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TWISTED ALEXANDER POLYNOMIALS OF GENUS ONE TWO-BRIDGE KNOTS

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Abstract

Morifuji [14] computed the twisted Alexander polynomial of twist knots for nonabelian representations. In this paper we compute the twisted Alexander polynomial and Reidemeister torsion of genus one two-bridge knots, a class of knots which includes twist knots. As an application, we give a formula for the Reidemeister torsion of the 3-manifold obtained by $\frac{1}{q}$-Dehn surgery on a genus one two-bridge knot.

1. Introduction

The twisted Alexander polynomial, a generalization of the Alexander polynomial, was introduced by Lin [10] for knots in S^3 and by Wada [19] for finitely presented groups. It was interpreted in terms of Reidemeister torsion by Kitano [9] and Kirk-Livingston [5]. Twisted Alexander polynomials have been extensively studied in the past ten years by many authors, see the survey papers [2, 13] and references therein.

In [14] Morifuji computed the twisted Alexander polynomial of twist knots for nonabelian representations. In this paper we will generalize his result to genus one two-bridge knots. In a related direction, Kitano [6] gave a formula for the Reidemeister torsion of the 3-manifold obtained by $\frac{1}{q}$-Dehn surgery on the figure eight knot. In [17] we generalized his result to twist knots. In this paper we will also compute the Reidemeister torsion of the 3-manifold obtained by $\frac{1}{q}$-Dehn surgery on a genus one two-bridge knot.

Let $J(k,l)$ be the knot/link in Figure 1, where k, l denote the numbers of half twists in the boxes. Positive (resp. negative) numbers correspond to right-handed (resp. left-handed) twists. Note that $J(k,l)$ is a knot if and only if kl is even. It is known that the set of all genus one two-bridge knots is the same as

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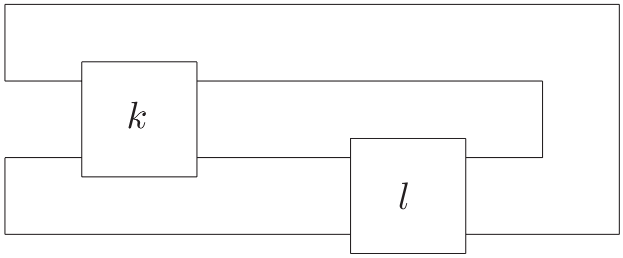

FIGURE 1. The knot/link $J(k,l)$.

the set of all the knots $J(2m,2n)$ with $mn \neq 0$, see e.g. [1]. The knots $J(2,2n)$ are known as twist knots. For more information on $J(k,l)$, see [3].

From now on we fix $K = J(2m,2n)$ with $mn \neq 0$. Let $X_K = S^3 \backslash K$ be the complement of K in S^3. The knot group of K, which is the fundamental group of X_K, has a presentation $\pi_1(X_K) = \langle a, b \mid w^n a = b w^n \rangle$ where a, b are meridians and $w = (ba^{-1})^m (b^{-1}a)^m$. A representation $\rho : \pi_1(X_K) \to SL_2(\mathbf{C})$ is called nonabelian if the image of ρ is a nonabelian subgroup of $SL_2(\mathbf{C})$. Suppose $\rho : \pi_1(X_K) \to SL_2(\mathbf{C})$ is a nonabelian representation. Up to conjugation, we may assume that

$$\rho(a) = \begin{bmatrix} s & 1 \\ 0 & s^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} s & 0 \\ 2-y & s^{-1} \end{bmatrix}$$

where $s \neq 0$ and $y \neq 2$ satisfy $\rho(w^n a) = \rho(b w^n)$. By [16] this matrix equation is equivalent to a single equation $\phi_K(s,y) = 0$, called the Riley equation of K. We also call $\phi_K(s,y) \in \mathbf{C}[s^{\pm 1}, y]$ the Riley polynomial of K. It will be computed explicitly in Section 2. Note that $y = \operatorname{tr} \rho(ab^{-1})$.

Let $S_k(v)$ be the Chebychev polynomials of the second kind defined by $S_0(v) = 1$, $S_1(v) = v$ and $S_k(v) = v S_{k-1}(v) - S_{k-2}(v)$ for all integers k.

Let $x := \operatorname{tr} \rho(a) = s + s^{-1}$ and $z := \operatorname{tr} \rho(w) = 2 + (y-2)(y+2-x^2) S_{m-1}^2(y)$.

THEOREM 1. *Let $K = J(2m,2n)$ with $mn \neq 0$. Suppose $\rho : \pi_1(X_K) \to SL_2(\mathbf{C})$ is a nonabelian representation. Then the twisted Alexander polynomial of K is given by*

$$\begin{aligned}\Delta_{K,\rho}(t) = (t^2 + 1 - tx)\left(\frac{S_m(y) - S_{m-2}(y) - 2}{y-2}\right)\left(\frac{S_n(z) - S_{n-2}(z) - 2}{z-2}\right) \\ + tx S_{m-1}(y) S_{n-1}(z).\end{aligned}$$

THEOREM 2. *Let $K = J(2m,2n)$ with $mn \neq 0$. Suppose $\rho : \pi_1(X_K) \to SL_2(\mathbf{C})$ is a nonabelian representation. If $x \neq 2$ then the Reidemeister torsion of K is given by*

$$\tau_\rho(K) = (2-x)\left(\frac{S_m(y) - S_{m-2}(y) - 2}{y-2}\right)\left(\frac{S_n(z) - S_{n-2}(z) - 2}{z-2}\right) + x S_{m-1}(y) S_{n-1}(z).$$

Now let M be the 3-manifold obtained by $\frac{1}{q}$-surgery on the genus one two-bridge knot $J(2m, 2n)$. The fundamental group $\pi_1(M)$ has a presentation

$$\pi_1(M) = \langle a, b \mid w^n a = b w^n, a\lambda^q = 1 \rangle,$$

where λ is the canonical longitude corresponding to the meridian $\mu = a$.

THEOREM 3. *Let $K = J(2m, 2n)$ with $mn \neq 0$. Suppose $\rho : \pi_1(X_K) \to SL_2(\mathbf{C})$ is a nonabelian representation which extends to a representation $\rho : \pi_1(M) \to SL_2(\mathbf{C})$. If $x \notin \{0, 2\}$ then the Reidemeister torsion of M is given by*

$$\tau_\rho(M) = \left\{ (2-x)\left(\frac{S_m(y) - S_{m-2}(y) - 2}{y-2}\right)\left(\frac{S_n(z) - S_{n-2}(z) - 2}{z-2}\right) + x S_{m-1}(y) S_{n-1}(z) \right\} \left(\frac{4 - x^2 + (y+2-x^2)(y-2)S_{m-1}^2(y)}{x^2 (y-2)^2 S_{m-1}^2(y)}\right).$$

Remark 1.1. (1) Theorem 1 generalizes the formula for the twisted Alexander polynomial of twist knots by Morifuji [14].

(2) Theorem 3 generalizes the formulas for the Reidemeister torsion of the 3-manifold obtained by $\frac{1}{q}$-surgery on the figure eight knot by Kitano [6] and on twist knots by the author [17].

The paper is organized as follows. In Section 2 we give a formula for the Riley polynomial of a genus one two-bridge knot, and compute the trace of a canonical longitude. In Section 3 we review the twisted Alexander polynomial and Reidemeister torsion of a knot. We prove Theorems 1, 2 and 3 in Section 4.

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2. Nonabelian representations

In this section we give a formula for the Riley polynomial of a genus one two-bridge knot. We also compute the trace of a canonical longitude.

2.1. Chebyshev polynomials. Recall that $S_k(v)$ are the Chebychev polynomials defined by $S_0(v) = 1$, $S_1(v) = v$ and $S_k(v) = vS_{k-1}(v) - S_{k-2}(v)$ for all integers k. The following lemma is elementary. We will use it many times without referring to it.

LEMMA 2.1. *We have $S_k^2(v) - vS_k(v)S_{k-1}(v) + S_{k-1}^2(v) = 1$.*

Let $P_k(v) := \sum_{i=0}^{k} S_i(v)$. The next two lemmas are proved in [17].

LEMMA 2.2. *We have* $P_k(v) = \dfrac{S_{k+1}(v) - S_k(v) - 1}{v - 2}$.

LEMMA 2.3. *Suppose* $V = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbf{C})$. *Then*

$$V^k = \begin{bmatrix} S_k(v) - dS_{k-1}(v) & bS_{k-1}(v) \\ cS_{k-1}(v) & S_k(v) - aS_{k-1}(v) \end{bmatrix}, \tag{2.1}$$

$$\sum_{i=0}^{k} V^i = \begin{bmatrix} P_k(v) - dP_{k-1}(v) & bP_{k-1}(v) \\ cP_{k-1}(v) & P_k(v) - aP_{k-1}(v) \end{bmatrix}, \tag{2.2}$$

where $v := \operatorname{tr} V = a + d$. *Moreover, we have*

$$\det\left(\sum_{i=0}^{k} V^i\right) = \frac{S_{k+1}(v) - S_{k-1}(v) - 2}{v - 2}. \tag{2.3}$$

2.2. The Riley polynomial. Recall that $K = J(2m, 2n)$ with $mn \neq 0$. The knot group of K has a presentation $\pi_1(X_K) = \langle a, b \mid w^n a = b w^n \rangle$ where a, b are meridians and $w = (ba^{-1})^m (b^{-1}a)^m$, see [3]. Suppose $\rho : \pi_1(X_K) \to SL_2(\mathbf{C})$ is a nonabelian representation. Up to conjugation, we may assume that

$$\rho(a) = \begin{bmatrix} s & 1 \\ 0 & s^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} s & 0 \\ 2 - y & s^{-1} \end{bmatrix}$$

where $s \neq 0$ and $y \neq 2$ satisfy $\rho(w^n a) = \rho(b w^n)$. By [16], this matrix equation is equivalent to the Riley equation $\phi_K(s, y) = 0$. We now compute $\phi_K(s, y)$.

Since $\rho(ba^{-1}) = \begin{bmatrix} 1 & -s \\ s^{-1}(2 - y) & y - 1 \end{bmatrix}$ and $y = \operatorname{tr} \rho(ba^{-1})$, by Lemma 2.3 we have

$$\rho((ba^{-1})^m) = \begin{bmatrix} S_m(y) - (y - 1)S_{m-1}(y) & -sS_{m-1}(y) \\ s^{-1}(2 - y)S_{m-1}(y) & S_m(y) - S_{m-1}(y) \end{bmatrix}.$$

Similarly

$$\rho((b^{-1}a)^m) = \begin{bmatrix} S_m(y) - (y - 1)S_{m-1}(y) & s^{-1}S_{m-1}(y) \\ s(y - 2)S_{m-1}(y) & S_m(y) - S_{m-1}(y) \end{bmatrix}.$$

Hence $\rho(w) = \rho((ba^{-1})^m (b^{-1}a)^m) = \begin{bmatrix} w_{11} & w_{12} \\ (2 - y)w_{12} & w_{22} \end{bmatrix}$ where

$$w_{11} = S_m^2(y) + (2 - 2y)S_m(y)S_{m-1}(y) + (1 + 2s^2 - 2y - s^2 y + y^2)S_{m-1}^2(y),$$

$$w_{12} = (s^{-1} - s)S_m(y)S_{m-1}(y) + (s^{-1} + s - s^{-1}y)S_{m-1}^2(y),$$

$$w_{22} = S_m^2(y) - 2S_m(y)S_{m-1}(y) + (1 + 2s^{-2} - s^{-2}y)S_{m-1}^2(y).$$

Let $z = \operatorname{tr} \rho(w)$. Since $S_m^2(y) - yS_m(y)S_{m-1}(y) + S_{m-1}^2(y) = 1$ (by Lemma 2.1), we have

$$\begin{aligned} z = w_{11} + w_{22} &= 2(S_m^2(y) - yS_m(y)S_{m-1}(y) + S_{m-1}^2(y)) \\ &\quad + (2s^2 + 2s^{-2} - 2y - s^2y - s^{-2}y + y^2)S_{m-1}^2(y) \\ &= 2 + (y-2)(y - s^2 - s^{-2})S_{m-1}^2(y). \end{aligned}$$

By Lemma 2.3 we have $\rho(w^n) = \begin{bmatrix} S_n(z) - w_{22}S_{n-1}(z) & w_{12}S_{n-1}(z) \\ (2-y)w_{12}S_{n-1}(z) & S_n(z) - w_{11}S_{n-1}(z) \end{bmatrix}$. Hence

$$\rho(w^n a - bw^n) = \begin{bmatrix} 0 & \phi_K(s,y) \\ (2-y)\phi_K(s,y) & 0 \end{bmatrix}$$

where $\phi_K(s,y)$

$$\begin{aligned} &= S_n(z) - \{(s - s^{-1})w_{12} + w_{22}\}S_{n-1}(z) \\ &= S_n(z) - \{S_m^2(y) - (s^2 + s^{-2})S_m(y)S_{m-1}(y) + (1 + s^2 + s^{-2} - y)S_{m-1}^2(y)\}S_{n-1}(z) \\ &= S_n(z) - \{1 + (y - s^2 - s^{-2})S_{m-1}(y)(S_m(y) - S_{m-1}(y))\}S_{n-1}(z). \end{aligned}$$

Remark 2.4. Similar formulas for $\phi_K(s,y)$ were already obtained in [11, 15].

2.3. Trace of the longitude. By [3] the canonical longitude of $K = J(2m, 2n)$ corresponding to the meridian $\mu = a$ is $\lambda = \overleftarrow{w}^n w^n$, where $\overleftarrow{w}$ is the word in the letters a, b obtained by writing w in the reversed order. We now compute its trace. This computation will be used in the proof of Theorem 3.

Let $\alpha = 1 + (y - s^2 - s^{-2})S_{m-1}(y)(S_m(y) - S_{m-1}(y))$.

LEMMA 2.5. *We have*

$$\alpha^2 - z\alpha + 1 = (y - s^2 - s^{-2})S_{m-1}^2(y)(2 - s^2 - s^{-2} + (y - s^2 - s^{-2})(y-2)S_{m-1}^2(y)).$$

Proof. By a direct calculation we have

$$\begin{aligned} \alpha^2 - z\alpha + 1 = (y - s^2 - s^{-2})S_{m-1}^2(y)\{2 - y + (y - s^2 - s^{-2}) \\ (S_m^2(y) - yS_m(y)S_{m-1}(y) + (y-1)S_{m-1}^2(y))\}. \end{aligned}$$

The lemma follows, since $S_m^2(y) - yS_m(y)S_{m-1}(y) + S_{m-1}^2(y) = 1$. □

LEMMA 2.6. *We have*

$$S_{n-1}^2(z) = \{(y - s^2 - s^{-2})S_{m-1}^2(y)(2 - s^2 - s^{-2} + (y - s^2 - s^{-2})(y-2)S_{m-1}^2(y))\}^{-1}.$$

Proof. Since $s \neq 0$ and $y \neq 2$ satisfy the Riley equation $\phi_K(s,y) = 0$, we have $S_n(z) = \alpha S_{n-1}(z)$. Hence

$$1 = S_n^2(z) - zS_n(z)S_{n-1}(z) + S_{n-1}^2(z) = (\alpha^2 - z\alpha + 1)S_{n-1}^2(z).$$

The lemma then follows from Lemma 2.5. □

Proposition 2.7. *We have*

$$\operatorname{tr}\rho(\lambda) = 2 - \frac{(s+s^{-1})^2(y-2)^2 S_{m-1}^2(y)}{2 - s^2 - s^{-2} + (y - s^2 - s^{-2})(y-2)S_{m-1}^2(y)}.$$

Proof. We have $\rho(\overleftarrow{w}) = \begin{bmatrix} \overleftarrow{w}_{11} & \overleftarrow{w}_{12} \\ (2-y)\overleftarrow{w}_{12} & \overleftarrow{w}_{22} \end{bmatrix}$ where

$$\overleftarrow{w}_{11} = S_m^2(y) - 2S_m(y)S_{m-1}(y) + (1 + 2s^2 - s^2 y)S_{m-1}^2(y),$$
$$\overleftarrow{w}_{12} = (s - s^{-1})S_m(y)S_{m-1}(y) + (s^{-1} + s - sy)S_{m-1}^2(y),$$
$$\overleftarrow{w}_{22} = S_m^2(y) + (2-2y)S_m(y)S_{m-1}(y) + (1 + 2s^{-2} - 2y - s^{-2}y + y^2)S_{m-1}^2(y).$$

By Lemma 2.3 we have

$$\rho(\overleftarrow{w}^n) = \begin{bmatrix} S_n(z) - \overleftarrow{w}_{22}S_{n-1}(z) & \overleftarrow{w}_{12}S_{n-1}(z) \\ (2-y)\overleftarrow{w}_{12}S_{n-1}(z) & S_n(z) - \overleftarrow{w}_{11}S_{n-1}(z) \end{bmatrix}.$$

By a direct calculation, using $S_m^2(y) - yS_m(y)S_{m-1}(y) + S_{m-1}^2(y) = 1$, we have

$$\begin{aligned} \operatorname{tr}\rho(\lambda) &= \operatorname{tr}(\rho(\overleftarrow{w}^n)\rho(w^n)) \\ &= 2S_n^2(z) - 2\{2 + (y-2)(y - s^2 - s^{-2})S_{m-1}^2(y)\}S_n(z)S_{n-1}(z) \\ &\quad + \{2 - (s+s^{-1})^2(y-2)^2(y - s^2 - s^{-2})S_{m-1}^4(y)\}S_{n-1}^2(z) \\ &= 2 - (s+s^{-1})^2(y-2)^2(y - s^2 - s^{-2})S_{m-1}^4(y)S_{n-1}^2(z). \end{aligned}$$

The lemma then follows from Lemma 2.6. □

3. Twisted Alexander polynomial and Reidemeister torsion

In this section we briefly review the twisted Alexander polynomial and the Reidemeister torsion of a knot. For more details, see [10, 19, 2, 13, 4, 12, 18].

3.1. Twisted Alexander polynomial of a knot. Let L be a knot in S^3 and $X_L = S^3 \backslash L$ its complement. We choose a Wirtinger presentation for the knot group of L:

$$\pi_1(X_L) = \langle a_1, \dots, a_l \mid r_1, \dots, r_{l-1} \rangle.$$

The abelianization homomorphism $f : \pi_1(X_L) \to H_1(X_L; \mathbf{Z}) \cong \mathbf{Z} = \langle t \rangle$ is given by $f(a_1) = \cdots = f(a_l) = t$. Here we specify a generator t of $H_1(X_L; \mathbf{Z})$ and denote the sum in $\mathbf{Z}$ multiplicatively.

Let $\rho : \pi_1(X_L) \to SL_2(\mathbf{C})$ be a representation. The maps ρ and f naturally induce two ring homomorphisms $\tilde{\rho} : \mathbf{Z}[\pi_1(X_L)] \to M_2(\mathbf{C})$ and $\tilde{f} : \mathbf{Z}[\pi_1(X_L)] \to \mathbf{Z}[t^{\pm 1}]$ respectively, where $\mathbf{Z}[\pi_1(X_L)]$ is the group ring of $\pi_1(X_L)$ and $M_2(\mathbf{C})$ is the matrix algebra of degree 2 over $\mathbf{C}$. Then $\Phi := \tilde{\rho} \otimes \tilde{f}$ defines a ring homomorphism $\mathbf{Z}[\pi_1(X_L)] \to M_2(\mathbf{C}[t^{\pm 1}])$.

Consider the $(l-1) \times l$ matrix A whose (i,j)-component is the 2×2 matrix

$$\Phi\left(\frac{\partial r_i}{\partial a_j}\right) \in M_2(\mathbf{Z}[t^{\pm 1}]),$$

where $\partial/\partial a$ denotes the Fox's free calculus. For $1 \le j \le l$, denote by A_j the $(l-1) \times (l-1)$ matrix obtained from A by removing the jth column. We regard A_j as a $2(l-1) \times 2(l-1)$ matrix with coefficients in $\mathbf{C}[t^{\pm 1}]$. Then Wada's twisted Alexander polynomial [19] of a knot L associated to a representation $\rho : \pi_1(X_L) \to SL_2(\mathbf{C})$ is defined to be

$$\Delta_{L,\rho}(t) = \frac{\det A_j}{\det \Phi(a_j - 1)}.$$

Note that $\Delta_{L,\rho}(t)$ is well-defined up to a factor t^{2k} $(k \in \mathbf{Z})$.

3.2. Torsion of a chain complex. Let C be a chain complex of finite dimensional vector spaces over $\mathbf{C}$:

$$C = (0 \longrightarrow C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0)$$

such that for each $i = 0, 1, \dots, m$ the followings hold

- the homology group $H_i(C)$ is trivial, and
- a preferred basis c_i of C_i is given.

Let $B_i \subset C_i$ be the image of ∂_{i+1}. For each i choose a basis b_i of B_i. The short exact sequence of $\mathbf{C}$-vector spaces

$$0 \to B_i \to C_i \xrightarrow{\partial_i} B_{i-1} \to 0$$

implies that a new basis of C_i can be obtained by taking the union of the vectors of b_i and some lifts $\tilde{b}_{i-1}$ of the vectors b_{i-1}. Define $[(b_i \cup \tilde{b}_{i-1})/c_i]$ to be the determinant of the matrix expressing $(b_i \cup \tilde{b}_{i-1})$ in the basis c_i. Note that this scalar does not depend on the choice of the lift $\tilde{b}_{i-1}$ of b_{i-1}.

The torsion of C is defined to be

$$\tau(C) := \prod_{i=0}^{m} [(b_i \cup \tilde{b}_{i-1})/c_i]^{(-1)^{i+1}} \in \mathbf{C} \backslash \{0\}.$$

Remark 3.1. Once a preferred basis of C is given, the torsion $\tau(C)$ is independent of the choice of $b_0, \dots, b_m$.

3.3. Reidemeister torsion of a CW-complex. Let M be a finite CW-complex and $\rho : \pi_1(M) \to SL_2(\mathbf{C})$ a representation. Denote by $\tilde{M}$ the universal covering of M. The fundamental group $\pi_1(M)$ acts on $\tilde{M}$ as deck transformations. Then the chain complex $C(\tilde{M}; \mathbf{Z})$ has the structure of a chain complex of left $\mathbf{Z}[\pi_1(M)]$-modules.

Let V be the 2-dimensional vector space $\mathbf{C}^2$ with the canonical basis $\{e_1, e_2\}$. Using the representation ρ, V has the structure of a right $\mathbf{Z}[\pi_1(M)]$-module which we denote by V_ρ. Define the chain complex $C(M; V_\rho)$ to be $V_\rho \otimes_{\mathbf{Z}[\pi_1(M)]} C(\tilde{M}; \mathbf{Z})$, and choose a preferred basis of $C(M; V_\rho)$ as follows. Let $\{u_1^i, \dots, u_{m_i}^i\}$ be the set of i-cells of M, and choose a lift $\tilde{u}_j^i$ of each cell. Then $\{e_1 \otimes \tilde{u}_1^i, e_2 \otimes \tilde{u}_1^i, \dots, e_1 \otimes \tilde{u}_{m_i}^i, e_2 \otimes \tilde{u}_{m_i}^i\}$ is chosen to be the preferred basis of $C_i(M; V_\rho)$.

The Reidemeister torsion $\tau_\rho(M)$ is defined as follows:

$$\tau_\rho(M) = \begin{cases} \tau(C(M; V_\rho)) & \text{if } \rho \text{ is acyclic,} \\ 0 & \text{otherwise.} \end{cases}$$

Here a representation ρ is called acyclic if all the homology groups $H_i(M; V_\rho)$ are trivial.

For a knot L in S^3 and a representation $\rho : \pi_1(X_L) \to SL_2(\mathbf{C})$, the Reidemeister torsion $\tau_\rho(L)$ of L is defined to be that of the knot complement X_L.

The following result which relates the Reidemeister torsion and the twisted Alexander polynomial of a knot is due to Johnson.

Theorem 3.2 ([4]). *Let* $\rho : \pi_1(X_L) \to SL_2(\mathbf{C})$ *be a representation such that* $\det(\rho(\mu) - I) \neq 0$, *where* μ *is a meridian of* L. *Then the Reidemeister torsion of* L *is given by*

$$\tau_\rho(L) = \Delta_{L,\rho}(1).$$

4. Proof of main results

4.1. Proof of Theorem 1. Recall that $K = J(2m, 2n)$ and $\pi_1(X_K) = \langle a, b \mid w^n a = b w^n \rangle$, where a, b are meridians and $w = (ba^{-1})^m (b^{-1}a)^m$.

Let $r = w^n a w^{-n} b^{-1}$. We have $\Delta_{K,\rho}(t) = \det \Phi\left(\frac{\partial r}{\partial a}\right) / \det \Phi(b - 1)$. It is easy to see that $\det \Phi(b - 1) = t^2 - t(s + s^{-1}) + 1 = t^2 - tx + 1$.

For an integer k and a word u (in 2 letters a, b), let $\delta_k(u) = 1 + u + \cdots + u^k$.

Lemma 4.1. *We have*

$$\frac{\partial r}{\partial a} = w^n \left(1 + (1 - a)\delta_{n-1}(w^{-1}) w^{-1} \frac{\partial w}{\partial a}\right)$$

where

$$w^{-1} \frac{\partial w}{\partial a} = (a^{-1}b)^m (\delta_{m-1}(b^{-1}a) b^{-1} - \delta_{m-1}(ab^{-1})).$$

Proof. The lemma follows from direct calculations. □

Let

$$\Omega_1 = \rho(\delta_{n-1}(w^{-1})(a^{-1}b)^m),$$
$$\Omega_2 = \{t^{-1}\rho(\delta_{m-1}(b^{-1}a)b^{-1}) - \rho(\delta_{m-1}(ab^{-1}))\}(I - t\rho(a)).$$

Then by Lemma 4.1 we have

$$\det \Phi\left(\frac{\partial r}{\partial a}\right) = \det(I + \Omega_1\Omega_2) = 1 + \operatorname{tr}(\Omega_1\Omega_2) + \det(\Omega_1\Omega_2).$$

LEMMA 4.2. *We have*

$$\Omega_1 = \begin{bmatrix} \beta P_{n-1}(z) - \gamma P_{n-2}(z) & -S_{m-1}(y)(s^{-1}P_{n-1}(z) - sP_{n-2}(z)) \\ (2-y)S_{m-1}(y)(sP_{n-1}(z) - s^{-1}P_{n-2}(z)) & \gamma P_{n-1}(z) - \beta P_{n-2}(z) \end{bmatrix}$$

where $\beta = S_m(y) - S_{m-1}(y)$ *and* $\gamma = S_m(y) - (y-1)S_{m-1}(y)$.

Proof. By Lemma 2.3 we have

$$\rho((a^{-1}b)^m) = \begin{bmatrix} S_m(y) - S_{m-1}(y) & -s^{-1}S_{m-1}(y) \\ -s(y-2)S_{m-1}(y) & S_m(y) - (y-1)S_{m-1}(y) \end{bmatrix}$$

and

$$\rho(\delta_{n-1}(w^{-1})) = \begin{bmatrix} P_{n-1}(z) - w_{11}P_{n-2}(z) & -w_{12}P_{n-2}(z) \\ (y-2)w_{12}P_{n-2}(z) & P_{n-1}(z) - w_{22}P_{n-2}(z) \end{bmatrix}.$$

The lemma then follows by a direct calculation. □

LEMMA 4.3. *We have*

$$\Omega_2 = \begin{bmatrix} (st + s^{-1}t^{-1} - 2)(P_{m-1}(y) - P_{m-2}(y)) & (t - s^{-1})P_{m-1}(y) + (t^{-1} - s)P_{m-2}(y) \\ (2-y)(st-1)(t^{-1}P_{m-1}(y) - s^{-1}P_{m-2}(y)) & (s^{-1}t + st^{-1} - y)(P_{m-1}(y) - P_{m-2}(y)) \end{bmatrix}.$$

Moreover

$$\det \Omega_2 = (t + t^{-1} - x)^2\left(\frac{S_m(y) - S_{m-2}(y) - 2}{y-2}\right).$$

Proof. By Lemma 2.3 we have

$$\rho(\delta_{m-1}(ab^{-1})) = \begin{bmatrix} P_{m-1}(y) - P_{m-2}(y) & sP_{m-2}(y) \\ s^{-1}(y-2)P_{m-2}(y) & P_{m-1}(y) - (y-1)P_{m-2}(y) \end{bmatrix},$$

$$\rho(\delta_{m-1}(b^{-1}a)) = \begin{bmatrix} P_{m-1}(y) - (y-1)P_{m-2}(y) & s^{-1}P_{m-2}(y) \\ s(y-2)P_{m-2}(y) & P_{m-1}(y) - P_{m-2}(y) \end{bmatrix}.$$

The formula for Ω_2 then follows by a direct calculation. The one for $\det \Omega_2$ is obtained by using the formula $P_k(y) = \dfrac{S_{k+1}(y) - S_k(y) - 1}{y-2}$ in Lemma 2.2. □

We now complete the proof of Theorem 1 by computing the determinant and the trace of the matrix $\Omega_1\Omega_2$. By Lemma 2.3 we have $\det \Omega_1 = \dfrac{S_n(z) - S_{n-2}(z) - 2}{z-2}$. Hence

$$\det(\Omega_1\Omega_2) = (t + t^{-1} - x)^2 \left(\frac{S_n(z) - S_{n-2}(z) - 2}{z-2}\right)\left(\frac{S_m(y) - S_{m-2}(y) - 2}{y-2}\right). \tag{4.1}$$

By a direct calcultion, using the matrix forms of Ω_1 and Ω_2 in Lemmas 4.2 and 4.3 and the formula $P_k(y) = \dfrac{S_{k+1}(y) - S_k(y) - 1}{y-2}$, we have

$$\begin{aligned}
\operatorname{tr}(\Omega_1\Omega_2) &= \{(t + t^{-1})x - x^2 + (x^2 - 2 - y)(S_m(y) - (y-1)S_{m-1}(y))\} \\
&\quad \times S_{m-1}(y)(P_{n-1}(z) - P_{n-2}(z)) + (2-y)(x^2 - 2 - y)S_{m-1}^2(y)P_{n-2}(z) \\
&= \{(t + t^{-1})x - x^2 + (x^2 - 2 - y)(S_m(y) - (y-1)S_{m-1}(y))\} \\
&\quad \times S_{m-1}(y)S_{n-1}(z) + (z-2)P_{n-2}(z) \\
&= \{(t + t^{-1})x - x^2 + (x^2 - 2 - y(S_m(y) - (y-1)S_{m-1}(y))\} \\
&\quad \times S_{m-1}(y)S_{n-1}(z) + S_{n-1}(z) - S_{n-2}(z) - 1.
\end{aligned}$$

Since $S_{n-2}(z) = \{1 - (y + 2 - x^2)S_{m-1}(y)(S_{m-1}(y) - S_{m-2}(y))\}S_{n-1}(z)$ we get

$$\operatorname{tr}(\Omega_1\Omega_2) = ((t + t^{-1})x - x^2)S_{m-1}(y)S_{n-1}(z) - 1. \tag{4.2}$$

Finally, by combining the equations (4.1), (4.2) and

$$\Delta_{K,\rho}(t) = \frac{1 + \operatorname{tr}(\Omega_1\Omega_2) + \det(\Omega_1\Omega_2)}{t^2 - tx + 1}$$

we obtain Theorem 1, since $\Delta_{K,\rho}(t)$ is defined up to multiplication by a factor t^{2k} $(k \in \mathbf{Z})$.

4.2. Proof of Theorem 2. Note that $\det(\rho(b) - I) = 2 - x$. Since $\tau_\rho(K) = \Delta_{K,\rho}(1)$ for $x \neq 2$, Theorem 2 follows directly from Theorem 1.

4.3. Proof of Theorem 3. Let M be the 3-manifold obtained by $\frac{1}{q}$-surgery on the genus one two-bridge knot $K = J(2m, 2n)$. Suppose $\rho : \pi_1(X_K) \to SL_2(\mathbf{C})$ is a nonabelian representation which extends to a representation $\rho : \pi_1(M) \to SL_2(\mathbf{C})$. Recall that λ is the canonical longitude corresponding to the meridian $\mu = a$. If $\operatorname{tr} \rho(\lambda) \neq 2$, then by [6] (see also [7, 8]) the Reidemeister torsion of M is given by

$$\tau_\rho(M) = \frac{\tau_\rho(K)}{2 - \operatorname{tr} \rho(\lambda)}. \tag{4.3}$$

By Theorem 2 we have

$$\tau_\rho(K) = (2-x)\left(\frac{S_m(y) - S_{m-2}(y) - 2}{y-2}\right)\left(\frac{S_n(z) - S_{n-2}(z) - 2}{z-2}\right) + xS_{m-1}(y)S_{n-1}(z)$$

if $x \neq 2$. By Proposition 2.7 we have

$$\operatorname{tr}\rho(\lambda) - 2 = -\frac{x^2(y-2)^2 S_{m-1}^2(y)}{4 - x^2 + (y+2-x^2)(y-2)S_{m-1}^2(y)}.$$

By Lemma 2.6 we have $S_{m-1}(y) \neq 0$. This implies that $\operatorname{tr}\rho(\lambda) \neq 2$ if and only if $x \neq 0$. Theorem 3 then follows from (4.3).

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