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UNIQUENESS THEOREM FOR MEROMORPHIC MAPPINGS WITH MULTIPLE VALUES

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Abstract

In this article, we will prove a uniqueness theorem for meromorphic mappings into complex projective space $\mathbf{P}^n(\mathbf{C})$ with different multiple values and a general condition on the intersections of the inverse images of these hyperplanes.

1. Introduction

In 1975, H. Fujimoto [5] proved a uniqueness theorem for linearly nondegenerate meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ which have the same inverse images of $3n+2$ hyperplanes in general position counted with multiplicities.

In 1983, L. Smiley [8] obtained a uniqueness theorem for meromorphic mappings which share $3n+2$ hyperplanes in $\mathbf{P}^n(\mathbf{C})$ in general position without counting multiplicities (i.e., they have the same inverse images of $3n+2$ hyperplanes and are identical on these inverse images) and satisfy an additional condition "codimension of the intersections of inverse images of two different hyperplanes are at least two".

The unicity problem of meromorphic mappings with truncated multiplicities has been extended and deepened by contribution of many authors. In [1], [2], [3], [7] and [9], the authors improved the result of L. Smiley by reducing the number of involving hyperplanes. In order to state some of them, we need the following.

Take a meromorphic mapping f of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ which is linearly nondegenerate over $\mathbf{C}^m$ such that for positive integers k, d $(1 \le d \le n)$ and q hyperplanes $H_1, \dots, H_q$ in $\mathbf{P}^n(\mathbf{C})$ in general position with

$$(1.1) \qquad \dim f^{-1}\left(\bigcap_{j=1}^{d+1} H_{i_j}\right) \le m-2 \quad (1 \le i_1 < \cdots < i_{d+1} \le q).$$

Let $\mathscr{F}(f, \{H_i\}_{i=1}^q, d, k)$ be the set of all linearly nondegenerate over $\mathbf{C}^m$ meromorphic maps $g : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ satisfying the conditions

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(a) $\min(\nu_{(f,H_j)}, k) = \min(\nu_{(g,H_j)}, k)$ $(1 \le j \le q)$,
(b) $f(z) = g(z)$ on $\bigcup_{j=1}^{q} f^{-1}(H_j)$.
Denote by $\#S$ the cardinality of the set S.

THEOREM A (Z. Chen - Q. Yan [3]). $\#\mathscr{F}(f, \{H_i\}_{i=1}^{2n+3}, 1, 1) = 1$.

In 2010, T. B. Cao and H. X. Yi proved a uniqueness theorem for linearly nondegenerate meromorphic mappings with different multiple values as follows.

THEOREM B (T. B. Cao and H. X. Yi [2]). *Let f and g be two linearly nondegenerate meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ and let $H_1, \ldots, H_q$ be q $(q \ge 2n)$ hyperplanes of $\mathbf{P}^n(\mathbf{C})$ in general position satisfying the condition* (1.1) *with $d = 1$. Take m_i $(1 \le i \le q)$ be positive integers or ∞ such that $m_1 \ge m_2 \ge \cdots \ge m_q \ge n$ with*
(a) $\min(\nu_{(f,H_j),\le m_j}, 1) = \min(\nu_{(g,H_j),\le m_j}, 1)$ $(1 \le j \le q)$,
(b) $f(z) = g(z)$ on $\bigcup_{j=1}^{q} \{z \in \mathbf{C}^m : 0 < \nu_{(f,H_j)}(z) \le m_j\}$.
If $\sum_{j=1}^{q} \dfrac{m_j}{m_j+1} > \dfrac{nq-q+n+1}{n} - \dfrac{4n-4}{q+2n-2} + \left(\dfrac{1}{m_1+1} + \dfrac{1}{m_2+1}\right)$ *then* $f \equiv g$.

In this theorem, numbers m_i are called the mutiple values of the sharing conditions (a) and (b) of f and g with respect to hyperplanes. However, in the all results of Z. Chen - Q. Yan, T. B. Cao - H. X. Yi and mentioned authors on unicity problem with truncated multiplicity, the case where $d = 1$ in the condition (1.1) is considered. Moreover their techniques do not work for case $d > 1$. In 2012, H. H. Giang, L. N. Quynh and S. D. Quang [4] introduced new techniques to treat the case $d \ge 1$. However, they only considered the case where the mappings f and g share all hyperplanes with the same multiple values. Thus, our purpose of this paper is to prove a uniqueness theorem which generalizes Theorem B by considering the general case where $d \ge 1$. Namely, we will prove a theorem as follows.

THEOREM 1.2. *Let f and g be two linearly nondegenerate meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. Let d $(1 \le d \le n)$ be positive integer and $H_1, \ldots, H_q$ be q $(q = (n+1)d + n + 2)$ hyperplanes of $\mathbf{P}^n(\mathbf{C})$ in general position. Let k_i $(1 \le i \le q)$ be positive integers or $+\infty$ such that*

$$\sum_{i=1}^{q} \frac{1}{k_i+1} < \frac{2n+1+d(n+1)}{2n(d+1)+1}.$$

Assume that
(a) $\dim f^{-1}(\bigcap_{j=1}^{d+1} H_{i_j}) \le m-2$ $(1 \le i_1 < \cdots < i_{d+1} \le q)$,
(b) $\min(\nu_{(f,H_j),\le k_i}, 1) = \min(\nu_{(g,H_j),\le k_i}, 1)$ $(1 \le j \le q)$,
(c) $f(z) = g(z)$ *on* $\bigcup_{j=1}^{q} \operatorname{Supp}\{z \in \mathbf{C}^m : \nu_{(f,H_j),\le k_i}(z)\}$.
Then $f \equiv g$.

2. Basic notions in Nevanlinna theory

The following definition are due to [4, 7, 9].

2.1. We set $\|z\| = (|z_1|^2 + \cdots + |z_m|^2)^{1/2}$ for $z = (z_1, \ldots, z_m) \in \mathbf{C}^m$ and define

$$B(r) := \{z \in \mathbf{C}^m : \|z\| < r\}, \quad S(r) := \{z \in \mathbf{C}^m : \|z\| = r\} \quad (0 < r < \infty).$$

Define

$$\sigma(z) := (dd^c\|z\|^2)^{m-1} \quad \text{and}$$
$$\eta(z) := d^c \log\|z\|^2 \wedge (dd^c \log\|z\|^2)^{m-1} \quad \text{on } \mathbf{C}^m \backslash \{0\}.$$

2.2. Let F be a nonzero holomorphic function on a domain Ω in $\mathbf{C}^m$. For an m-tuple $\alpha = (\alpha_1, \ldots, \alpha_m)$ of nonnegative integers, we set $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\mathscr{D}^\alpha F = \dfrac{\partial^{|\alpha|} F}{\partial^{\alpha_1} z_1 \cdots \partial^{\alpha_m} z_m}$. We define the map $v_F : \Omega \to \mathbf{Z}$ by

$$v_F(z) := \max\{l : \mathscr{D}^\alpha F(z) = 0 \text{ for all } \alpha \text{ with } |\alpha| < l\} \quad (z \in \Omega).$$

We mean by a divisor on a domain Ω in $\mathbf{C}^m$ a map $v : \Omega \to \mathbf{Z}$ such that, for each $a \in \Omega$, there are nonzero holomorphic functions F and G on a connected neighborhood $U \subset \Omega$ of a such that $v(z) = v_F(z) - v_G(z)$ for each $z \in U$ outside an analytic set of dimension $\leq m - 2$. Two divisors are regarded as the same if they are identical outside an analytic set of dimension $\leq m - 2$. For a divisor v on Ω we set $|v| := \overline{\{z : v(z) \neq 0\}}$, which is a purely $(m-1)$-dimensional analytic subset of Ω or empty set.

Take a nonzero meromorphic function φ on a domain Ω in $\mathbf{C}^m$. For each $a \in \Omega$, we choose nonzero holomorphic functions F and G on a neighborhood $U \subset \Omega$ such that $\varphi = \dfrac{F}{G}$ on U and $\dim(F^{-1}(0) \cap G^{-1}(0)) \leq m - 2$, and we define the divisors v_φ, v_φ^∞ by $v_\varphi := v_F$, $v_\varphi^\infty := v_G$, which are independent of choices of F and G and so globally well-defined on Ω.

2.3. For a divisor v on $\mathbf{C}^m$ and for positive integers k, M or $M = \infty$, we define the counting function of v by

$$v^{(M)}(z) = \min\{M, v(z)\},$$
$$v_{\leq k}^{(M)}(z) = \begin{cases} 0 & \text{if } v(z) > k, \\ v^{(M)}(z) & \text{if } v(z) \leq k, \end{cases}$$
$$v_{>k}^{(M)}(z) = \begin{cases} 0 & \text{if } v(z) \leq k, \\ v^{(M)}(z) & \text{if } v(z) > k, \end{cases}$$

$$n(t) = \begin{cases} \int_{B(t)} \nu(z)\sigma & \text{if } m \geq 2 \\ \sum\limits_{|z| \leq t} \nu(z) & \text{if } m = 1. \end{cases}$$

Similarly, we define $n^{[M]}(t)$, $n^{[M]}_{\leq k}(t)$, $n^{[M]}_{>k}(t)$.

Define

$$N(r, \nu) = \int_1^r \frac{n(t)}{t^{2m-1}}\, dt \quad (1 < r < \infty).$$

Similarly, we define $N(r, \nu^{[M]})$, $N(r, \nu^{[M]}_{\leq k})$, $N(r, \nu^{[M]}_{>k})$ and denote them by $N^{[M]}(r, \nu)$, $N^{[M]}_{\leq k}(r, \nu)$, $N^{[M]}_{>k}(r, \nu)$ respectively.

Let $\varphi : \mathbf{C}^m \to \mathbf{C}$ be a meromorphic function. Define

$$N_\varphi(r) = N(r, \nu_\varphi), \quad N^{[M]}_\varphi(r) = N^{[M]}(r, \nu_\varphi),$$
$$N^{[M]}_{\varphi, \leq k}(r) = N^{[M]}_{\leq k}(r, \nu_\varphi), \quad N^{[M]}_{\varphi, >k}(r) = N^{[M]}_{>k}(r, \nu_\varphi).$$

For brevity, we will omit the character $^{[M]}$ if $M = \infty$.

2.4. Let $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a meromorphic mapping. For an arbitrarily fixed homogeneous coordinate $(w_0 : \cdots : w_n)$ on $\mathbf{P}^n(\mathbf{C})$, we take a reduced representation $f = (f_0 : \cdots : f_n)$, where each f_i is a holomorphic function on $\mathbf{C}^m$ and $f(z) = (f_0(z) : \cdots : f_n(z))$ outside the analytic set $I(f) = \{f_0 = \cdots = f_n = 0\}$ of codimension ≥ 2. Set $\|f\| = (|f_0|^2 + \cdots + |f_n|^2)^{1/2}$.

The characteristic function of f is defined by

$$T_f(r) = \int_{S(r)} \log\|f\|\eta - \int_{S(1)} \log\|f\|\eta.$$

Let H be a hyperplane in $\mathbf{P}^n(\mathbf{C})$ given by $H = \{a_0\omega_0 + \cdots + a_n\omega_n = 0\}$, where $a := (a_0, \dots, a_n) \neq (0, \dots, 0)$. We set $(f, H) = \sum_{i=0}^n a_i f_i$. We define the corresponding divisor f^*H by $f^*H(z) = \nu_{(f,H)}(z)$ $(z \in \mathbf{C}^m)$, which is independent of the choice of the reduced representation of f. The proximity function of f with respect to H is defined by

$$m_{f,H}(r) = \int_{S(r)} \log \frac{\|f\| \cdot \|H\|}{|(f, H)|}\eta - \int_{S(1)} \log \frac{\|f\| \cdot \|H\|}{|(f, H)|}\eta,$$

where $\|H\| = (\sum_{i=0}^n |a_i|^2)^{1/2}$.

2.5. Let φ be a nonzero meromorphic function on $\mathbf{C}^m$, which is occasionally regarded as a meromorphic map into $\mathbf{P}^1(\mathbf{C})$. The proximity function of φ is defined by

$$m(r, \varphi) := \int_{S(r)} \log^+|\varphi|\eta,$$

where $\log^+ t = \max\{0, \log t\}$ for $t > 0$. The Nevanlinna characteristic function of φ is defined by

$$T(r, \varphi) = N_{1/\varphi}(r) + m(r, \varphi).$$

Then

$$T_\varphi(r) = T(r, \varphi) + O(1).$$

The meromorphic function φ is said to be "small" with respect to f if $\| \ T(r, \varphi) = o(T_f(r))$.

2.6. Jensen formula. *Let φ be a nonzero meromorphic function in $\mathbf{C}^m$. The Jensen formula is stated as follows.*

$$N_\varphi(r) - N_{1/\varphi}(r) = \int_{S(r)} \log|\varphi|\eta - \int_{S(1)} \log|\varphi|\eta.$$

2.7. As usual, by the notation "$\| \ P$", we mean the assertion P holds for all $r \in [0, \infty)$ excluding a Borel subset E of the interval $[0, \infty)$ with $\int_E dr < \infty$.

The following play essential roles in Nevanlinna theory (see [6]).

2.8. The first main theorem. *Let $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a meromorphic mapping and let H be a hyperplane in $\mathbf{P}^n(\mathbf{C})$ such that $f(\mathbf{C}^m) \not\subset H$. Then*

$$N_{(f,H)}(r) + m_{f,H}(r) = T_f(r) \quad (r > 1).$$

2.9. The second main theorem. *Let $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a linearly nondegenerate meromorphic mapping and $H_1, \dots, H_q$ be q hyperplanes in general position in $\mathbf{P}^n(\mathbf{C})$. Then*

$$\| \ (q - n - 1) T_f(r) \leq \sum_{i=1}^{q} N_{(f,H_i)}^{[n]}(r) + o(T_f(r)).$$

3. Proof of Theorem 1.2

In order to prove Theorem 1.2, we need the following.

LEMMA 3.1 (Lemma 2.2 [2]). *Let f be a linearly nondegenerate meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. Let H be a hyperplane in $\mathbf{P}^n(\mathbf{C})$ in general position and $k(\geq n)$ be a positive integer. Then*

$$N_{(f,H)}^{[n]}(r) \leq n\left(1 - \frac{n}{k+1}\right) N_{(f,H),\leq k}^{[1]}(r) + \frac{n}{k+1} N_{(f,H)}(r)$$

and

$$N_{(f,H)}^{[n]}(r) \leq n\left(1 - \frac{n}{k+1}\right) N_{(f,H),\leq k}^{[1]}(r) + \frac{n}{k+1} T_f(r) + o(T_f(r)).$$

LEMMA 3.2. *Let f and g be nonconstant meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. Let $\{H_i\}_{i=1}^q$ $(q \geq n+2)$ be hyperplanes in $\mathbf{P}^n(\mathbf{C})$ in general position. Asumme that*

$$\min\{v_{(f,H_i),\leq k_i}, 1\} = \min\{v_{(g,H_i),\leq k_i}, 1\}, \quad \text{for all } 1 \leq j \leq q.$$

If $\sum_{i=1}^q \frac{1}{k_i+1} < \frac{q-n-1}{n}$, *then* $\| \; T_g(r) = O(T_f(r))$ *and* $\| \; T_f(r) = O(T_g(r))$.

Proof. By the Second Main Theorem and Lemma 3.1, we have

$$\begin{aligned}
\| \; (q-n-1)T_g(r) &\leq \sum_{i=1}^q N^{[n]}_{(g,H_i)}(r) + o(T_g(r)) \\
&\leq \sum_{i=1}^q \left(n\left(1 - \frac{n}{k_i+1}\right) N^{[1]}_{(g,H),\leq k_i}(r) + \frac{n}{k_i+1} T_g(r) \right) + o(T_g(r)) \\
&\leq \sum_{i=1}^q nN^{[1]}_{(f,H),\leq k_i}(r) + \frac{n}{k_i+1} T_g(r) + o(T_g(r)) \\
&\leq qnT_f(r) + n\sum_{i=1}^q \frac{1}{k_i+1} T_g(r) + o(T_g(r)).
\end{aligned}$$

Thus

$$\| \left(q-n-1-n\sum_{i=1}^q \frac{1}{k_i+1} \right) T_g(r) \leq qnT_f(r) + o(T_g(r)).$$

Hence $\| \; T_g(r) = O(T_f(r))$. Similarly, we get $\| \; T_f(r) = O(T_g(r))$. □

Proof of Theorem 1.2. Assume that $\sum_{i=1}^q \frac{1}{k_i+1} < \frac{2n+1+d(n+1)}{2n(d+1)+1}$. Suppose contrarily that $f \not\equiv g$. By changing indices if necessary, we may assume that

$$\underbrace{\frac{(f,H_1)}{(g,H_1)} \equiv \frac{(f,H_2)}{(g,H_2)} \equiv \cdots \equiv \frac{(f,H_{v_1})}{(g,H_{v_1})}}_{\text{group 1}} \not\equiv \underbrace{\frac{(f,H_{v_1+1})}{(g,H_{v_1+1}} \equiv \cdots \equiv \frac{(f,H_{v_2})}{(g,H_{v_2})}}_{\text{group 2}}$$

$$\not\equiv \underbrace{\frac{(f,H_{v_2+1})}{(g,H_{v_2+1})} \equiv \cdots \equiv \frac{(f,H_{v_3})}{(g,H_{v_3})}}_{\text{group 3}} \not\equiv \cdots \not\equiv \underbrace{\frac{(f,H_{v_{s-1}+1})}{(g,H_{v_{s-1}+1})} \equiv \cdots \equiv \frac{(f,H_{v_s})}{(g,H_{v_s})}}_{\text{group } s},$$

where $v_s = q$.

For each $1 \le i \le q$, we set

$$\sigma(i) = \begin{cases} i+n & \text{if } i+n \le q, \\ i+n-q & \text{if } i+n > q \end{cases}$$

and

$$P_i = (f, H_i)(g, H_{\sigma(i)}) - (g, H_i)(f, H_{\sigma(i)}).$$

Since $f \not\equiv g$, the number of elements of each group is at most n. Then $\dfrac{(f, H_i)}{(g, H_i)}$ and $\dfrac{(f, H_{\sigma(i)})}{(g, H_{\sigma(i)})}$ belong to distinct groups. Therefore $P_i \not\equiv 0$ $(1 \le i \le q)$. We set

$$P = \prod_{i=1}^{q} P_i \not\equiv 0$$

and

$$S = \bigcup_{1 \le i_1 < \cdots < i_{d+1} \le q} f^{-1}\left(\bigcap_{j=1}^{d+1} H_{i_j}\right).$$

Then S is an analytic set of codimension at most 2. We set

$$\begin{aligned} \mathscr{V} &= \{z \in \mathbf{C}^m : \nu^0_{(f, H_{i_1}), \le k_{i_1}} \cdot \nu^0_{(f, H_{i_2}), \le k_{i_2}} \cdots \nu^0_{(f, H_{i_t}), \le k_{i_t}} > 0\} \\ &\quad (1 \le i_1 < \cdots < i_t \le q, t < d) \\ \mathscr{D} &= \bigcup_{j \notin \{i_1, \dots i_t\}} \{z \in \mathbf{C}^m : \nu^0_{(f, H_j), \le k_j} > 0\} \end{aligned}$$

Fix a point $z \notin I(f) \cup I(g) \cup S$. We assume that $z \in \mathscr{V}$. For an index $i \in \{1, \dots, q\}$, we distinguish the following four cases:

Case 1. $i, \sigma(i) \notin \{i_1, \dots, i_t\}$. Then z is a zero point of P_i with multiplicity at least 1, since $f(z) = g(z)$. We denote $v(z)$ the number of indices i in this case. It is easy to see that $v(z) \ge q - 2t = (n+1)d + n + 2 - 2t$.

Case 2. $i \in \{i_1, \dots, i_t\}$ and $\sigma(i) \notin \{i_1, \dots, i_t\}$. Then z is a zero point of P_i with multiplicity at least $\min\{\nu_{(f, H_i), \le k_i}, \nu_{(g, H_i), \le k_i}\}$.

Case 3. $\sigma(i) \in \{i_1, \dots, i_t\}$ and $i \notin \{i_1, \dots, i_t\}$. Then z is a zero point of P_i with multiplicity at least $\min\{\nu_{(f, H_{\sigma(i)}), \le k_i}, \nu_{(g, H_{\sigma(i)}), \le k_i}\}$.

Case 4. $i, \sigma(i) \in \{i_1, \dots, i_t\}$. Then z is a zero point of P_i with multiplicity at least $\min\{\nu_{(f, H_i), \le k_i}, \nu_{(g, H_i), \le k_i}\} + \min\{\nu_{(f, H_{\sigma(i)}), \le k_i}, \nu_{(g, H_{\sigma(i)}), \le k_i}\}$.

Therefore, from the above four cases, it follows that

$$\begin{aligned}
\nu_P(z) &\geq 2\sum_{j=1}^{t}\min\{\nu_{(f,H_{i_j}),\leq k_{i_j}}(z),\nu_{(g,H_{i_j}),\leq k_{i_j}}(z)\}+v(z)\\
&\geq \left(2-\frac{q-1}{(n+1)d}\right)\sum_{j=1}^{t}\min\{\nu_{(f,H_{i_j}),\leq k_{i_j}}(z),\nu_{(g,H_{i_j}),\leq k_{i_j}}(z)\}\\
&\quad+\frac{q-1}{(n+1)d}\sum_{j=1}^{t}\min\{\nu_{(f,H_{i_j}),\leq k_{i_j}}(z),\nu_{(g,H_{i_j}),\leq k_{i_j}}(z)\}+v(z)\\
&\geq \left(2-\frac{q-1}{(n+1)d}\right)\sum_{j=1}^{t}\min\{\nu_{(f,H_{i_j}),\leq k_{i_j}}(z),\nu_{(g,H_{i_j}),\leq k_{i_j}}(z)\}\\
&\quad+\frac{q-1}{(n+1)d}\sum_{j=1}^{t}(\min\{\nu_{(f,H_{i_j}),\leq k_{i_j}}(z),n\}+\min\{\nu_{(g,H_{i_j}),\leq k_{i_j}}(z),n\}-n)+v(z)\\
&\geq \frac{d-1}{d}\sum_{j=1}^{t}\min\{\nu_{(f,H_{i_j}),\leq k_{i_j}}(z),\nu_{(g,H_{i_j}),\leq k_{i_j}}(z)\}\\
&\quad+\frac{d+1}{d}\sum_{j=1}^{t}(\min\{\nu_{(f,H_{i_j}),\leq k_{i_j}}(z),n\}+\min\{\nu_{(g,H_{i_j}),\leq k_{i_j}}(z),n\}-n)+v(z)\\
&\geq \frac{d-1}{d}t+\frac{d+1}{d}\sum_{j=1}^{t}(\min\{\nu_{(f,H_{i_j}),\leq k_{i_j}}(z),n\}+\min\{\nu_{(g,H_{i_j}),\leq k_{i_j}}(z),n\})\\
&\quad-\frac{d+1}{d}nt+(n+1)d+n+2-2t\\
&\geq \frac{d+1}{d}\sum_{j=1}^{t}(\min\{\nu_{(f,H_{i_j}),\leq k_{i_j}}(z),n\}+\min\{\nu_{(g,H_{i_j}),\leq k_{i_j}}(z),n\})\\
&\quad+d-1-n(d+1)+(n+1)d+n+2-2d\\
&=\frac{d+1}{d}\sum_{j=1}^{t}(\min\{\nu_{(f,H_{i_j}),\leq k_{i_j}}(z),n\}+\min\{\nu_{(g,H_{i_j}),\leq k_{i_j}}(z),n\})+1\\
&\geq \left(\frac{d+1}{d}+\frac{1}{2nd}\right)\sum_{i=1}^{q}(\min\{\nu_{(f,H_i),\leq k_{i_j}}(z),n\}+\min\{\nu_{(g,H_i),\leq k_{i_j}}(z),n\}),
\end{aligned}$$

for all z outside the analytic set $I(f)\cup I(g)\cup S$.

Set $T(r)=T_f(r)+T_g(r)$. Integrating both sides of the above inequality and using the Second Main Theorem, we have

$$\begin{aligned}
N_P(r) &\geq \left(\frac{d+1}{d}+\frac{1}{2nd}\right)\sum_{i=1}^{q}(N^{(n)}_{(f,H_i),\leq k_{i_j}}(r)+N^{(n)}_{(g,H_i),\leq k_{i_j}}(r))\\
&= \left(\frac{d+1}{d}+\frac{1}{2nd}\right)\sum_{i=1}^{q}(N^{(n)}_{(f,H_i)}(r)-N^{(n)}_{(f,H_i),>k_{i_j}}(r)+N^{(n)}_{(g,H_i)}(r)-N^{(n)}_{(g,H_i),\leq k_{i_j}}(r))\\
&\geq \left(\frac{d+1}{d}+\frac{1}{2nd}\right)\sum_{i=1}^{q}\left(N^{(n)}_{(f,H_i)}(r)+N^{(n)}_{(g,H_i)}(r)-\frac{n}{k_i+1}(T_f(r)+T_g(r))\right)\\
&\geq \left(\frac{d+1}{d}+\frac{1}{2nd}\right)\left(((n+1)d+1)T(r)-\sum_{i=1}^{q}\frac{n}{k_i+1}T(r)\right)+o(T(r))\\
&= \Bigg((n+1)d+n+2+\frac{1}{d}+\frac{1}{2nd}+\frac{n+1}{2n}\\
&\qquad -\frac{2n(d+1)+1}{2nd}\sum_{i=1}^{q}\frac{n}{k_i+1}\Bigg)T(r)+o(T(r)).
\end{aligned}$$

On the other hand, by the Jensen formula, we have

$$\begin{aligned}
N_P(r) &= \int_{S(r)}\log|P|\eta+O(1)=\sum_{i=1}^{q}\int_{S(r)}\log|P_i|\eta+O(1)\\
&\leq \sum_{i=1}^{q}\int_{S(r)}\log(|(f,H_i)|^2+|(f,H_{\sigma(i)})|^2)^{1/2}\eta\\
&\quad+\sum_{i=1}^{q}\int_{S(r)}\log(|(g,H_i)|^2+|(g,H_{\sigma(i)})|^2)^{1/2}\eta+O(1)\\
&\leq q(T_f(r)+T_g(r))+o(T_f(r)+T_g(r))=qT(r)+o(T(r)).
\end{aligned}$$

This implies that

$$\begin{aligned}
qT(r) &\geq \left((n+1)d+n+2+\frac{1}{d}+\frac{1}{2nd}+\frac{n+1}{2n}-\frac{2n(d+1)+1}{2d}\sum_{i=1}^{q}\frac{1}{k_i+1}\right)T(r)\\
&\quad+o(T(r)).
\end{aligned}$$

Letting $r\to\infty$, we get

$$q\geq\left((n+1)d+n+2+\frac{1}{d}+\frac{1}{2nd}+\frac{n+1}{2n}-\frac{2n(d+1)+1}{2d}\sum_{i=1}^{q}\frac{1}{k_i+1}\right),$$

i.e., $\dfrac{2n+1+d(n+1)}{2n(d+1)+1}\leq\sum_{i=1}^{q}\dfrac{1}{k_i+1}$. This is a contradiction.

Then the supposition is impossible. Hence the theorem is proved. □

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