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A REGULATOR MAP FOR 1-CYCLES WITH MODULUS

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Abstract

Let k be a field of characteristic 0. We define a map from the additive higher Chow group of 1-cycles with strong sup m-modulus $CH_1(A_k(m), n)_{ssup}$ to the module of absolute Kähler differentials of k with twisted k^*-action $\Omega_k^{n-2}\langle\omega\rangle$ of weight ω. We will call the map a *regulator map*, and we show that the regulator map is surjective if k is an algebraically closed field. In case $\omega = m+1$, this map specializes to Park's regulator map. We study a relationship between the cyclic homology and the additive higher Chow group with strong sup modulus by using our regulator map.

1. Introduction

Let k be a field. Bloch and Esnault ([3], [4]) introduced the additive higher Chow group of k as an additive version of Bloch's higher Chow group $CH_d(k, n)$ of k. It was generalized and studied further by Rülling, Park, Krishna, Levine ([8], [9], [13], [14], [16]). In particular the additive higher Chow group of 0-cycles of k is well-understood and related to the group of big de Rham-Witt forms of k (see [16]). In this paper we study the additive higher Chow group of 1-cycles of k.

The additive higher Chow group is defined as the homology group of a certain complex, called an additive cycle complex, which is built up from algebraic cycles satisfying two conditions: One is the face condition which was already used in the definition of Bloch's higher Chow group. The other new condition is called the modulus condition. In this paper, we use two kinds of modulus conditions, the sup modulus condition and the strong sup modulus condition. The corresponding additive higher Chow groups of k are denoted by

$$CH_d(A_k(m), n)_{sup} \quad \text{and} \quad CH_d(A_k(m), n)_{ssup},$$

where m indicates the modulus (see §2 for definitions). There is a natural map

$$(1.1) \qquad CH_d(A_k(m), n)_{ssup} \to CH_d(A_k(m), n)_{sup}.$$

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Let k be a field. For a k-vector space V and an integer ω, we define $V\langle\omega\rangle$ to be the k-vector space which has the same underlying additive group and is equipped with the twisted k^*-action of weight ω given by

$$a \star x := a^{\omega}x, \quad a \in k^*,\ x \in V.$$

A group homomorphism between groups with k^*-actions which preserves the k^*-actions is called *weight-preserving*. The additive higher Chow group has a natural k^*-action and Park ([13]) defined the following weight-preserving map, called a regulator map, from the additive higher Chow group of 1-cycles to the module of absolute Kähler differentials of k with a twisted k^*-action:

$$R_{2,m} : CH_1(A_k(m), n)_{sup} \to \Omega_k^{n-2}\langle m+1\rangle.$$

The first main result of this paper generalizes this result.

Theorem 1.1 (See Corollary 3.20). *Let k be a field of characteristic zero. Let $m \geq 2$ be an integer. For each integer c with $m \leq c < 2m$, there exists a weight-preserving map*

$$L_c^n : CH_1(A_k(m), n)_{ssup} \to \Omega_k^{n-2}\langle c\rangle$$

such that the composite of Park's regulator map $R_{2,m}$ and (1.1) *coincides with L_{m+1}^n. Moreover, if k is an algebraically closed field, the map*

$$L^n = \bigoplus_{m \leq c < 2m} L_c^n : CH_1(A_k(m), n)_{ssup} \to \bigoplus_{m \leq c < 2m} \Omega_k^{n-2}\langle c\rangle$$

is surjective.

In case of $n = 2$, the above theorem provides the map

$$L^2 = \bigoplus_{m \leq c < 2m} L_c^2 : CH_1(A_k(m), 2)_{ssup} \to \bigoplus_{m \leq c < 2m} k\langle c\rangle.$$

We expect that this map is related to the following map:

$$K_3(k[\varepsilon], \varepsilon)^{(2)} \to B_2(k[\varepsilon]) \xrightarrow{Li} \bigoplus_{m < \omega < 2m} k\langle\omega\rangle,$$

where $k[\varepsilon] := k[x]/x^m$ is the truncated polynomial ring, and $K_3(k[\varepsilon], \varepsilon)^{(2)}$ is the l^2-eigenspace for the l-th Adams operator (for any integer $l > 1$) of the relative algebraic K-theory ([10, §11.2.19]), and $B_2(k[\varepsilon])$ is the Bloch group of $k[\varepsilon]$ ([18, §1.3]), and Li is an additive dilogarithm defined by Ünver ([18, Thm. 1.3.2]), who proved that the composite of the above maps is an isomorphism.

The second main result of this paper concerns a relationship between $CH_1(A_k(m), 2)_{sup}$ and the cyclic homology of the truncated polynomial ring. Recall that the cyclic homology of a truncated polynomial ring has a natural k^*-action which induces a decomposition, called the weight decomposition

([10, Def. 2.1.3], see also §4.2). We will see in Corollary 4.9 *the following isomorphism*:

$$HC_2(k[x]/x^m, (x)) \cong x^{m+1}k[x]/x^{2m}, \tag{1.2}$$

where the left hand side is the relative cyclic homology of the truncated polynomial ring ([10, §2.1.15]). We prove the following:

THEOREM 1.2 (See Theorem 4.10). *Let k be a number field and $m \geq 2$ be an integer. Then there exists a weight-preserving map*

$$\Phi : HC_2(k[x]/x^m, (x)) \to CH_1(A_k(m), 2)_{sup}.$$

We also construct a map (see Definition 4.5 and Remark 4.7)

$$\Phi' : x^m k[x] \to CH_1(A_k(m), 2)_{ssup}$$

such that the composed map

$$x^m k[x] \xrightarrow{\Phi'} CH_1(A_k(m), 2)_{ssup} \to CH_1(A_k(m), 2)_{sup}$$

factor through $x^m k[x]/x^{2m}$:

$$\begin{tikzcd}
x^m k[x] \arrow[r, "\Phi'"] \arrow[d] & CH_1(A_k(m), 2)_{ssup} \arrow[d] \\
x^m k[x]/x^{2m} \arrow[r] & CH_1(A_k(m), 2)_{sup}.
\end{tikzcd}$$

The composition of the lower horizontal map with (1.2) is the map Φ of Theorem 1.2. We also show that the composed map

$$x^m k[x] \xrightarrow{\Phi'} CH_1(A_k(m), 2)_{ssup} \xrightarrow{\tilde{L}} \bigoplus_{m \leq \omega < 2m} k\langle\omega\rangle \cong x^m k[x]/x^{2m},$$

where $\tilde{L}$ is a direct sum of modifications of the regulator maps L_c^n, is a natural quotient map. In particular Φ' is not a trivial map (see Corollary 4.12).

There is a folklore conjecture that the additive higher Chow group is related to the motivic cohomology group (still conjectural) $H^*_{\mathscr{M}}(k[\varepsilon], (\varepsilon); \mathbf{Z}(r))$ of the relative truncated polynomial ring. It is expected to relate to the relative algebraic K-group $K_n(k[\varepsilon], (\varepsilon))$ by a spectral sequence of Atiyah-Hirzebruch type as the motivic cohomology of smooth schemes X relates to the algebraic K-group $K_*(X)$. On the other hand, there are isomorphisms $K_{n+1}(k[\varepsilon], (\varepsilon)) \cong \bigoplus_{p \geq 0} (\Omega_k^{n-2p})^{m-1}$ and $K_n(k[\varepsilon], (\varepsilon)) \cong HC_{n-1}(k[\varepsilon], (\varepsilon))$, where the first isomorphism was proved by Hesselholt [7], and the second isomorphism was proved by Goodwillie [6]. Thus Theorems 1.1 and 1.2 give a modest evidence toward the above conjecture.

This paper is organized as follows.

In section 2, we give the definition of the additive higher Chow groups with m-modulus by using the strong sup and the sup modulus condition ([9]). We adopt the (strong) sup modulus condition (not the sum condition) to define the regulator map by using the residue theory developed in [19], [12] and [13]. In subsection 2.2, we give some examples of additive cycles and give some relations. These examples will be used in later sections.

In section 3, we construct a weight-preserving regulator map from the group of 1-cycles with the sup m-modulus condition to the module of absolute Kähler differentials of k with the twisted k^*-actions. This map is a generalization of Park's regulator map ([13]). We show that this regulator map is surjective. We then show that this regulator map induces a map from the additive higher Chow group of 1-cycles with strong modulus condition by using an argument similar to that of Park ([13]).

In section 4, we relate the additive higher Chow groups and the cyclic homology of a truncated polynomial ring over a number field k. Firstly, using certain 2-cycles satisfying the sup modulus condition, we find some relations in the additive higher Chow group of 1-cycles with sup modulus. Secondly, after recalling the natural k^*-action on the cyclic homology of the truncated polynomial ring, we construct the following weight-preserving isomorphism via Hochschild homology by using a technique of Loday ([10]):

$$HC_2(k[x]/x^m, (x)) \cong x^{m+1}k[x]/x^{2m}.$$

Finally, using this isomoprhism, we define a weight-preserving map from the cyclic homology to the additive higher Chow group of 1-cycles with sup m-modulus.

In the appendix, we summarize some results for the residue theory from [12], [13] and [19].

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2. Additive higher Chow groups over a field

2.1. Definition of additive higher Chow groups

We fix a base field k. Set $\mathbf{P}^1 = \mathbf{P}^1_k = \operatorname{Proj} k[Y_0, Y_1]$, and let $y = Y_1/Y_0$. Let $\square = \mathbf{P}^1_k - \{1\}$. We denote by O the origin of $\mathbf{A}^1_k$. Letting $q_i : (\mathbf{P}^1_k)^n \to \mathbf{P}^1_k$ be the i-th projection, we use the coordinate system $(y_1, \dots, y_n)$ on $\square^n$, with $y_i = y \circ q_i$. For a scheme X, let $X_{(d)}$ denote the set of all integral closed subschemes of dimension d on X. A face of $\square^n$ is a closed subscheme F defined by equations of the form

$$y_{i_1} = \varepsilon_1, \dots, y_{i_r} = \varepsilon_r, \quad \varepsilon_j \in \{0, \infty\}.$$

For $1 \le i \le n$, one denotes by $F_{n,i}$ the Cartier divisor on $(\mathbf{P}^1_k)^n$ defined by $\{y_i = 1\}$. We omit the subscript n and write it simply as F_i whenever it is clear from the context. For each $i \in \{1, \dots, n\}$ and $\varepsilon \in \{0, \infty\}$, we have the codimension 1 face maps

$$\iota_i^{\varepsilon} = \iota_{i,\varepsilon}^n : \square^n \to \square^{n+1}$$

with

$$(y_1, \dots, y_n) \mapsto (y_1, \dots, y_{i-1}, \varepsilon, y_i, \dots, y_n).$$

For a Weil divisor D on X, let $D_Y = \mathrm{ord}_Y(D)$ denote the coefficient of the prime divisor Y on X.

Definition 2.1 ([13, Def. 2.1]). Let $D_1, \dots, D_n$ be Weil divisors on X. Express $D_i = \sum_Y (D_i)_Y [Y]$, where Y runs over all prime Weil divisors on X. We define the supremum of $D_1, \dots, D_n$ as

$$\sup_{1 \le i \le n} D_i := \sum_Y \left(\max_{1 \le i \le n} (D_i)_Y \right) [Y].$$

Definition 2.2 ([9, Def. 2.3]). Let k be a field, and $m \ge 2$ be an integer. Let $Y \in ((\mathbf{A}_k^1 \backslash O) \times \square^n)_{(d)}$ be an integral subscheme. Let $\overline{Y}$ be the Zariski closure of Y in $\mathbf{A}_k^1 \times (\mathbf{P}_k^1)^n$, and $\nu = \nu_Y : \overline{Y}^N \to \mathbf{A}_k^1 \times (\mathbf{P}_k^1)^n$ be its normalization:

$$\begin{array}{ccccc} & & Y & \hookrightarrow & \mathbf{A}_k^1 \times \square^n \\ & & \downarrow & & \downarrow \\ \nu : \overline{Y}^N & \longrightarrow & \overline{Y} & \hookrightarrow & \mathbf{A}_k^1 \times (\mathbf{P}^1)^n. \end{array}$$

The integral subscheme Y is said to satisfy a *sup m-modulus condition* if the following inequality as Weil divisors on $\overline{Y}^N$ holds:

$$m \cdot \nu^*(\{x = 0\} \times (\mathbf{P}^1)^n) \le \sup_{1 \le i \le n} \nu^*(\mathbf{A}_k^1 \times F_i).$$

The integral subscheme Y is said to satisfy a *strong sup m-modulus condition* if there exists an integer i such that the following inequality as Cartier divisors on $\overline{Y}^N$ holds:

$$m \cdot \nu^*(\{x = 0\} \times (\mathbf{P}^1)^n) \le \nu^*(\mathbf{A}_k^1 \times F_i).$$

Definition 2.3 ([9, Def. 2.6, Def. 2.7]). Let k be a field, and let $m \ge 2$ be an integer. For any integer $d \ge 0$, let $C_d(A_k(m), n)_{ssup}$ be the set of all integral closed subschemes $W \in ((\mathbf{A}_k^1 \backslash O) \times \square^n)_{(d)}$ which satisfy the following two conditions:

(i) W intersects properly with $(\mathbf{A}_k^1 \backslash O) \times F$ for each face $F \subset \square^n$.

(ii) W satisfies the strong sup m-modulus condition.

Let $z_d(A_k(m), n)_{ssup}$ be the free abelian group on the set $C_d(A_k(m), n)_{ssup}$. The correspondence

$$\underline{n} \to z_{n-c}(A_k(m), n)_{ssup}, \quad \underline{n} := \{0, \infty\}^n$$

gives rises to a cubical object in the category of abelian groups. The associated non-degenerate complex is called an additive cycle complex and denoted by $\bar{z}_{n-c}(A_k(m),n)_{ssup}$. The boundary map of the complex $\bar{z}_{n-c}(A_k(m),n)_{ssup}$ is given by

$$\partial = \sum_{1 \le i \le n} (-1)^i (\partial_i^0 - \partial_i^\infty),$$

where ∂_i^ε is the pullback along the face map ι_i^ε. The homology group at $\bar{z}_d(A_k(m),n)_{ssup}$ is denoted by $CH_d(A_k(m),n)_{ssup}$:

$$CH_d(A_k(m),n)_{ssup} := H_0(\bar{z}_{d+\bullet}(A_k(m),n+\bullet)_{ssup},\partial).$$

We call $CH_d(A_k(m),n)_{ssup}$ the additive higher Chow group over the field k with strong sup m-modulus.

Remark 2.4. Similarly we can define an additive higher Chow group over a field k with sup m-modulus ([13, §2]). Let $C_d(A_k(m),n)_{sup}$ be the set of all integral closed subschemes $W \in ((\mathbf{A}_k^1 \backslash O) \times \square^n)_{(d)}$ which satisfy the following three conditions:

(i) W intersects properly with $(\mathbf{A}_k^1 \backslash O) \times F$ for each face $F \subset \square^n$.
(ii) W satisfies the sup m-modulus condition.
(iii) For any codimension r face F, the associated cycle of the scheme $F \cap W$ lies in the group $z_{d-r}(A_k(m),n-r)_{sup}$.

Let $z_d(A_k(m),n)_{sup}$ be the free abelian group on the set $C_d(A_k(m),n)_{sup}$. Similarly we define an associated non-degenerate complex $\bar{z}_{n-c}(A_k(m),n)_{sup}$ and its homology group $CH_d(A_k(m),n)_{sup}$. The latter group is called an additive higher Chow group over the field k with sup m-modulus. We note that there is a natural map

$$z_d(A_k(m),n)_{ssup} \to z_d(A_k(m),n)_{sup}.$$

Example 2.5. We consider the conditions (i), (ii) for the set of 0-cycles $C_0(A_k(m),n)_{ssup}$. Let $p \in C_0(A_k(m),n)_{ssup}$ be a 0-cycle. The first condition (i) implies that the closed point $p \in (\mathbf{A}_k^1 \times \square^n)_{(0)}$ does not lie on the closed subscheme $(\mathbf{A}_k^1 \backslash O) \times F$ for each face $F \subset \square^n$. The second condition (ii) implies that the closed point $p \in (\mathbf{A}_k^1 \times \square^n)_{(0)}$ does not lie on the closed subscheme $\{x = 0\} \subset \mathbf{A}_k^1 \times \square^n$.

Remark 2.6. Recall that the higher Chow group $CH^d(\mathbf{A}_k^1,n)$ is defined as the n-th homology of the complex $z^d(\mathbf{A}_k^1,*)/z^d(\mathbf{A}_k^1,*)_{deg}$, where the group $z^d(\mathbf{A}_k^1,n)$ is built out of the codimension d-cycles on $\mathbf{A}_k^1 \times \square^n$ which intersect properly with $\mathbf{A}_k^1 \times F$ for each face $F \subset \square^n$, and the complex $z^d(\mathbf{A}_k^1,*)_{deg}$ is the subcomplex of degenerate cycles of $z^d(\mathbf{A}_k^1,*)$ (see [17, pp. 178–181]). By the condition (i), the group $z_d(A_k(m),n)_{ssup}$ is naturally viewed as the subgroup of the group $z^{n+1-d}(\mathbf{A}_k^1,n)$, and this induces a natural morphism

$$CH_d(A_k(m),n)_{ssup} \to CH^{n+1-d}(\mathbf{A}_k^1,n).$$

Remark 2.7. There is an another modulus condition called a sum modulus condition ([9]). By using the sum modulus condition instead of (ii) in Definition 2.3, we define an additive higher Chow group with sum m-modulus and the group is denoted by $CH_d(A_k(m),n)_{sum}$. We note that there are natural maps

$$CH_d(A_k(m),n)_{ssup} \to CH_d(A_k(m),n)_{sup} \to CH_d(A_k(m),n)_{sum}.$$

If $d=0$, the above maps are isomorphism. It is not known whether these three groups coincide in general.

Remark 2.8. Bloch-Esnault and Rülling ([4], [16]) studied the additive higher Chow group of zero cycles over a field with the sum m-modulus. Let $m \geq 2$ be an integer. Let k be a field of char $k \neq 2$. By Bloch, Esnault, and Rülling ([4], [16]), we have an isomorphism

$$CH_0(A_k(m),n)_{sum} \cong \mathbf{W}_{m-1}\Omega_k^n$$

where $\mathbf{W}_{m-1}\Omega_k^n$ is the big de Rham-Witt group of k.

Remark 2.9. Krishna, Levine, and Park generalized the definition of an additive higher Chow group for any k-scheme X (see [8], [9]).

Let $C \subset \mathbf{A}_k^1 \times \square^n$ be an integral closed subscheme and $\nu : \bar{C}^N \to \mathbf{A}_k^1 \times (\mathbf{P}_k^1)^n$ be a normalization of its Zariski closure in $\mathbf{A}_k^1 \times (\mathbf{P}_k^1)^n$. If C satisfies the sup modulus condition, for any irreducible component p of $\nu^*(\{x=0\} \times (\mathbf{P}_k^1)^n)$ seen as a prime Weil divisor, there exists an index $i \in \{1,\dots,n\}$ such that

$$m \cdot \nu^*(\{x=0\} \times (\mathbf{P}^1)^n)_p \leq \nu^*(\mathbf{A}_k^1 \times \{y_i = 1\})_p$$

and we say that C satisfies the m-modulus condition on y_i along p. If C satisfies the strong sup modulus condition, we can choose an index i independently of a choice of an irreducible component p.

2.2. Examples of additive cycles and their properties

We fix an integer $m \geq 2$. Let O denote the origin of $\mathbf{A}_k^1$. In this subsection, we give some examples of additive cycles with strong sup modulus.

Example 2.10. Let V be a 0-cycle on $\mathbf{A}_k^1$ which satisfies $O \notin |V|$, where $|V|$ is the support of V. Then V satisfies the strong sup m-modulus condition for any $m \geq 2$.

Example 2.11. Let $Z \subset \mathbf{A}_k^1$ be a Zariski closed subset such that $O \notin Z$. Let $Y \subset Z \times \square^n$ be a closed subscheme which intersects with $Z \times F$ properly for each face $F \subset \square^n$. Then Y satisfies the strong sup m-modulus condition for any $m \geq 2$.

Let C be the following curve, whose projections to $\square^2$ was used in the paper of Totaro ([17])

$$C : t \mapsto \left(a, t, \frac{b_1(t - b_2)}{t - b_1 b_2}\right) \in \mathbf{A}_k^1 \times \square^2,$$

where $a, b_1, b_2 \in k^*$ are constants.

Then C is a cycle in $z_1(A_k(m), 2)_{ssup}$ and the boundary of C is

$$\partial C = (a, b_1) + (a, b_2) - (a, b_1 b_2) \in z_0(A_k(m), 1)_{ssup}.$$

Therefore, we get the following lemma.

LEMMA 2.12. *Let $a, b_1, b_2 \in k^*$. In the group $z_0(A_k(m), 1)_{ssup}/\partial z_1(A_k(m), 2)_{ssup}$ we have the following relation*:

$$(a, b_1 b_2) = (a, b_1) + (a, b_2).$$

Example 2.13. Let C be the following parametric curve used in the paper of Bloch and Esnault ([4])

$$C : t \mapsto \left(t, \frac{(1 - a_1 t)(1 - a_2 t)}{1 - (a_1 + a_2)t}, c\right) \in \mathbf{A}_k^1 \times \square^2,$$

where $a_1, a_2, c \in k^*$ are constants. This C is a normal irreducible curve, and this cycle satisfies the strong sup modulus condition for $m = 2$. The boundary of C is

$$\partial C = -\left(\frac{1}{a_1}, c\right) - \left(\frac{1}{a_2}, c\right) + \left(\frac{1}{a_1 + a_2}, c\right) \in z_0(A_k(m), 1)_{ssup}.$$

Therefore, we get the following lemma.

LEMMA 2.14. *Let $a_1, a_2 \in k^*$ and put $m = 2$. In the group $z_0(A_k(m), 1)_{ssup}/\partial z_1(A_k(m), 2)_{ssup}$ we have the following relation*:

$$\left(\frac{1}{a_1}, c\right) + \left(\frac{1}{a_2}, c\right) = \left(\frac{1}{a_1 + a_2}, c\right).$$

Example 2.15. Let $g(t) \in k(t)^*$, $c \in k^*$, and suppose $g(0) \in k^*$. Let C be the parametric curves of the form

$$C : t \mapsto (t, 1 - t^m g(t), c) \in \mathbf{A}_k^1 \times \square^2.$$

Then C satisfies the strong sup m-modulus condition.

Let $a \in k^*$. We suppose that k has all m-th root of a. By putting $g(t) = a^{-1}$, $c = a$ in Example 2.15, we have the following parametric curve

$$C : t \mapsto \left(t, 1 - \frac{t^m}{a}, a\right) \in \mathbf{A}_k^1 \times \square^2,$$

and $C \in z_1(A_k(m), 2)_{ssup}$. The boundary of C is

$$\partial C = \sum_{\zeta^m = a} (\zeta, a) \in z_0(A_k(m), 1)_{ssup}. \tag{2.1}$$

Similarly, the following parametric curve

$$C' : t \mapsto \left(t, 1 - \frac{t^m}{a}, t\right) \in \mathbf{A}_k^1 \times \square^2$$

satisfies the strong sup m-modulus condition. The boundary of C' is

$$\partial C' = \sum_{\zeta^m = a} (\zeta, \zeta) \in z_0(A_k(m), 1)_{ssup}. \tag{2.2}$$

PROPOSITION 2.16. *Let C, C' be as above. Let $\alpha \in k^*$, and put $a = \alpha^m$. Suppose that k has a primitive m-th root of unity. In the group $z_0(A_k(m), 1)_{ssup}/\partial z_1(A_k(m), 2)_{ssup}$ we have the following relation*:

$$\partial C = m \partial C'.$$

Proof. It follows from Lemma 2.12 and the above arguments. □

3. A regulator map

3.1. A regulator map and its properties

Let k be a perfect field, and let $m, s \geq 2$ be integers. Let $F \subset (\mathbf{P}^1)^n$ be a union of all faces $\{y_i = \varepsilon\} \subset (\mathbf{P}^1)^n$ for $i = 1, 2, \ldots, n$, $\varepsilon \in \{0, \infty\}$. For $1 \leq i \leq n$, let $\omega_i^{n,s} \in \Omega^{n-1}_{\mathbf{A}_k^1 \times (\mathbf{P}^1)^n/\mathbf{Z}}(\log F)(*\{x = 0\})$ be the following absolute Kähler differential $(n-1)$-forms similar to the ones used in the paper of Park ([13]):

$$\begin{aligned}
\omega_1^{n,s} &= \frac{1 - y_1}{x^s} \frac{dy_2}{y_2} \cdots \frac{dy_n}{y_n}, \\
\omega_i^{n,s} &= \frac{1 - y_i}{x^s} \frac{dy_{i+1}}{y_{i+1}} \cdots \frac{dy_n}{y_n} \frac{dy_1}{y_1} \cdots \frac{dy_{i-1}}{y_{i-1}} \quad (1 < i < n), \\
\omega_n^{n,s} &= \frac{1 - y_n}{x^s} \frac{dy_1}{y_1} \cdots \frac{dy_{n-1}}{y_{n-1}}.
\end{aligned} \tag{3.1}$$

We omit the superscripts n or s whenever it is clear from the context.

In this subsection, we define the following map called a regulator map

$$L_s : z_1(A_k(m), n)_{ssup} \to \Omega_k^{n-2}$$

for $n \geq 2$ by using the arguments similar to those in [13]. We use the residue theory to define the regulator map. Let C be a normal curve over a perfect field k, and let $p \in C$ be a closed point. For any rational absolute Kähler differential form ω on C, we denote by $\mathrm{res}_p(\omega)$ the residue value of ω at p. The residue theory was studied by El Zein, Beilinson, Parshin, Lomadze, Yekutieli ([1], [5], [11], [12], [15], [19]), and generalized to higher dimensions. In §5, we summarize some results for the generalized residue theory.

Definition 3.1. Let $C \subset \mathbf{A}_k^1 \times \square^n$ be an irreducible curve and let $\nu : \bar{C}^N \to \bar{C}$ be a normalization of its Zariski closure in $\mathbf{A}_k^1 \times (\mathbf{P}^1)^n$:

$$\begin{tikzcd} & C \arrow[r, hook] \arrow[d, hook] & \mathbf{A}^1 \times \square^n \arrow[d, hook] \\ \nu : \bar{C}^N \arrow[r] & \bar{C} \arrow[r, hook] & \mathbf{A}^1 \times (\mathbf{P}^1)^n. \end{tikzcd}$$

For any closed point $p \in \nu^{-1}(\bar{C} \cap \{x = 0\})$, we put

$$R_i^s(C, p) := (-1)^{(i-1)} \operatorname{res}_p(\nu^* \omega_i^s) \in \Omega_k^{n-2}.$$

If C satisfies the sup modulus condition, we can find $i \in \{1, \dots, n\}$ such that

$$m \cdot \nu^*(\{x = 0\} \times (\mathbf{P}_k^1)^n)_p \leq \nu^*(\mathbf{A}_k^1 \times \{y_i = 1\})_p. \tag{3.2}$$

Moreover if C satisfies the strong sup modulus condition, there exists some i such that (3.2) works for all p.

Under this strong sup modulus condition on C, we will prove the following Lemma 3.3. If $s < 2m$, then $R_i^s(C, p)$ does not depend on the choice of such i. Hence if $s < 2m$, we omit the subscript i and write

$$R^s(C, p) := R_i^s(C, p).$$

For simplicity, we denote by $|\nu^*\{x = 0\}|$ the set of all closed points of a support of the Weil divisor $\nu^*(\{x = 0\} \times (\mathbf{P}^1)^n)$.

Definition 3.2. Let $s < 2m$ be an integer. For each irreducible curve $C \in z_1(A_k(m), n)_{ssup}$, let $\nu : \bar{C}^N \to \bar{C}$ be a normalization of its Zariski closure in $\mathbf{A}_k^1 \times (\mathbf{P}^1)^n$. We define a map

$$L_s^n : z_1(A_k(m), n)_{ssup} \to \Omega_k^{n-2}$$

by

$$L_s^n(C) := \sum_{p \in |\nu^*\{x=0\}|} R^s(C, p) \quad \text{for } C \in C_1(A_k(m), n)$$

and we extend it $\mathbf{Z}$-linearly. We omit the superscript n whenever it is clear from the context.

Lemma 3.3. *Let $C \in z_1(A_k(m), n)_{ssup}$ be an irreducible curve and let $\nu : \bar{C}^N \to \bar{C}$ be a normalization of its Zariski closure in $\mathbf{A}_k^1 \times (\mathbf{P}^1)^n$. Then*

(1) *For $s < 2m$, $R_i^s(C, p)$ does not depend on a choice of i.*
(2) *For $s < m$, L_s^n is the zero map.*

Proof. For simplicity, we only show the case of $n = 2$. A proof of the general case is similar.

(1) Let $C \in z_1(A_k(m), 2)_{ssup}$ be an irreducible curve, and take a closed point $p \in |v^*\{x = 0\}|$. Suppose that C satisfies the modulus condition both on y_1 and y_2 along p. It is enough to show that $R_1^s(C, p) = R_2^s(C, p) = 0$ for $s < 2m$. Let x', y_1', y_2' be images of the coordinate functions $x, y_1, y_2 \in \mathcal{O}_{\mathbf{A}_k^1 \times (\mathbf{P}^1)^2}$ in the DVR $A = \mathcal{O}_{\bar{C}^N, p}$. By using a uniformizing parameter t on A, we can write

$$(x', y_1', y_2') = (t^r u, 1 - t^{rm} f, 1 - t^{rm} g)$$

for some $u \in A^*$, $f, g \in A$. Since p is a closed point on the support of $v^*(\{x = 0\} \times (\mathbf{P}^1)^n)$, we have $r \geq 1$. By a direct computation,

$$\begin{aligned} v^*\left(\frac{1 - y_1}{x^s} \frac{dy_2}{y_2}\right) &= \frac{t^{rm} f}{t^{rs} u^s} \frac{d(1 - t^{rm} g)}{1 - t^{rm} g} \\ &= \frac{-t^{r(m-s)} f}{u^s} \frac{rmt^{rm-1} g \, dt + t^{rm} g' \, dt}{1 - t^{rm} g}. \end{aligned}$$

Hence we can write

$$v^*\left(\frac{1 - y_1}{x^s} \frac{dy_2}{y_2}\right) = \alpha t^{r(m-s)+(rm-1)} \, dt = \alpha t^{2rm-rs} \frac{dt}{t}$$

for some $\alpha \in A$. Since

$$\begin{aligned} 2rm - rs \geq 1 &\Leftrightarrow 2rm - 1 \geq rs \\ &\Leftrightarrow 2m - \frac{1}{r} \geq s, \end{aligned}$$

and $r \geq 1$, it follows that $R_1^s(C, p) = 0$ for $2m > s$. Similarly, we have $R_2^s(C, p) = 0$ for $2m > s$. This proves Lemma 3.3 (1).

(2) Let $C \in z_1(A_k(m), 2)_{ssup}$ be an irreducible curve, and take a closed point $p \in |v^*\{x = 0\}|$. We may assume that C satisfies the modulus condition on y_1 by (1). By using the same notation as above, we can write

$$(x', y_1', y_2') = (t^r u, 1 - t^{rm} f, g)$$

for some $u \in A^*$, $f, g \in A$, $r \geq 1$. By a direct computation,

$$\begin{aligned} v^*\left(\frac{1 - y_1}{x^s} \frac{dy_2}{y_2}\right) &= \frac{t^{rm} f}{t^{rs} u^s} \frac{dg}{g} \\ &= \frac{t^{r(m-s)} f}{u^s} \frac{g' \, dt}{g}. \end{aligned}$$

Let v_t be the discrete valuation of A. Since $v_t\left(\dfrac{g'}{g}\right) \geq -1$, there exists $\beta \in A$ such that

$$v^*\left(\frac{1 - y_1}{x^s} \frac{dy_2}{y_2}\right) = t^{r(m-s)} \beta \frac{dt}{t}.$$

Since $s < m$, we get $r(m-s) \geq r \geq 1$. Therefore we have $R_1^s(C, p) = 0$. This completes the proof of Lemma 3.3 (2). $\square$

Definition 3.4. Let k be a field. Let V be a k-vector space and let $s > 0$ be an integer. We define $V\langle s\rangle$ to be the k-vector space which has the same underlying additive group and is equipped with the twisted k^*-action of weight s given by

$$a \star v := a^s v, \quad a \in k^*,\ v \in V.$$

k^* acts on $\mathbf{A}_k^1 \times \square^n$ by

$$a \star (x, y_1, \dots, y_n) = \left(\frac{x}{a}, y_1, \dots, y_n\right),$$

and this action induces a k^*-action on the complex $z_{\bullet-c}(A_k(m), \bullet)_{ssup}$, hence on $CH_d(A_k(m), n)_{ssup}$.

By a direct computation, we have the following lemma.

Lemma 3.5. *Let $s < 2m$ and let $\alpha \in k^*$. Then we have*

$$L_s^n(\alpha \star C) = \alpha^s L_s^n(C).$$

Therefore, L_s^n defines a map of k^*-set:

$$L_s^n : z_1(A_k(m), n)_{ssup} \to \Omega_k^{n-2}\langle s\rangle.$$

Definition 3.6. We define a homomorphism L^n by

$$L^n = \bigoplus_{m \leq s < 2m} L_s^n : z_1(A_k(m), n)_{ssup} \to \bigoplus_{m \leq s < 2m} \Omega_k^{n-2}\langle s\rangle.$$

L^n is compatible with the k^*-action. We denote by $K_s^n := \ker L_s^n$ and $K^n := \ker L^n = \bigcap_{s<2m} K_s^n$.

Remark 3.7. By the same arguments, we can define the map from the group of additive cycles with sup modulus:

$$L^n = \bigoplus_{m \leq s < 2m} L_s^n : z_1(A_k(m), n)_{sup} \to \bigoplus_{m \leq s < 2m} \Omega_k^{n-2}\langle s\rangle.$$

By a direct computation, we get the following similar to Example 2.11.

Lemma 3.8. *Let $a, b_1, b_2, \dots, b_n \in k^*$. Let C be the parametric curve of the form*

$$C : t \mapsto \left(a, t, \frac{b_1(t-b_2)}{t-b_1 b_2}, b_3, \dots, b_n\right) \in \mathbf{A}_k^1 \times \square^n.$$

Then C satisfies the strong sup modulus condition and $C \in K^n$.

COROLLARY 3.9. *Under the same notations as in Lemma* 3.8, *we have the following relation in* $z_0(A_k(m), n-1)_{ssup}/\partial K^n$:

$$(a, b_1 b_2, b_3, \dots, b_n) = (a, b_1, b_3, \dots, b_n) + (a, b_2, b_3, \dots, b_n).$$

Proof. It follows from Lemma 2.12 and Lemma 3.8. □

LEMMA 3.10. *Let* $a \neq 0$, $b_1, \dots, b_n \in k^* - \{1\}$. *Then following cycles lie in* K^n:

(1) *A* 1*-cycle* W *on* $\mathbf{A}_k^1 \times \square^n$ *which intersects properly with* $\mathbf{A}_k^1 \times F$ *for each face* $F \subset \square^n$ *and is contained in* $\{a\} \times \square^n$.
(2) *An additive cycle* $W \in z_1(A_k(m), n)$ *contained in* $\mathbf{A}_k^1 \times \square \times \{b_1\} \times \cdots \times \{b_{n-1}\}$.

Proof. This is just a direct computation. □

DEFINITION 3.11. We denote by

$$\bar{z}_d(A_k(m), n)_{ssup,\partial} := \ker(\bar{z}_d(A_k(m), n)_{ssup} \xrightarrow{\partial} \bar{z}_{d-1}(A_k(m), n-1)_{ssup}).$$

By the definition of additive higher Chow group, we get

$$CH_d(A_k(m), n)_{ssup} = \bar{z}_d(A_k(m), n)_{ssup,\partial}/\partial\bar{z}_{d+1}(A_k(m), n+1)_{ssup}.$$

DEFINITION 3.12. Let $c \geq 2$ be an integer. Let $(k^*)^c_{\mathbf{Z}}$ denote the $\mathbf{Z}$-submodule of k generated by the set

$$(k^*)^c := \{a^c \in k^* \mid a \in k^*\}.$$

For a field k, we consider the following condition $\spadesuit_c$:
($\spadesuit_c$) $(k^*)^c_{\mathbf{Z}} = k$.

LEMMA 3.13. *k has the property $\spadesuit_c$ when k satisfies one of the following conditions.*

(i) *k is an algebraically closed field.*
(ii) *$c < p = \mathrm{char}\, k$ or $\mathrm{char}\, k = 0$.*
(iii) *k is a finite field, and $\gcd(p-1, c) = 1$.*

Proof. The cases of (i) and (iii) are clear since $k^* = (k^*)^c$. We consider the case of (ii). Let $x \in k$ and let $a \in \mathbf{Z}$. Since

$$(x+a)^c = x^c + \binom{c}{1} x^{c-1} a + \cdots + \binom{c}{c-1} x a^{c-1} + a^c,$$

we have

$$\binom{c}{1} x^{c-1} a + \cdots + \binom{c}{c-1} x a^{c-1} \equiv 0 \bmod (k^*)^c_{\mathbf{Z}}.$$

Therefore, by putting

$$A = \begin{pmatrix} \binom{c}{1}a_1 & \binom{c}{2}a_1^2 & \cdots & \binom{c}{c-1}a_1^{c-1} \\ \binom{c}{1}a_2 & \binom{c}{2}a_2^2 & \cdots & \binom{c}{c-1}a_2^{c-1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{c}{1}a_{c-1} & \binom{c}{2}a_{c-1}^2 & \cdots & \binom{c}{c-1}a_{c-1}^{c-1} \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x^{c-1} \\ x^{c-2} \\ \vdots \\ x \end{pmatrix},$$

we have

$$A\boldsymbol{x} \equiv \mathbf{0} \bmod (k^*)^c_{\mathbf{Z}}.$$

Then A is the $(c-1) \times (c-1)$ matrix with entries in $\mathbf{Z}$, and

$$\det A = \prod_{l=1}^{c-1} \binom{c}{l} \prod_{1 \le i < j \le c-1} (a_j - a_i).$$

Put $a_i = i$. For the case of $\operatorname{char} k = p > 0$, we have $\det A \not\equiv 0 \bmod p$. Hence we get

$$\boldsymbol{x} \equiv \mathbf{0} \bmod (k^*)^c_{\mathbf{Z}}.$$

Especially $x \equiv 0 \bmod (k^*)^c_{\mathbf{Z}}$, we have $x \in (k^*)^c_{\mathbf{Z}}$.

For the case of $\operatorname{char} k = 0$, put $\alpha = \det A$. Then we have

$$\alpha\boldsymbol{x} \equiv \mathbf{0} \bmod (k^*)^c_{\mathbf{Z}}.$$

Especially, $\alpha x \equiv 0 \bmod (k^*)^c_{\mathbf{Z}}$. By replacing x by $\dfrac{x}{\alpha}$, we get $x \equiv 0 \bmod (k^*)^c_{\mathbf{Z}}$ and $x \in (k^*)^c_{\mathbf{Z}}$. This concludes the proof. $\square$

PROPOSITION 3.14. *Let $m \le c < 2m$ be an integer, and suppose that k satisfies the condition $\spadesuit_c$. Suppose that k contains the primitive c-th root of unity. Then the following map is surjective*:

$$L_c^n : z_1(A_k(m), n)_{ssup, \partial} \to \Omega_k^{n-2}\langle c \rangle.$$

Proof. Let $\alpha \in k^*$, $b_1, \dots, b_n \in k^* - \{1\}$. Put $a = \alpha^c$. Let C be the parametric curve of the form

$$C : t \mapsto \left(t, 1 - \frac{t^c}{a}, a, b_1, \dots, b_{n-2}\right) \in \mathbf{A}^1 \times \square^n.$$

By Lemma 3.10, we have $C \in K^n$.

By the same argument as in (2.1) in §2.2, this C satisfies the strong sup modulus condition when $c \geq m$, and the boundary of C is

$$\partial C = \sum_{\zeta^c = 1} (\zeta\alpha, a, b_1, \dots, b_{n-2}).$$

Let C' be the parametric curve of the form

$$C' : t \mapsto \left(t, 1 - \frac{t^c}{a}, t, b_1, \dots, b_{n-2}\right) \in \mathbf{A}^1 \times \square^n.$$

If $c \geq m$, by the same argument as in (2.2) in §2.2, this C' satisfies the strong sup modulus condition and $C' \in z_1(A_k(m), n)_{ssup}$. The boundary of C' is

$$\partial C' = \sum_{\zeta^c = 1} (\zeta\alpha, \zeta\alpha, b_1, \dots, b_{n-2}).$$

By Corollary 3.9, we have

$$\begin{aligned} c\partial C' &= \sum_{\zeta^s = 1} c(\zeta\alpha, \zeta\alpha, b_1, \dots, b_{n-2}) \\ &\equiv \sum_{\zeta^c = 1} (\zeta\alpha, \zeta^c\alpha^c, b_1, \dots, b_{n-2}) \bmod \partial K^n \\ &= \sum_{\zeta^c = 1} (\zeta\alpha, a, b_1, \dots, b_{n-2}) \\ &= \partial C. \end{aligned}$$

Since $C \in K^n$, there exists $C'' \in K^n$ such that

$$cC' - C'' \in z_1(A(m), n)_{ssup, \partial}. \tag{3.3}$$

On the other hand,

$$\begin{aligned} L_s^n(C') &= \operatorname{res}_{t=0} v^*\left(\frac{1 - y_1}{x^s} \frac{dy_2}{y_2} \frac{dy_3}{y_3} \cdots \frac{dy_n}{y_n}\right) \\ &= \operatorname{res}_{t=0}\left(\frac{t^c}{at^s} \frac{dt}{t}\right) \frac{db_1}{b_1} \cdots \frac{db_{n-2}}{b_{n-2}} \\ &= \begin{cases} a^{-1} \dfrac{db_1}{b_1} \cdots \dfrac{db_{n-2}}{b_{n-2}} & (s = c) \\ 0 & (s \neq c). \end{cases} \end{aligned} \tag{3.4}$$

By putting $s = c$, we have

$$L_c^n(cC' - C'') = \frac{c}{a} \frac{db_1}{b_1} \cdots \frac{db_{n-2}}{b_{n-2}}.$$

Since k satisfies the condition $\spadesuit_c$ and $a \in (k^*)^c$, we have

$$k \supset c(k^*)^c_{\mathbf{Z}} = ck = k.$$

Hence we get

$$\operatorname{Im} L^n_c = \Omega^{n-2}_k.$$

This implies that L_c is surjective. $\square$

Corollary 3.15. *Suppose that k satisfies the condition $\spadesuit_c$ for all integers $m \leq c < 2m$. Suppose that k contains the primitive c-th root of unity for all integers $m \leq c < 2m$. Then the following map is surjective*:

$$L^n : z_1(A_k(m), n)_{ssup,\partial} \to \bigoplus_{m \leq c < 2m} \Omega^{n-2}_k \langle c \rangle.$$

Proof. By the equations (3.3), (3.4) in the proof of previous Proposition 3.14 and the condition $\spadesuit_c$ (Definition 3.12, see also Lemma 3.13), for any $a_1, \ldots, a_m \in k$, $b_{c,1}, \ldots, b_{c,n-2} \in k^*$, there exist cycles $C^s \in z_1(A_k(m), n)_{ssup,\partial}$ $(m \leq s < 2m)$ such that

$$L^n_c(C^s) = \begin{cases} a_c \dfrac{db_{c,1}}{b_{c,1}} \cdots \dfrac{db_{c,n-2}}{b_{c,n-2}} & (s = c) \\ 0 & (s \neq c). \end{cases}$$

By putting

$$C = \sum_{m \leq s < 2m} C^s,$$

we have

$$L^n_c(C) = a_c \frac{db_{c,1}}{b_{c,1}} \cdots \frac{db_{c,n-2}}{b_{c,n-2}} \quad (m \leq c < 2m).$$ $\square$

3.2. A regulator map from the additive higher Chow group of 1-cycles

In this subsection, we will show that the map $L = L^n$ induces a surjective map

$$L : CH_1(A_k(m), n)_{ssup} \to \bigoplus_{m \leq s < 2m} \Omega^{n-2}_k \langle s \rangle$$

by using an argument similar to that of Park ([13]). We will prove this using the residue theorem of the generalized residue theory ([19]). First, we summarize the notation used in the proof.

We define the map $\operatorname{sgn} : \mathbf{Z} \cup \{\infty\} \to \{\pm 1\}$ in the following way: if i is an integer, then $\operatorname{sgn}(i) = (-1)^i$. For $i = \infty$, we define $\operatorname{sgn}(\infty) = -1$. We denote $\operatorname{sgn}(a, b, \ldots) = \operatorname{sgn}(a) \operatorname{sgn}(b) \cdots$ for simplicity.

Denote by $F^n_{i,\varepsilon}$ the face of $\square^n$ defined by the equation of the form $y_i = \varepsilon$. When it does not cause confusions, we omit the superscript n.

In what follows we assume $\mathrm{char}(k) = 0$. We fix natural numbers n, c. Recall that we defined the differential $(n-1)$-forms ω_i^n on $\mathbf{A}_k^1 \times (\mathbf{P}^1)^n$ for $1 \le i \le n$ in (3.1) in §3.1. Here, we denote by ω_i the differential $(n-1)$-form ω_i^n on $\mathbf{A}_k^1 \times (\mathbf{P}^1)^n$, and denote by η_i the differential n-form ω_i^{n+1} on $\mathbf{A}_k^1 \times (\mathbf{P}^1)^{n+1}$.

Let $\pi_i : \mathbf{A}^1 \times (\mathbf{P}^1)^{n+1} \to \mathbf{A}^1 \times (\mathbf{P}^1)^n$ be the projection that contracts the i-th factor on $(\mathbf{P}^1)^{n+1}$. For $1 \le i \le n+1$, we put $\omega_l(i) = \pi_i^* \omega_l$ the differential $(n-1)$-forms on $\mathbf{A}_k^1 \times (\mathbf{P}^1)^{n+1}$.

By a direct computation, we have the equality

$$\eta_{\alpha_l(i)} = \mathrm{sgn}(i, l) \omega_l(i) \wedge \frac{dy_i}{y_i}, \tag{3.5}$$

where α_l is the unique order preserving injective map

$$\alpha_l : \{1, \dots, n\} \to \{1, \dots, n+1\} \backslash \{l\}.$$

We use the equality (3.5) in the following proof of Theorem 3.16.

For an irreducible closed subvariety $W \subset \mathbf{A}_k^1 \times \square^n$, we denote by $v = v_W : \overline{W}^N \to \overline{W} \to \mathbf{A}_k^1 \times (\mathbf{P}^1)^n$ a normalization of its Zariski closure in $\mathbf{A}_k^1 \times (\mathbf{P}^1)^n$. If W satisfies the sup modulus condition, for any prime Weil divisor Y on $\overline{W}^N$, we denote by $S(W, Y)$ the set of all integers i such that the following inequality holds:

$$m \cdot \mathrm{ord}_Y\, v^*\{x = 0\} \le \mathrm{ord}_Y\, v^*\{y_i = 1\}.$$

We denote by $S(W)$ the set of all integers i such that the above inequality holds for all prime Weil divisors Y on $\overline{W}^N$:

$$S(W) := \bigcap_Y S(W, Y).$$

If W satisfies the sup modulus condition, we have $S(W, Y) \ne \emptyset$ for any prime Weil divisor Y on $\overline{W}^N$. Moreover if W satisfies the strong sup modulus condition, we have $S(W) \ne \emptyset$.

For any birational surjective morphism $\phi : \tilde{W} \to \overline{W}$ from a normal variety $\tilde{W}$, we define the set $S_\phi(W, Y)$ similarly:

$$S_\phi(W, Y) = \{i \,|\, m\, \mathrm{ord}_Y(\phi^*\{x = 0\}) \le \mathrm{ord}_Y(\phi^*\{y_i = 1\})\},$$
$$S_\phi(W) = \bigcap_Y S_\phi(W, Y).$$

By the universality of normalization, the map ϕ factors through the map v:

$$\begin{tikzcd} \overline{W}^N \arrow[r, "v"] & \overline{W} \arrow[r] & \mathbf{A}_k^1 \times (\mathbf{P}^1)^n \\ & \tilde{W}. \arrow[ul, dashed] \arrow[u, "\phi"'] & \end{tikzcd}$$

Hence the set $S_\phi(W)$ is not empty if W satisfies the strong sup modulus condition. We omit the subscript ϕ whenever it is clear from the context.

This paper's extension to general weights of the regulator maps of [13] for strong sup 1-cycles is checked.

THEOREM 3.16. *Let $W \in z_1(A_k(m), n+1)_{ssup}$ be an irreducible surface over k. Let $m \leq c < 2m$ be an integer. Then $L_c^n(\partial W) = 0$.*

Proof. For simplicity, we denote $L = L_c^n$. For an integer i and $\varepsilon \in \{0, \infty\}$, we denote by $(\partial_i^\varepsilon W)^{(0)}$ the set of all prime Weil divisors appearing with non-zero coefficient.

By definition, we have

$$\begin{aligned}
\partial W &= \sum_{i=1}^{n+1} \sum_{\varepsilon} \operatorname{sgn}(i, \varepsilon) \partial_i^\varepsilon W \\
&= \sum_{i=1}^{n+1} \sum_{\varepsilon} \sum_{Y \in (\partial_i^\varepsilon W)^{(0)}} \operatorname{sgn}(i, \varepsilon) \operatorname{ord}_Y(\partial_i^\varepsilon W) Y \\
&= \sum_{i=1}^{n+1} \sum_{\varepsilon} \sum_{Y \in (\partial_i^\varepsilon W)^{(0)}} \operatorname{sgn}(i, \varepsilon) \operatorname{ord}_{\overline{Y}}(\overline{\partial_i^\varepsilon W}) Y
\end{aligned}$$

where ε runs over the set $\{0, \infty\}$. For any $Y \in (\partial_i^\varepsilon W)^{(0)}$, we choose an integer $l(Y) \in S_{\nu_Y}(Y)$. Then we have

$$\begin{aligned}
L(\partial W) &= \sum_{i=1}^{n+1} \sum_{\varepsilon} \sum_{Y \in (\partial_i^\varepsilon W)^{(0)}} \operatorname{sgn}(i, \varepsilon) \operatorname{ord}_{\overline{Y}}(\overline{\partial_i^\varepsilon W}) L(Y) \\
&= -\sum_{i=1}^{n+1} \sum_{\varepsilon} \sum_{Y \in (\partial_i^\varepsilon W)^{(0)}} \operatorname{sgn}(i, \varepsilon) \operatorname{ord}_{\overline{Y}}(\overline{\partial_i^\varepsilon W}) \\
&\quad \times \sum_{p \in \nu_Y^*\{x=0\}} (-1)^{l(Y)} \operatorname{Res}_{(\overline{Y}^N, p)}(\nu_Y^* \omega_{l(Y)}(i)) \\
&= -\sum_{i=1}^{n+1} \sum_{\varepsilon} \sum_{Y \in (\partial_i^\varepsilon W)^{(0)}} \sum_{p \in \nu_Y^*\{x=0\}} \operatorname{sgn}(i, \varepsilon, l(Y)) \operatorname{ord}_{\overline{Y}}(\overline{\partial_i^\varepsilon W}) \\
&\quad \times \operatorname{Res}_{(\overline{Y}^N, p)}(\nu_Y^* \omega_{l(Y)}(i)).
\end{aligned}$$

We can regard Y as a closed subscheme of W naturally. We define the morphisms ϕ_1, ϕ_2, ϕ, ψ, ϕ_Y, ν_Y as following. Let $\phi_1 : W_1 \to \overline{W}$ be a composition of a sequence of blow-ups such that the strict transforms of all $Y \in (\partial_i^\varepsilon W)^{(0)}$ are smooth. Let $\nu : \overline{W}^N \to \overline{W}$ be a normalization of $\overline{W}$. Let $\phi_2 : W_2 := W_1^N \to W_1$ be a normalization of W_1, and let $\psi : W_2 = W_1^N \to \overline{W}^N$ be an induced morphism by the universality of normalization. Let $\phi = \phi_1 \circ \phi_2$ be the composite map. Let $\phi_{1,Y} : \tilde{Y} \to \overline{Y}$ be the strict transform of $\overline{Y}$ under the blow-up ϕ_1. For simplicity, we use the same notation ϕ_1 instead of $\phi_{1,Y}$. Let $\nu_Y : \overline{Y}^N \to \overline{Y}$ be a normalization of $\overline{Y}$ and let $\psi_Y : \tilde{Y} \to \overline{Y}^N$ be an induced morphism by the universality of normalization.

$$\begin{tikzcd} W_2 = W_1^N \arrow[r, "\phi_2"] \arrow[rr, bend left, "\phi"] \arrow[dr, "\psi"'] & W_1 \arrow[r, "\phi_1"] & \overline{W} \\ & \overline{W}^N \arrow[ur, "\nu"'] & \end{tikzcd} \qquad \begin{tikzcd} \tilde{Y} \arrow[r, "\phi_1"] \arrow[d, "\psi_Y"'] & \overline{Y} \\ \overline{Y}^N \arrow[ur, "\nu_Y"'] & \end{tikzcd}$$

We note that all these morphisms are proper surjective birational.

By using the projection theorem (Theorem 5.11) in the residue theory for the morphisms $\psi_Y : \tilde{Y} \to \overline{Y}^N$, $\phi_1 : W_1 \to W$, we get

$$\mathrm{Res}_{(\overline{Y}^N,p)}(\nu_Y^*\omega_l(i)) = \sum_{q \to p} \mathrm{Res}_{(\tilde{Y},q)}(\phi_1^*\omega_l(i)).$$

Hence we have

$$\begin{aligned} L(\partial W) &= -\sum_{i=1}^{n+1}\sum_{\varepsilon}\sum_{Y \in (\partial_i^{\varepsilon} W)^{(0)}}\sum_{p \in \nu_Y^*\{x=0\}} \mathrm{sgn}(i,\varepsilon,l(Y)) \\ &\quad \times \mathrm{ord}_{\overline{Y}}(\overline{\partial_i^{\varepsilon} W})\, \mathrm{Res}_{(\overline{Y}^N,p)}(\nu_Y^*\omega_{l(Y)}(i)) \\ &= -\sum_{i=1}^{n+1}\sum_{\varepsilon}\sum_{Y \in (\partial_i^{\varepsilon} W)^{(0)}}\sum_{p \in \phi_1^*\{x=0\}} \mathrm{sgn}(i,\varepsilon,l(Y)) \\ &\quad \times \mathrm{ord}_{\tilde{Y}}(\widetilde{\partial_i^{\varepsilon} W})\, \mathrm{Res}_{(\tilde{Y},p)}(\phi_1^*\omega_{l(Y)}(i)) \\ &= -\sum_{i=1}^{n+1}\sum_{\varepsilon}\sum_{Y \in (\partial_i^{\varepsilon} W)^{(0)}}\sum_{p \in \phi_1^*\{x=0\}} \mathrm{sgn}(i,l(Y)) \\ &\quad \times \mathrm{Res}_{(W_1,\tilde{Y})}\left(\phi_1^*\frac{dy_i}{y_i}\right) \mathrm{Res}_{(\tilde{Y},p)}(\phi_1^*\omega_{l(Y)}(i)), \end{aligned}$$

where the last equality follows from a direct computation of a residue values at the normal variety $\tilde{Y}$. By a direct computation, we have the following lemma.

LEMMA 3.17. *With the above notations,*

$$\begin{aligned} \mathrm{sgn}(i,l)\left(\mathrm{Res}_{(W_1,\tilde{Y})}\left(\phi_1^*\frac{dy_i}{y_i}\right)\right)\phi_1^*\omega_l(i) &= \mathrm{Res}_{(W_1,\tilde{Y})}\left(\mathrm{sgn}(i,l)\phi_1^*\omega_l(i) \wedge \phi_1^*\frac{dy_i}{y_i}\right) \\ &= \mathrm{Res}_{(W_1,\tilde{Y})}(\phi_1^*\eta_{\alpha_l(i)}). \end{aligned}$$

By using this lemma, we have

$$
\begin{aligned}
(3.6)\quad L(\partial W) &= -\sum_{i=1}^{n+1}\sum_{\varepsilon}\sum_{Y\in(\partial_i^{\varepsilon}W)^{(0)}}\sum_{p\in\phi_1^*\{x=0\}} \operatorname{sgn}(i,l(Y)) \\
&\quad\times \operatorname{Res}_{(W_1,\tilde{Y})}\left(\phi_1^*\frac{dy_i}{y_i}\right)\operatorname{Res}_{(\tilde{Y},p)}(\phi_1^*\omega_{l(Y)}(i)) \\
&= -\sum_{i=1}^{n+1}\sum_{\varepsilon}\sum_{Y\in(\partial_i^{\varepsilon}W)^{(0)}}\sum_{p\in\phi_1^*\{x=0\}} \operatorname{Res}_{(\tilde{Y},p)}(\operatorname{Res}_{(W_1,\tilde{Y})}(\phi_1^*\eta_{\alpha_{l(Y)}(i)})) \\
&= -\sum_{i=1}^{n+1}\sum_{\varepsilon}\sum_{Y\in(\partial_i^{\varepsilon}W)^{(0)}}\sum_{p\in\phi_1^*\{x=0\}} \operatorname{Res}_{(W_1,\tilde{Y},p)}(\phi_1^*\eta_{\alpha_{l(Y)}(i)}) \\
&= -\sum_{i=1}^{n+1}\sum_{\varepsilon}\sum_{Y\in(\partial_i^{\varepsilon}W)^{(0)}}\sum_{p\in\phi_1^*\{x=0\}} \\
&\quad\times \sum_{(W_2,C,q)\xrightarrow{\phi_2}(W_1,\tilde{Y},p)} \operatorname{Res}_{(W_2,C,q)}(\phi^*\eta_{\alpha_{l(Y)}(i)}),
\end{aligned}
$$

where we use the transitivity of residue maps (Theorem 5.9) and the projection theorem (Theorem 5.11) for the map $\phi_2 : W_2 \to W_1$.

Consider a chain $\xi = (W_2, C, q)$ on W_2 satisfying $\operatorname{Res}_{\xi}(\eta_l) \neq 0$. By the shape of the differential form η_l, we notice that $\phi(C)$ is a subset of $W \cap \{x = 0\}$ or a subset of $\partial_i^{\varepsilon} W$ for some i, ε. We note that all the chain $\xi = (W_2, C, q)$ in the equation (3.6) satisfying that $\phi(C)$ is a subset of $\partial_i^{\varepsilon} W$ for some i, ε.

By using the residue theorem for varying curves (Theorem 5.10(1)), we have

$$
\begin{aligned}
L(\partial W) &= -\sum_{i=1}^{n+1}\sum_{\varepsilon}\sum_{Y\in(\partial_i^{\varepsilon}W)^{(0)}}\sum_{p\in\phi_1^*\{x=0\}}\sum_{(W_2,C,q)\xrightarrow{\phi_2}(W_1,\tilde{Y},p)} \operatorname{Res}_{(W_2,C,q)}(\phi^*\eta_{\alpha_{l(Y)}(i)}) \\
&= \sum_{i=1}^{n+1}\sum_{\varepsilon}\sum_{Y\in(\partial_i^{\varepsilon}W)^{(0)}}\sum_{p\in\phi_1^*\{x=0\}}\sum_{q\in D,\,\phi(q)=p} \operatorname{Res}_{(W_2,D,q)}(\phi^*\eta_{\alpha_{l(Y)}(i)}),
\end{aligned}
$$

where all chains (W_2, D, q) satisfying that $\phi(D)$ is a subset of $W \cap \{x = 0\}$.

The following lemma follows from a direct computation.

Lemma 3.18. *Under the same notations as above, let $D_2 \subset \phi^*\{x = 0\}$ be an irreducible component, and put $D = \psi(D_2) \subset v^*\{x = 0\}$. Let $q \in C \cap D_2$ be a closed point, and put*

$$p_1 = \phi_2(q) \in \tilde{Y}, \quad p = \psi_Y(p_2).$$

Then we have

$$S(W,D) = S_\phi(W,D_2) \subset S_\phi(Y,q) = S_{\phi_1}(Y,p_1) = S(Y,p).$$

Especially we can replace the index $l(Y)$ by an element of $S(W,D)$. Since W satisfies the strong sup modulus condition, we have $\emptyset \neq S(W) \subset S(W,D)$. Hence we can choose $l = l(Y)$ independently of Y, so we get

$$L(\partial W) = \sum_{i=1}^{n+1}\sum_{\varepsilon}\sum_{Y\in(\partial_i^\varepsilon W)^{(0)}}\sum_{p\in\phi_1^*\{x=0\}}\sum_{q\in D,\phi(q)=p}\mathrm{Res}_{(W_2,D,q)}(\phi^*\eta_{\alpha_l(i)}). \tag{3.7}$$

By using the residue theorem for varying closed points (Theorem 5.10(2)) and the transitivity of residue maps (Theorem 5.9), the right hand side in equation (3.7) is equal to 0. Hence we get $L(\partial W) = 0$. □

Remark 3.19. If one attempts the above argument for the sup modulus cycles, then one cannot necessarily choose $l = l(Y)$ independently of Y in the sentence just above (3.7). The referee had informed the author that the main theorem of [13] is probably incorrect for the sup modulus 1-cycles that do not satisfy the strong sup modulus condition. According to the referee via a private communication with the author of [13], the author of *ibid.* knows about the problem and said he obtained a counterexample as well for sup modulus condition, which will be available in a forthcoming paper on this subject.

COROLLARY 3.20. *L_c^n induces a map*

$$L_c^n : CH_1(A_k(m),n)_{ssup} \to \Omega_k^{n-2}\langle c\rangle.$$

We have a surjective map

$$L^n : CH_1(A_k(m),n)_{ssup} \to \bigoplus_{m\le s<2m}\Omega_k^{n-2}\langle s\rangle.$$

4. A weight structure of the cyclic homology and the additive higher Chow group

4.1. Preliminary

Let k be a field k. For $f(x), g(x) \in k(x)$, define $[x,f,g]$ to be the parametric curve of the form

$$t \mapsto (t, f(t), g(t)) \in \mathbf{A}^1 \times (\mathbf{P}^1)^2.$$

We naturally regard it as a 1-cycle on $\mathbf{A}^1 \times \square^2$. For $f(x) \in k[x]$, define C_f to be the parametric curve of the form

$$C_f : t \mapsto (t, f(t), 1 - f(t)) \in \mathbf{A}^1 \times \square^2. \tag{4.1}$$

Let v_x be the valuation of the DVR $k[x]_{(x)}$. If $v_x(f(x)) \ge m$, we have $C_f \in z_1(A_k(m),2)_{sup,\partial}$.

Recall that (the cubical version of) Bloch's higher Chow groups $CH^d(K,n)$ of a field K are defined as follows. Let $z^d(K,n)$ be the group of codimension d-cycles on $\mathrm{Spec}\,K \times \square^n$ which intersect properly with $\mathrm{Spec}\,K \times F$ for each face $F \subset \square^n$. We define the boundary $\partial = \sum_{i=1}^n (-1)^i(\partial_i^0 - \partial_i^\infty) : z^d(K,n) \to z^d(K,n-1)$ and get the complex of abelian groups $(z^d(K,*),\partial)$. The n-th homology of the associated non-degenerate complex is the Bloch's higher Chow group $CH^p(K,n)$ (see [17, pp. 178–181]). In the following Lemma 4.1, we use the fact that $CH^1(K,2) = 0$ ([2, Thm. 6.1]).

Lemma 4.1. *Let $\mathfrak{p} \in \mathbf{A}_k^1 \backslash O$ be a closed point, and let $C \in z_1(A_k(m),2)_{sup}$ be an irreducible curve satisfying the condition $C \subset \mathfrak{p} \times \square^2$. Then we have*

$$[C] = 0 \in CH_1(A_k(m),2)_{sup}.$$

Proof. Let $\iota_{\mathfrak{p}} : \mathrm{Spec}\,k(\mathfrak{p}) \to \mathbf{A}^1$ be the closed immersion. Then $\iota_{\mathfrak{p}}$ induces a closed immersion

$$\phi_{\mathfrak{p}} : \mathrm{Spec}\,k(\mathfrak{p}) \times \square^q \to \mathbf{A}_k^1 \times \square^q,$$

hence it induces the push-forward $\phi_{\mathfrak{p}} : z^{q-r}(k(\mathfrak{p}),q) \to z_r(\mathbf{A}_k^1 \times \square^q)$, where $z_r(\mathbf{A}_k^1 \times \square^q)$ is a group of r-cycles on $\mathbf{A}_k^1 \times \square^q$. Since this map factor through $z_r(A_k(m),q)_{sup}$, we have

$$\begin{tikzcd} z^{q-r}(k(\mathfrak{p}),q) \arrow[r, "\phi_{\mathfrak{p}}"] \arrow[dr, dashed, "\phi_{\mathfrak{p}}^q"'] & z_r(\mathbf{A}_k^1 \times \square^q) \\ & z_r(A_k(m),q)_{sup}. \arrow[u, hook] \end{tikzcd}$$

By the following commutative diagram

$$\begin{tikzcd} z^1(k(\mathfrak{p}),3) \arrow[r] \arrow[d, "\phi_{\mathfrak{p}}^3"'] & z^1(k(\mathfrak{p}),2) \arrow[r] \arrow[d, "\phi_{\mathfrak{p}}^2"'] & z^1(k(\mathfrak{p}),1) \arrow[d, "\phi_{\mathfrak{p}}^1"'] \\ z_2(A_k(m),3)_{sup} \arrow[r] & z_1(A_k(m),2)_{sup} \arrow[r] & z_0(A_k(m),1)_{sup}, \end{tikzcd}$$

we have

$$\phi_{\mathfrak{p}}^2 : CH^1(k(\mathfrak{p}),2) \to CH_1(A_k(m),2)_{sup}.$$

Since $CH^1(k(\mathfrak{p}),2) = 0$, we notice that $\phi_{\mathfrak{p}}^2$ is the zero map. We can easily check that $[C] \in \mathrm{Im}\,\phi_{\mathfrak{p}}$, hence we get

$$[C] = 0 \in CH_1(A_k(m),2)_{sup}. \qquad \square$$

This lemma says if our 1-cycle is constant on the first coordinate $\mathbf{A}_k^1$, this 1-cycle is the boundary of some 2-cycle. Hence we can disregard 1-cycles which are constant on $\mathbf{A}_k^1$.

Remark 4.2. Lemma 4.1 is motivated by the fact

$$L(C) = 0$$

where $C \subset \mathbf{A}_k^1 \times \square^2$ is a curve satisfying $C \subset \{a\} \times \square^2$.

COROLLARY 4.3. *Let $f \in x^m k[x]$, $g, h \in k(x)$. Then we have the following relation in $CH_1(A_k(m), 2)_{sup}$:*

$$[x, 1 - f, g] + [x, 1 - f, h] = [x, 1 - f, gh].$$

Proof. We consider the parametric 2-cycle of the form

$$C : (x, y) \mapsto \left(x, 1 - f(x), \frac{g(x)(y - h(x))}{(y - g(x)h(x))}, y \right) \in \mathbf{A}^1 \times \square^3.$$

On easily sees that C satisfies the sup modulus condition. By Lemma 4.1, we get

$$\partial C \equiv -[x, 1 - f, h] + [x, 1 - f, gh] - [x, 1 - f, g]$$

since the solutions of the equation $1 - f(x) = 0$ define closed points of $\mathbf{A}_k^1$. $\square$

PROPOSITION 4.4. *Let $f \in x^{2m} k[x]$. Then we have*

$$[C_f] = 0 \in CH_1(A_k(m), 2)_{sup}.$$

Proof. Put $\phi = \phi(x) \in k(x)$ so that $f = x^{2m}\phi(x)$. Let S be the parametric 2-cycle of the form

$$S : (x, y) \mapsto \left(x, 1 - \frac{x^m \phi(x)}{y}, 1 - x^m y, y \right) \in \mathbf{A}_k^1 \times \square^3.$$

We must show that it satisfies the sup modulus condition. The scheme $\mathbf{A}_k^1 \times \square^3$ is covered by the standard affine open sets, such as $\operatorname{Spec} k[x, y_1, y_2, y_3]$, $\operatorname{Spec} k[x, y_1^{-1}, y_2, y_3]$, and so on. In any affine open sets, if there exists i such that $\dfrac{y_i - 1}{x^m}$ is integral on S, the 2-cycle S satisfies the sup modulus condition.

On $\operatorname{Spec} k[x, y_1, y_2, y_3]$, the 2-cycle S is given by the equations of the form

$$y_1 = 1 - \frac{x^m \phi}{y}, \quad y_2 = 1 - x^m y, \quad y_3 = y.$$

These equations are equal to

$$y_1 = 1 - \frac{x^m \phi}{y_3}, \quad y_2 = 1 - x^m y_3.$$

Hence in this coordinate, we have

$$\frac{y_2 - 1}{x^m} = -y_3,$$

so S satisfies the modulus condition on $\operatorname{Spec} k[x, y_1, y_2, y_3]$. Let $y_3' = y_3^{-1}$ and consider the modulus condition on $\operatorname{Spec} k[x, y_1, y_2, y_3']$. In this case, the 2-cycle S is given by the equations of the form

$$y_1 = 1 - x^m \phi y_3', \quad y_2 = 1 - \frac{x^m}{y_3'}.$$

Hence we have

$$\frac{y_1 - 1}{x^m} = -\phi y_3',$$

so S satisfies the modulus condition in this coordinate. We can easily check that S satisfies the modulus condition for any other coordinates, thus we have $S \in z_2(A_k(m), 3)_{sup}$.

The boundary of S is calculated by using Corollary 4.3 as follows:

$$\begin{aligned}\partial S &= [x, 1 - x^{2m}\phi, x^m \phi] - \left[x, 1 - x^{2m}\phi, \frac{1}{x^m}\right] \\ &\equiv [x, 1 - f, f].\end{aligned}$$

Thus we get the desired relation. □

Recall C_f is the parametric curve defined in the equation (4.1) in §4.1. This C_f satisfies the sup modulus condition if $v_x(f) \geq m$ where v_x is a valuation of $k[x]_{(x)}$.

Definition 4.5. We define the map $\Phi : x^m k[x] \to z_1(A_k(m), 2)_{sup}$ as follows. For a homogeneous element $f_c = ax^c \in x^m k[x]$ where $a \in k$ and $c \geq m$ is an integer, we define

$$\Phi(f_c) = C_{f_c},$$

and extend it linearly. By Proposition 4.4, we get the following:

Corollary 4.6. *The map* Φ *induces*

$$\Phi : x^m k[x]/(x^{2m}) \to CH_1(A_k(m), 2)_{sup}.$$

The abelian group $x^m k[x]/(x^{2m})$ has a natural k^*-action defined by

$$a \star f(x) \mapsto f(ax).$$

Hence we have the decomposition of k^*-set

$$x^m k[x]/(x^{2m}) \cong \bigoplus_{m \leq s < 2m} k \cdot x^s \cong \bigoplus_{m \leq s < 2m} k\langle s \rangle.$$

If $f \in x^m k[x]$, we have

$$a \star C_f = \left[\frac{x}{a}, f(x), 1 - f(x)\right] = [x, f(ax), 1 - f(ax)]$$
$$= [a, a \star f(x), 1 - a \star f(x)] = C_{a \star f}.$$

Hence the map Φ is compatible with k^*-actions.

Remark 4.7. For $f \in x^m k[x]$, we can check that C_f satisfies the strong sup modulus condition. Hence we can define the map similarly:

$$\Phi' : x^m k[x] \to z_1(A_k(m), 2)_{ssup} \to CH_1(A_k(m), 2)_{ssup}.$$

However the irreducible surface which is used in the proof of Proposition 4.4 does not satisfy the strong sup modulus condition in general.

4.2. A weight structure of the Hochschild homology and the cyclic homology

Let k be a number field. In this subsection, we study a weight structure of the Hochschild homology and the cyclic homology of the truncated polynomial rings over k. We calculate a weight decomposition of the cyclic homology via the Hochschild homology by using a technique of Loday ([10]).

Recall [10, §1.1.3] that the Hochschild complex $C(A) = C(A/\mathbf{Q})$ of $\mathbf{Q}$-algebra A is defined by

$$C_n(A) = A^{\otimes(n+1)} \tag{4.2}$$
$$b(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n)$$
$$+ (-1)^n (a_n a_0 \otimes \cdots \otimes a_{n-1}),$$

where $\otimes = \otimes_{\mathbf{Q}}$. The n-th homology group $H_n(C_*(A))$ of this complex is called the n-th Hochschild homology group and denoted by $HH_n(A)$. The polynomial ring $k[x]$ has the natural k^*-action defined by

$$\lambda \star f(x) := f(\lambda x). \tag{4.3}$$

This action induces an action on the truncated polynomial ring $A = k[\varepsilon] = k[x]/x^m$, hence it induces an action on the Hochschild homology $HH_n(A)$ and the cyclic homology $HC_n(A)$. Let $1 \neq \lambda \in k^*$. λ defines a linear map

$$HH_n(A) \to HH_n(A); \quad f \mapsto \lambda \star f.$$

For any integer ω, we denote by $HH_n(A)_\omega$ the eigenspace of the above linear map associated with λ^ω.

For a homogeneous element $a\varepsilon^n \in A$ where $a \in k^*$, we define a new weight as follows:

$$|a\varepsilon^n| := n.$$

This weight is called an x-*weight* and it induces an x-weight on $C_n(A) = A^{n+1}$ defined by

$$|a_0 \otimes \cdots \otimes a_n| := \sum |a_i|$$

where a_i is a homogeneous element of A. Hence we have a natural weight decomposition of the Hochschild complex

$$C(A) = \bigoplus_{\omega \geq 0} C(A)_\omega,$$

where $C(A)_\omega$ consists of all elements which is 0 or whose x-weight is ω. Hence $HH_n(A)_\omega$ is an n-th homology group of the complex $C(A)_\omega$ and $HH_n(A)_\omega$ does not depend on the choice of λ. Similarly, we define a weight structure on the cyclic homology $HC_n(A)$. The cyclic homology is defined to be the total homology of certain bicomplex $CC(A)$, called the cyclic bicomplex. This is the bicomplex $CC(A)$ whose component in bidegree (p,q) is $CC_{pq}(A) = C_q(A) = A^{\otimes q+1}$. We will not explane the definitions of vertical and horizontal differentials of $CC(A)$; we refer to [10, Def. 2.1.3] for more details.

Let I be an ideal of A. The relative Hochschild homology groups $HH_n(A,I)$ are defined to be the homology groups of the complex $\mathrm{Ker}(C(A) \to C(A/I))$. Similarly, we define the relative cyclic homology groups $HC_n(A,I)$ to be the homology groups of $\mathrm{Tot}(\mathrm{Ker}(CC(A) \to CC(A/I)))$ (see [10, §1.1.16 and §2.1.15]). Hence we have a following long exact sequence

$$\cdots \to HC_n(A,I) \to HC_n(A) \to HC_n(A/I) \to HC_{n-1}(A,I) \to \cdots. \tag{4.4}$$

Let k be a number field and consider the Hochschild and cyclic homologies of the truncated polynomial ring $k[\varepsilon] := k[x]/x^m$. Then we have isomorphisms ([10, E.1.1.8, E.4.4.3])

$$HH_{2n}(k[\varepsilon]) \xrightarrow{\simeq} HH_{2n}(\mathbf{Q}[\varepsilon]) \otimes_{\mathbf{Q}} k,$$
$$HC_{2n}(k[\varepsilon]) \xrightarrow{\simeq} HC_{2n}(\mathbf{Q}[\varepsilon]) \otimes_{\mathbf{Q}} k.$$

Hence it is sufficient to assume that $k = \mathbf{Q}$ to calculate the weight structure of the cyclic homology $HC_2(k[\varepsilon])$. We can easily check that $HC_{2n-1}(k) = 0$, $HC_{2n}(k) = k$. By using a long exact sequence (4.4), we have a split exact sequence

$$0 \to HC_2(k[\varepsilon],(\varepsilon)) \to HC_2(k[\varepsilon]) \to k \to 0.$$

By using the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & HH_2(k[\varepsilon],(\varepsilon)) \arrow[r, "\cong"] \arrow[d] & HH_2(k[\varepsilon]) \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & HC_2(k[\varepsilon],(\varepsilon)) \arrow[r] & HC_2(k[\varepsilon]) \arrow[r] & k \arrow[r] & 0
\end{tikzcd} \tag{4.5}$$

and isomorphisms $HH_2(k[\varepsilon]) \cong k^{m-1}$, $HC_2(k[\varepsilon]) \cong k^m$ ([10, E.4.1.8]), we have

$$HC_2(k[\varepsilon]) \cong HC_2(k[\varepsilon],(\varepsilon)) \oplus k \cong HH_2(k[\varepsilon]) \oplus k. \tag{4.6}$$

Now we calculate the weight structure of the Hochschild homology $HH_2(k[\varepsilon])$ by using the technique of Loday.

Let $k = \mathbf{Q}$ and consider the Hochschild homology of $k[x]/x^m$. Let $V = k \cdot x \oplus k \cdot y$ be the graded free k-module of rank 2 with $|x| = \deg x := 0$, $|y| = \deg y := 1$. Then the graded symmetric algebra over V is

$$\bigwedge V \cong k[x,y]/y^2 = k[x] \oplus k[x]y.$$

We define the differential δ on $\bigwedge V$ by the assignment

$$x \mapsto 0, \quad y \mapsto x^m.$$

We see immediately that $\delta^2 = 0$ and that it satisfies the Leibniz rule. By using the Leibniz rule, we get an endomorphism of $\bigwedge V$. Then $(\bigwedge V, \delta)$ becomes a commutative differential graded algebra. We consider $k[x]/x^m$ as a commutative differential graded algebra with the trivial differential. Then the following commutative diagram

$$\begin{tikzcd}
k[x] \arrow[d, "p"'] & k[x]y \arrow[l, "\delta"'] \arrow[d] & 0 \arrow[l, "0"'] \arrow[d] \\
k[x]/x^m & 0 \arrow[l, "0"'] & 0, \arrow[l, "0"']
\end{tikzcd}$$

where p is a natural quotient map, gives a quasi-isomorphism of complexes. Hence we get isomorphisms ([10, Theorem 5.3.5])

$$HH_n(k[x]/x^m) \simeq HH_n(k[x]/x^m, 0) \simeq HH_n(k[x,y]/y^2, \delta), \tag{4.7}$$

where the groups $HH_n(k[x]/x^m, 0)$ and $HH_n(k[x,y]/y^2, \delta)$ are the Hochschild homology of differential graded algebra ([10, §5.3.2]), which is defined as follows. For any differential graded k-algebra (A,δ), let $(A,\delta)^{\otimes n}$ be the iterated tensor product of the complex (A,δ). Similarly as in the equation (4.2), we define the map

$$b : (A,\delta)^{\otimes n+1} \to (A,\delta)^{\otimes n},$$

$$b(a_0, a_1, \ldots, a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0, \ldots, a_i a_{i+1}, \ldots, a_n) + (-1)^n (-1)^{|a_n|(|a_0|+\cdots+|a_n|)} (a_n a_0, \ldots, a_{n-1}),$$

where the elements a_i are all homogeneous of (A,δ) of degree $|a_i|$. The Hochschild complex $C_*(A,\delta)$ is the total complex of the bicomplex $(A,\delta)^{\bullet}$, whose component in bidegree (p,q) is $((A,\delta)^{\otimes q+1})_p$. The Hochschild homology $HH_*(A,\delta)$ is defined to be its homology.

We calculate the x-weight structure of the last group $HH_n(k[x,y]/y^2,\delta)$ in the isomorphisms (4.7). For this we need to recall some definitions about the module of differentials of graded commutative algebra and related topics ([10, §5.4]).

Let A be a graded commutative algebra. We define a graded A-module $\Omega^1_{A/k}$ as follows. Let I be the kernel of the multiplication $\mu : A \otimes A \to A$. Then I is a graded A-bimodule, and we define $\Omega^1_{A/k} := I/I^2$. The group $\Omega^1_{A/k}$ is generated as a graded A-module by the set of all elements $\{da \mid a \in A\}$ where da is the image of the following map

$$d : A \to \Omega^1_{A/k}, \quad a \mapsto da := 1 \otimes a - a \otimes 1 \bmod I^2.$$

Note that the map d preserves the homogeneous degrees, so we have an equality of homogeneous degrees $|a| = |da|$ for any homogeneous element $a \in A$. We define the graded module of the n-th differentials $\Omega^n_{A/k}$ as the quotient of the n-fold tensor product $(\Omega^1_{A/k})^{\otimes n}$ by the submodule generated by

$$da \otimes db + (-1)^{|da|\,|db|} db \otimes da \tag{4.8}$$

for all homogeneous elements $da, db \in \Omega^1_{A/k}$. The n-th differentials $\Omega^n_{A/k}$ is the graded module and we denote by $(\Omega^n_{A/k})_q$ the homogeneous submodule of degree q. Moreover if A is a graded commutative differential algebra with differential δ, there is an obvious extension of the differential map δ to $\Omega^n_{A/k}$:

$$\begin{aligned}\delta(a_0\, da_1 \cdots da_n) = (-1)^n(&\delta a_0\, da_1 \cdots da_n + (-1)^{|a_0|} a_0\, d(\delta a_1) da_2 \cdots da_n \\ &+ \cdots + (-1)^{|a_0|+\cdots+|a_{n-1}|} a_0\, da_1 \cdots d(\delta a_n)).\end{aligned}$$

So we get the complex

$$((\Omega^n_{A/k})_*, \delta) : \cdots \to (\Omega^n_{A/k})_q \xrightarrow{\delta_q} (\Omega^n_{A/k})_{q-1} \to \cdots.$$

Put $A = k[x,y]/y^2$. Then we have ([10, Proposition 5.4.6])

$$HH_n(A,\delta) \simeq \bigoplus_{i \geq 0} H_{n-i}((\Omega^i_{A/k})_*, \delta),$$

where $\Omega^i_{A/k}$ is the i-th differentials. Note that $\Omega^1_{A/k}$ is generated by symbols dx, dy as an A-module with degrees $|dx| = 0$, $|dy| = 1$. By the definition of $\Omega^*_{A/k}$ (see also (4.8)), the module of differentials $\Omega^*_{A/k}$ is generated by the symbols dx, dy as an A-module with the relations

$$dxdx = 0, \quad dxdy = -dydx.$$

Hence $\Omega^q_{A/k}$ is generated by

$$x^i(dy)^q,\ x^i\, dx(dy)^{q-1},\ x^i y(dy)^q,\ x^i y\, dx(dy)^{q-1}$$

as a k-module. The complex $((\Omega^q_{A/k})_*, \delta)$ is the form

$$0 \longrightarrow (\Omega^q_{A/k})_{q+1} \xrightarrow{\delta_{q+1}} (\Omega^q_{A/k})_q \xrightarrow{\delta_q} (\Omega^q_{A/k})_{q-1} \xrightarrow{\delta_{q-1}} 0.$$

Hence we have an isomorphism

$$HH_2(A,\delta) \cong H_1((\Omega^1_{A/k})_2 \xrightarrow{\delta_2} (\Omega^1_{A/k})_1 \xrightarrow{\delta_1} (\Omega^1_{A/k})_0).$$

By a direct computation, we have

$$\begin{aligned}
\delta(x^i(dy)^q) &= x^i\delta((dy)^q) = qmx^{i+m-1}\, dx(dy)^{q-1}\\
\delta(x^i\, dx(dy)^{q-1}) &= x^i\, dx\delta((dy)^{q-1}) = x^i\, dx(q-1)mx^{m-1}\, dx(dy)^{q-2} = 0\\
\delta(x^iy(dy)^q) &= x^i\delta(y)(dy)^q + x^i(-1)^{|y|}y\delta((dy)^q)\\
&= x^{i+m}(dy)^q - x^iyqmx^{m-1}\, dx(dy)^{q-1}\\
&= x^{i+m}(dy)^q - qmx^{i+m-1}y\, dx(dy)^{q-1}\\
\delta(x^iy\, dx(dy)^{q-1}) &= x^i\delta(y)\, dx(dy)^{q-1} + x^i(-1)^{|y|}y\delta(dx(dy)^{q-1}) = x^{i+m}\, dx(dy)^{q-1}.
\end{aligned}$$

Hence if we set $v_i = x^i\, dy - mx^{i-1}y\, dx$, we get

$$\operatorname{Ker}\delta_1 = \bigoplus_{0<i} kv_i$$
$$\operatorname{Im}\delta_2 = \bigoplus_{m\le i} kv_i.$$

Since $kv_i \simeq k\langle m+i\rangle$, we get

$$HH_2(k[x]/x^m) \simeq \bigoplus_{0<i<m} kv_i \simeq \bigoplus_{m<\omega<2m} k\langle\omega\rangle.$$

This is the desired weight decomposition.

PROPOSITION 4.8. *Let k be a number field. Then the weight decomposition of the Hochschild homology induces an isomorphism*

$$HH_2(k[x]/x^m) = \bigoplus_{m<\omega<2m} HH_2(k[x]/x^m)_\omega \cong \bigoplus_{m<\omega<2m} k\langle\omega\rangle. \tag{4.9}$$

Hence there exists a weight preserving isomorphism

$$HH_2(k[x]/x^m) \cong x^{m+1}k[x]/x^{2m}, \tag{4.10}$$

where the weight structure of $x^{m+1}k[x]/x^{2m}$ is induced by (4.3).

COROLLARY 4.9. *Let k be a number field. Then the weight decomposition of the cyclic homology induces isomorphisms*

$$HC_2(k[x]/x^m,(x)) = \bigoplus_{m<\omega<2m} HC_2(k[x]/x^m)_\omega \cong \bigoplus_{m<\omega<2m} k\langle\omega\rangle \cong x^{m+1}k[x]/x^{2m}.$$

Proof. It follows immediately from the isomorphisms (4.6), (4.9) and (4.10). □

4.3. The cyclic homology, the additive higher Chow group, and the regulator map

THEOREM 4.10. *There exists a weight preserving map*

$$\Phi : HC_2(k[x]/x^m, (x)) \to CH_1(A_k(m), 2)_{sup}.$$

Proof. By Corollary 4.9 we have a weight preserving isomorphism

$$HC_2(k[x]/x^m, (x)) \xrightarrow{\simeq} x^{m+1}k[x]/x^{2m}.$$

By Corollary 4.6 we have a weight preserving map

$$x^m k[x]/x^{2m} \to CH_1(A_k(m), 2)_{sup}.$$

By composing above maps and the natural inclusion $x^{m+1}k[x]/x^{2m} \hookrightarrow x^m k[x]/x^{2m}$, we get a weight preserving map

$$\Phi : HC_2(k[x]/x^m, (x)) \to CH_1(A_k(m), 2)_{sup}. \qquad \square$$

COROLLARY 4.11. *We have the following commutative diagram*

$$\begin{array}{ccc} x^m k[x] & \xrightarrow{\Phi'} & CH_1(A_k(m), 2)_{ssup} \\ \downarrow & & \downarrow \\ x^m k[x]/x^{2m} & \xrightarrow[\Phi]{} & CH_1(A_k(m), 2)_{sup}, \end{array}$$

where Φ' is from Remark 4.7, and the map Φ' is a nontrivial homomorphism.

Proof. By definition $\Phi'(x^{m+i})$ is the parametric curve C of the form

$$C : t \mapsto (t, t^{m+i}, 1 - t^{m+i}) \in \mathbf{A}^1 \times \square^2$$

and C satisfies the strong sup modulus condition on y_2. By an easy computation, we have

$$L^2_{m+i}(C) = -\mathrm{res}_{t=0}\, v^*\left(\frac{1 - y_2}{x^{m+i}} \frac{dy_1}{y_1}\right) = -(m+i) \in k\langle m+i\rangle,$$

where $v : \bar{C}^N \to \bar{C}$ is a normalization of its Zariski closure in $\mathbf{A}^1_k \times (\mathbf{P}^1)^2$. $\square$

COROLLARY 4.12. *Let $\tilde{L}^2 = \bigoplus_{m<\omega<2m} \dfrac{-1}{\omega} L^2_\omega$ be a direct sum of modifications of the regulator maps L^n_ω. Then the composed map*

$$x^{m+1}k[x] \xrightarrow{\Phi'} CH_1(A_k(m), 2)_{ssup} \xrightarrow{\tilde{L}^2} \bigoplus_{m<\omega<2m} k\langle\omega\rangle$$

$$\simeq HC_2(k[x]/x^m, (x)) \simeq x^{m+1}k[x]/x^{2m}$$

is the natural quotient map.

5. Appendix: The residue theory

In this section, we summarize some results for the residue theory from [12], [13] and [19]. For details, see *ibids.* In what follows all fields which appear are perfect.

DEFINITION 5.1 ([19, Def. 3.1.1]). Let X be a scheme. A saturated chain of length n is a sequence $\xi = (x_0 > x_1 > \cdots > x_n)$ of points such that x_i is an immediate specialization of x_{i-1}. Denote by $C_n(X)$ the set of all saturated chains of length n. Denote by $C(X) := \bigcup_n C_n(X)$ the set of all saturated chains.

For simplicity, instead of a saturated chain we will simply say a chain.

DEFINITION 5.2 ([19, Def. 3.1.2]). Let $\xi \in C(X)$ be a chain and let $\mathscr{F}$ be a quasi-coherent sheaf. Then we can define the Beilinson completion of $\mathscr{F}$ along ξ ([19, Def. 3.1.2]). We denote by $\mathscr{F}_\xi$ the Beilinson completion of $\mathscr{F}$ along ξ. For any chain $\xi = (x_0, \dots, x_n) \in C(X)$, we shall write $k(\xi) := k(x_0)_\xi = \mathscr{O}_{X,\xi}/(\mathfrak{m}_{x_0})_\xi$ the residue field of Beilinson completion.

Remark 5.3. If $\xi = (x)$ is a chain of length 0, the Beilinson completion $\mathscr{F}_\xi = \mathscr{F}_{(x)}$ coincides with the $\mathfrak{m}_x$-adic completion of $\mathscr{F}_x$. In general, we can calculate the Beilinson completion by an n-fold zig-zag of inverse and direct limits.

DEFINITION 5.4 ([19, Thm. 2.4.3]). Let k be a perfect field and $f : K \to L$ be a morphism of topological local fields and set $n = \dim(f) := \dim L - \dim K$. Then there is a homomorphism

$$\mathrm{Res}_{L/K} = \mathrm{Res}_f : \Omega^{*,sep}_{L/k} \to \Omega^{*-n,sep}_{K/k}$$

of semi-topological differential graded left $\Omega^{*,sep}_{K/k}$-modules of degree $-n$. (For a proof and the definition of $\Omega^{*,sep}_{K/k}$, see [19, Thm. 2.4.3 and Def. 1.5.3].) We call $\mathrm{Res}_{L/K}$ a residue map.

Remark 5.5. If $L = K((t_1, \dots, t_n))$, we can calculate the residue map by

$$\mathrm{Res}_{L/K}\left(\frac{dt_n}{t_n} \wedge \cdots \wedge \frac{dt_1}{t_1}\right) = 1.$$

In general, since any morphism $K \to L$ factors as $K \to K((t)) \to L$ with $K((t)) \to L$ finite, we can calculate the residue by using the natural trace map.

Remark 5.6. More generally, for any morphism $f : A \to B$ of cluster of topological local fields which are reduced (see [19, Def. 2.2.1 and p. 52]), we can define a residue map similarly. (For detail, see [19, Cor. 2.4.20].)

Definition 5.7 ([19, Def. 4.1.3]). Let X be a scheme of finite type over a perfect field k. Let $\xi = (x, \dots, y) \in C(X)$ be a chain of length n and let $\sigma : k(y) \to \mathcal{O}_{X,(y)}$ be a coefficient field. Then there is a natural homomorphism

$$\bar{\sigma} : k(y) \xrightarrow{\sigma} \mathcal{O}_{X,(y)} \to \mathcal{O}_{X,\xi} \twoheadrightarrow k(\xi).$$

This map is a morphism of cluster of topological local fields of dimension n (see [19, Def. 2.2.1]). Define

$$\mathrm{Res}_{\xi,\sigma} : \Omega^*_{k(x)/k} \longrightarrow \Omega^{*,sep}_{k(\xi)/k} \xrightarrow{\mathrm{Res}_{\bar{\sigma}}} \Omega^{*-n}_{k(y)/k}.$$

We say that $\mathrm{Res}_{\xi,\sigma}$ is a residue map.

For simplicity, we will often omit the subscript σ if no confusion arises.

Let X be a variety over a field k. Then we have $\Omega^*_{k(X)} \cong \Omega^*_k \otimes_k \Omega^*_{k(X)/k}$. By using this isomorphism, we define an absolute residue map as follows.

Definition 5.8 ([13, §1.3]). Let X be a d-dimensional variety over a perfect field and let $\xi = (x, \dots, y) \in C(X)$ be a chain of length r. We define an absolute residue map of degree $-r$

$$\mathrm{Res}_\xi : \Omega^n_{k(x)} \to \Omega^{n-r}_{k(y)}$$

as follows. For $n \geq d$, we define Res_ξ as a composite of

$$\Omega^n_{k(x)} \longrightarrow \Omega^{n-d}_k \otimes_k \Omega^d_{k(x)/k} \xrightarrow{1 \otimes \mathrm{Res}_\xi} \Omega^{n-d}_{k(y)} \otimes_k \Omega^{d-r}_{k(y)/k} \hookrightarrow \Omega^{n-r}_{k(y)}.$$

For $n < d$, we define Res_ξ as a composite of

$$\Omega^n_{k(x)} \longrightarrow \Omega^n_{k(x)/k} \xrightarrow{\mathrm{Res}_\xi} \Omega^{n-r}_{k(y)/k} \hookrightarrow \Omega^{n-r}_{k(y)}.$$

Under appropriate assumptions, the residue map satisfies the transitivity and reciprocity (Theorem 5.9, Theorem 5.10). Clearly the absolute residue map inherits these properties.

Theorem 5.9 ([19, Cor. 4.1.16]). *Let $\xi = (x, \dots, y)$, $\eta = (y, \dots, z) \in C(X)$ be chains and let σ, τ be coefficient fields of y, z respectively. We assume σ and τ are compatible coefficient fields for η* ([19, p. 87]). *Then*

$$\mathrm{Res}_{\xi \vee \eta} = \mathrm{Res}_\eta \circ \mathrm{Res}_\xi : \Omega^*_{k(x)} \to \Omega^*_{k(z)},$$

where $\xi \vee \eta = (x, \dots, y, \dots, z)$ is the concatenation of chains.

Theorem 5.10 ([19, Thm. 4.2.15]). (1) *Let W be a surface and let $p \in W$ be a closed point. Then*

$$\sum_{W > ? > p} \mathrm{Res}_{W > ? > p} = 0.$$

(2) *Let C be a proper curve. Then*

$$\sum_{C>?} \mathrm{Res}_{C>?} = 0.$$

Theorem 5.11 ([10, Thm. 2]). *Let X, Y be n-dimensional varieties over a perfect field k. Let $f: X \to Y$ be a surjective birational proper morphism. Let $\xi \in C(Y)$ be a chain of length n. Then $K(Y) = K(X)$ and*

$$\mathrm{Res}_{\xi}^{Y} = \sum_{f:\eta\to\xi} \mathrm{Res}_{\eta}^{X} : \Omega^{*}_{k(X)} \to \Omega^{*-n}_{k}.$$

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