operatorname{Tr}_g(\nabla_g v^*_g \cdot v^*_g) + \frac{1}{2} d|v|^2_g \\ &= 2v \nabla^{*\Omega}_g v^*_g - 2g \nabla^{*\Omega}_g (v^*_g)^2 + \frac{1}{2} d|v|^2_g. \end{aligned}$$

Then using the variation formula (5) we infer the fundamental identity

$$2[(D_{g,\Omega} \nabla^{*\bullet}_\bullet)(v, V)] v^*_g = \frac{1}{2} \nabla_g |v|^2_g - 2v^*_g \cdot (\nabla^{*\Omega}_g v^*_g + \nabla_g V^*_\Omega). \tag{20}$$

The variation formula for the $\bar{\partial}_{T_{X,J_t}}$-operator acting on vector fields in lemma 1 of [7] writes as

$$2 \frac{d}{dt} (\bar{\partial}_{T_{X,J_t}} \xi) = \xi \neg \nabla_{g_t} \dot{g}^*_t - [\partial^{g_t}_{T_{X,J_t}} \xi, \dot{g}^*_t] + [\bar{\partial}_{T_{X,J_t}} \xi, \dot{g}^*_t].$$

Using this, the variation formula (20) and the assumption on the initial speed of the curve (g_t, Ω_t), we infer

$$\begin{aligned} 2 \frac{d}{dt_{|_{t=0}}} (\bar{\partial}_{T_{X,J_t}} \nabla^{*\Omega_t}_{g_t} \dot{g}^*_t) = & \nabla^{*\Omega}_g \dot{g}^*_0 \neg \nabla_g \dot{g}^*_0 \\ & - [\partial^g_{T_{X,J}} \nabla^{*\Omega}_g \dot{g}^*_0, \dot{g}^*_0] + [\bar{\partial}_{T_{X,J}} \nabla^{*\Omega}_g \dot{g}^*_0, \dot{g}^*_0] \\ & + \frac{1}{2} \bar{\partial}_{T_{X,J}} \nabla_g |\dot{g}_0|^2_g + 2 \bar{\partial}_{T_{X,J}} \nabla^{*\Omega}_g \frac{d}{dt_{|_{t=0}}} \dot{g}^*_t. \end{aligned}$$

Using this equality, the elementary identity

$$\frac{d}{dt} A^T_{g_t} = [A^T_{g_t}, \dot{g}^*_t],$$

for arbitrary endomorphism section A of T_X and time deriving the identity (18), we obtain the required conclusion. (Notice that the endomorphism section $\partial^g_{T_{X,J}}\nabla^{*_\Omega}_g \dot{g}^*_0$ is g-symmetric thanks to the assumption $\nabla^{*_\Omega}_g \dot{g}^*_0 = -\nabla_g \dot{\Omega}^*_0$). □

Corollary 2. *Let $(J_t)_t \subset \mathscr{J}_\omega$ be a smooth curve such that $\dot{J}_0 \in \mathscr{H}^{0,1}_{g,\Omega}(T_{X,J})$ then*

$$\nabla^{*_\Omega}_g(\dot{J}_0 \neg \nabla^{1,0}_{g,J}\dot{J}_0) = [\nabla^{*_\Omega}_g(\dot{J}_0 \neg \nabla^{1,0}_{g,J}\dot{J}_0)]^T_g.$$

Proof. The identity (17) implies

$$\bar{\partial}_{T_{X,J}}\nabla^{*_\Omega}_g \frac{d}{dt}_{|_{t=0}} \dot{g}^*_t = \Delta^{\Omega,-J}_{T_{X,g}} \frac{d}{dt}_{|_{t=0}} \dot{g}^*_t - \nabla^{*_\Omega}_g(\dot{g}^*_0 \neg \nabla^{1,0}_{g,J}\dot{g}^*_0).$$

Plunging this in the equality (19) and using the fact that the Laplacian term is g-symmetric (see [6], identity (14.7), section 14), we infer the required conclusion. □

Lemma 5. *Let (X, J, g, ω) be a Fano manifold such that $\omega = \mathrm{Ric}_J(\Omega)$, with $\Omega \in \mathscr{V}_1$ and let $(J_t)_t \subset \mathscr{J}_\omega$ be a smooth curve such that $J_0 = J$ and $\nabla^{*_\Omega}_g \dot{J}_0 = 0$. Then there exists unique $(\psi, A_1) \in \Lambda^{\Omega,\perp}_{g,J} \oplus \mathscr{H}^{0,1}_{g,\Omega}(T_{X,J})$ such that*

$$\frac{d}{dt}_{|_{t=0}} \dot{g}^*_t + \nabla^{*_\Omega}_g(\Delta^{\Omega,-J}_{T_{X,g}})^{-1}(\dot{J}_0 \neg \nabla^{1,0}_{g,J}\dot{J}_0) = \bar{\partial}_{T_{X,J}}\nabla_{g,J}\bar{\psi} + A_1.$$

Proof. The identity (17) implies

$$\bar{\partial}_{T_{X,J}}\left[\frac{d}{dt}_{|_{t=0}} \dot{g}^*_t + \nabla^{*_\Omega}_g(\Delta^{\Omega,-J}_{T_{X,g}})^{-1}(\dot{J}_0 \neg \nabla^{1,0}_{g,J}\dot{J}_0)\right] = 0.$$

Moreover the endomorphism

$$\begin{aligned}
&\frac{d}{dt}_{|_{t=0}} \dot{g}^*_t + \nabla^{*_\Omega}_g(\Delta^{\Omega,-J}_{T_{X,g}})^{-1}(\dot{J}_0 \neg \nabla^{1,0}_{g,J}\dot{J}_0)\\
&\qquad = \frac{d}{dt}_{|_{t=0}} \dot{g}^*_t + (\Delta^{\Omega,-J}_{T_{X,g}})^{-1}\nabla^{*_\Omega}_g(\dot{J}_0 \neg \nabla^{1,0}_{g,J}\dot{J}_0),
\end{aligned}$$

is g-symmetric thanks to corollary 2, lemma 13 in [6] and identity (14.7) in section 14 of [6]. By corollary 3 in section 14 of [6], we infer the required conclusion. Notice that (ψ, A_1) is uniquely determined by $\dot{J}_0$ and $\ddot{J}_0$. □

7. Proof of theorem 1

For any smooth family $(g_t, \Omega_t)_t \subset \mathscr{S}_\omega$, with $(g_0, \Omega_0) = (g, \Omega)$, we consider the smooth curve $t \mapsto \gamma_t := \underline{H}_{g_t,\Omega_t}\Omega_t/\Omega \in C^\infty_\Omega(X, \mathbf{R})_0$. Then $(g_t, \Omega_t)_t \equiv (J_t, \omega)_t$ is a family of Kähler-Ricci solitons if and only if $\gamma_t \equiv 0$. We assume this identity and we notice that $0 = \dot{\gamma}_0 = D_{g,\Omega}\underline{H}(\dot{g}_0, \dot{\Omega}_0)$. We write

$$\dot{g}^*_0 = -J\dot{J}_0 = \bar{\partial}_{T_{X,J}}\nabla_{g,J}\bar{\theta} + 2A,$$

with $(\theta, A) \in \Lambda^{\Omega,\perp}_{g,J} \oplus \mathscr{H}^{0,1}_{g,\Omega}(T_{X,J})$. The properties of the first variation of $\underline{H}$ imply $\theta \in \mathbf{O}^J_{g,\Omega}$. According to the isomorphism τ in remark 1, we pick the unique $u \in \Lambda^{\Omega,\perp}_{g,\mathbf{R}}$ such that $\theta - iu \in \Lambda^{\Omega}_{g,J}$ and we consider the one parameter sub-group of ω-symplectomorphisms $(\Psi_t)_t$, $\Psi_0 = \mathrm{id}_X$, given by $2\dot{\Psi}_t = -(\omega^{-1} du) \circ \Psi_t$. Then $(\Psi_t^* J_t, \omega)_t$ is still a family of Kähler-Ricci solitons and

$$\begin{aligned} \frac{d}{dt}_{|_{t=0}} \Psi_t^* J_t &= \dot{J}_0 - \frac{1}{2} L_{\omega^{-1} du} J \\ &= J \bar{\partial}_{T_{X,J}} \nabla_{g,J} \overline{(\theta - iu)} + 2JA \\ &= 2JA. \end{aligned}$$

Thus we can assume, without loss of generality in the statement of the theorem 1, that the family of Kähler-Ricci solitons $(J_t, \omega)_t$ satisfies $\dot{J}_0 \in \mathscr{H}^{0,1}_{g,\Omega}(T_{X,J})$. Using this assumption, we explicit the second variation of the map $(g, \Omega) \mapsto \underline{H}_{g,\Omega}$. The fact that $\dot{g}_0^* = 2A$, implies $\dot{\Omega}_0 = 0$, thanks to the equations defining the space $\mathbf{T}^J_{g,\Omega}$. Thus

$$2 \frac{d^2}{dt^2}_{|_{t=0}} \underline{H}_{g_t,\Omega_t} = 2\nabla_G D\underline{H}(g, \Omega)(\dot{g}_0, 0; \dot{g}_0, 0) + 2D_{g,\Omega}\underline{H}(\xi, \Xi),$$

with

$$\begin{aligned} \xi_g^* &:= \frac{d}{dt}_{|_{t=0}} \dot{g}_t^*, \\ \Xi_\Omega^* &:= \frac{d}{dt}_{|_{t=0}} \dot{\Omega}_t^* + \frac{1}{4}|\dot{g}_0|_g^2 - \frac{1}{4} G_{g,\Omega}(\dot{g}_0, 0; \dot{g}_0, 0). \end{aligned}$$

Using the fact that (g, Ω) is a soliton and the first and second variation formulas for Perelman's functions H (see [8], the identity (1.5) in section 1 of [6] and corollary 1), and $\mathscr{W}$ (see the end of the proof of lemma 7 in section 7 of [6]), we infer

$$\begin{aligned} 2 \frac{d^2}{dt^2}_{|_{t=0}} \underline{H}_{g_t,\Omega_t} &= \nabla_G D(2H - \mathscr{W})(g, \Omega)(\dot{g}_0, 0; \dot{g}_0, 0) + 2D_{g,\Omega}H(\xi, \Xi) \\ &= -2\langle \mathscr{L}_g^\Omega A, A \rangle_g - (\Delta_g^\Omega - 2\mathbf{I})|A|_g^2 - 2\int_X |A|_g^2 \Omega \\ &\quad + 2\int_X |A|_g^2 F\Omega + 2(\Delta_g^\Omega - \mathbf{I})\Xi_\Omega^* - \mathrm{div}^\Omega \nabla_g^{*_\Omega} \xi_g^* \\ &= 2\int_X |A|_g^2 F\Omega - 2\langle \mathscr{L}_g^\Omega A, A \rangle_g + \Delta_g^\Omega |A|_g^2 \\ &\quad + 2(\Delta_g^\Omega - \mathbf{I}) \frac{d}{dt}_{|_{t=0}} \dot{\Omega}_t^* - \mathrm{div}^\Omega \nabla_g^{*_\Omega} \frac{d}{dt}_{|_{t=0}} \dot{g}_t^*. \end{aligned}$$

Using lemma 5 and the weighted complex Bochner formula (13.9) in section 13 of [6], we obtain

$$\nabla_g^{*\Omega} \frac{d}{dt}_{|_{t=0}} \dot{g}_t^* = \bar{\partial}_{T_{X,J}}^{*_{g,\Omega}} \bar{\partial}_{T_{X,J}} \nabla_{g,J} \bar{\psi} = \frac{1}{2} \nabla_{g,J} \overline{(\Delta_{g,J}^{\Omega} - 2\mathbf{I})\psi}, \tag{21}$$

and thus

$$\begin{aligned} -\mathrm{div}^{\Omega}\, \nabla_g^{*\Omega} \frac{d}{dt}_{|_{t=0}} \dot{g}_t^* &= \frac{1}{2} \Delta_g^{\Omega} R_\psi + \frac{1}{2} B_{g,J}^{\Omega} I_\psi, \\ R_\psi &:= \mathrm{Re}[(\Delta_{g,J}^{\Omega} - 2\mathbf{I})\psi], \\ I_\psi &:= \mathrm{Im}[(\Delta_{g,J}^{\Omega} - 2\mathbf{I})\psi]. \end{aligned}$$

(Here we use the notation $z = \mathrm{Re}\, z + i\, \mathrm{Im}\, z$, for any $z \in \mathbf{C}$). Differentiating the tangential identity $2dd_{J_t}^c \dot{\Omega}_t^* = -d[\nabla_{g_t}^{*\Omega_t} \dot{g}_t^* \neg \omega]$, we obtain,

$$2dd_J^c \frac{d}{dt}_{|_{t=0}} \dot{\Omega}_t^* = -d\left[\frac{d}{dt}_{|_{t=0}} (\nabla_{g_t}^{*\Omega_t} \dot{g}_t^*) \neg \omega\right].$$

Using the variation formula (20), and the identity (21) we obtain

$$\begin{aligned} \frac{d}{dt}_{|_{t=0}} (\nabla_{g_t}^{*\Omega_t} \dot{g}_t^*) &= \frac{1}{4} \nabla_g |\dot{g}_0|_g^2 + \nabla_g^{*\Omega} \frac{d}{dt}_{|_{t=0}} \dot{g}_t^* \\ &= \nabla_g |A|_g^2 + \frac{1}{2} \nabla_{g,J} \overline{(\Delta_{g,J}^{\Omega} - 2\mathbf{I})\psi}, \end{aligned}$$

and thus

$$\frac{d}{dt}_{|_{t=0}} \dot{\Omega}_t^* = -\frac{1}{2} R_\psi - |A|_g^2 + \int_X |A|_g^2 \Omega.$$

We obtain in conclusion the variation formula

$$\begin{aligned} 2\frac{d^2}{dt^2}_{|_{t=0}} \underline{H}_{g_t,\Omega_t} &= 2\int_X |A|_g^2 F\Omega - 2\langle \mathscr{L}_g^{\Omega} A, A\rangle_g \\ &\quad - (\Delta_g^{\Omega} - 2\mathbf{I})|A|_g^2 - 2\int_X |A|_g^2 \Omega \\ &\quad - \frac{1}{2}(\Delta_g^{\Omega} - 2\mathbf{I}) R_\psi + \frac{1}{2} B_{g,J}^{\Omega} I_\psi \\ &= -2\langle J\nabla_g f \neg \nabla_g A, JA\rangle_g + 4\langle \nabla_g^2 f, A^2\rangle_g + 2\int_X |A|_g^2 F\Omega \\ &\quad - (\Delta_g^{\Omega} - 2\mathbf{I})|A|_g^2 - 2\int_X |A|_g^2 \Omega - \frac{1}{2} P_{g,J}^{\Omega}\, \mathrm{Re}\, \psi, \end{aligned}$$

thanks to identity (16) and a computation in the proof of lemma 25 in section 19 of [6]. We denote respectively by π_1 and π_2 the projection to the first and second factor of the decomposition (4). Then the identity

$$0 = \pi_2 \ddot{\gamma}_0 = \pi_2 \frac{d^2}{dt^2}_{|_{t=0}} \underline{H}_{g_t, \Omega_t},$$

is equivalent to the identity

$$\int_X u_1 [4\langle \nabla_g^2 f, A^2 \rangle_g - 2\langle J\nabla_g f \neg \nabla_g A, JA \rangle_g - (\Delta_g^\Omega - 2\mathbf{I})|A|_g^2] \Omega = 0, \tag{22}$$

for any $u = u_1 + iu_2 \in \Lambda_{g,J}^\Omega$, with u_1, u_2, real valued. We observe now the equalities

$$\begin{aligned}
\int_X u_1 (\Delta_g^\Omega - 2\mathbf{I})|A|_g^2 \Omega &= -\int_X B_{g,J}^\Omega u_2 |A|_g^2 \Omega \\
&= \int_X u_2 B_{g,J}^\Omega |A|_g^2 \Omega \\
&= \int_X u_2 (J\nabla_g f).|A|_g^2 \Omega \\
&= 2\int_X u_2 \langle J\nabla_g f \neg \nabla_g A, A \rangle_g \Omega.
\end{aligned}$$

We conclude that the identity (22) is equivalent to

$$2\int_X u_1 \langle \nabla_g^2 f, A^2 \rangle_g \Omega = \int_X \langle J\nabla_g f \neg \nabla_g A, i\bar{u} \times_J A \rangle_g \Omega,$$

which shows the required conclusion.

8. Appendix

8.1. Proof of the identities (11) and (12)

By definition of the Ω-divergence operator and using the symmetry of A we infer

$$\begin{aligned}
\operatorname{div}^\Omega (A \cdot \xi) &= g(\nabla_{g,e_k}(A \cdot \xi), e_k) - g(A \cdot \xi, \nabla_g f) \\
&= g(\nabla_{g,e_k} A \cdot \xi + A \cdot \nabla_{g,e_k} \xi, e_k) - g(\xi, A \cdot \nabla_g f) \\
&= g(\xi, \nabla_{g,e_k} A \cdot e_k - A \cdot \nabla_g f) + g(\nabla_{g,e_k} \xi, Ae_k),
\end{aligned}$$

and thus the identity (11). We expand now the term

$$\begin{aligned}
\operatorname{div}^\Omega \operatorname{Tr}_g(\nabla_g A \cdot A) &= \operatorname{div}^\Omega (\nabla_{g,e_k} A \cdot Ae_k) \\
&= g(\nabla_{g,e_l}(\nabla_{g,e_k} A \cdot Ae_k), e_l) - g(\nabla_{g,e_k} A \cdot Ae_k, \nabla_g f) \\
&= g(\nabla_{g,e_l} \nabla_{g,e_k} A \cdot Ae_k + \nabla_{g,e_k} A \cdot \nabla_{g,e_l} A \cdot e_k, e_l) \\
&\quad - g(Ae_k, \nabla_{g,e_k} A \cdot \nabla_g f).
\end{aligned}$$

Expanding further we infer

$$\begin{aligned}\operatorname{div}^{\Omega}\operatorname{Tr}_g(\nabla_g A\cdot A) &= g(Ae_k, \nabla_{g,e_l}\nabla_{g,e_k}A\cdot e_l) + g(\nabla_{g,e_l}A\cdot e_k, \nabla_{g,e_k}A\cdot e_l)\\ &\quad - g(Ae_k, \nabla_{g,e_k}A\cdot\nabla_g f)\\ &= g(Ae_k, \nabla_{g,e_l}\widehat{\nabla_g A}(e_l,e_k) - \widehat{\nabla_g A}(\nabla_g f, e_k))\\ &\quad + \langle\widehat{\nabla_g A}, \nabla_g A\rangle_g,\end{aligned}$$

and thus the identity (12).

8.2. The Maurer-Cartan equation in the Kähler case

We observe that for any vector spaces V and E, we can define a contraction operation

$$\begin{aligned}\neg : (\Lambda^p V^* \otimes V)\times(\Lambda^q V^*\otimes E) &\to \Lambda^{p+q-1}V^*\otimes E\\ (\alpha,\beta) &\mapsto \alpha\neg\beta,\end{aligned}$$

by the expression

$$(\alpha\neg\beta)(\xi) := \sum_{|I|=\deg\alpha}\varepsilon_I\beta(\alpha(\xi_I), \xi_{\complement I}).$$

This map restricts to

$$\neg : \mathscr{E}^{0,p}(T_X^{1,0})\times\mathscr{E}^{r,q}\to\mathscr{E}^{r-1,p+q}.$$

We notice indeed the identity $\alpha\neg\beta = \bar{\zeta}_I^*\wedge(\alpha_I\neg\beta)$, where $\alpha = \alpha_I\otimes\bar{\zeta}_I^*$, with $(\zeta_k)_k\subset C^\infty(U, T_{X,J}^{1,0})$ a local frame. (We use from now on the Einstein convention for sums). Obviously, the contraction operation $\neg$, generalizes the one used in the previous sections.

Lemma 6 (Expression of the exterior Lie product). *Let (X,J,ω) be a Kähler manifold and let $\alpha,\beta\in C^\infty(X,\Lambda_J^{0,\bullet}T_X^*\otimes_{\mathbf{C}}T_{X,J}^{1,0})$. Then hold the identity*

$$[\alpha,\beta] = \alpha\neg\partial^{\omega}_{T^{1,0}_{X,J}}\beta - (-1)^{|\alpha|\,|\beta|}\beta\neg\partial^{\omega}_{T^{1,0}_{X,J}}\alpha.$$

Proof. In the case $|\alpha| = |\beta| = 0$, the identity follows from an elementary computation in geodesic holomorphic coordinates. In order to show the general case, let $(\zeta_k)_k\subset\mathcal{O}(U, T_{X,J}^{1,0})$ be a local frame. We consider the local expressions $\alpha = \alpha_K\otimes\bar{\zeta}_K^*$, $\beta = \beta_L\otimes\bar{\zeta}_L^*$. Then

$$\begin{aligned}[\alpha,\beta] &= [\alpha_K,\beta_L]\otimes(\bar{\zeta}_K^*\wedge\bar{\zeta}_L^*)\\ &= (\alpha_K\neg\partial^{\omega}_{T^{1,0}_{X,J}}\beta_L - \beta_L\neg\partial^{\omega}_{T^{1,0}_{X,J}}\alpha_K)\otimes(\bar{\zeta}_K^*\wedge\bar{\zeta}_L^*).\end{aligned}$$

The identity $\bar{\partial}_{T^{1,0}_{X,J}}\zeta_k = 0$ implies $\partial_J\bar{\zeta}_K^* = 0$. We infer

$$\partial^{\omega}_{T^{1,0}_{X,J}}\alpha = \partial^{\omega}_{T^{1,0}_{X,J}}\alpha_K\wedge\bar{\zeta}_K^*,$$

and a similar local expression for β. Thus using the identity

$$\alpha \neg \gamma = \bar{\zeta}_K^* \wedge (\alpha_K \neg \gamma),$$

with γ arbitrary, we deduce

$$\begin{aligned}\alpha \neg \partial^{\omega}_{T^{1,0}_{X,J}} \beta &= (\alpha_K \neg \partial^{\omega}_{T^{1,0}_{X,J}} \beta_L) \otimes (\bar{\zeta}_K^* \wedge \bar{\zeta}_L^*),\\ \beta \neg \partial^{\omega}_{T^{1,0}_{X,J}} \alpha &= (\beta_L \neg \partial^{\omega}_{T^{1,0}_{X,J}} \alpha_K) \otimes (\bar{\zeta}_L^* \wedge \bar{\zeta}_K^*)\\ &= (-1)^{|\alpha|\,|\beta|} (\beta_L \neg \partial^{\omega}_{T^{1,0}_{X,J}} \alpha_K) \otimes (\bar{\zeta}_K^* \wedge \bar{\zeta}_L^*),\end{aligned}$$

and thus the required conclusion. □

We deduce that over a Kähler manifold the Maurer-Cartan equation

$$\bar{\partial}_{T^{1,0}_{X,J_0}} \theta + \frac{1}{2}[\theta, \theta] = 0,$$

writes as

$$\bar{\partial}_{T^{1,0}_{X,J}} \theta + \theta \neg \partial^{\omega}_{T^{1,0}_{X,J}} \theta = 0. \tag{23}$$

We show below that we can rewrite the Maurer-Cartan equation in equivalent real terms as

$$\bar{\partial}_{T_{X,J}} \mu + \mu \neg \nabla^{1,0}_{g,J} \mu = 0, \tag{24}$$

or in more explicit terms

$$(\mathbf{I} + \mu) \neg J \nabla_g \mu = (\mathbf{I} + \mu) J \neg \nabla_g \mu.$$

In order to show (24) we expand, for any $u, v \in T_X$, the term

$$\begin{aligned}(\theta \neg \partial^{\omega}_{T^{1,0}_{X,J}} \theta)(u, v) &= \partial^{\omega}_{T^{1,0}_{X,J}} \theta(\theta u, v) + \partial^{\omega}_{T^{1,0}_{X,J}} \theta(u, \theta v)\\ &= \nabla^{1,0}_{g,J} \theta(\theta u, v) - \nabla^{1,0}_{g,J} \theta(v, \theta u)\\ &\quad + \nabla^{1,0}_{g,J} \theta(u, \theta v) - \nabla^{1,0}_{g,J} \theta(\theta v, u).\end{aligned}$$

Expanding further we obtain

$$\begin{aligned}2(\theta \neg \partial^{\omega}_{T^{1,0}_{X,J}} \theta)(u, v) &= \nabla_g \theta(\theta u, v) - i \nabla_g \theta(J \theta u, v)\\ &\quad - \nabla_g \theta(v, \theta u) + i \nabla_g \theta(J v, \theta u)\\ &\quad + \nabla_g \theta(u, \theta v) - i \nabla_g \theta(J u, \theta v)\\ &\quad - \nabla_g \theta(\theta v, u) + i \nabla_g \theta(J \theta v, u).\end{aligned}$$

Using the fact that θ takes values in $T^{1,0}_{X,J}$ we obtain

$$2(\theta \neg \partial^{\omega}_{T^{1,0}_{X,J}} \theta)(u,v) = 2\nabla_g \theta(\theta u, v) - \nabla_g \theta(v, \theta u) + i\nabla_g \theta(Jv, \theta u)$$
$$- 2\nabla_g \theta(\theta v, u) + \nabla_g \theta(u, \theta v) - i\nabla_g \theta(Ju, \theta v).$$

Replacing on the right hand side of this equality the identity $2\theta = \mu - iJ\mu$ and adding the conjugate of both sides we infer

$$\begin{aligned} 8(\theta \neg \partial^{\omega}_{T^{1,0}_{X,J}} \theta)(u,v) &+ 8\overline{(\theta \neg \partial^{\omega}_{T^{1,0}_{X,J}} \theta)(u,v)} \\ &= 4\nabla_g \mu(\mu u, v) - 4J\nabla_g \mu(J\mu u, v) \\ &\quad - 2\nabla_g \mu(v, \mu u) + 2J\nabla_g \mu(v, J\mu u) \\ &\quad + 2\nabla_g \mu(Jv, J\mu u) + 2J\nabla_g \mu(Jv, \mu u) \\ &\quad + 2\nabla_g \mu(u, \mu v) - 2J\nabla_g \mu(u, J\mu u) \\ &\quad - 2\nabla_g \mu(Ju, J\mu v) - 2J\nabla_g \mu(Ju, \mu v) \\ &\quad - 4\nabla_g \mu(\mu v, u) + 4J\nabla_g \mu(J\mu v, u). \end{aligned}$$

Using the anti J-linearity of $\nabla_{g,\xi}\mu$ we deduce

$$\begin{aligned} 8(\theta \neg \partial^{\omega}_{T^{1,0}_{X,J}} \theta)(u,v) &+ 8\overline{(\theta \neg \partial^{\omega}_{T^{1,0}_{X,J}} \theta)(u,v)} \\ &= 4\nabla_g \mu(\mu u, v) - 4J\nabla_g \mu(J\mu u, v) \\ &\quad - 4\nabla_g \mu(\mu v, u) + 4J\nabla_g \mu(J\mu v, u) \\ &= 8\nabla^{1,0}_{g,J} \mu(\mu u, v) - 8\nabla^{1,0}_{g,J} \mu(\mu v, u) \\ &= 8(\mu \neg \nabla^{1,0}_{g,J} \mu)(u,v). \end{aligned}$$

The latter combined with

$$\bar{\partial}_{T^{1,0}_{X,J}} \theta(u,v) + \overline{\bar{\partial}_{T^{1,0}_{X,J}} \theta(u,v)} = \bar{\partial}_{T_{X,J}} \mu(u,v),$$

and (23) implies the required identity (24).

References

[1] A. L. Besse, Einstein manifolds, Springer-Verlag, 2007.

[2] D. G. Ebin, The manifolds of Riemannian metrics, Proceedings of symposia on pure marthematics **15**, 1970.

[3] A. Futaki, Kähler-Einstein metrics and Integral Invariants, Lecture notes in mathematics **1314**. Springer-Verlag, Berlin, 1988, 437–443.

[4] S. Hall and T. Murphy, Variation of complex structures and the stability of Kähler-Ricci Solitons, arXiv:1206.4922, Pacific J. Math. **265** (2013), 441–454.

[5] N. Pali, The total second variation of Perelman's $\mathcal{W}$-functional, arXiv:1201.0969, Calc. Var. Partial Differential Equations **50** (2014), 115–144.

[6] N. Pali, The Soliton-Ricci Flow vith variable volume forms, arXiv:1406.0806, Complex Manifolds **3** (2016), 41–144.

[7] N. Pali, Variation formulas for the complex components of the Bakry-Emery-Ricci endomorphism, arXiv:1406.0805, Complex Var. Elliptic Equ. **60** (2015), 635–667.

[8] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, arXiv:math/0211159.

[9] F. Podestà and A. Spiro, On moduli spaces of Ricci solitons, arXiv:1302.4307, J. Geom. Anal. **25** (2015), 1157–1174.

Nefton Pali
Université Paris Sud
Département de Mathématiques
Bâtiment 425 F91405 Orsay
France
E-mail: nefton.pali@math.u-psud.fr